Assignment 4

1. Spectral Decomposition

   (a) Spectral decompositions of the obserables X, Z, Y
   Calculate the eigenvalues of X:

$$det(\lambda I - X) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = det(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix})$$
$$\lambda^2 - 1 = 0$$
$$\lambda_1 = 1, \lambda_2 = -1$$

Calculate the eigenvectors of X:

$$(X - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-x + y = 0$$
$$x - y = 0$$
$$x = 1, y = 1$$
$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(X - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x + y = 0$$
$$x + y = 0$$
$$x = 1, y = -1$$
$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Spectral decomposition of X

$$X = \sum_{i=1}^{2} \lambda_i |\lambda_i\rangle \langle\lambda_i| = \lambda_1 |\lambda_1\rangle \langle\lambda_1| + \lambda_2 |\lambda_2\rangle \langle\lambda_2|$$
$$= (+1)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} + (-1)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}$$
$$= (+1)\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (-1)\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Calculate the eigen values of Z:

$$det(\lambda I - Z) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = det(\begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 1 \end{bmatrix})$$
$$(\lambda - 1)(\lambda + 1) = 0$$
$$\lambda_1 = 1, \lambda_2 = -1$$

$$(Z - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
Apply Gaussian elimination
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$y = 0, x = 1$$
$$|\lambda_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(Z - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
Apply Gaussian elimination
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x = 0, y = 1$$
$$|\lambda_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Spectral decomposition of Z

$$\begin{aligned} Z &= \sum_{i=1}^{2} \lambda_i \left|\lambda_i\right\rangle \left\langle\lambda_i\right| = \lambda_1 \left|\lambda_1\right\rangle \left\langle\lambda_1\right| + \lambda_2 \left|\lambda_2\right\rangle \left\langle\lambda_2\right| \\ &= (+1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ &= (+1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Calculate the eigenvalues of Y

$$det(\lambda I - Y) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}) = det(\begin{bmatrix} \lambda & i \\ -i & \lambda \end{bmatrix})$$

$$\lambda^2 - (-i * i) = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = 1, \lambda_2 = -1$$

$$(Y - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$

$$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply Gaussian elimination

$$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = -i, y = 1$$

$$\left|\lambda_1\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$(Y - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$

$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply Gaussian elimination

$$\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = i, y = 1$$

$$\left|\lambda_2\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Spectral decomposition of Y

$$\begin{aligned} Y &= \sum_{i=1}^{2} \lambda_i \left|\lambda_i\right\rangle \left\langle\lambda_i\right| = \lambda_1 \left|\lambda_1\right\rangle \left\langle\lambda_1\right| + \lambda_2 \left|\lambda_2\right\rangle \left\langle\lambda_2\right| \\ &= (+1)\frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 1 \end{bmatrix} + (-1)\frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \end{bmatrix} \\ &= (+1)\frac{1}{2} \begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix} + (-1)\frac{1}{2} \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix} \end{aligned}$$

(b) Show that $\langle\psi|P_r|\psi\rangle \geq 0$ and moreover, $\sum_r \langle\psi|P_r|\psi\rangle = 1$
We know that $P^2 = P$ for any projector P. We know that $P^\intercal =$ P. We also know that the inner product of any normalized state is 1. Since

$$\left|\psi\right\rangle = \alpha \left|0\right\rangle + \beta \left|1\right\rangle$$

$$\langle\psi|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \alpha & \beta \end{pmatrix} = |\alpha|^2 + |\beta|^2 = 1$$

So we can say that

$$\langle\psi|P_r|\psi\rangle = \langle\psi|P_r^\intercal P_r|\psi\rangle = (\left\langle\psi\right| P_r^\intercal)(P_r \left|\psi\right\rangle) = \langle\phi|\phi\rangle \geq 0$$

$$\therefore \sum_r \langle\psi|P_r|\psi\rangle = \langle\psi| \sum_r P_r|\psi\rangle = \langle\psi|I|\psi\rangle = \langle\psi|\psi\rangle = 1$$

