MSAS – Assignment #2: Modeling

Niccolò Bardazzi, 10800456

1 Questions

Question 1

1) List the stages of dynamic investigation and their meaning. 2) When going from the *real system* to the *physical model* a number of assumptions are made; report the most important ones along with their mathematical implications. 3) For each of the assumptions below, shortly state what sort of simplification may result: i) The gravity torque on a pendulum is taken proportional to the pendulum angle θ; ii) Only wind forces and gravity are assumed in studying the motion of an aircraft; iii) A temperature sensor is assumed to report the temperature exactly; iv) The pressure in a hydraulic actuator is assumed uniform throughout the chamber. 4) List the *effort* and *flow* variables for the domains treated and discuss their similarity.

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1) Stages of dynamic investigation:

- *Real system*: real-life system which behaves with an exact dynamics.
- *Physical model*: abstraction of the real system according to the engineering's perception.
- *Mathematical model*: system representation through algebraic and differential equations.

2) The most important implications from the *real system* to the *physical model* are:

- Neglecting small effects:
  - reduce the number of variables
  - reduce the number of equations
  - reduce the overall complexity of the system
- The environment surrounding is independent from system behaviour:
  - reduce the overall complexity of the system
- Lumping:
  - shifts the problem from PDEs to ODEs, which implies lower computational burden
  - reduce the overall complexity of the system
- Linearity:
  - reduce the overall complexity of the system
- Constant parameters:
  - reduce the overall complexity of the system
- Neglecting uncertainties and noise:
  - allows to use a deterministic approach

3-i) **Linearity assumption**: Gravity torque on a pendulum depends on $\sin\theta$, however performing a TSE till the 1^{st} order in the neighbourhood of the equilibrium point it is possible to approximate sine with a linear function.
ii) **Neglecting small effects**: Even if the motion of an aircraft is subjected to solar wind, gravity gradient or magnetic field, these effects are relatively smaller than wind and gravity forces in some order of magnitude, therefore they can be neglected.
iii) **Neglecting uncertainties and noise**: Temperature sensors have always a delay and in addition since they are sensors, they provide measures with variations due to the quality of the signal and its mechanical/electrical properties as well, therefore this assumption allows to use a deterministic approach neglecting the noise.
iv) **Lumping**: Liquids can be treated as incompressible fluids whenever the difference of pressure is not so remarked along the chamber, hence a constant pressure is a good approximation since it avoids PDEs in favour of ODEs, with almost no loss in accuracy.

4) All the systems can be represented by mean of two variables which are the effort and the flow. In Table 1 these two variables have been listed for all the domains.

Table 1: Systems similarities.

Domain	Effort	Flow
Mechanical	F	v
Fluid	p	$\dot{Q}$
Electrical	V	i
Thermal	T	Q

This system similarity is helpful to describe the resistance, capacitance and inductance of every specific domain. In particular the definition of these 3 elements has to obey to the general rules:

$$effort = resistance \times flow \tag{1}$$

$$flow = capacitance \times \frac{d}{dt}(effort) \tag{2}$$

$$effort = inductance \times \frac{d}{dt}(flow) \tag{3}$$

It has to be noticed that thermal domain is the only exception because it does not respect Equation 3, hence no inductance element is defined for this domain.

Question 2

1) Briefly discuss the physical meaning of the bulk modulus; show how the *effective* bulk modulus is computed. 2) Under what circumstances the fluid resistance yields a linear relation between effort and flow variables? 3) Find the expression of the leakage through a thin annular gap starting from the balance between shear stress and pressure drop.

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1) The bulk modulus $\beta = \rho_0 \left(\frac{\partial p}{\partial \rho}\right)_{p_0,T_0} = -\frac{1}{V_0}\left(\frac{\partial V_0}{\partial p}\right)_{p_0,T_0}$ represents the capacity of the fluid to be compressed (fluid's elasticity). The effective bulk modulus adds also the effect container's and trapped gas' elasticity:

$$\frac{1}{\beta_e} = \frac{1}{\beta_l} + \frac{V_g}{V_t}\frac{1}{\beta_g} + \frac{1}{\beta_c}$$

where:

- β_g : bulk modulus of gas
- β_l : bulk modulus of liquid
- β_c : bulk modulus of container

2) The fluid resistance yields a linear relation only for laminar regions of the flow:, where the friction coefficient is: $f = \frac{64}{Re}$ with $Re = \frac{vD\rho}{\mu}$. Therefore since distributed losses have been defined as: $\Delta p = \frac{1}{2}\frac{L}{D}\rho v^2 f$ the relation between effort and flow becomes linear. On the other hand, from transition on f starts depending always more on the relative roughness of the surface, thus this linear dependency starts losing its value.

3) Balance condition:

$$2r\pi D\,\Delta p = -2L\pi D\mu\frac{dv}{dr}$$

By integrating from r to $\frac{g}{4}$ and knowing that $v(\frac{g}{4}) = 0$:

$$\int_r^{\frac{g}{4}} \frac{2r\pi D\,\Delta p}{2L\pi D\mu}dr = -\int_r^{\frac{g}{4}} dv \quad \Rightarrow \quad v(r) = \frac{\Delta p}{2L\mu}\left(\frac{g^2}{16} - r^2\right)$$

At the end, the total flow has been found integrating $dQ = v(r)\pi D\,dy$:

$$Q = \int_{-\frac{g}{4}}^{\frac{g}{4}} dQ = \frac{\pi D g^3}{96L\mu}\Delta p$$

Question 3

1) Derive from scratch the mathematical model for RC and RL circuits and express the system response in closed form. 2) Consider a real DC motor and a) sketch its physical model (list the assumptions made); b) derive its mathematical model; c) show how the motor constant depends on the physical parameters.

