Assignment 2

1. Consider the Bell state $|\beta_{00}\rangle$, suppose both qubits are measured in the diagonal basis. What are the outcomes?

   We know that $|\beta_{00}\rangle$ is equivalent to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in order to mesaure in the diagonal basis. To do this, we need to express $|\beta_{00}\rangle$ in terms of $\{|+\rangle, |-\rangle\}$ rather than $\{|0\rangle, |1\rangle\}$. Luckily we know that $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and that $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$[1].

   Subsituting this questions into $|\beta_{00}\rangle$:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}[(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)) + (\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle))]$$

$$= \frac{1}{\sqrt{2}}[\frac{1}{2}((|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle)) + \frac{1}{2}((|+\rangle - |-\rangle) \otimes (|+\rangle - |-\rangle))]$$

$$= \frac{1}{\sqrt{2}}\frac{1}{2}[(|{++}\rangle + |{+-}\rangle + |{-+}\rangle + |{--}\rangle) + (|{++}\rangle - |{+-}\rangle - |{-+}\rangle + |{--}\rangle)]$$

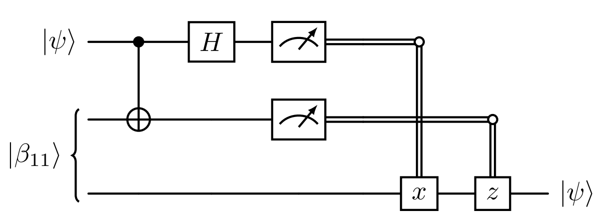
$$= \frac{1}{\sqrt{2}}[(|{++}\rangle + |{--}\rangle)]$$

   Messuring the state:

$$\begin{cases} |{++}\rangle & \text{w.p. } \frac{1}{2} \\ |{+-}\rangle & \text{w.p. } 0 \\ |{-+}\rangle & \text{w.p. } 0 \\ |{--}\rangle & \text{w.p. } \frac{1}{2} \end{cases}$$

2. Describe a quantum teleportation circuit when ALice and Bob shared the Bell state $|\beta_{11}\rangle$ instead of $|\beta_{00}\rangle$.

   The following circuit will successfully teleport an arbitrary qubit, $|\psi\rangle$, between Alice and Bob.



[1]T. Wong, Introduction to Classical and Quantum Computing, Rooted Grove, 2022. Page 87

Proof:

$$\begin{aligned}
|\psi\rangle \otimes |\beta_{11}\rangle &= (a|0\rangle + b|1\rangle) \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} \\
&= a|0\rangle \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} + b|1\rangle \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}[a|001\rangle - a|010\rangle + b|101\rangle - b|110\rangle] \\
&\xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}[a|001\rangle - a|010\rangle + b|111\rangle - b|100\rangle] \\
&\xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}[a\frac{|0\rangle + |1\rangle}{\sqrt{2}}(|01\rangle - |10\rangle) + b\frac{|0\rangle - |1\rangle}{\sqrt{2}}(|11\rangle - |00\rangle)] \\
&= \frac{1}{2}[(a|0\rangle + a|1\rangle)(|01\rangle - |10\rangle) + (b|0\rangle - b|1\rangle)(|11\rangle - |00\rangle)] \\
&= \frac{1}{2}[a|001\rangle - a|010\rangle + a|101\rangle - a|110\rangle + b|011\rangle - b|000\rangle + b|111\rangle - b|100\rangle] \\
&= \frac{1}{2}[|00\rangle(a|1\rangle - b|0\rangle) - |01\rangle(a|0\rangle - b|1\rangle) + |10\rangle(a|1\rangle + b|0\rangle) - |11\rangle(a|0\rangle + b|1\rangle)]
\end{aligned}$$

Now messuring the first two qubits gets us:

$$\begin{cases}
|00\rangle XZ|\psi\rangle & \text{w.p. } \frac{1}{4} \\
|01\rangle Z|\psi\rangle & \text{w.p. } \frac{1}{4} \\
|10\rangle X|\psi\rangle & \text{w.p. } \frac{1}{4} \\
|11\rangle |\psi\rangle & \text{w.p. } \frac{1}{4}
\end{cases}$$

3. Suppose Alice and Bob shared a Bell state $|\beta\rangle$ and wanted to share a secret clasical bit. Explain how they can do this without any commincation. What if they forget whether to messure in the rectilinear or diagonal basis? Can they still share the secret bit?