(c) Compute $\langle X \rangle$ and $\langle Z \rangle$ under $|+\rangle$

Compute $\langle X \rangle$ under $|+\rangle$

$$\langle X \rangle_{|+\rangle} = \langle +|X|+\rangle = \langle +|(|+\rangle \langle +| - |-\rangle \langle -|)|+\rangle$$
$$= \langle +|+\rangle \langle +|+\rangle - \langle +|-\rangle \langle -|+\rangle$$
$$= 1 * 1 - 0 * 0 = 1$$

Compute $\langle Z \rangle$ under $|+\rangle$

$$\langle Z \rangle_{|+\rangle} = \langle +|Z|+\rangle = \langle +|(|0\rangle \langle 0| - |1\rangle \langle 1|)|+\rangle$$
$$= \langle +|0\rangle \langle 0|+\rangle - \langle +|1\rangle \langle 1|+\rangle$$
$$= (\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}) - (\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{1}{2} = 0$$

(d) Compute $\langle X_1 Z_2 \rangle$ under $|\beta_{00}\rangle$

$$\langle X_1 Z_2 \rangle_{|\beta_{00}\rangle} = \frac{1}{2}[\langle 00| + \langle 11| \, (X_1 Z_2) \, |00\rangle + |11\rangle]$$
$$= \frac{1}{2}[\langle 00|X_1 Z_2|00\rangle + \langle 00|X_1 Z_2|11\rangle + \langle 11|X_1 Z_2|00\rangle + \langle 1|X_1 Z_2|11\rangle]$$
$$= \frac{1}{2}[0 * 1 + 1 * 0 + 1 * 0 + 0 * -1] = 0$$

2. CHSH Game

(a) (T/F)$\langle \mathcal{O} \rangle$ holds. True.

(b) (T/F) $\langle QS \rangle = \frac{1}{\sqrt{2}}$.

$$QS = Z_1 \otimes \frac{-1}{\sqrt{2}}(Z_2 + X_2) = \frac{-1}{\sqrt{2}}(Z_1 Z_2 + Z_1 X_2)$$
$$\langle QS \rangle_{|\beta_{11}\rangle} = \langle \beta_{11}|QS|\beta_{11}\rangle = \frac{\langle 01| - \langle 10|}{\sqrt{2}}[\frac{-1}{\sqrt{2}}(Z_1 Z_2 + Z_1 X_2)]\frac{|01\rangle - |10\rangle}{\sqrt{2}}$$
$$= \frac{-1}{2\sqrt{2}}[(\langle 01| - \langle 10|)Z_1 Z_2(|01\rangle - \langle 10|) + (\langle 01| - \langle 10|)Z_1 X_2(|01\rangle - \langle 10|)]$$
$$= \frac{-1}{2\sqrt{2}}[\langle 01|Z_1 Z_2|01\rangle - \langle 01|Z_1 Z_2|10\rangle - \langle 10|Z_1 Z_2|01\rangle + \langle 10|Z_1 Z_2|10\rangle$$
$$+ \langle 01|Z_1 X_2|01\rangle - \langle 01|Z_1 X_2|10\rangle - \langle 10|Z_1 X_2|01\rangle + \langle 10|Z_1 X_2|10\rangle]$$
$$= \frac{-1}{2\sqrt{2}}[-1 - 0 - 0 + (-1) + 0 - 0 - 0 + 0] = \frac{-1}{2\sqrt{2}} * -2 = \frac{1}{\sqrt{2}}$$

True.