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1) **RC**

$$i = \frac{V_0 - V_C}{R} \quad \text{and} \quad i = C\frac{dV_C}{dt}$$

Substituting:

$$\frac{dV_C}{dt} + \frac{1}{RC}V_C = \frac{V_0}{RC} \quad \text{whose solution is:} \quad V_C = V_0(1 - e^{-\frac{t}{RC}})$$

RL

$$i = \frac{V_0 - V_L}{R} \quad \text{and} \quad V_L = L\frac{di}{dt}$$

Substituting:

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_0}{L} \quad \text{whose solution is:} \quad i = \frac{V_0}{R}(1 - e^{-t\frac{R}{L}})$$

2)

a) A reasonable physical model, as sketched in Figure 1, consists of representing an armature coil, through which a current i flows, as a resistor and an inductance plus an additional voltage drop proportional to the angular velocity ω_m.

b-c) Since $\mathbf{i}$ is always perpendicular to $\mathbf{B}$, then $F_e = ilB$. Applying F_e twice on each winding, the torque is $T = 2NF_er = Ki$, where $K = 2NlBr$. While since also the velocity $\mathbf{u}$ is perpendicular to $\mathbf{B}$, the back electromotive force $e_m = \omega_m lBr$, applied on each winding: $e_m = 2NlBr\omega_m = K\omega_m$. The motor constant K is clearly dependent on the geometry of the motor: N, l, r but also on the physics of it: B. Using Kirchoff law and torque equilibrium the system has been modelled like:

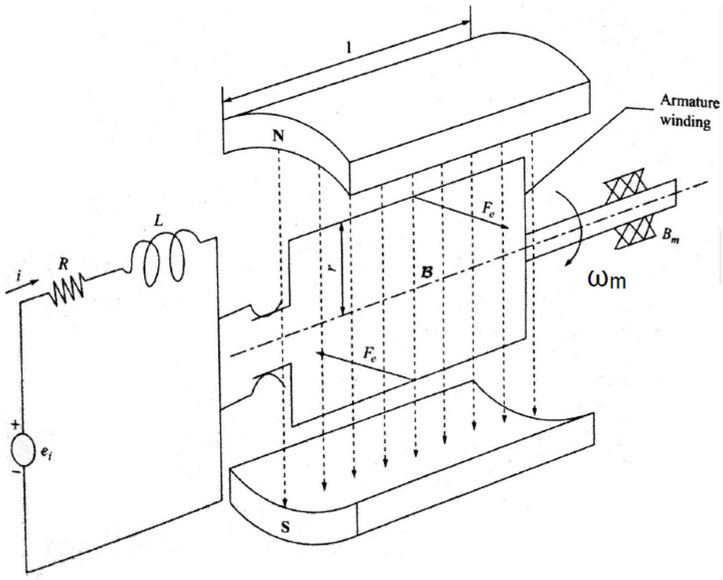

Figure 1: DC motor sketch.

$$\begin{cases} e_1 - e_2 = Ri \\ L\frac{di}{dt} = e_2 - e_m \\ J\dot{\omega}_m + b\omega_m = T_m - T_L \end{cases} \quad \text{with:} \quad \begin{cases} T_m = Ki \\ e_m = K\omega_m \end{cases}$$

$$\Downarrow$$

$$\begin{cases} L\frac{di}{dt} + Ri + K\omega_m = e_1 \\ J\dot{\omega}_m + b\omega_m - Ki = -T_L \end{cases}$$

Question 4

1) Write down the Fourier law and show how it is specialized in the case of conduction through a thin plate; discuss the concept of thermal resistance. 2) Report the equation for thermal radiation in case of a) black body and b) real body and discuss them.

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1) Fourier equation:

$$\frac{Q}{A} = -k(T)\nabla T$$

where A is the cross sectional area. Assuming thermal flow along 1 direction only (x), the Fourier law becomes:

$$\frac{Q}{A} = -k(T)\frac{dT}{dx}$$

Taking L as the thickness of the plate and supposing also k(T) as constant with respect to T and Q as constant along x:

$$\int_0^L \frac{Q}{A}\,dx = -\int_{T_1}^{T_2} k(T)\,dT \quad \Rightarrow \quad \frac{Q}{A}L = -k(T_2 - T_1)$$

The thermal resistance R is the coefficient that multiplied by the flow (Q: heat transferred) gives the effort (T: temperature). In this case, $R = \frac{L}{Ak}$ but for the thermal domain it strongly depends on the geometry of the object passed by the heat, e.g. for cylindrical objects: $R = \frac{\ln\left(\frac{r_{out}}{r_{in}}\right)}{2\pi k L}$.

2) Thermal radiation equations for:
a) Black body:

$$Q = AF_v\sigma(T^4 - T_E^4) \tag{4}$$

b) Real body:

$$Q = AF_eF_v\sigma(T^4 - T_E^4) \tag{5}$$

where:

- F_v is the view factor
- σ is the Stefan-Boltzmann constant
- T_E is the temperature of the environment
- F_e is the emissivity factor

As highlighted by Equation 4 and Equation 5 the only difference regards the emissivity factor which is equal to 1 for black body and $0 < F_e < 1$ for real bodies.

2 Exercises

Exercise 1

A miniaturized reaction wheel can be modeled as a couple of massive disks, connected with a flexible shaft (Figure 2). The first disk is driven by a rigid shaft, linked to an electric motor. The motor provides a variable torque, while the rigid shaft is subjected to viscous friction, due to motor internal mechanisms. At $t_0 = 0$, the motor provides the torque $T(t) = T_0$.
1) Write down the mathematical model from first principles. 2) Using the data given in the figure caption, and guessing a value for the flexible shaft stiffness k and the viscous friction coefficient b, compute the system response from t_0 to $t_f = 10$ s. 3) Two accelerometers are placed on the two disks recorded samples at 100 Hz, which were saved in the file `samples.txt`; the samples are affected by measurement noise. Determine the value of k and b that allows retracing the experimental data, so avoiding parametric errors.

