Assignment 4

1. Spectral Decomposition

   (a) Spectral decompositions of the obserables X, Z, Y

   Calculate the eigenvalues of X:

$$det(\lambda I - X) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = det(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix})$$
$$\lambda^2 - 1 = 0$$
$$\lambda_1 = 1, \lambda_2 = -1$$

   Calculate the eigenvectors of X:

$$(X - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-x + y = 0$$
$$x - y = 0$$
$$x = 1, y = 1$$
$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(X - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x + y = 0$$
$$x + y = 0$$
$$x = 1, y = -1$$
$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   Spectral decomposition of X

$$X = \sum_{i=1}^{2} \lambda_i |\lambda_i\rangle \langle\lambda_i| = \lambda_1 |\lambda_1\rangle \langle\lambda_1| + \lambda_2 |\lambda_2\rangle \langle\lambda_2|$$
$$= (+1)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} + (-1)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}$$
$$= (+1)\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (-1)\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

   Calculate the eigen values of Z:

$$det(\lambda I - Z) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = det(\begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 1 \end{bmatrix})$$
$$(\lambda - 1)(\lambda + 1) = 0$$
$$\lambda_1 = 1, \lambda_2 = -1$$

$$(Z - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Apply Gaussian elimination

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$y = 0, x = 1$$
$$|\lambda_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(Z - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

   Apply Gaussian elimination

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x = 0, y = 1$$
$$|\lambda_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Spectral decomposition of Z

$$
\begin{aligned}
Z = \sum_{i=1}^{2} \lambda_i \left|\lambda_i\right\rangle \left\langle\lambda_i\right| &= \lambda_1 \left|\lambda_1\right\rangle \left\langle\lambda_1\right| + \lambda_2 \left|\lambda_2\right\rangle \left\langle\lambda_2\right| \\
&= (+1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
&= (+1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

Calculate the eigenvalues of Y

$$
det(\lambda I - Y) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}) = det(\begin{bmatrix} \lambda & i \\ -i & \lambda \end{bmatrix})
$$

$$
\lambda^2 - (-i * i) = 0
$$

$$
\lambda^2 - 1 = 0
$$

$$
\lambda_1 = 1, \lambda_2 = -1
$$

$$
(Y - \lambda_1 I) \cdot \vec{\lambda_1} = 0
$$

$$
\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Apply Gaussian elimination

$$
\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$$
x = -i, y = 1
$$

$$
\left|\lambda_1\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}
$$

$$
(Y - \lambda_2 I) \cdot \vec{\lambda_2} = 0
$$

$$
\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Apply Gaussian elimination

$$
\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

$$
x = i, y = 1
$$

$$
\left|\lambda_2\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}
$$

Spectral decomposition of Y

$$
\begin{aligned}
Y = \sum_{i=1}^{2} \lambda_i \left|\lambda_i\right\rangle \left\langle\lambda_i\right| &= \lambda_1 \left|\lambda_1\right\rangle \left\langle\lambda_1\right| + \lambda_2 \left|\lambda_2\right\rangle \left\langle\lambda_2\right| \\
&= (+1) \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 1 \end{bmatrix} + (-1) \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \end{bmatrix} \\
&= (+1) \frac{1}{2} \begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix} + (-1) \frac{1}{2} \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}
\end{aligned}
$$

(b) Show that $\langle\psi|P_r|\psi\rangle \geq 0$ and moreover, $\sum_r \langle\psi|P_r|\psi\rangle = 1$

(c) Compute $\langle X\rangle$ and $\langle Z\rangle$ under $|+\rangle$

Compute $\langle X\rangle$ under $|+\rangle$

$$
\begin{aligned}
\langle X\rangle_{|+\rangle} = \langle +|X|+\rangle &= \langle +|(\left|+\right\rangle \left\langle+\right| - \left|-\right\rangle \left\langle-\right|)|+\rangle \\
&= \langle +|+\rangle \langle +|+\rangle - \langle +|-\rangle \langle -|+\rangle \\
&= 1 * 1 - 0 * 0 = 1
\end{aligned}
$$

Compute $\langle Z\rangle$ under $|+\rangle$

$$
\begin{aligned}
\langle Z\rangle_{|+\rangle} = \langle +|Z|+\rangle &= \langle +|(\left|0\right\rangle \left\langle0\right| - \left|1\right\rangle \left\langle1\right|)|+\rangle \\
&= \langle +|0\rangle \langle 0|+\rangle - \langle +|1\rangle \langle 1|+\rangle \\
&= (\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}) - (\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{1}{2} = 0
\end{aligned}
$$

(d) Compute $\langle X_1 Z_2 \rangle$ under $|\beta_{00}\rangle$

$$\begin{aligned}\langle X_1 Z_2 \rangle_{|\beta_{00}\rangle} &= \frac{1}{2}[\langle 00| + \langle 11| \, (X_1 Z_2) \, |00\rangle + |11\rangle] \\ &= \frac{1}{2}[\langle 00|X_1 Z_2|00\rangle + \langle 00|X_1 Z_2|11\rangle + \langle 11|X_1 Z_2|00\rangle + \langle 1|X_1 Z_2|11\rangle] \\ &= \frac{1}{2}[0 * 1 + 1 * 0 + 1 * 0 + 0 * -1] = 0\end{aligned}$$

2. CHSH Game

(a) (T/F)$\langle \mathcal{O} \rangle$ holds

(b) (T/F) $\langle QS \rangle = \frac{1}{\sqrt{2}}$.

