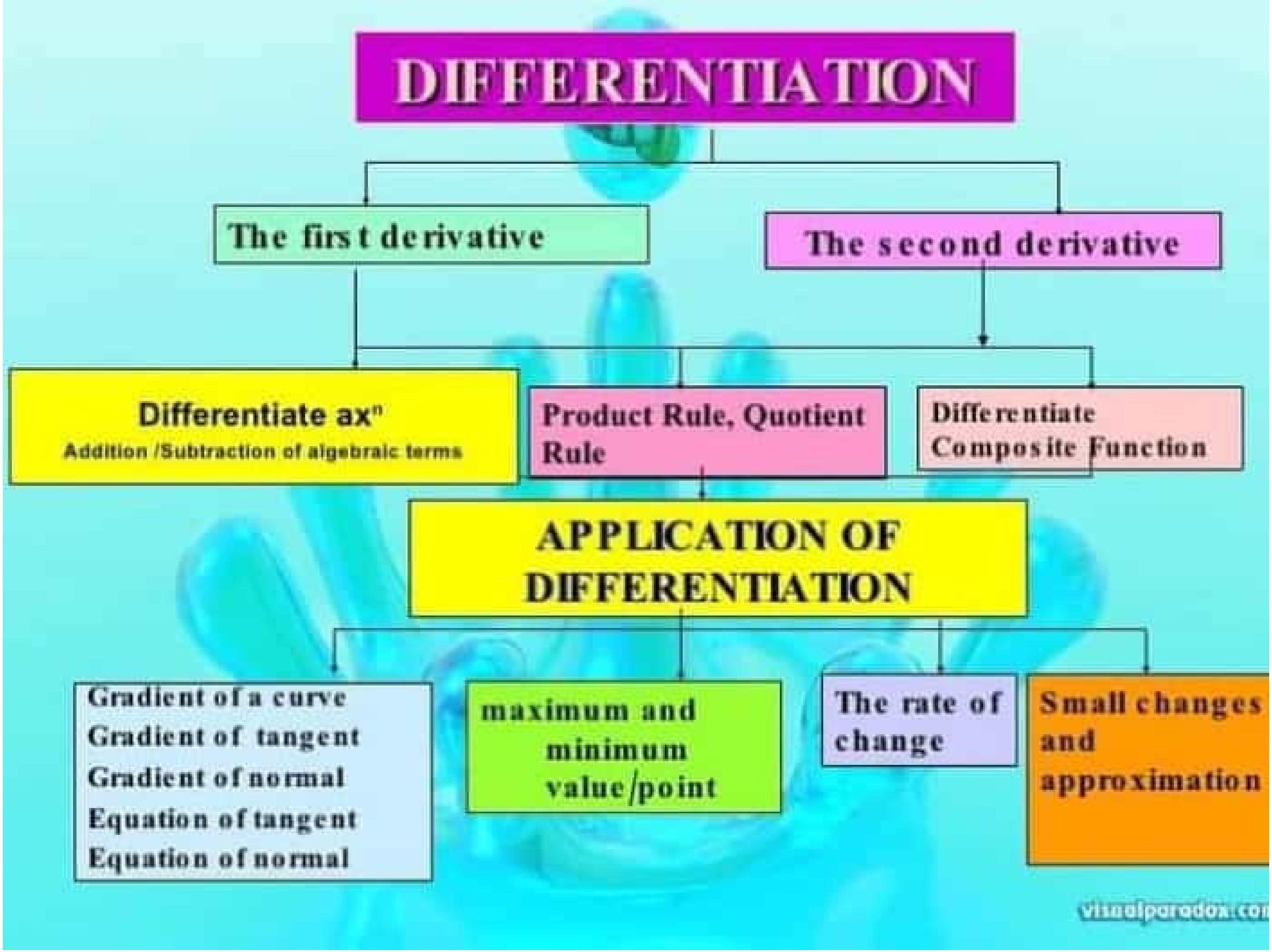
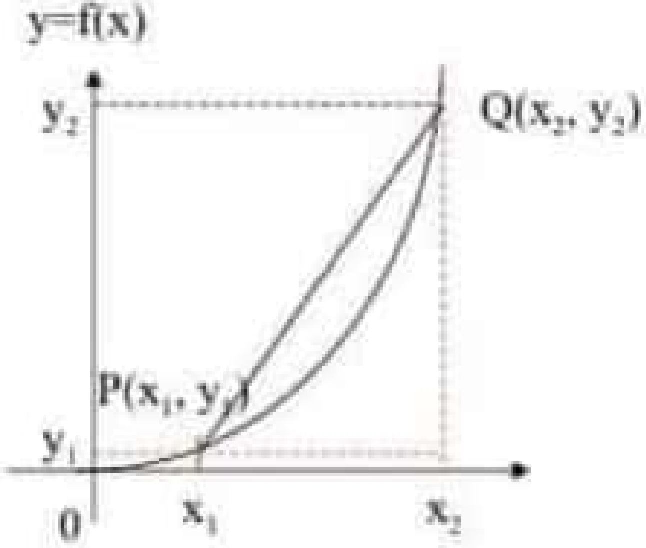