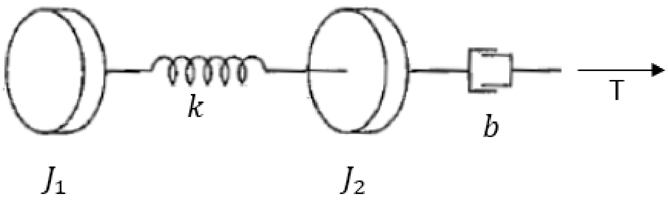

Figure 2: Physical model ($J_1 = 0.2\,\text{kg·m}^2$; $J_2 = 0.1\,\text{kg·m}^2$; $T_0 = 0.1\,\text{Nm}$).

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1) Using torque equilibrium:

$$\begin{cases} J_2\ddot{\theta}_2 = -k(\theta_2 - \theta_1) - sign(\dot{\theta}_2)b\dot{\theta}_2^{\,2} + T(t) \\ J_1\ddot{\theta}_1 = -k(\theta_1 - \theta_2) \end{cases} \quad \text{whose I.Cs.:} \quad \begin{cases} \theta_1(t_0) = 0 \\ \theta_2(t_0) = 0 \\ \dot{\theta}_1(t_0) = 0 \\ \dot{\theta}_2(t_0) = 0 \end{cases}$$

2) The guessed value for k has been derived from $k = \frac{GI_{xx}}{L}$ where it has been used aluminum and a bar with $L = 28\,\text{mm}, D = 50\,\mu\text{m}$, which leads to a $k = 22.77\,\text{N}\backslash\text{m}$, while the friction coefficient b has been supposed: $b = 4\,\text{kg·m}^2$. Figure 3 shows the system response, in particular it is possible to notice a delay of $\dot{\theta}_1$ at the very beginning with respect to $\dot{\theta}_2$ due to the inertia J_2 and then a series of oscillations in angular velocities because of the spring behaviour, however, at the steady-state the two disks rotate at the same angular velocity.

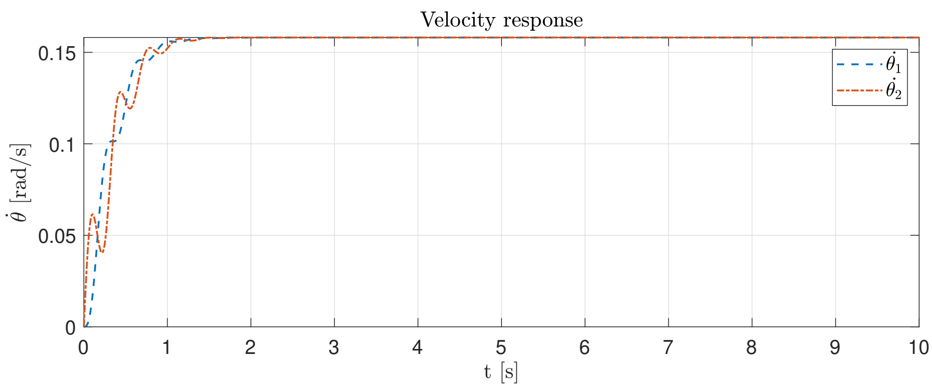

Figure 3: System response in velocity.

Since the problem does not involve two domains and J_1, J_2 have the same order of magnitude the whole system is not supposed to be stiff, for this reason it has been used the `ode113` integrator of `MATLAB`.

3) The retracing of k, b has been achieved through a minimization scheme solved with `fminsearch` and written as:

$$\min_{k,b} ||\mathbf{f}|| \quad \text{where:} \quad \mathbf{f} = \begin{bmatrix} ||\ddot{\theta}_1 - \ddot{\theta}_{sample_1}|| \\ ||\ddot{\theta}_2 - \ddot{\theta}_{sample_2}|| \end{bmatrix}$$

This time it has been preferred to perform the integration with an `ode15s` which is a stiff integrator with variable order method from 1 to 5. The reason is that the minimization function spans over an undefined domain, and the `ode113` does not work for all the k, b iterated with the initial guesses previously mentioned. At the end it has been achieved the minimum of $\mathbf{f}$ for $b = 2.7205\,\text{kg}\cdot\text{m}^2$ and $k = 3.1418\,\text{N}\backslash\text{m}$, indeed they make a nice match in accelerations as shown in Figure 4.

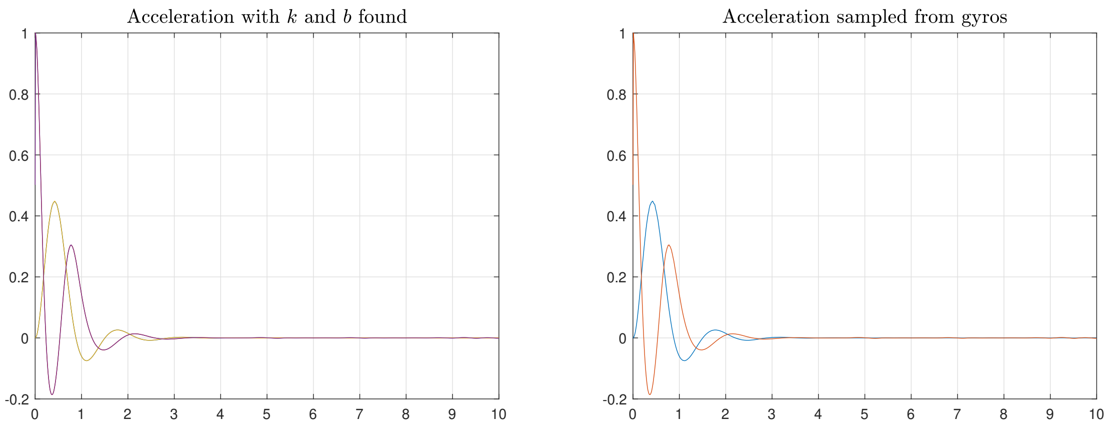

Figure 4: Comparison between accelerations achieved with k, b with the ones in `sample.txt`.

