

## Assignment 2

1. Consider the Bell state  $|\beta_{00}\rangle$ , suppose both qubits are measured in the diagonal basis. What are the outcomes?

We know that  $|\beta_{00}\rangle$  is equivalent to  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  in order to measure in the diagonal basis. To do this, we need to express  $|\beta_{00}\rangle$  in terms of  $\{|+\rangle, |-\rangle\}$  rather than  $\{|0\rangle, |1\rangle\}$ . Luckily we know that  $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$  and that  $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$ <sup>1</sup>.

Substituting these questions into  $|\beta_{00}\rangle$ :

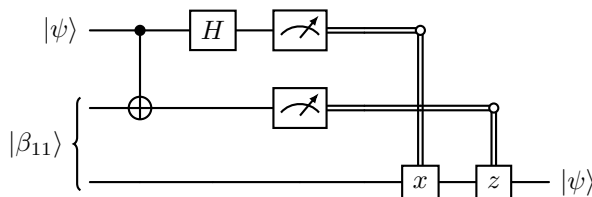
$$\begin{aligned}
 & \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
 = & \frac{1}{\sqrt{2}} \left[ \left( \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \right) \otimes \left( \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \right) + \left( \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \right) \otimes \left( \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \right) \right] \\
 = & \frac{1}{\sqrt{2}} \left[ \frac{1}{2}((|+\rangle + |-\rangle) \otimes (|+\rangle + |-\rangle)) + \frac{1}{2}((|+\rangle - |-\rangle) \otimes (|+\rangle - |-\rangle)) \right] \\
 = & \frac{1}{\sqrt{2}} \frac{1}{2} [(|++\rangle + |+-\rangle + |-+\rangle + |--\rangle) + (|++\rangle - |+-\rangle - |-+\rangle + |--\rangle)] \\
 = & \frac{1}{\sqrt{2}} [(|++\rangle + |--\rangle)]
 \end{aligned}$$

Messuring the state:

$$\begin{cases} |++\rangle & \text{w.p. } \frac{1}{2} \\ |+-\rangle & \text{w.p. } 0 \\ |-+\rangle & \text{w.p. } 0 \\ |--\rangle & \text{w.p. } \frac{1}{2} \end{cases}$$

2. Describe a quantum teleportation circuit when Alice and Bob shared the Bell state  $|\beta_{11}\rangle$  instead of  $|\beta_{00}\rangle$ .

The following circuit will successfully teleport an arbitrary qubit,  $|\psi\rangle$ , between Alice and Bob.



<sup>1</sup>T. Wong, Introduction to Classical and Quantum Computing, Rooted Grove, 2022. Page 87

Proof:

$$\begin{aligned}
|\psi\rangle \otimes |\beta_{11}\rangle &= (a|0\rangle + b|1\rangle) \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} \\
&= a|0\rangle \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} + b|1\rangle \otimes \frac{|01\rangle - |10\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}[a|001\rangle - a|010\rangle + b|101\rangle - b|110\rangle] \\
&\xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}[a|001\rangle - a|010\rangle + b|111\rangle - b|100\rangle] \\
&\xrightarrow{H \otimes I \otimes I} \frac{1}{\sqrt{2}}[a \frac{|0\rangle + |1\rangle}{\sqrt{2}}(|01\rangle - |10\rangle) + b \frac{|0\rangle - |1\rangle}{\sqrt{2}}(|11\rangle - |00\rangle)] \\
&= \frac{1}{2}[(a|0\rangle + a|1\rangle)(|01\rangle - |10\rangle) + (b|0\rangle - b|1\rangle)(|11\rangle - |00\rangle)] \\
&= \frac{1}{2}[a|001\rangle - a|010\rangle + a|101\rangle - a|110\rangle + b|011\rangle - b|000\rangle + b|111\rangle - b|100\rangle] \\
&= \frac{1}{2}[|00\rangle(a|1\rangle - b|0\rangle) - |01\rangle(a|0\rangle - b|1\rangle) + |10\rangle(a|1\rangle + b|0\rangle) - |11\rangle(a|0\rangle + b|1\rangle)]
\end{aligned}$$

Now measuring the first two qubits gets us:

$$\begin{cases} |00\rangle XZ|\psi\rangle & \text{w.p. } \frac{1}{4} \\ |01\rangle Z|\psi\rangle & \text{w.p. } \frac{1}{4} \\ |10\rangle X|\psi\rangle & \text{w.p. } \frac{1}{4} \\ |11\rangle |\psi\rangle & \text{w.p. } \frac{1}{4} \end{cases}$$