Introduction
to Differentiation

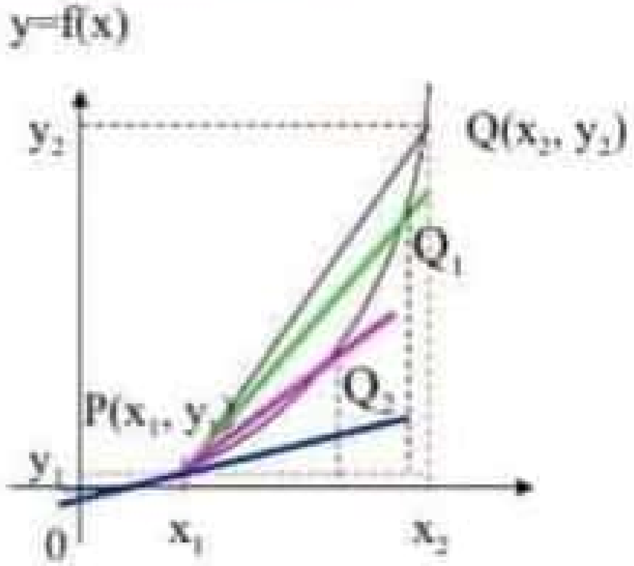


DIFFERENTIATION
The first derivative
The second derivative
Differentiate ax^n
Addition /Subtraction of algebraic terms
Product Rule, Quotient Rule
Differentiate Composite Function
APPLICATION OF DIFFERENTIATION
Gradient of a curve
Gradient of tangent
Gradient of normal
Equation of tangent
Equation of normal
maximum and minimum value/point
The rate of change
Small changes and approximation

CONCEPT OF DIFFERENTIATION



Gradient of chord = $\dfrac{y_2 - y_1}{x_2 - x_1}$



When point Q approaches point P (i.e $x_2 \longrightarrow x_1$)

Then $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\delta y}{\delta x}$

When $x_2 \longrightarrow x_1$, $\delta x \longrightarrow 0$

Then $$\lim_{x_2 \to x_1} \left(\frac{y_2 - y_1}{x_2 - x_1} \right) = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Differentiation Technicques

Differentiate ax^n

(b) If y = a, a is a constant ---- $\frac{dy}{dx} = 0$

(d) If y = ax, a is a constant---- $\frac{dy}{dx} = a$

- **If y= ax^n, a is a constant ----** $\frac{dy}{dx} = nax^{n-1}$

(d) Differentiate Addition, Subtraction of algebraic terms.

If $f(x) = p(x) \pm q(x)$, then

$$f'(x) = p'(x) \pm q'(x)$$

Differentiate Product/ Quotient of two Polynomials

- (a) If y = uv, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

- (b) If $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Differentiate Composite Function

If $y = f(u)$ and $u = g(x)$,

then, the composite function

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

or

$$\frac{d}{dx}(ax+b)^n = an(ax+b)^{n-1}$$

The Second Derivative

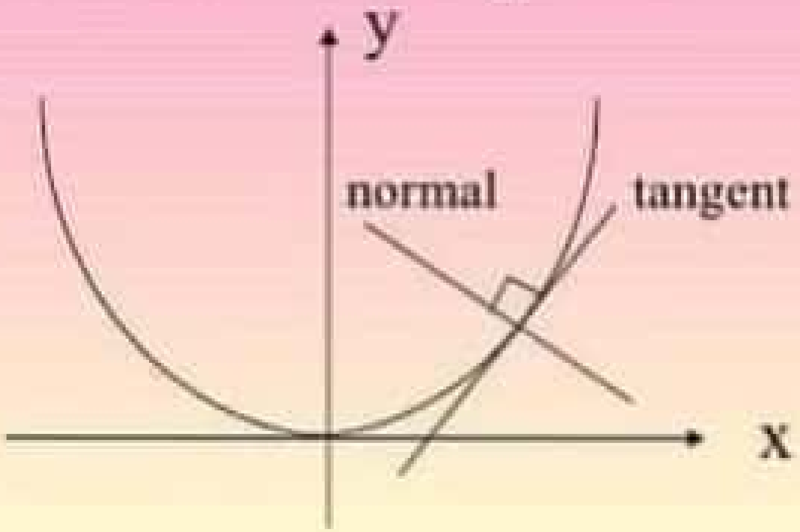
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$$

