Assignment 1

1. Show that the Hadamard gate *almost* commutes with the Pauli gates.

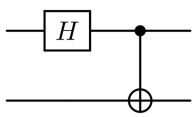
   Using the Pauli gates X and Z we can show that the Hadamard gate almost commutes.

$$X{*}H = H{*}Z$$

$$X * H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$H * Z = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

2. Show that the Bell states form a basis of all two-qubit states.



   Bell states $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$ can be combind to form $|00\rangle$.

$$\begin{aligned}
|00\rangle &= |\beta_{00}\rangle + |\beta_{10}\rangle \\
&= \frac{|00\rangle + |11\rangle}{\sqrt{2}} + \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle - \frac{1}{\sqrt{2}}|11\rangle \\
&= \frac{2}{\sqrt{2}}|00\rangle + 0\,|11\rangle
\end{aligned}$$

   Normalize the qubit.

$$\begin{aligned}
&= 1\,|00\rangle + 0\,|11\rangle \\
&= |00\rangle
\end{aligned}$$

   Bell states $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$ can be combind to form $|11\rangle$.

$$\begin{aligned}
|11\rangle &= |\beta_{00}\rangle - |\beta_{10}\rangle \\
&= \frac{|00\rangle + |11\rangle}{\sqrt{2}} - \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle - (\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle) \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle - \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= 0\,|00\rangle + \frac{2}{\sqrt{2}}|11\rangle
\end{aligned}$$

Normalize the qubit.

$$= 0\,|00\rangle + 1\,|11\rangle$$
$$= |11\rangle$$

Similary, $|\beta_{01}\rangle$ and $|\beta_{11}\rangle$ can be combind to form $|01\rangle$ and $|10\rangle$.

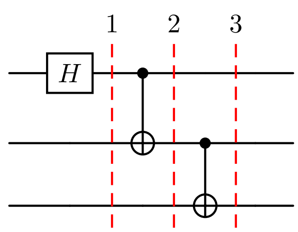
$$|01\rangle = |\beta_{01}\rangle + |\beta_{11}\rangle$$
$$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) + \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
$$= \frac{2}{\sqrt{2}}\,|01\rangle + 0\,|10\rangle$$

$$|10\rangle = |\beta_{01}\rangle - |\beta_{11}\rangle$$
$$= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) - \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$
$$= 0\,|01\rangle + \frac{2}{\sqrt{2}}\,|10\rangle$$

3. Explain what does it mean to perform a Bell basis measurement of a two-qubit state...
4. Describe a quantum circuit that starting with $|000\rangle$ prepares the output state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

The following 3 stage circuit will produce the desiried output state:



- Stage 1 - Hadamard Gate:

$$|0\rangle \otimes |0\rangle \otimes |0\rangle \xrightarrow{H\otimes I\otimes I} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle)$$

- Stage 2 - 1^{st} Control Not:
Just looking at the first two qubits of the superposition. The first qubit is the control bit and second qubit is the bit we may modifiy.

$$\frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)$$

- Stage 3 - 2^{nd} Control Not:
Just looking at the last two qubits of the superposition. The second qubit is the control bit and the third qubit is the bit we may modify

$$\frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$