Assignment 3

1. Finding a hidden parity function:

   (a) Truth table of f and its $\pm$ 1 version:

x_1	x_2	f	F
0	0	1	-1
0	1	0	1
1	0	1	-1
1	1	0	1

   (b) Truth table of the Walsh basis function:

	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$
$\|00\rangle$	1	1	1	1
$\|01\rangle$	1	-1	1	-1
$\|10\rangle$	1	1	-1	-1
$\|11\rangle$	1	-1	-1	1

   (c) F as a linear combination of the Walsh basis functions:

   (d) Complete trace

$$|00\rangle \xrightarrow{H\otimes H} \frac{1}{2}\Sigma_{x\in\{0,1\}^2}|x\rangle \xrightarrow{\hat{U}_f} \frac{1}{2}\Sigma_{x\in\{0,1\}^2}(-1)^{f(x)}|x\rangle$$
$$\xrightarrow{H\otimes H} \frac{1}{4}\Sigma_{y\in\{0,1\}^2}(\Sigma_{x\in\{0,1\}^2}(-1)^{fx}(-1)^{y\cdot x})|y\rangle$$
$$\text{Let } F(x) = (-1)^{f(x)}$$
$$\text{Let } \chi_y(x) = (-1)^{y\cdot x}$$
$$= \Sigma_{y\in\{0,1\}^2}(\frac{1}{4}\Sigma_{x\in\{0,1\}^2}F(x)\chi_y(x))|y\rangle$$

We can now use the tables from parts (a) and (b) to solve the equation:

$$= [\frac{1}{4}[F(|00\rangle)\chi_{|00\rangle(|00\rangle)} + F(|01\rangle)\chi_{|00\rangle(|01\rangle)} + F(|10\rangle)\chi_{|00\rangle(|10\rangle)} + F(|11\rangle)\chi_{|00\rangle(|11\rangle)}]|00\rangle +$$
$$\frac{1}{4}[F(|00\rangle)\chi_{|01\rangle(|00\rangle)} + F(|01\rangle)\chi_{|01\rangle(|01\rangle)} + F(|10\rangle)\chi_{|01\rangle(|10\rangle)} + F(|11\rangle)\chi_{|01\rangle(|11\rangle)}]|01\rangle +$$
$$\frac{1}{4}[F(|00\rangle)\chi_{|10\rangle(|00\rangle)} + F(|01\rangle)\chi_{|10\rangle(|01\rangle)} + F(|10\rangle)\chi_{|10\rangle(|10\rangle)} + F(|11\rangle)\chi_{|10\rangle(|11\rangle)}]|10\rangle +$$
$$\frac{1}{4}[F(|00\rangle)\chi_{|11\rangle(|00\rangle)} + F(|01\rangle)\chi_{|11\rangle(|01\rangle)} + F(|10\rangle)\chi_{|11\rangle(|10\rangle)} + F(|11\rangle)\chi_{|11\rangle(|11\rangle)}]|11\rangle]$$
$$= [\frac{1}{4}[-1\cdot 1 + 1\cdot 1 + -1\cdot 1 + 1\cdot 1]|00\rangle + \frac{1}{4}[-1\cdot 1 + 1\cdot -1 + -1\cdot 1 + 1\cdot -1]|01\rangle +$$
$$\frac{1}{4}[-1\cdot 1 + 1\cdot 1 + -1\cdot -1 + 1\cdot -1]|10\rangle + \frac{1}{4}[-1\cdot 1 + 1\cdot -1 + -1\cdot -1 + 1\cdot 1]|11\rangle]$$
$$= [0\cdot|00\rangle - 1\cdot|01\rangle + 0\cdot|10\rangle + 0\cdot|11\rangle] = |01\rangle$$

   (e) (T/F) The output of the circuit reveals what f is.
   False the output does not reveal what f is.

2. Finding a hidden shift:

   (a) Show the truth table f and its ± version:

x_1	x_2	f	F
0	0	0	1
0	1	1	-1
1	0	1	-1
1	1	0	1

   (b) Complete trace:

$$|00\rangle \xrightarrow{H\otimes H} \frac{1}{2}\Sigma_{x\in\{0,1\}^2}|x\rangle \xrightarrow{\hat{U}_f} \frac{1}{2}\Sigma_{x\in\{0,1\}^2}(-1)^{f(x)}|x\rangle$$
$$\xrightarrow{H\otimes H} \frac{1}{4}\Sigma_{y\in\{0,1\}^2}(\Sigma_{x\in\{0,1\}^2}(-1)^{fx}(-1)^{y\cdot x})|y\rangle$$
$$\text{Let } F(x) = (-1)^{f(x)}$$
$$\text{Let } \chi_y(x) = (-1)^{y\cdot x}$$
$$= \Sigma_{y\in\{0,1\}^2}(\frac{1}{4}\Sigma_{x\in\{0,1\}^2}F(x)\chi_y(x))|y\rangle$$