Application of Differentiation

1. **The gradient of the curve y= f(x) at a point is the derivative of y with respect to x, i.e. $\frac{dy}{dx}$ or f'(x).**

2. **The gradient of tangent at point A is the value of $\frac{dy}{dx}$ at point A**

3. **(Gradient of normal) x (gradient of tangen) = -1**



Equation of Tangent and Equation of Normal

Equation of tangent at point (x_1, y_1) with gradient m is

$$y - y_1 = m(x - x_1)$$

Equation of normal at point (x_1, y_1) is

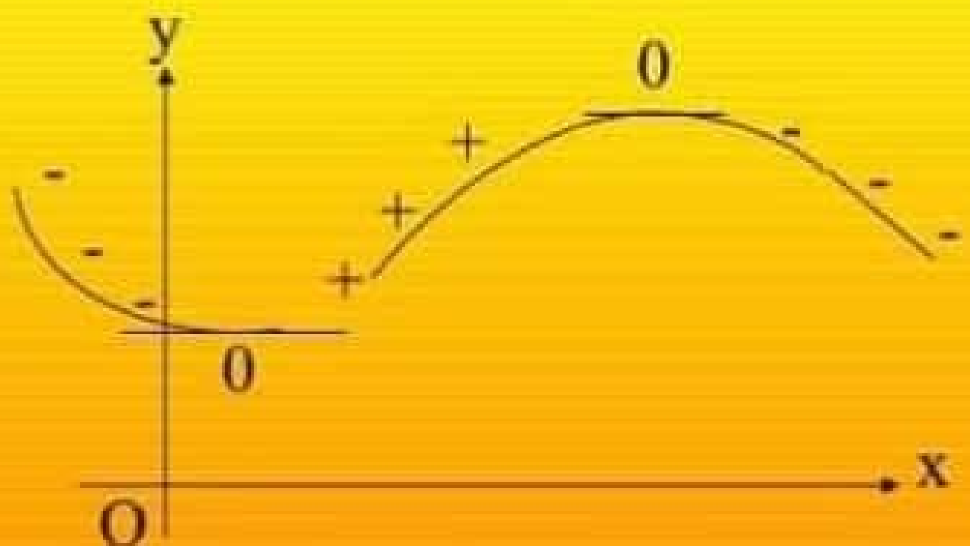
$$y - y_1 = -\frac{1}{m}(x - x_1)$$

Maximum and Minimum Point/Value

At the turning point (stationary point), $\frac{dy}{dx} = 0$

¥ **For maximum point** $\frac{d^2y}{dx^2} < 0$

¥ **For minimum point** $\frac{d^2y}{dx^2} > 0$



The Rate of Change

If $y = f(x)$, then $$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

is the rate of change of y with respect to time, t

SMALL CHANGES AND APPROXIMATION

- If $y = f(x)$ and ∂y is a small change in y corresponding with ∂x, a small change in x, then

$$\frac{\partial y}{\partial x} \approx \frac{dy}{dx}$$

$$\partial y \approx \frac{dy}{dx} . \partial x$$

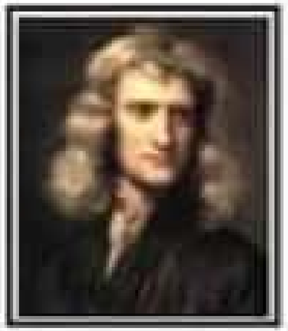
NOTE

The History of Differentiation

Differentiation is part of the science of Calculus, and was first developed in the 17th century by two different Mathematicians.