   They can share a secret classical bit simply by messuring the each particle of the bell state at the same time. If they shared bell state $|\beta_{00}\rangle$ then their is a 50% chance that each of them measure $|0\rangle$ and a 50% chance that each of them mesaure $|1\rangle$.

   If they forget to measure in the rectilinear or diagonal basis, then they won't be able to share a bit. Suppose Alice mesaures in the rectilinear basis and mesaures a $|0\rangle$, if Bob measures in the diagonal basis he would either measure a $|+\rangle$ or a $|-\rangle$ since $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ Without any commincation there would be no way for Alice and Bob to end up with the same result.

4. Gate teleportation

   Input

$$\begin{aligned}
&|\alpha\rangle \otimes |\beta_{00}\rangle \otimes |\beta_{00}\rangle \otimes |\omega\rangle \\
&= (\alpha_1|0\rangle + \alpha_2|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes (\omega_1|0\rangle + \omega_2|1\rangle) \\
&= \frac{1}{\sqrt{2}}[\alpha_1|000\rangle + \alpha_1|011\rangle + \alpha_2|100\rangle + \alpha_2|111\rangle] \otimes \frac{1}{\sqrt{2}}[\omega_1|000\rangle + \omega_2|001\rangle + \omega_1|110\rangle + \omega_2|111\rangle]
\end{aligned}$$

   Bell Basis Messurement circuit

$$\xrightarrow{\text{CNOT(S)}} \frac{1}{\sqrt{2}}[\alpha_1 |000\rangle + \alpha_1 |011\rangle + \alpha_2 |110\rangle + \alpha_2 |101\rangle] \otimes \frac{1}{\sqrt{2}}[\omega_1 |000\rangle + \omega_2 |011\rangle + \omega_1 |110\rangle + \omega_2 |101\rangle]$$

$$= \frac{1}{\sqrt{2}}[\alpha_1 |0\rangle (|00\rangle + |11\rangle) + \alpha_2 |1\rangle (|10\rangle + |01\rangle)] \otimes \frac{1}{\sqrt{2}}[(|00\rangle + |11\rangle)\omega_1 |0\rangle + (|01\rangle + |10\rangle)\omega_2 |1\rangle]$$

$$\xrightarrow{H\otimes I\otimes I\otimes I\otimes I\otimes H} \frac{1}{\sqrt{2}}[\alpha_1(\frac{|0\rangle + |1\rangle}{\sqrt{2}})(|00\rangle + |11\rangle) + \alpha_2(\frac{|0\rangle - |1\rangle}{\sqrt{2}})(|10\rangle + |01\rangle)]$$

$$\otimes\frac{1}{\sqrt{2}}[(|00\rangle + |11\rangle)\omega_1(\frac{|0\rangle + |1\rangle}{\sqrt{2}}) + (|01\rangle + |10\rangle)\omega_2(\frac{|0\rangle - |1\rangle}{\sqrt{2}})]$$

$$= \frac{1}{2}[\alpha_1 |000\rangle + \alpha_1 |011\rangle + \alpha_1 |100\rangle + \alpha_1 |111\rangle + \alpha_2 |010\rangle + \alpha_2 |001\rangle - \alpha_2 |110\rangle - \alpha_2 |101\rangle]$$

$$\otimes\frac{1}{2}[\omega_1 |000\rangle + \omega_1 |001\rangle + \omega_1 |110\rangle + \omega_1 |111\rangle + \omega_2 |010\rangle - \omega_2 |011\rangle + \omega_2 |100\rangle - \omega_2 |101\rangle]$$

$$= \frac{1}{2}[|00\rangle (\alpha_1 |0\rangle + \alpha_2 |1\rangle) + |01\rangle (\alpha_1 |1\rangle + \alpha_2 |0\rangle) + |10\rangle (\alpha_1 |0\rangle - \alpha_2 |1\rangle) + |11\rangle (\alpha_1 |1\rangle - \alpha_2 |0\rangle)]$$