Exercise 2

The hydraulic system in Figure 5 consists of an accumulator, a check valve, a distributor, an actuator, and a tank, plus delivery and return lines. At $t = t_{-\infty}$, the accumulator contains nitrogen only. To charge it, the nitrogen undergoes an isothermal transformation from $\{p_{N_2}(t_{-\infty}), V_{N_2}(t_{-\infty})\}$ to $\{p_{N_2}(t_0) = p_0, V_{N_2}(t_0) = V_0\}$, t_0 being the initial time. 1) Assuming incompressible fluid, adiabatic discharge of the accumulator, and no leakage in the actuator, write down a mathematical model that allows computing pressures and flow rates in the sections labeled. 2) Considering the distributor command z in Figure 5, carry out a simulation to show the system response in $[t_0, t_f]$. 3) Determine the time t_e that takes the piston to reach the maximum stroke, $x(t_e) = x_{\max}$, starting from $x(t_0) = x_0$, $\dot{x}(t_0) = v_0$.

- Fluid: Skydrol, $\rho = 890$ kg/m^3.
- Accumulator: $V_{N_2}(t_{-\infty}) = 10$ dm^3, $p_{N_2}(t_{-\infty}) = 2.5$ MPa, $p_0 = 21$ MPa, adiabatic exponent $\gamma = 1.2$.
- Delivery: Coefficient of pressure drop[1] at accumulator outlet $k_A = 1.12$, coefficient of pressure drop across the check valve $k_{cv} = 2$, diameter of the delivery line $D_{23} = 18$ mm; Branch 2–3: Length $L_{23} = 2$ m, friction factor[2] $f_{23} = 0.032$.
- Distributor: Coefficient of pressure drop across the distributor $k_d = 12$, circular cross section, diameter $d_o = 5$ mm.
- Actuator: Diameter of the cylinder $D_c = 50$ mm, diameter of the rod $D_r = 22$ mm, mass of the piston $m = 2$ kg, maximum stroke $x_{\max} = 200$ mm; Load: $F(x) = F_0 + kx$, $F_0 = 1$ kN, $k = 120$ kN/m.
- Return: Diameter of the return line $D_r = 18$ mm; Branch 6–7: Length $L_{67} = 15$ m, friction factor $f_{67} = 0.035$.
- Tank: Pressure $p_T = 0.1$ MPa, initial volume $V_T(t_0) = 1$ dm^3, coefficient of pressure drop at tank inlet $k_T = 1.12$.
- Initial time: $t_0 = 0$, $x_0 = 0$, $v_0 = 0$; $t_1 = 1$ s, $t_2 = 1.5$ s; final time $t_f = 3$ s.

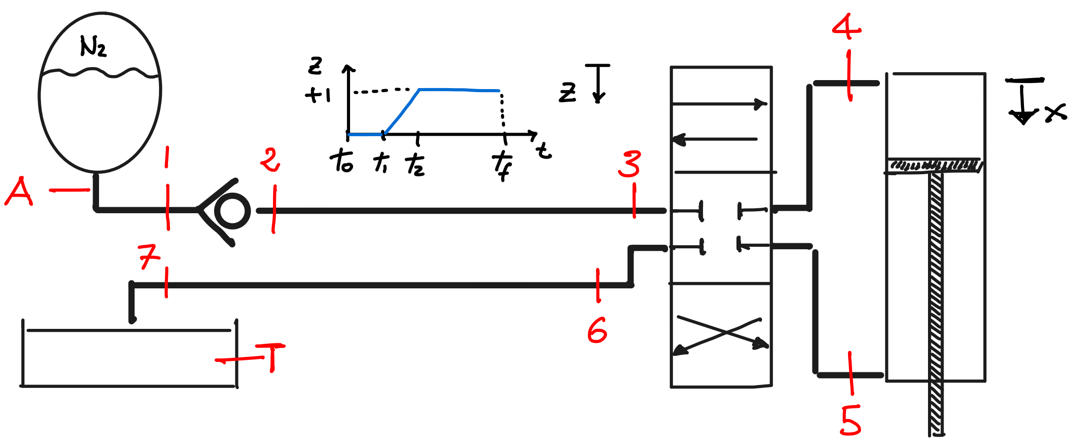

Figure 5: Hydraulic system physical model; assume any other missing data.

[1] $\Delta p = 1/2 k\rho v^2$.
[2] $\Delta p = fL/D 1/2\rho v^2$.

1) The initial volume of water has been computed through an isothermal transformation (charging), $PV = const.$:

$$V_{N_2}(t_0) = V_{N_2}(t_{-\infty})\frac{p_{N_2}(t_{-\infty})}{p_0}$$

Therefore:

$$V_{acc_0} = V_{N_2}(t_{-\infty}) - V_{N_2}(t_0)$$

while for all the other time instants the transformation is an adiabatic one (discharging), $PV^\gamma = const.$:

$$V_{N_2}(t_0) = V_{N_2}(t_{-\infty})\left(\frac{p_{N_2}(t_{-\infty})}{p_0}\right)^{\frac{1}{\gamma}}$$

The overall mathematical model has been written in the following form:

$$\begin{cases} \dot{x}_k = v_k \\ \dot{v}_k = -F_0 - k\,x_k + P_4\,A_c - P_5\,(A_c - A_r) \\ \dot{V}_{acc} = Q_{acc} \\ \dot{V}_T = v_k(A_c - A_r) \end{cases} \quad \text{with:} \quad \begin{cases} x(t_1) = 0 \\ v(t_1) = 0 \\ V_{acc}(t_1) = V_{acc_0} \\ V_T(t_1) = V_T(t_0) \end{cases}$$

with controls on velocity and acceleration at the maximum stroke and at the minimum stroke as well.