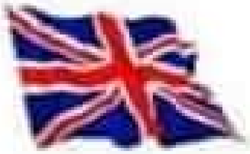
Gottfried Leibniz
(1646-1716)

Germany



Sir Isaac Newton
(1642-1727)

England



Differentiation, or finding the instantaneous rate of change, is an essential part of:

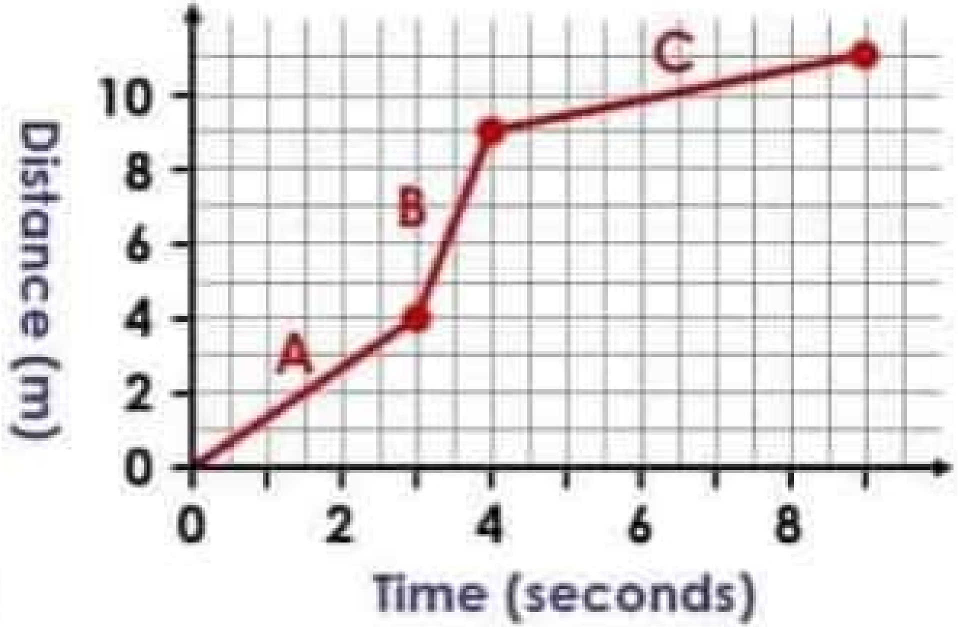
- Mathematics and Physics
- Chemistry
- Biology
- Computer Science
- Engineering
- Navigation and Astronomy

NOTE

Calculating Speed

Example

Calculate the speed for each section of the journey opposite.



$$\text{speed in A} = \frac{4}{3} \approx 1.33 \text{ m/s}$$

$$\text{speed in B} = \frac{5}{1} = 5 \text{ m/s}$$

$$\text{speed in C} = \frac{2}{5} = 0.4 \text{ m/s}$$

$$\text{average speed} = \frac{11}{9} \approx 1.22 \text{ m/s}$$

Notice the following things:

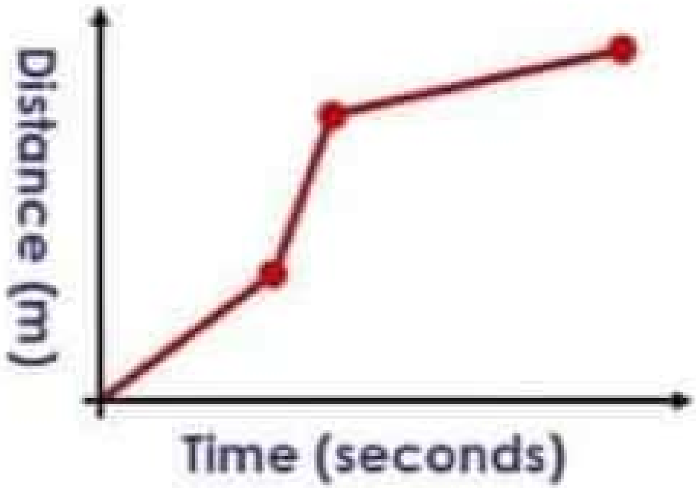
- the speed at each instant is not the same as the average
- speed is the same as gradient

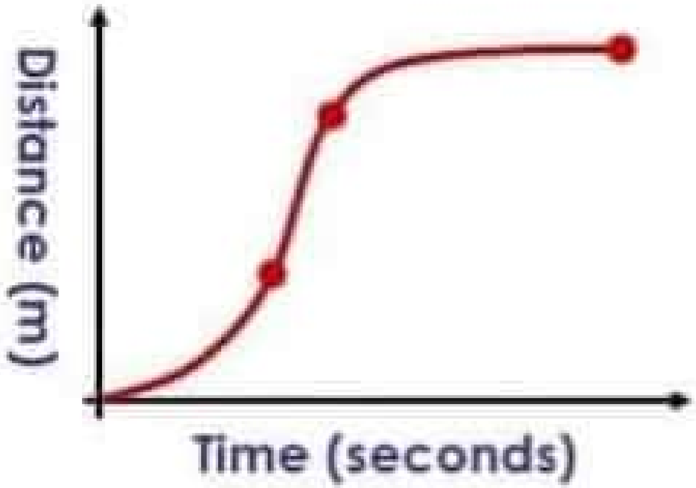
$$S = \frac{D}{T} = \frac{\Delta y}{\Delta x} = m$$

