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CS469/569 Quantum Information and Computation. Final Exam (Spring 2024)

Submitted work must be your own. Please prepare your solution in TeX using quantikz.

The *stabilizers* of a 5-qubit quantum error correction are given by

$$M_1 = Z_2X_3X_4Z_5, \quad M_2 = Z_3X_4X_5Z_1, \quad M_3 = Z_4X_5X_1Z_2, \quad M_4 = Z_5X_1X_2Z_3.$$

The *encodings* of the basis states $|0\rangle$ and $|1\rangle$ are defined by

$$|\overline{0}\rangle := \frac{(I+M_1)}{\sqrt{2}}\frac{(I+M_2)}{\sqrt{2}}\frac{(I+M_3)}{\sqrt{2}}\frac{(I+M_4)}{\sqrt{2}}|00000\rangle, \quad |\overline{1}\rangle := \frac{(I+M_1)}{\sqrt{2}}\frac{(I+M_2)}{\sqrt{2}}\frac{(I+M_3)}{\sqrt{2}}\frac{(I+M_4)}{\sqrt{2}}|11111\rangle.$$

By linearity, the qubit $|\psi\rangle = a|0\rangle + b|1\rangle$ is encoded as $|\overline{\psi}\rangle = a|\overline{0}\rangle + b|\overline{1}\rangle$.

1. (T/F) The stabilizers *commute* with each other: $M_jM_k = M_kM_j$, for $j,k = 1,2,3,4$.

$$\begin{gathered}
M_0 \text{ and } M_1 \text{ commute}\\
M_0 = I \otimes Z \otimes X \otimes X \otimes Z\\
M_1 = Z \otimes I \otimes Z \otimes X \otimes X\\
M_0 * M_1 = +1 * +1 * -1 * +1 * -1 = +1
\end{gathered}$$

$$\begin{gathered}
M_0 \text{ and } M_2 \text{ commute}\\
M_0 = I \otimes Z \otimes X \otimes X \otimes Z\\
M_2 = X \otimes Z \otimes I \otimes Z \otimes X\\
M_0 * M_2 = +1 * +1 * +1 * -1 * -1 = 1
\end{gathered}$$

$$\begin{gathered}
M_0 \text{ and } M_3 \text{ commute}\\
M_0 = I \otimes Z \otimes X \otimes X \otimes Z\\
M_3 = X \otimes X \otimes Z \otimes I \otimes Z\\
M_0 * M_3 = +1 * -1 * -1 * +1 * +1 = +1
\end{gathered}$$

$$\begin{gathered}
M_1 \text{ and } M_2 \text{ commute}\\
M_1 = Z \otimes I \otimes Z \otimes X \otimes X\\
M_2 = X \otimes Z \otimes I \otimes Z \otimes X\\
M_1 * M_2 = -1 * +1 * +1 * -1 * +1 = +1
\end{gathered}$$

$$\begin{gathered}
M_1 \text{ and } M_3 \text{ commute}\\
M_1 = Z \otimes I \otimes Z \otimes X \otimes X\\
M_3 = X \otimes X \otimes Z \otimes I \otimes Z\\
M_1 * M_3 = -1 * +1 * +1 * +1 * -1 = +1
\end{gathered}$$

$$M_2 \text{ and } M_3 \text{ commute}$$
$$M_2 = X \otimes Z \otimes I \otimes Z \otimes X$$
$$M_3 = X \otimes X \otimes Z \otimes I \otimes Z$$
$$M_2 * M_3 = +1 * -1 * +1 * +1 * -1 = +1$$

True, all of the stabilizers commute with eachother.