$$\otimes\frac{1}{2}[(\omega_1 |0\rangle + \omega_2 |1\rangle) |00\rangle + (\omega_1 |0\rangle - \omega_2 |1\rangle) |01\rangle + (\omega_1 |1\rangle + \omega_2 |0\rangle) |10\rangle + (\omega_1 |1\rangle - \omega_2 |0\rangle) |11\rangle]$$

$$= \frac{1}{2}[|00\rangle |\alpha\rangle + |01\rangle (\mathrm{X} |\alpha\rangle) + |10\rangle (\mathrm{Z} |\alpha\rangle) + |11\rangle (\mathrm{XZ} |\alpha\rangle)] \otimes \frac{1}{2}[|\omega\rangle |00\rangle + (\mathrm{X} |\omega\rangle) |01\rangle$$

$$+(\mathrm{Z} |\omega\rangle) |10\rangle + (\mathrm{XZ} |\omega\rangle) |11\rangle]$$

$$= \frac{1}{4}[|00\rangle |\alpha\rangle |\omega\rangle |00\rangle + |00\rangle |\alpha\rangle (\mathrm{X} |\omega\rangle) |01\rangle + |00\rangle |\alpha\rangle (\mathrm{Z} |\omega\rangle) |10\rangle +$$

$$|00\rangle |\alpha\rangle (\mathrm{XZ} |\omega\rangle) |11\rangle + |01\rangle (\mathrm{X} |\alpha\rangle) |\omega\rangle |00\rangle + |01\rangle (\mathrm{X} |\alpha\rangle)(\mathrm{Z} |\omega\rangle) |01\rangle +$$

$$|01\rangle (\mathrm{X} |\alpha\rangle)(\mathrm{X} |\omega\rangle) |10\rangle + |01\rangle (\mathrm{X} |\alpha\rangle)(\mathrm{XZ} |\omega\rangle) |11\rangle + |10\rangle (\mathrm{Z} |\alpha\rangle) |\omega\rangle |00\rangle + |10\rangle (\mathrm{Z} |\alpha\rangle)(\mathrm{Z} |\omega\rangle) |01\rangle +$$

$$|10\rangle (\mathrm{Z} |\alpha\rangle)(\mathrm{X} |\omega\rangle) |10\rangle + |10\rangle (\mathrm{Z} |\alpha\rangle)(\mathrm{XZ} |\omega\rangle) |11\rangle + |11\rangle (\mathrm{XZ} |\alpha\rangle) |\omega\rangle |00\rangle + |11\rangle (\mathrm{XZ} |\alpha\rangle)(\mathrm{Z} |\omega\rangle) |01\rangle +$$

$$|11\rangle (\mathrm{XZ} |\alpha\rangle)(\mathrm{X} |\omega\rangle) |10\rangle + |11\rangle (\mathrm{XZ} |\alpha\rangle)(\mathrm{XZ} |\omega\rangle) |11\rangle]$$