2-3)

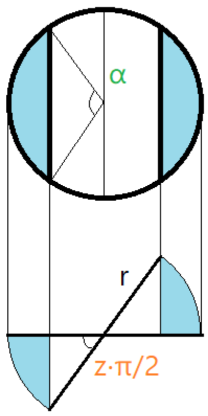

Figure 6: Butterfly valve model.

The valve has been modelled like a butterfly valve with the control directly acting on the angle of opening: $z{\cdot}\pi/2$ (Figure 6), where z is the linear ramp shown in Figure 5. From geometry it is possible to find a relation between z and α:

$$r \cdot \cos z = r \cdot \left(1 - \cos\frac{\alpha}{2}\right)$$

Therefore:

$$\alpha = 2 \arccos\left(1 - \cos z\right)$$

The blue area of is the opened area of interest where the water is passing through:

$$A_{open} = \pi r^2 - 2\left(\frac{r^2\alpha}{2} - r^2 \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$$

Since the system is a combination of two domains, we are expecting the problem to be a stiff one, hence `ode15s` has been used as integration scheme. Indeed it has been also tested the explicit integrator `ode113` that takes too long, confirming that the system is not suited to be integrated explicitly. Controls have been set on position, velocity and acceleration in dependence of: maximum and minimum stroke, maximum and minimum values of the control variable and minimum opened area (threshold set at $10^{-9}\,\mathrm{m}^2$). In Figure 7 it has been shown the evolution of the state of the piston in time.

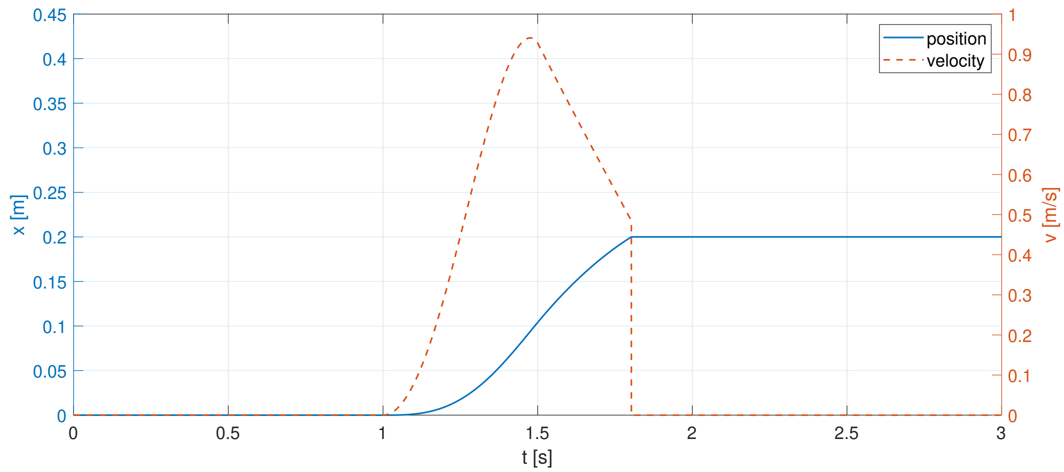

Figure 7: State of the piston in time.

After the peak in velocity which coincides with the end of the valve opening, no matter what the value of the elasticity of the spring is, velocity goes down because Δp_{45} starts increasing less (saddle point, Figure 8) due to the fact that the increment in pressure is given only by the decreasing in the velocity of the fluid inside pipes.

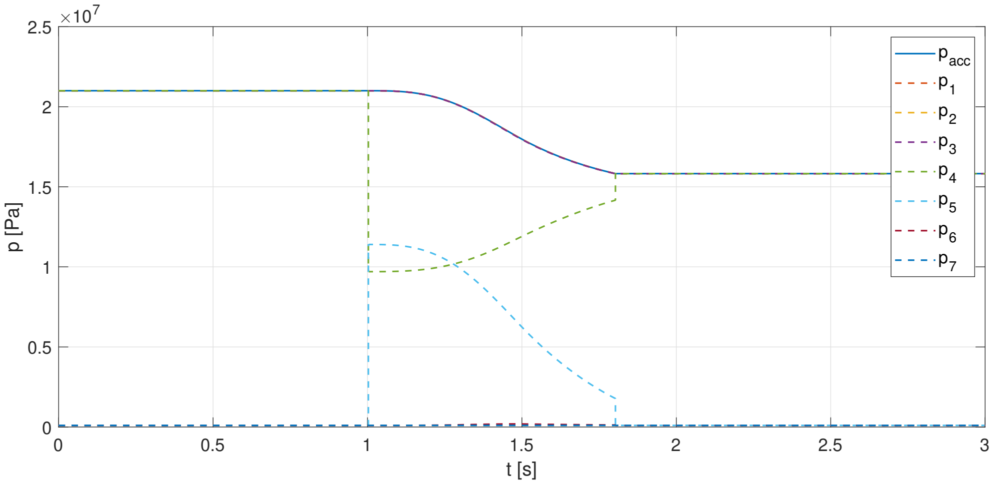

Figure 8: Pressure evolution in time.

As clearly visible from the two figures above, the time needed to reach the maximum stroke is $t = 1.803\,\mathrm{s}$

Exercise 3

Consider the ideal physical model network shown in Figure 9. The switch has been open for a long time. The capacitor is charged and has a voltage drop between its ends equal to 1 V. Then, at $t = 0$, the switch is closed. 1) Plot the subsequent time history of the voltage V_C across the capacitor. 2) Assume a voltage source characterized by $v(t) = \sin(2\pi f t)\arctan(t)$ having the positive terminal downward inserted in place of the switch. What is in this case the voltage history across the capacitor?