(c) (T/F) $\langle RS \rangle = \frac{1}{\sqrt{2}}$.

$$RS = X_1 \otimes \frac{-1}{\sqrt{2}}(Z_2 + X_2) = \frac{-1}{\sqrt{2}}(X_1Z_2 + X_1X_2)$$

$$\langle RS \rangle_{|\beta_{11}\rangle} = \langle \beta_{11}|RS|\beta_{11}\rangle = \frac{\langle 01| - \langle 10|}{\sqrt{2}}[\frac{-1}{\sqrt{2}}(X_1Z_2 + X_1X_2)]\frac{|01\rangle - |10\rangle}{\sqrt{2}}$$
$$= \frac{-1}{2\sqrt{2}}[(\langle 01| - \langle 10|)X_1Z_2(|01\rangle - \langle 10|) + (\langle 01| - \langle 10|)X_1X_2(|01\rangle - \langle 10|)]$$
$$= \frac{-1}{2\sqrt{2}}[\langle 01|X_1Z_2|01\rangle - \langle 01|X_1Z_2|10\rangle - \langle 10|X_1Z_2|01\rangle + \langle 10|X_1Z_2|10\rangle$$
$$+ \langle 01|X_1X_2|01\rangle - \langle 01|X_1X_2|10\rangle - \langle 10|X_1X_2|01\rangle + \langle 10|X_1X_2|10\rangle]$$
$$= \frac{-1}{2\sqrt{2}}[0 - 0 - 0 + 0 + 0 - 1 - 1 + 0] = \frac{-1}{2\sqrt{2}} * -2 = \frac{1}{\sqrt{2}}$$

True.

(d) (T/F) $\langle RT \rangle = \frac{1}{\sqrt{2}}$.

$$RT = X_1 \otimes \frac{1}{\sqrt{2}}(Z_2 - X_2) = \frac{1}{\sqrt{2}}(X_1Z_2 - X_1X_2)$$

$$\langle RT \rangle_{|\beta_{11}\rangle} = \frac{1}{2\sqrt{2}}[(\langle 01| - \langle 10|)X_1Z_2(|01\rangle - |10\rangle) - (\langle 01| - \langle 10|)X_1X_2(|01\rangle - |10\rangle)]$$

From parts b and d we know that $\langle X_1Z_2 \rangle_{|\beta_{11}\rangle} = 0$ and that $\langle X_1X_2 \rangle_{|\beta_{11}\rangle} = -2$.

$$= \frac{1}{2\sqrt{2}}[0 - (-2)] = \frac{1}{2\sqrt{2}} * 2 = \frac{1}{\sqrt{2}}$$

True.

(e) (T/F) $\langle QT \rangle = -\frac{1}{\sqrt{2}}$.

$$QT = Z_1 \otimes \frac{1}{\sqrt{2}}(Z_2 - X_2) = \frac{1}{\sqrt{2}}(Z_1Z_2 - Z_1X_2)$$

$$\langle QT \rangle_{|\beta_{11}\rangle} = \frac{1}{2\sqrt{2}}[(\langle 01| - \langle 10|)Z_1Z_2(|01\rangle - |10\rangle) - (\langle 01| - \langle 10|)Z_1X_2(|01\rangle - |10\rangle)]$$

From parts b and d we know that $\langle Z_1Z_2 \rangle_{|\beta_{11}\rangle} = -2$ and that $\langle Z_1X_2 \rangle_{|\beta_{11}\rangle} = 0$.

$$= \frac{1}{2\sqrt{2}}[(-2) - 0] = \frac{1}{2\sqrt{2}} * -2 = \frac{-1}{\sqrt{2}}$$

True.

(f) (T/F) If Q, R, S, T are random variables with ± 1 values then
$\mathbb{E}$[QS] + $\mathbb{E}$[RS] + $\mathbb{E}$[RT] - $\mathbb{E}$[QT] $\leq$ 2.

$$QS + RS + RT - QT$$
$$(Q + R)S + (R - Q)T$$

Either Q+R or R-Q evaluates to zero, then the other term must evaluate to ± 2. Then the whole expression must evaluate to ± 2. ∴ True, the expression evaluates to $\leq$ 2.

(g) (T/F) $\langle \mathcal{O} \rangle = 2\sqrt{2}$ (Does this mean Quantum 1, Einstein 0?)

$$\langle \mathcal{O} \rangle = \langle QS \rangle + \langle RS \rangle + \langle RT \rangle - \langle QT \rangle$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{-1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

True, this does mean Quantum 1, Einstein 0, since we were able to calculate a value that is > 2 using the quantum states.