Assignment 1

1. Show that the Hadamard gate *almost* commutes with the Pauli gates.

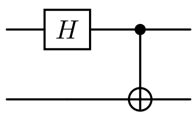
   Using the Pauli gates X and Z we can show that the Hadamard gate almost commutes.

   $$\text{X*H} = \text{H*Z}$$

   $$X * H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

   $$H * Z = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

2. Show that the Bell states form a basis of all two-qubit states.

   Bell states $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$ can be combind to form $|00\rangle$.

   $$\begin{aligned} &|\beta_{00}\rangle + |\beta_{10}\rangle \\ &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} + \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\ &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle - \frac{1}{\sqrt{2}}|11\rangle \\ &= \frac{2}{\sqrt{2}}|00\rangle + 0\,|11\rangle \end{aligned}$$

   Normalize with $\frac{1}{\sqrt{2}}$:

   $$\begin{aligned} &\frac{1}{\sqrt{2}} * (\frac{2}{\sqrt{2}}|00\rangle + 0\,|11\rangle) \\ &= \frac{2}{2}|00\rangle + 0\,|11\rangle \end{aligned}$$

   $$\therefore |00\rangle = \frac{1}{\sqrt{2}}(|\beta_{00}\rangle + |\beta_{10}\rangle)$$

Bell states $|\beta_{00}\rangle$ and $|\beta_{10}\rangle$ can be combind to form $|11\rangle$.

$$\begin{aligned}
&|\beta_{00}\rangle - |\beta_{10}\rangle \\
&= \frac{|00\rangle + |11\rangle}{\sqrt{2}} - \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle - (\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle) \\
&= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle - \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
&= 0\,|00\rangle + \frac{2}{\sqrt{2}}|11\rangle
\end{aligned}$$

Normalize with $\frac{1}{\sqrt{2}}$:

$$\begin{aligned}
&\frac{1}{\sqrt{2}} * (0\,|00\rangle + \frac{2}{\sqrt{2}}|11\rangle) \\
&= 0\,|00\rangle + \frac{2}{2}|11\rangle
\end{aligned}$$

$$\therefore |11\rangle = \frac{1}{\sqrt{2}}(|\beta_{00}\rangle - |\beta_{10}\rangle)$$

Similary, $|\beta_{01}\rangle$ and $|\beta_{11}\rangle$ can be combind to form $|01\rangle$ and $|10\rangle$.

$$\begin{aligned}
&|\beta_{01}\rangle + |\beta_{11}\rangle \\
&= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) + \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\
&= \frac{2}{\sqrt{2}}|01\rangle + 0\,|10\rangle
\end{aligned}$$

$$\therefore |01\rangle = \frac{1}{\sqrt{2}}(|\beta_{01}\rangle + |\beta_{11}\rangle)$$

$$\begin{aligned}
&|\beta_{01}\rangle - |\beta_{11}\rangle \\
&= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) - \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\
&= 0\,|01\rangle + \frac{2}{\sqrt{2}}|10\rangle
\end{aligned}$$

$$\therefore |10\rangle = \frac{1}{\sqrt{2}}(|\beta_{01}\rangle - |\beta_{11}\rangle)$$

3. Explain what does it mean to perform a Bell basis measurement of a two-qubit state.

   Performing a Bell basis measurement on an arbitray two-qubit state would determine which of the four bell states those qubits are in (with a certain probability for each β). For example if we put the state $|00\rangle$ through the circuit we can get:

$$|00\rangle \xrightarrow{\text{CNOT}} |00\rangle \xrightarrow{H\otimes I} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}[|00\rangle + |10\rangle]$$

   Since we know we can express any 2 qubit state as a linear combination of two Bell states we can re-write the result to be in this form.

$$\frac{1}{\sqrt{2}}[|00\rangle + |10\rangle] = \frac{1}{\sqrt{2}}[\frac{1}{\sqrt{2}}(|\beta_{00}\rangle + |\beta_{10}\rangle) + \frac{1}{\sqrt{2}}(|\beta_{01}\rangle - |\beta_{11}\rangle)]$$
$$= \frac{1}{2}[|\beta_{00}\rangle + |\beta_{10}\rangle + |\beta_{01}\rangle - |\beta_{11}\rangle]$$