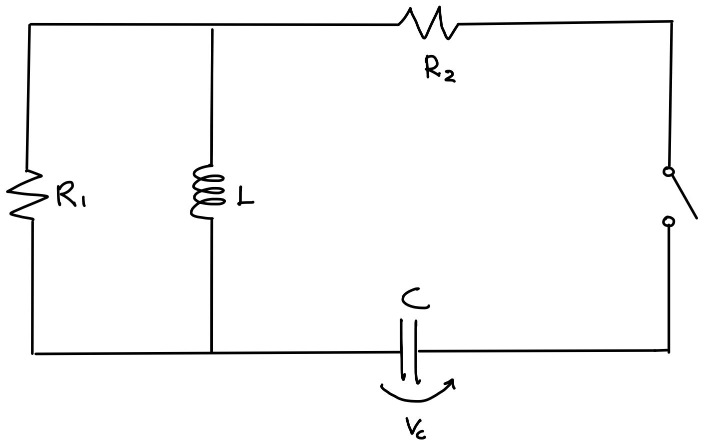

Figure 9: Circuit physical model ($R_1 = 1000\,\Omega$; $R_2 = 100\,\Omega$; $L = 1\,\text{mH}$; $C = 1\,\text{mF}$; $f = 5\,\text{Hz}$.)

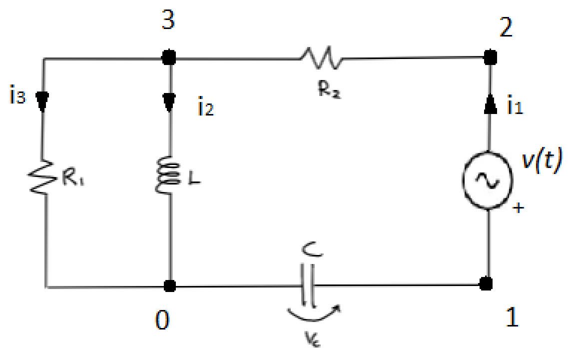

Figure 10: Circuit physical model.

1) In Figure 10 it has been drawn the general case with the generator $v(t)$ which in the case of simple discharging can be set equal to 0. The mathematical model has been written with voltages differences with respect to V_0 and it has been found imposing:

$$\begin{cases} i_1 = i_2 + i_3 \\ V_3 = L\frac{di_2}{dt} \\ V_3 = R_2 i_1 + V_2 \\ R_1 i_3 = V_2 + R_2 i_1 \\ i_1 = -C\frac{dV_C}{dt} \\ v(t) = V_C - V_2 \end{cases}$$

At the end it is possible to write the system in the linear form $\dot{x} = Ax + b$:

$$\begin{bmatrix} \dot{V}_C \\ \ddot{V}_C \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC+\frac{R_2LC}{R_1}} & -\frac{R_2C+\frac{L}{R_1}}{\frac{LC+R_2LC}{R_1}} \end{bmatrix} \begin{bmatrix} V_C \\ \dot{V}_C \end{bmatrix} + \begin{bmatrix} v(t)\frac{1}{LC+\frac{R_2LC}{R_1}} \\ -\frac{dv(t)}{dt}\frac{\frac{L}{R_1}}{LC+\frac{R_2LC}{R_1}} \end{bmatrix}$$

whose eigenvalues of A have been found in order to choose the proper integration scheme:

$$\lambda = \begin{bmatrix} -10.00 \\ -90900.00 \end{bmatrix}$$

The problem is clearly a stiff one, thus the integration has been performed with the `MATLAB` stiff integrator `ode15s`. As displayed in Figure 11 the capacitor is exhausting its voltage in a brief amount of time with an exponential decay.

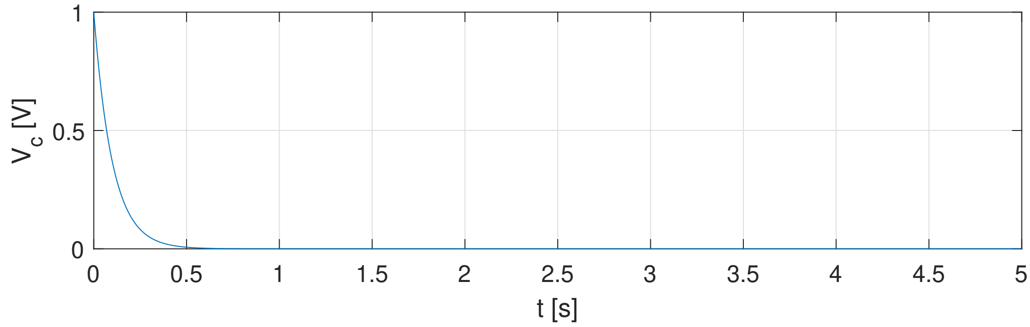

Figure 11: Capacitor discharge.

2) In Figure 12 it has been shows the behaviour of V_C with a voltage source $v(t) = \sin(2\pi f t)\arctan(t)$.

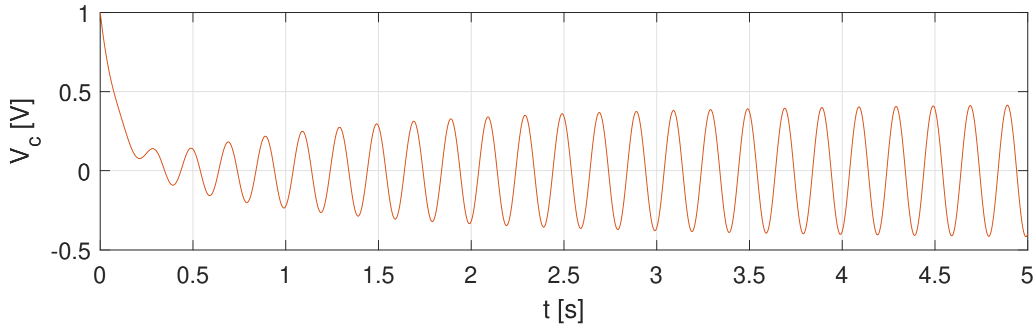

Figure 12: Voltage source $v(t) = \sin(2\pi f t)\arctan(t)$ and capacitor discharge.