2. (T/F) $(I + M_k)^2 = 2(I + M_k)$ and $M_k(I + M_k) = I + M_k$ for $k = 1, 2, 3, 4$.

$$(I + M_k)^2 = (I + M_k)(I + M_k) = I^2 + I * M_k + M_k * I + M_k^2$$
$$= I + M_k + M_k + I = 2I + 2M_k = 2(I + M_k)$$

$$M_k(I + M_k) = M_k * I + M_k^2 = M_k + I = I + M_k$$

True

3. (T/F) $|\overline{0}\rangle$ and $|\overline{1}\rangle$ are $+1$-*eigenvectors* of the stabilizers: $M_k|\overline{0}\rangle = |\overline{0}\rangle$ and $M_k|\overline{1}\rangle = |\overline{1}\rangle$. So, any encoded state $|\overline{\psi}\rangle$ is a joint $(+1)$-eigenvector of the stabilizers.

Since we know that

$$|\overline{0}\rangle := \frac{(I + M_1)}{\sqrt{2}} \frac{(I + M_2)}{\sqrt{2}} \frac{(I + M_3)}{\sqrt{2}} \frac{(I + M_4)}{\sqrt{2}} |00000\rangle$$
$$= \frac{1}{4}(I + M_1)(I + M_2)(I + M_3)(I + M_4)|00000\rangle$$

We can say that for any k = 1,2,3,4 $M_k|\overline{0}\rangle = |\overline{0}\rangle$. For example M_1

$$M_1|\overline{0}\rangle = M_1 \frac{1}{4}(I + M_1)(I + M_2)(I + M_3)(I + M_4)|00000\rangle$$
$$= \frac{1}{4} M_1(I + M_1)(I + M_2)(I + M_3)(I + M_4)|00000\rangle = \frac{1}{4}(M_1 + M_1^2)(I + M_2)(I + M_3)(I + M_4)|00000\rangle$$
$$= \frac{1}{4}(M_1 + I)(I + M_2)(I + M_3)(I + M_4)|00000\rangle = |\overline{0}\rangle$$

Similarly, we can say that for any k = 1,2,3,4 $M_k|\overline{1}\rangle = |\overline{1}\rangle$. For example M_3

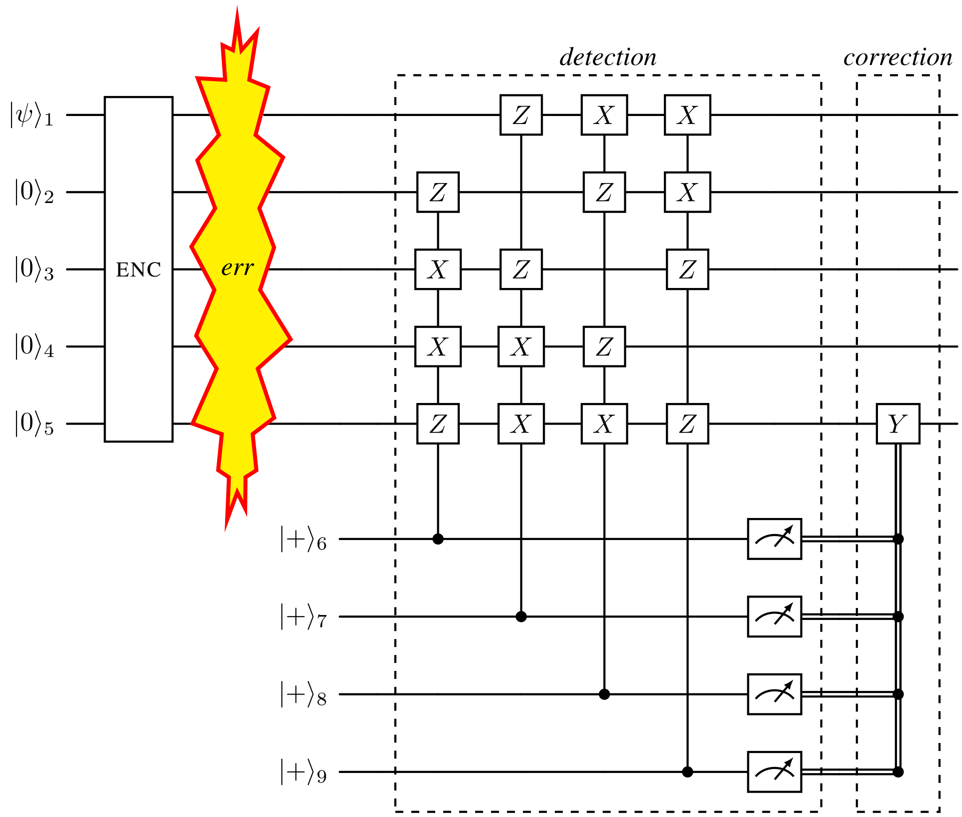
$$M_3|\overline{1}\rangle = M_3 \frac{1}{4}(I + M_1)(I + M_2)(I + M_3)(I + M_4)|11111\rangle$$
$$= \frac{1}{4}(I + M_1)(I + M_2)M_3(I + M_3)(I + M_4)|11111\rangle = \frac{1}{4}(I + M_1)(I + M_2)(M_3 + M_3^2)(I + M_4)|11111\rangle$$
$$= \frac{1}{4}(I + M_1)(I + M_2)(M_3 + I)(I + M_4)|11111\rangle = |\overline{1}\rangle$$