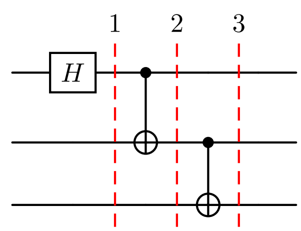
   We can also rewrite the measurement:

$$\begin{cases} |\beta_{00}\rangle & \text{w.p. } \frac{1}{4} \\ |\beta_{01}\rangle & \text{w.p. } \frac{1}{4} \\ |\beta_{10}\rangle & \text{w.p. } \frac{1}{4} \\ |\beta_{11}\rangle & \text{w.p. } \frac{1}{4} \end{cases}$$

4. Describe a quantum circuit that starting with $|000\rangle$ prepares the output state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

   The following 3 stage circuit will produce the desiried output state:



   - Stage 1 - Hadamard Gate:

$$|0\rangle \otimes |0\rangle \otimes |0\rangle \xrightarrow{H\otimes I\otimes I} \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}(|000\rangle + |100\rangle)$$

   - Stage 2 - 1^{st} Control Not:

     Just looking at the first two qubits of the superposition. The first qubit is the control bit and second qubit is the bit we may modifiy.

$$\frac{1}{\sqrt{2}}(|000\rangle + |100\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)$$

- Stage 3 - 2[nd] Control Not:

  Just looking at the last two qubits of the superposition. The second qubit is the control bit and the third qubit is the bit we may modify

$$\frac{1}{\sqrt{2}}(|000\rangle + |110\rangle) \xrightarrow{\text{CNOT}} \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

5. Suppose that Alice and Bob share a Bell state, and that Bob and Charlie share another a different Bell state. Confirm (or deny) that the protocol below where we perform a Bell measurement on Bob's particles will create an entangled pair between Alice and Charlie.

   Yes, this circuit will produce an entangled pair between Alice and Charlie. We can observe the following with using the input $|0000\rangle$:

$$|00\rangle \otimes |00\rangle \xrightarrow{\text{create bell states}} \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle] \otimes \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$$
$$= \frac{1}{\sqrt{2}}[|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle]$$

Rewrite middle states in terms of Bell states.

$$\frac{1}{2\sqrt{2}}[|0\rangle\left(|\beta_{00}\rangle + |\beta_{10}\rangle\right)|0\rangle + |0\rangle\left(|\beta_{01}\rangle + |\beta_{11}\rangle\right)|1\rangle + |1\rangle\left(|\beta_{01}\rangle - |\beta_{11}\rangle\right)|0\rangle + |1\rangle\left(|\beta_{00}\rangle - |\beta_{10}\rangle\right)|1\rangle]$$
$$= \frac{1}{2\sqrt{2}}[|0\beta_{00}0\rangle + |0\beta_{10}0\rangle + |0\beta_{01}1\rangle + |0\beta_{11}1\rangle + |1\beta_{01}0\rangle - |1\beta_{11}0\rangle + |1\beta_{00}1\rangle + |1\beta_{10}1\rangle]$$
$$= \frac{1}{2\sqrt{2}}[|\beta_{00}\rangle\left(|00\rangle + |11\rangle\right) + |\beta_{01}\rangle\left(|01\rangle + |10\rangle\right) + |\beta_{10}\rangle\left(|00\rangle - |11\rangle\right) + |\beta_{11}\rangle\left(|01\rangle - |10\rangle\right)]$$
$$= \frac{1}{2}[|\beta_{00}\rangle\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$
$$+ |\beta_{01}\rangle\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$+ |\beta_{10}\rangle\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$
$$+ |\beta_{11}\rangle\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)]$$

The next step in the circuit would be to messure the middle two qubits in the Bell basis. As you can see no matter which Bell state we messure the result will always be an entangled pair with the first and last qubits.