Assignment 2

1. Consider the Bell state $|\beta_{00}\rangle$, suppose both qubits are measured in the diagonal basis. What are the outcomes?

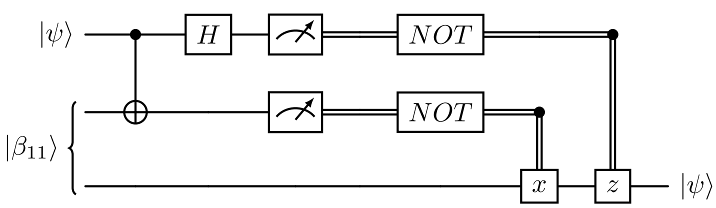
   We know that $|\beta_{00}\rangle$ is equivalent to $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ in order to mesaure in the diagonal basis. To do this, we need to express $|\beta_{00}\rangle$ in terms of $\{|+\rangle, |-\rangle\}$ rather than $\{|0\rangle, |1\rangle\}$. Luckily we know that $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and that $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$[1].

   Subsituting this questions into $|\beta_{00}\rangle$:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$= \frac{1}{\sqrt{2}}[(\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)) + (\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle))]$$
$$= \frac{1}{\sqrt{2}}[\frac{1}{2}((|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle)) + \frac{1}{2}((|+\rangle - |-\rangle) \otimes (|+\rangle - |-\rangle))]$$
$$= \frac{1}{\sqrt{2}}\frac{1}{2}[(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle) + (|++\rangle - |+-\rangle - |-+\rangle + |--\rangle)]$$
$$= \frac{1}{\sqrt{2}}[(|++\rangle + |--\rangle)]$$

2. Describe a quantum teleportation circuit when ALice and Bob shared the Bell state $|\beta_{11}\rangle$ instead of $|\beta_{00}\rangle$.

   The following circuit will successfully teleport an arbitrary qubit, $|\psi\rangle$, between Alice and Bob.



   Proof:

[1] T. Wong, Introduction to Classical and Quantum Computing, Rooted Grove, 2022. Page 87

$$
\begin{aligned}
|\psi\rangle \otimes |\beta_{11}\rangle &= (a|0\rangle + b|1\rangle) \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} \\
&= a|0\rangle \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} + b|1\rangle \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}[a|001\rangle - a|010\rangle + b|101\rangle - b|110\rangle] \\
&\xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}[a|001\rangle - a|010\rangle + b|111\rangle - b|100\rangle] \\
&\xrightarrow{H\otimes I\otimes I} \frac{1}{\sqrt{2}}[a\frac{|0\rangle + |1\rangle}{\sqrt{2}}(|01\rangle - |10\rangle) + b\frac{|0\rangle - |1\rangle}{\sqrt{2}}(|11\rangle - |00\rangle)] \\
&= \frac{1}{2}[(a|0\rangle + a|1\rangle)(|01\rangle - |10\rangle) + (b|0\rangle - b|1\rangle)(|11\rangle - |00\rangle)] \\
&= \frac{1}{2}[a|001\rangle - a|010\rangle + a|101\rangle - a|110\rangle + b|011\rangle - b|000\rangle + b|111\rangle - b|100\rangle] \\
&= \frac{1}{2}[|00\rangle(a|1\rangle - b|0\rangle) - |01\rangle(a|0\rangle - b|1\rangle) + |10\rangle(a|1\rangle + b|0\rangle) - |11\rangle(a|0\rangle + b|1\rangle)]
\end{aligned}
$$

Now messuring the first two qubits gets us:

$$
\begin{cases}
|00\rangle XZ|\psi\rangle & \text{w.p. } \frac{1}{4} \\
|01\rangle Z|\psi\rangle & \text{w.p. } \frac{1}{4} \\
|10\rangle X|\psi\rangle & \text{w.p. } \frac{1}{4} \\
|11\rangle |\psi\rangle & \text{w.p. } \frac{1}{4}
\end{cases}
$$