It is clear that at the very beginning, the capacitor discharge drives mainly V_C, however after a period of $\sim 3\,s$ the initial charge influence is no more present and V_C starts depending only on $v(t)$.

Exercise 4

The rocket engine in Figure 13 is fired in laboratory conditions. With reference to Figure 13, the nozzle is made up of an inner lining (k_1), an inner layer having specific heat c_2 and high conductivity k_2, an insulating layer having specific heat c_4 and low conductivity k_4, and an outer coating (k_5). The interface between the conductor and the insulator layers has thermal conductivity k_3. 1) Select the materials of which the nozzle is made of[3], and therefore determine the values of k_i ($i = 1, \dots, 5$), c_2, and c_4. Assign also the values of ℓ_i (i=1,...,5), L, and A in Figure 13. 2) Derive a physical model and the associated mathematical model using one node per each of the five layers and considering that only the conductor and insulator layers have thermal capacitance. The inner wall temperature, T_i, as well as the outer wall temperature, T_o, are assigned. 3) Using the mathematical model at point 2), carry out a dynamic simulation to show the temperature profiles across the different sections. At initial time, $T_i(t_0) = T_o(t) = 20$ C°. When the rocket is fired, $T_i(t) = 1000$ C°, $t \in [t_1, t_f]$, following a ramp profile in $[t_0, t_1]$. Integrate the system using $t_1 = 1$ s and $t_f = 60$ s. 4) Repeat the simulation in point 3) using a mathematical model implementing two nodes for the conductor and insulator layers.

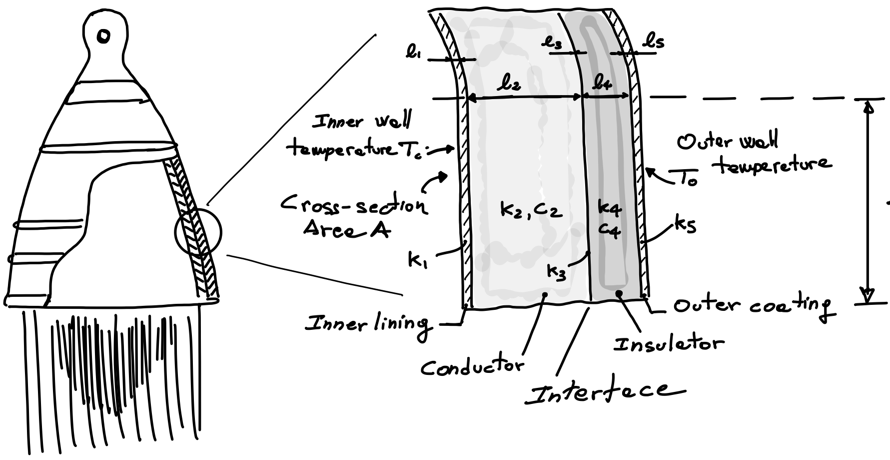

Figure 13: Real thermal system.

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1) Materials chosen are:

1. **graphite phenolic**, $l_1 = 7\,\text{mm}$, $k_1 = 200\,\text{W/K m}$
2. **silica phenolic**, $l_2 = 20\,\text{mm}$, $k_2 = 16.3\,\text{W/K m}$, $c_2 = 680\,\text{J/kg K}$, $\rho_2 = 1600\,\text{kg/m}^3$
3. **fiberglass overlap**, $l_3 = 0.5\,\text{mm}$, $k_3 = 0.04\,\text{W/K m}$
4. **cork**, $l_4 = 25\,\text{mm}$, $k_4 = 0.4\,\text{W/K m}$, $c_4 = 1900\,\text{J/kg K}$, $\rho_4 = 200\,\text{kg/m}^3$
5. **steel**, $l_5 = 10\,\text{mm}$, $k_5 = 45\,\text{W/K m}$

$L = 500\,\text{mm}$, $D = 525\,\text{mm}$

[3] The interface layer is not made of a physically existing material, though it produces a thermal resistance. For this layer, the value of the thermal resistance R_3 can be direcly assumed, so avoiding to choose k_3 and ℓ_3.

2)

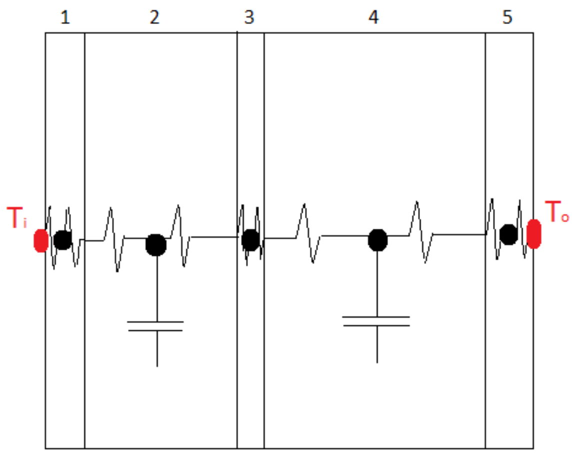

Figure 14: Physical model of nozzle's layers with 1 capacitance point.