   We can now use the table from part (a), and our knowledge of χ, to solve the equation:

$$= [\frac{1}{4}[F(|00\rangle)\chi_{|00\rangle(|00\rangle)} + F(|01\rangle)\chi_{|00\rangle(|01\rangle)} + F(|10\rangle)\chi_{|00\rangle(|10\rangle)} + F(|11\rangle)\chi_{|00\rangle(|11\rangle)}]|00\rangle +$$
$$\frac{1}{4}[F(|00\rangle)\chi_{|01\rangle(|00\rangle)} + F(|01\rangle)\chi_{|01\rangle(|01\rangle)} + F(|10\rangle)\chi_{|01\rangle(|10\rangle)} + F(|11\rangle)\chi_{|01\rangle(|11\rangle)}]|01\rangle +$$
$$\frac{1}{4}[F(|00\rangle)\chi_{|10\rangle(|00\rangle)} + F(|01\rangle)\chi_{|10\rangle(|01\rangle)} + F(|10\rangle)\chi_{|10\rangle(|10\rangle)} + F(|11\rangle)\chi_{|10\rangle(|11\rangle)}]|10\rangle +$$
$$\frac{1}{4}[F(|00\rangle)\chi_{|11\rangle(|00\rangle)} + F(|01\rangle)\chi_{|11\rangle(|01\rangle)} + F(|10\rangle)\chi_{|11\rangle(|10\rangle)} + F(|11\rangle)\chi_{|11\rangle(|11\rangle)}]|11\rangle]$$
$$= [\frac{1}{4}[1\cdot 1 + -1\cdot 1 + -1\cdot 1 + 1\cdot 1]|00\rangle + \frac{1}{4}[1\cdot 1 + -1\cdot -1 + -1\cdot 1 + 1\cdot -1]|01\rangle +$$
$$\frac{1}{4}[1\cdot 1 + -1\cdot 1 + -1\cdot -1 + 1\cdot -1]|10\rangle + \frac{1}{4}[1\cdot 1 + -1\cdot -1 + -1\cdot -1 + 1\cdot 1]|11\rangle]$$
$$= [0\cdot|00\rangle + 0\cdot|01\rangle + 0\cdot|10\rangle - 1\cdot|11\rangle] = |11\rangle$$

   (c) The final output of the circuit is $|00\rangle$.

   (d) (T/F) The circuit reveals the hidden string s?
   True the circuit does output the hidden string.

3. Quantum Fourier Transform

   (a) (T/F) The following bit-level QFT derivation for $N = 4$ is correct.
   True the derivation for $N = 4$ is correct.

   (b) (T/F) $R_k H|0\rangle = \frac{(|0\rangle + e^{\frac{2\pi i}{2^k}}|1\rangle)}{\sqrt{2}}$

$$
\begin{aligned}
R_k H|0\rangle &= \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^k}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot \frac{1}{\sqrt{2}} & 1 \cdot \frac{1}{\sqrt{2}} + 0 \cdot -\frac{1}{\sqrt{2}} \\ 0 \cdot \frac{1}{\sqrt{2}} + e^{\frac{2\pi i}{2^k}} \cdot \frac{1}{\sqrt{2}} & 0 \cdot \frac{1}{\sqrt{2}} + e^{\frac{2\pi i}{2^k}} \cdot -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{e^{\frac{2\pi i}{2^k}}}{\sqrt{2}} & -\frac{e^{\frac{2\pi i}{2^k}}}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \cdot 1 + \frac{1}{\sqrt{2}} \cdot 0 \\ \frac{e^{\frac{2\pi i}{2^k}}}{\sqrt{2}} \cdot 1 + -\frac{e^{\frac{2\pi i}{2^k}}}{\sqrt{2}} \cdot 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{e^{\frac{2\pi i}{2^k}}}{\sqrt{2}} \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{2^k}} \end{pmatrix} = \frac{(|0\rangle + e^{\frac{2\pi i}{2^k}}|1\rangle)}{\sqrt{2}}
\end{aligned}
$$

True. //

(c) (T/F) The following is a correct QFT circuit for N = 4.

True, the circuit is correct because $N = 4 = 2^2$ so $n = 2$. Where (starting from the bottom wire) the pattern of gates is as follows $H\Sigma_{i=2}^{n} R_i$ for all lines $|j_n\rangle \dots |j_1\rangle$. Then for line $|j_0\rangle$ just the Hadamard gate is applied.