$$\begin{cases}
|00\rangle |\alpha\rangle |\omega\rangle |00\rangle & \text{w.p. } \frac{1}{16} \\
|00\rangle |\alpha\rangle (\mathrm{Z} |\omega\rangle) |01\rangle & \text{w.p. } \frac{1}{16} \\
|00\rangle |\alpha\rangle (\mathrm{X} |\omega\rangle) |10\rangle & \text{w.p. } \frac{1}{16} \\
|00\rangle |\alpha\rangle (\mathrm{XZ} |\omega\rangle) |11\rangle & \text{w.p. } \frac{1}{16} \\
|01\rangle (\mathrm{X} |\alpha\rangle) |\omega\rangle |00\rangle & \text{w.p. } \frac{1}{16} \\
|01\rangle (\mathrm{X} |\alpha\rangle)(\mathrm{Z} |\omega\rangle) |01\rangle & \text{w.p. } \frac{1}{16} \\
|01\rangle (\mathrm{X} |\alpha\rangle)(\mathrm{X} |\omega\rangle) |10\rangle & \text{w.p. } \frac{1}{16} \\
|01\rangle (\mathrm{X} |\alpha\rangle)(\mathrm{XZ} |\omega\rangle) |11\rangle & \text{w.p. } \frac{1}{16} \\
|10\rangle (\mathrm{Z} |\alpha\rangle) |\omega\rangle |00\rangle & \text{w.p. } \frac{1}{16} \\
|10\rangle (\mathrm{Z} |\alpha\rangle)(\mathrm{Z} |\omega\rangle) |01\rangle & \text{w.p. } \frac{1}{16} \\
|10\rangle (\mathrm{Z} |\alpha\rangle)(\mathrm{X} |\omega\rangle) |10\rangle & \text{w.p. } \frac{1}{16} \\
|10\rangle (\mathrm{Z} |\alpha\rangle)(\mathrm{XZ} |\omega\rangle) |11\rangle & \text{w.p. } \frac{1}{16} \\
|11\rangle (\mathrm{XZ} |\alpha\rangle) |\omega\rangle |00\rangle & \text{w.p. } \frac{1}{16} \\
|11\rangle (\mathrm{XZ} |\alpha\rangle)(\mathrm{Z} |\omega\rangle) |01\rangle & \text{w.p. } \frac{1}{16} \\
|11\rangle (\mathrm{XZ} |\alpha\rangle)(\mathrm{X} |\omega\rangle) |10\rangle & \text{w.p. } \frac{1}{16} \\
|11\rangle (\mathrm{XZ} |\alpha\rangle)(\mathrm{XZ} |\omega\rangle) |11\rangle & \text{w.p. } \frac{1}{16}
\end{cases}$$

Now lets take a look at a couple of these cases to confirm or deny that the circuit will teleport the CNOT gate.

$$|00\rangle\,|\alpha\rangle\,|\omega\rangle\,|00\rangle$$

Just look the middle two qubits for now:

$$|\alpha\rangle\,|\omega\rangle = (\alpha_1\,|0\rangle + \alpha_2\,|1\rangle)(\omega_1\,|0\rangle + \omega_2\,|1\rangle) = \alpha_1\omega_1\,|00\rangle + \alpha_1\omega_2\,|01\rangle + \alpha_2\omega_1\,|10\rangle + \alpha_2\omega_2\,|11\rangle$$

$$\xrightarrow{\text{CNOT}} \alpha_1\omega_1\,|00\rangle + \alpha_1\omega_2\,|01\rangle + \alpha_2\omega_1\,|11\rangle + \alpha_2\omega_2\,|10\rangle$$

Since the other four qubits were all zero we don't need to apply any gates.

$$|10\rangle\,(\mathrm{Z}\,|\alpha\rangle)(\mathrm{X}\,|\omega\rangle)\,|10\rangle$$

$$(\mathrm{Z}\,|\alpha\rangle)(\mathrm{X}\,|\omega\rangle) = (\alpha_1\,|0\rangle - \alpha_2\,|1\rangle)(\omega_1\,|1\rangle + \omega_2\,|0\rangle) = \alpha_1\omega_1\,|01\rangle + \alpha_1\omega_2\,|00\rangle - \alpha_2\omega_1\,|11\rangle - \alpha_2\omega_2\,|10\rangle$$

$$\xrightarrow{\text{CNOT}} \alpha_1\omega_1\,|01\rangle + \alpha_1\omega_2\,|00\rangle - \alpha_2\omega_1\,|10\rangle - \alpha_2\omega_2\,|11\rangle$$

$$= \alpha_1\,|0\rangle\,(\omega_1\,|1\rangle + \omega_2\,|0\rangle) - \alpha_2\,|1\rangle\,(\omega_1\,|0\rangle + \omega_2\,|1\rangle)$$

We can see that the first qubit will always be $\mathrm{Z}\,|\alpha\rangle$

However, the second qubit could either be $\mathrm{X}\,|\omega\rangle$ or $|\omega\rangle$

Next we need to apply the gates to each qubit, however, they will be applied incorrectly

$$\xrightarrow{XZ\otimes Z} \mathrm{ZXZ}\,|\alpha\rangle \otimes (\mathrm{ZX}\,|\omega\rangle \text{ or } \mathrm{Z}\,|\omega\rangle)$$

This is not correct, so no the circuit does not perform gate teleportation. In order for the circuit to be correct the control wires for the X and Z gates would need to be switched.