The resistances on each layer have been modelled assuming the nozzle as a cylindrical nozzle instead of a conic one and the problem has been solved with finite differences and ODE integration:
No capacitance node:

$$T_i = \frac{\frac{T_{i-1}}{R_1} + \frac{T_{i+1}}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \tag{6}$$

Capacitance node:
FD:

$$A\,\Delta x\,\rho\,c\frac{T_i - T_i^{-1}}{\Delta t} = \frac{T_{i-1} - T_i}{R_1} + \frac{T_{i+1} - T_i}{R_2}$$

Therefore:

$$T_i = \frac{\frac{R_1 R_2 A \Delta x \rho c}{\Delta t} T_i^{-1} + R_2 T_{i-1} + R_1 T_{i+1}}{R_1 + R_2 + \frac{R_1 R_2 A \Delta x \rho c}{\Delta t}} \tag{7}$$

where:

- T^{-1}: temperature at instant before
- T_{i-1}, T_{i+1}: temperatures of previous and following nodes respectively
- R_1, R_2: resistances of between the previous and following nodes respectively

The system has been written in the linear form: $Ax = b$ where x is the temperature vector 7×1 to be recovered, A is a tridiagonal matrix 7×7 with all the resistances and b is a 7×1 vector with 1 as first and last element since T_i and T_o are constants.

ODE:

$$\dot{T}_i = \frac{1}{R_1 C}(T_{i-1} - T_i) - \frac{1}{R_2 C}(T_i - T_{i+1}) \tag{8}$$

where: R_1, R_2 have been taken planar.
This integration has been performed for just the capacitance nodes, while for the others it has been applied the Equation 6. The results have been displayed for both the methods in Figure 15.

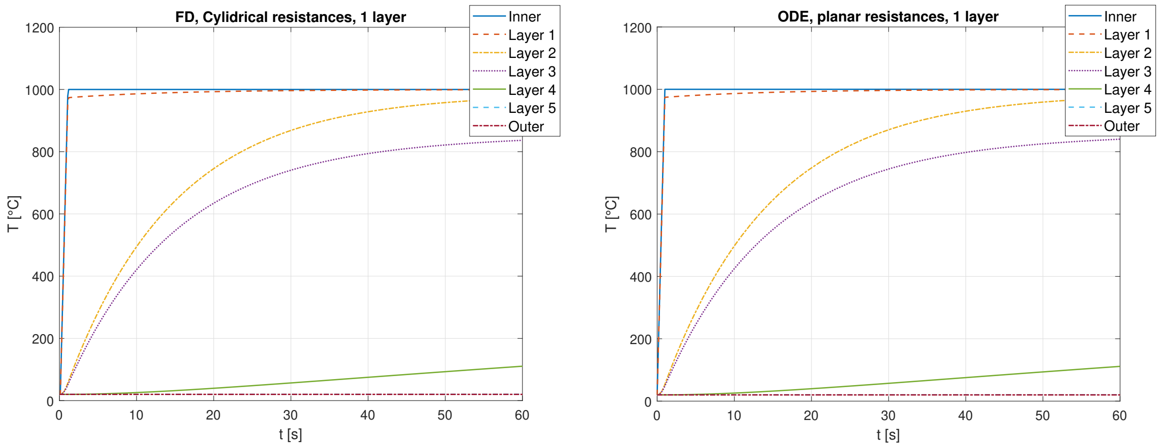

Figure 15: Comparison between finite differences and ODE with 1 node per capacitance layer.

Considering $x = [T_1\ T_3\ T_5\ T_7]^{\mathrm{T}}$, the eigenvalues are:

$$\lambda = \begin{bmatrix} -0.0033 \\ -0.0720 \\ 0.0000 \\ 0.0000 \end{bmatrix}$$

Therefore the integrator that has been used is `ode113` since the system is not stiff.
3) Since 2 nodes are requested for the conductor and the insulator, the resistance of this 2 layers has been divided in 3. It has been reconstructed the integration with finite differences and ODE schemes with Equation 6, 7, 8. Results have been shown in Figure 16.

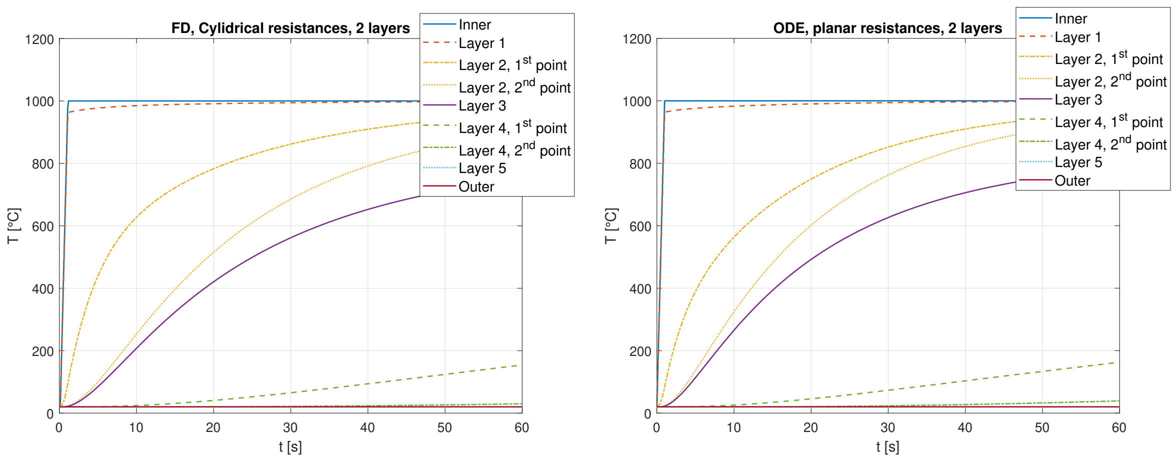

Figure 16: Comparison between finite differences and ODE with 2 nodes per capacitance layer.

It is clear that the model improved its accuracy in both cases indeed there are more differences in the two models. This is most likely due to the approximation of the layer like planar surface in one case while a cylindrical one for the other. With $x = [T_1\ T_3\ T_5\ T_7\ T_9]^{\mathrm{T}}$, the system eigenvalues are:

$$\lambda = \begin{bmatrix} -0.0285 \\ -0.0080 \\ 0.2918 \\ -0.5015 \\ 0.0000 \\ 0.0000 \end{bmatrix}$$