True, since $|\overline{0}\rangle$ and $|\overline{1}\rangle$ are $+1$-*eigenvectors* of the stabilizers, we can describe any encoded state $|\overline{\psi}\rangle$ as a linear combination of the $(+1)$-eigenvectors of the stabilizers.

4. Complete the following *commutation* table: use 0 for commuting and 1 for anti-commuting.

	X_1	Y_1	Z_1	X_2	Y_2	Z_2	X_3	Y_3	Z_3	X_4	Y_4	Z_4	X_5	Y_5	Z_5	I
$M_1 = Z_2X_3X_4Z_5$	0	0	0	1	1	0	0	1	1	0	1	1	1	1	0	0
$M_2 = Z_3X_4X_5Z_1$	1	1	0	0	0	0	1	1	0	0	1	1	0	1	1	0
$M_3 = Z_4X_5X_1Z_2$	0	1	1	1	1	0	0	0	0	1	1	0	0	1	1	0
$M_4 = Z_5X_1X_2Z_3$	0	1	1	0	1	1	1	1	0	0	0	0	1	1	0	0

5. Consider the following circuit for the 5-qubit quantum error correction code. Assume the circuit ENC correctly encodes the qubit $|\psi\rangle$ into its encoded state $|\overline{\psi}\rangle$. Assume all measurements are in the Z basis (rectilinear basis).



(a) (T/F) The *detection* module is correct.