3. Suppose Alice and Bob shared a Bell state  $|\beta\rangle$  and wanted to share a secret classical bit. Explain how they can do this without any commincation. What if they forget whether to measure in the rectilinear or diagonal basis? Can they still share the secret bit?
4. Gate teleportation

Input

$$\begin{aligned}
&|\alpha\rangle \otimes |\beta_{00}\rangle \otimes |\beta_{00}\rangle \otimes |\omega\rangle \\
&= (\alpha_1|0\rangle + \alpha_2|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes (\omega_1|0\rangle + \omega_2|1\rangle) \\
&= \frac{1}{\sqrt{2}}[\alpha_1|000\rangle + \alpha_1|011\rangle + \alpha_2|100\rangle + \alpha_2|111\rangle] \otimes \frac{1}{\sqrt{2}}[\omega_1|000\rangle + \omega_2|001\rangle + \omega_1|110\rangle + \omega_2|111\rangle]
\end{aligned}$$

Bell Basis Measurement circuit

$$\begin{aligned}
& \xrightarrow{\text{CNOT(S)}} \frac{1}{\sqrt{2}}[\alpha_1 |000\rangle + \alpha_1 |011\rangle + \alpha_2 |110\rangle + \alpha_2 |101\rangle] \otimes \frac{1}{\sqrt{2}}[\omega_1 |000\rangle + \omega_2 |011\rangle + \omega_1 |110\rangle + \omega_2 |101\rangle] \\
&= \frac{1}{\sqrt{2}}[\alpha_1 |0\rangle (|00\rangle + |11\rangle) + \alpha_2 |1\rangle (|10\rangle + |01\rangle)] \otimes \frac{1}{\sqrt{2}}[(|00\rangle + |11\rangle)\omega_1 |0\rangle + (|01\rangle + |10\rangle)\omega_2 |1\rangle] \\
&\quad \xrightarrow{H \otimes I \otimes I \otimes I \otimes I \otimes H} \frac{1}{\sqrt{2}}[\alpha_1 (\frac{|0\rangle + |1\rangle}{\sqrt{2}})(|00\rangle + |11\rangle) + \alpha_2 (\frac{|0\rangle - |1\rangle}{\sqrt{2}})(|10\rangle + |01\rangle)] \\
&\quad \otimes \frac{1}{\sqrt{2}}[(|00\rangle + |11\rangle)\omega_1 (\frac{|0\rangle + |1\rangle}{\sqrt{2}}) + (|01\rangle + |10\rangle)\omega_2 (\frac{|0\rangle - |1\rangle}{\sqrt{2}})] \\
&= \frac{1}{2}[\alpha_1 |000\rangle + \alpha_1 |011\rangle + \alpha_1 |100\rangle + \alpha_1 |111\rangle + \alpha_2 |001\rangle + \alpha_2 |010\rangle - \alpha_2 |101\rangle - \alpha_2 |110\rangle] \\
&\quad \otimes \frac{1}{2}[\omega_1 |000\rangle + \omega_1 |001\rangle + \omega_1 |110\rangle + \omega_1 |111\rangle + \omega_2 |010\rangle - \omega_2 |011\rangle + \omega_2 |100\rangle - \omega_2 |101\rangle] \\
&= \frac{1}{2}[|00\rangle (\alpha_1 |0\rangle + \alpha_2 |1\rangle) + |01\rangle (\alpha_1 |1\rangle + \alpha_2 |0\rangle) + |10\rangle (\alpha_1 |0\rangle - \alpha_2 |1\rangle) + |11\rangle (\alpha_1 |1\rangle - \alpha_2 |0\rangle)] \\
&\quad \otimes \frac{1}{2}[(\omega_1 |0\rangle + \omega_2 |1\rangle) |00\rangle + (\omega_1 |0\rangle - \omega_2 |1\rangle) |01\rangle + (\omega_1 |1\rangle + \omega_2 |0\rangle) |10\rangle + (\omega_1 |1\rangle - \omega_2 |0\rangle) |11\rangle] \\
&= \frac{1}{2}[|00\rangle |\alpha\rangle + |01\rangle (X |\alpha\rangle) + |10\rangle (Z |\alpha\rangle) + |11\rangle (XZ |\alpha\rangle)] \otimes \frac{1}{2}[|\omega\rangle |00\rangle + (Z |\omega\rangle) |01\rangle \\
&\quad + (X |\omega\rangle) |10\rangle + (XZ |\omega\rangle) |11\rangle]
\end{aligned}$$