$$\begin{aligned}QS &= Z_1 \otimes \frac{-1}{\sqrt{2}}(Z_2 + X_2) = \frac{-1}{\sqrt{2}}(Z_1 Z_2 + Z_1 X_2) \\ \langle QS \rangle_{|\beta_{11}\rangle} &= \langle \beta_{11}|QS|\beta_{11}\rangle = \frac{\langle 01| - \langle 10|}{\sqrt{2}}[\frac{-1}{\sqrt{2}}(Z_1 Z_2 + Z_1 X_2)]\frac{|01\rangle - |10\rangle}{\sqrt{2}} \\ &= \frac{-1}{2\sqrt{2}}[(\langle 01| - \langle 10|)Z_1 Z_2(|01\rangle - \langle 10|) + (\langle 01| - \langle 10|)Z_1 X_2(|01\rangle - \langle 10|)] \\ &= \frac{-1}{2\sqrt{2}}[\langle 01|Z_1 Z_2|01\rangle - \langle 01|Z_1 Z_2|10\rangle - \langle 10|Z_1 Z_2|01\rangle + \langle 10|Z_1 Z_2|10\rangle \\ &\quad + \langle 01|Z_1 X_2|01\rangle - \langle 01|Z_1 X_2|10\rangle - \langle 10|Z_1 X_2|01\rangle + \langle 10|Z_1 X_2|10\rangle] \\ &= \frac{-1}{2\sqrt{2}}[-1 - 0 - 0 + (-1) + 0 - 0 - 0 + 0] = \frac{-1}{2\sqrt{2}} * -2 = \frac{1}{\sqrt{2}}\end{aligned}$$

True.

(c) (T/F) $\langle RS \rangle = \frac{1}{\sqrt{2}}$.

$$\begin{aligned}RS &= X_1 \otimes \frac{-1}{\sqrt{2}}(Z_2 + X_2) = \frac{-1}{\sqrt{2}}(X_1 Z_2 + X_1 X_2) \\ \langle RS \rangle_{|\beta_{11}\rangle} &= \langle \beta_{11}|RS|\beta_{11}\rangle = \frac{\langle 01| - \langle 10|}{\sqrt{2}}[\frac{-1}{\sqrt{2}}(X_1 Z_2 + X_1 X_2)]\frac{|01\rangle - |10\rangle}{\sqrt{2}} \\ &= \frac{-1}{2\sqrt{2}}[(\langle 01| - \langle 10|)X_1 Z_2(|01\rangle - \langle 10|) + (\langle 01| - \langle 10|)X_1 X_2(|01\rangle - \langle 10|)] \\ &= \frac{-1}{2\sqrt{2}}[\langle 01|X_1 Z_2|01\rangle - \langle 01|X_1 Z_2|10\rangle - \langle 10|X_1 Z_2|01\rangle + \langle 10|X_1 Z_2|10\rangle \\ &\quad + \langle 01|X_1 X_2|01\rangle - \langle 01|X_1 X_2|10\rangle - \langle 10|X_1 X_2|01\rangle + \langle 10|X_1 X_2|10\rangle] \\ &= \frac{-1}{2\sqrt{2}}[0 - 0 - 0 + 0 + 0 - 1 - 1 + 0] = \frac{-1}{2\sqrt{2}} * -2 = \frac{1}{\sqrt{2}}\end{aligned}$$

True.

(d) (T/F) $\langle RT \rangle = \frac{1}{\sqrt{2}}$.

$$RT = X_1 \otimes \frac{1}{\sqrt{2}}(Z_2 - X_2) = \frac{1}{\sqrt{2}}(X_1 Z_2 - X_1 X_2)$$

$$\langle RT \rangle_{|\beta_{11}\rangle} = \frac{1}{2\sqrt{2}}[(\langle 01| - \langle 10|)X_1 Z_2(|01\rangle - |10\rangle) - (\langle 01| - \langle 10|)X_1 X_2(|01\rangle - |10\rangle)]$$

From parts b and d we know that $\langle X_1 Z_2 \rangle_{|\beta_{11}\rangle} = 0$ and that $\langle X_1 X_2 \rangle_{|\beta_{11}\rangle} = -2$.

$$= \frac{1}{2\sqrt{2}}[0 - (-2)] = \frac{1}{2\sqrt{2}} * 2 = \frac{1}{\sqrt{2}}$$

True.

(e) (T/F) $\langle QT \rangle = -\frac{1}{\sqrt{2}}$.

$$QT = Z_1 \otimes \frac{1}{\sqrt{2}}(Z_2 - X_2) = \frac{1}{\sqrt{2}}(Z_1 Z_2 - Z_1 X_2)$$

$$\langle QT \rangle_{|\beta_{11}\rangle} = \frac{1}{2\sqrt{2}}[(\langle 01| - \langle 10|)Z_1 Z_2(|01\rangle - |10\rangle) - (\langle 01| - \langle 10|)Z_1 X_2(|01\rangle - |10\rangle)]$$

From parts b and d we know that $\langle Z_1 Z_2 \rangle_{|\beta_{11}\rangle} = -2$ and that $\langle Z_1 X_2 \rangle_{|\beta_{11}\rangle} = 0$.

$$= \frac{1}{2\sqrt{2}}[(-2) - 0] = \frac{1}{2\sqrt{2}} * -2 = \frac{-1}{\sqrt{2}}$$

True.

(f) (T/F) If Q, R, S, T are random variables with ± 1 values then $\mathbb{E}$[QS] + $\mathbb{E}$[RS] + $\mathbb{E}$[RT] - $\mathbb{E}$[QT] ≤ 2

(g) (T/F) $\langle \mathcal{O} \rangle = 2\sqrt{2}$ (Does this mean Quantum 1, Einstein 0?)