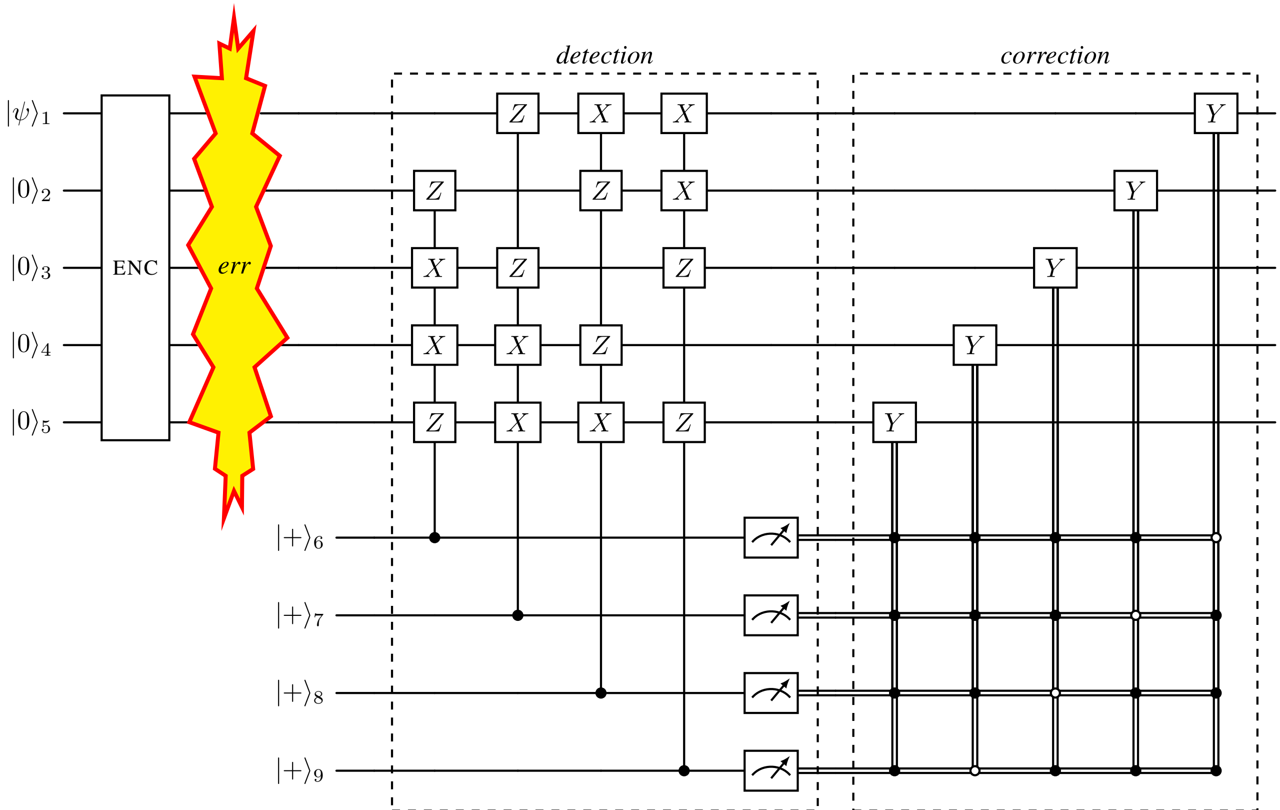
True the detection module is correct since it applies M_1 - M_4. Given that

$$M_1 = I \otimes Z \otimes X \otimes X \otimes Z$$
$$M_2 = Z \otimes I \otimes Z \otimes X \otimes X$$
$$M_3 = X \otimes Z \otimes I \otimes Z \otimes X$$
$$M_4 = X \otimes X \otimes Z \otimes I \otimes Z$$

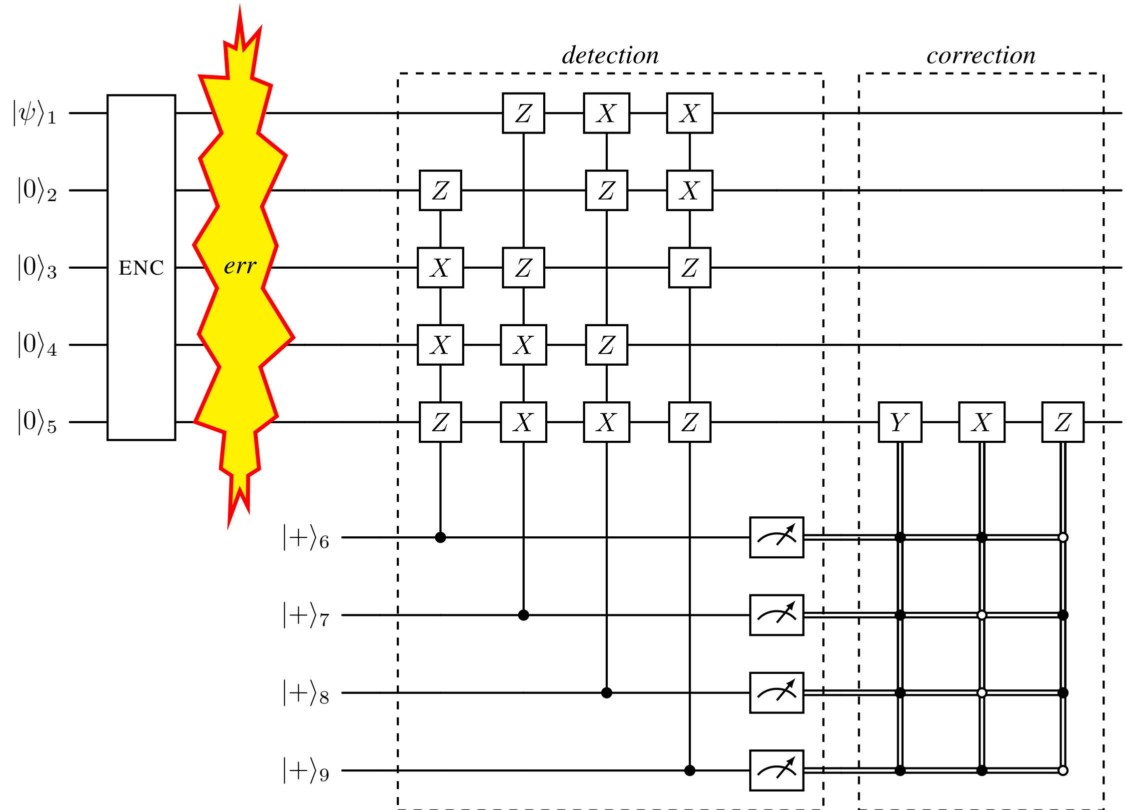
(b) (T/F) The *correction* module is correct for error Y_5.

