Assignment 4

1. Spectral Decomposition

   (a) Spectral decompositions of the obserables X, Z, Y
   Calculate the eigenvalues of X:

$$det(\lambda I - X) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = det(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix})$$
$$\lambda^2 - 1 = 0$$
$$\lambda_1 = 1, \lambda_2 = -1$$

   Calculate the eigenvectors of X:

$$(X - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$
$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-x + y = 0$$
$$x - y = 0$$
$$x = 1, y = 1$$
$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(X - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x + y = 0$$
$$x + y = 0$$
$$x = 1, y = -1$$
$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

   Spectral decomposition of X

$$X = \sum_{i=1}^{2} \lambda_i |\lambda_i\rangle \langle\lambda_i| = \lambda_1 |\lambda_1\rangle \langle\lambda_1| + \lambda_2 |\lambda_2\rangle \langle\lambda_2|$$
$$= (+1)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} + (-1)\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \end{bmatrix}$$
$$= (+1)\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + (-1)\frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

   Calculate the eigen values of Z:

$$det(\lambda I - Z) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = det(\begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 1 \end{bmatrix})$$
$$(\lambda - 1)(\lambda + 1) = 0$$
$$\lambda_1 = 1, \lambda_2 = -1$$

$$(Z - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$
$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
Apply Gaussian elimination
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$y = 0, x = 1$$
$$|\lambda_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(Z - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$
$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
Apply Gaussian elimination
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x = 0, y = 1$$
$$|\lambda_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Spectral decomposition of Z

$$Z = \sum_{i=1}^{2} \lambda_i |\lambda_i\rangle \langle\lambda_i| = \lambda_1 |\lambda_1\rangle \langle\lambda_1| + \lambda_2 |\lambda_2\rangle \langle\lambda_2|$$
$$= (+1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$= (+1) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Calculate the eigenvalues of Y

$$det(\lambda I - Y) = det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}) = det(\begin{bmatrix} \lambda & i \\ -i & \lambda \end{bmatrix})$$
$$\lambda^2 - (-i * i) = 0$$
$$\lambda^2 - 1 = 0$$
$$\lambda_1 = 1, \lambda_2 = -1$$

$$(Y - \lambda_1 I) \cdot \vec{\lambda_1} = 0$$
$$\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply Gaussian elimination

$$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x = -i, y = 1$$
$$|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$(Y - \lambda_2 I) \cdot \vec{\lambda_2} = 0$$
$$\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply Gaussian elimination

$$\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x = i, y = 1$$
$$|\lambda_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Spectral decomposition of Y

$$Y = \sum_{i=1}^{2} \lambda_i |\lambda_i\rangle \langle\lambda_i| = \lambda_1 |\lambda_1\rangle \langle\lambda_1| + \lambda_2 |\lambda_2\rangle \langle\lambda_2|$$
$$= (+1)\frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -i & 1 \end{bmatrix} + (-1)\frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 \end{bmatrix}$$
$$= (+1)\frac{1}{2} \begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix} + (-1)\frac{1}{2} \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}$$

(b) Show that $\langle\psi|P_r|\psi\rangle \geq 0$ and moreover, $\sum_r \langle\psi|P_r|\psi\rangle = 1$

(c) Compute $\langle X\rangle$ and $\langle Z\rangle$ under $|+\rangle$

Compute $\langle X\rangle$ under $|+\rangle$

$$\langle X\rangle_{|+\rangle} = \langle +|X|+\rangle = \langle +|(|+\rangle \langle +| - |-\rangle \langle -|)|+\rangle$$
$$= \langle +|+\rangle \langle +|+\rangle - \langle +|-\rangle \langle -|+\rangle$$
$$= 1 * 1 - 0 * 0 = 1$$

Compute $\langle Z\rangle$ under $|+\rangle$

$$\langle Z\rangle_{|+\rangle} = \langle +|Z|+\rangle = \langle +|(|0\rangle \langle 0| - |1\rangle \langle 1|)|+\rangle$$
$$= \langle +|0\rangle \langle 0|+\rangle - \langle +|1\rangle \langle 1|+\rangle$$
$$= (\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}) - (\frac{1}{\sqrt{2}} * \frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{1}{2} = 0$$

(d) Compute $\langle X_1 Z_2 \rangle$ under $|\beta_{00}\rangle$

$$\langle X_1 Z_2 \rangle_{|\beta_{00}\rangle} = \langle \beta_{00}| X_1 Z_2 |\beta_{00}\rangle = \frac{\langle 00| + \langle 11|}{\sqrt{2}} (X_1 \otimes Z_2) \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$
$$= [\langle 0|X_1|0\rangle + \langle 0|X_1|1\rangle + \langle 1|X_1|0\rangle + \langle 1|X_1|1\rangle] \otimes [\langle 0|Z_2|0\rangle + \langle 0|Z_2|1\rangle + \langle 1|Z_2|0\rangle + \langle 1|Z_2|1\rangle]$$
$$= [\langle 0|(|+\rangle\langle +| - |-\rangle\langle -|)|0\rangle + \langle 0|(|+\rangle\langle +| - |-\rangle\langle -|)|1\rangle$$
$$+ \langle 1|(|+\rangle\langle +| - |-\rangle\langle -|)|0\rangle + \langle 1|(|+\rangle\langle +| - |-\rangle\langle -|)|1\rangle]$$
$$\otimes$$
$$[\langle 0|(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle + \langle 0|(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle$$
$$+ \langle 1|(|0\rangle\langle 0| - |1\rangle\langle 1|)|0\rangle + \langle 1|(|0\rangle\langle 0| - |1\rangle\langle 1|)|1\rangle]$$
$$= [\langle 0|+\rangle\langle +|0\rangle - \langle 0|-\rangle\langle -|0\rangle + \langle 0|+\rangle\langle +|1\rangle - \langle 0|-\rangle\langle -|1\rangle$$
$$+ \langle 1|+\rangle\langle +|0\rangle - \langle 1|-\rangle\langle -|0\rangle + \langle 1|+\rangle\langle +|1\rangle - \langle 1|-\rangle\langle -|1\rangle]$$
$$\otimes$$
$$[\langle 0|0\rangle\langle 0|0\rangle - \langle 0|1\rangle\langle 1|0\rangle + \langle 0|0\rangle\langle 0|1\rangle - \langle 0|1\rangle\langle 1|1\rangle$$
$$+ \langle 1|0\rangle\langle 0|0\rangle - \langle 1|1\rangle\langle 1|0\rangle + \langle 1|0\rangle\langle 0|1\rangle - \langle 1|1\rangle\langle 1|1\rangle]$$
$$= [\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2}] \otimes [1 - 0 + 0 - 0 + 0 - 0 + 0 - 1] = 0$$

2. CHSH Game