True the correction Y gate will only get activated if all of the stabilizers measure $|-\rangle$.

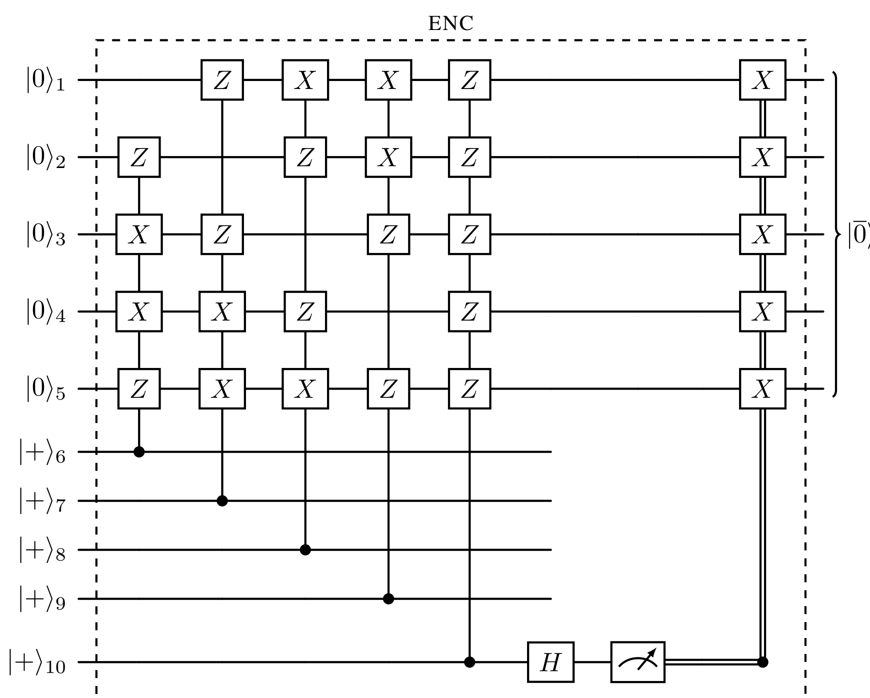
(c) Expand the correction module to handle *all* other Y errors.



(d) Expand the correction module to handle *all* other errors on the 5th qubit.



6. (CS569) Consider the following encoding circuit for the 5-qubit quantum error correction code.



(a) Let $\overline{Z} = Z_1Z_2Z_3Z_4Z_5$.

i. (T/F) $\overline{Z}$ commutes with all M_k, $k = 1, 2, 3, 4$.

$$M_0 \text{ and } \overline{Z} \text{ commute}$$
$$M_0 = I \otimes Z \otimes X \otimes X \otimes Z$$
$$\overline{Z} = Z \otimes Z \otimes Z \otimes Z \otimes Z$$
$$M_0 * \overline{Z} = +1 * +1 * -1 * -1 * +1 = +1$$

$$M_1 \text{ and } \overline{Z} \text{ commute}$$
$$M_1 = Z \otimes I \otimes Z \otimes X \otimes X$$
$$\overline{Z} = Z \otimes Z \otimes Z \otimes Z \otimes Z$$
$$M_1 * \overline{Z} = +1 * +1 * +1 * -1 * -1 = +1$$

$$M_2 \text{ and } \overline{Z} \text{ commute}$$
$$M_2 = X \otimes Z \otimes I \otimes Z \otimes X$$
$$\overline{Z} = Z \otimes Z \otimes Z \otimes Z \otimes Z$$
$$M_2 * \overline{Z} = -1 * +1 * +1 * +1 * -1 = +1$$

$$M_3 \text{ and } \overline{Z} \text{ commute}$$
$$M_3 = X \otimes X \otimes Z \otimes I \otimes Z$$
$$\overline{Z} = Z \otimes Z \otimes Z \otimes Z \otimes Z$$
$$M_3 * \overline{Z} = -1 * -1 * +1 * +1 * +1 = +1$$

True, $\overline{Z}$ commutes with all of the stabilizers.

ii. (T/F) $|\overline{0}\rangle$ is a $(+1)$-eigenvector of $\overline{Z}$ but $|\overline{1}\rangle$ is a (-1)-eigenvector of $\overline{Z}$.

$$|\overline{0}\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$$
$$\overline{Z} = Z \otimes Z \otimes Z \otimes Z \otimes Z$$
$$\overline{Z}|\overline{0}\rangle = +1 * +1 * +1 * +1 * +1 = +1$$

$$|\overline{1}\rangle = |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle \otimes |1\rangle$$
$$\overline{Z} = Z \otimes Z \otimes Z \otimes Z \otimes Z$$
$$\overline{Z}|\overline{1}\rangle = -1 * -1 * -1 * -1 * -1 = -1$$

True, $\overline{Z}|\overline{0}\rangle = |\overline{0}\rangle$ and $\overline{Z}|\overline{1}\rangle = -|\overline{1}\rangle$

(b) (T/F) The circuit will encode qubit $|0\rangle$ into its encoded state $|\overline{0}\rangle$.

$$\begin{aligned}
|00000\rangle \otimes |+++++\rangle &= |00000\rangle \otimes (\frac{|0\rangle + |1\rangle}{\sqrt{2}})(\frac{|0\rangle + |1\rangle}{\sqrt{2}})(\frac{|0\rangle + |1\rangle}{\sqrt{2}})(\frac{|0\rangle + |1\rangle}{\sqrt{2}})(\frac{|0\rangle + |1\rangle}{\sqrt{2}}) \\
&= |00000\rangle \otimes (\frac{I + M_1}{\sqrt{2}})(\frac{I + M2}{\sqrt{2}})(\frac{I + M_3}{\sqrt{2}})(\frac{I + M_4}{\sqrt{2}})(\frac{Z + ZX}{\sqrt{2}}) \\
&= |\overline{0}\rangle(\frac{Z + ZX}{\sqrt{2}})
\end{aligned}$$

Next, we can perform the Z parity messurement if the result is +1, we don't apply X. If the result is -1 we apply X, since we had $|\overline{1}\rangle$ instead of $|\overline{0}\rangle$.

True.

(c) Describe how the above circuit can be used to encode $|1\rangle$ into the encoded state $|\overline{1}\rangle$.

The same circuit could be using to encode $|1\rangle$ into $|\overline{1}\rangle$ by simply swapping qu-bit 1 for $|1\rangle$.