NOTE

Instantaneous Speed

$$S = \frac{D}{T} = \frac{\Delta y}{\Delta x} = m$$





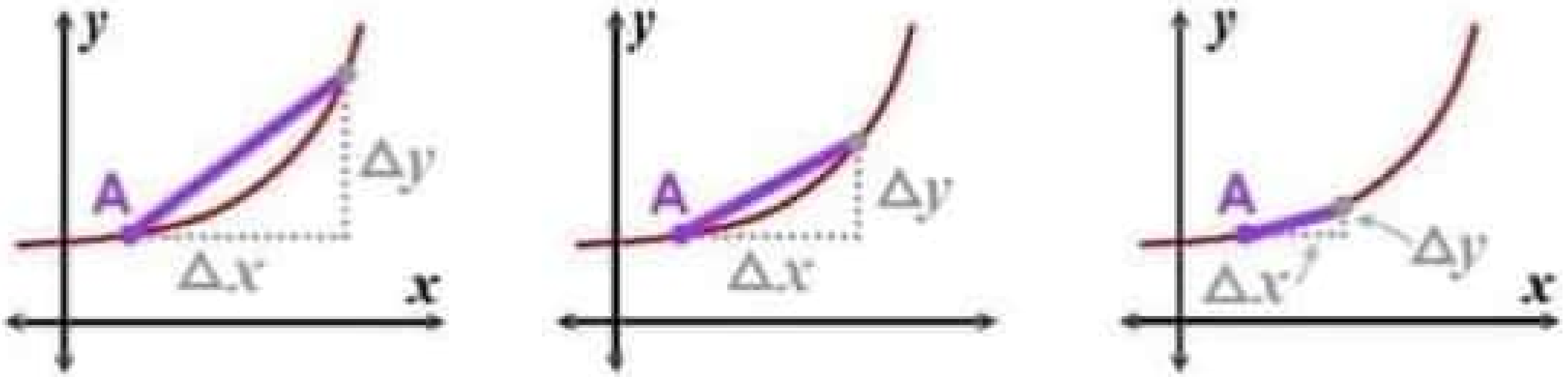
In reality speed does not often change instantly. The graph on the right is more realistic as it shows a gradually changing curve.

The journey has the same average speed, but the instantaneous speed is different at each point because the gradient of the curve is constantly changing. How can we find the instantaneous speed?

NOTE

Estimating the Instantaneous Rate of Change

The diagrams below show attempts to estimate the instantaneous gradient (the rate of change of y with respect to x) at the point A.



Notice that the accuracy improves as Δx gets closer to zero.

The instantaneous rate of change is written as:

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} \text{ as } \Delta x \text{ approaches } 0.$$

NOTE

Basic Differentiation

The instant rate of change of y with respect to x is written as $\frac{dy}{dx}$.

By long experimentation, it is possible
to prove the following:

If $y = x^n$

then $\boxed{\frac{dy}{dx} = nx^{n-1}}$

How to Differentiate:

- multiply by the power
- reduce the power by one

Note that $\frac{dy}{dx}$ describes both the rate of change and the gradient.

NOTE

Differentiation of Expressions with Multiple Terms

The basic process of differentiation can be applied to every x-term in an algebraic expession.

$$y = ax^m + bx^n + \ldots$$

$$\frac{dy}{dx} = amx^{m-1} + bnx^{n-1} + \ldots$$

Important

Expressions must be written as the sum of individual terms before differentiating.

How to Differentiate:

- multiply every x-term by the power
- reduce the power of every x-term by one

NOTE

Examples of Basic Differentiation

Example 1

Find $\frac{dy}{dx}$ for $y = 3x^4 - 5x^3 + \frac{7}{x^2} + 9$

$$y = 3x^4 - 5x^3 + 7x^{-2} + 9$$

$$\therefore \frac{dy}{dx} = 12x^3 - 15x^2 - 14x^{-3}$$

$$= \underline{\underline{12x^3 - 15x^2 - \frac{14}{x^3}}}$$

this disappears because

$9 = 9x^0$

(multiply by zero)

NOTE

The Derived Function

It is also possible to express differentiation using function notation.

If $f(x) = x^n$

then $f'(x) = nx^{n-1}$

Newton

$f'(x)$ and $\dfrac{dy}{dx}$ Leibniz

mean exactly the same thing written in different ways.

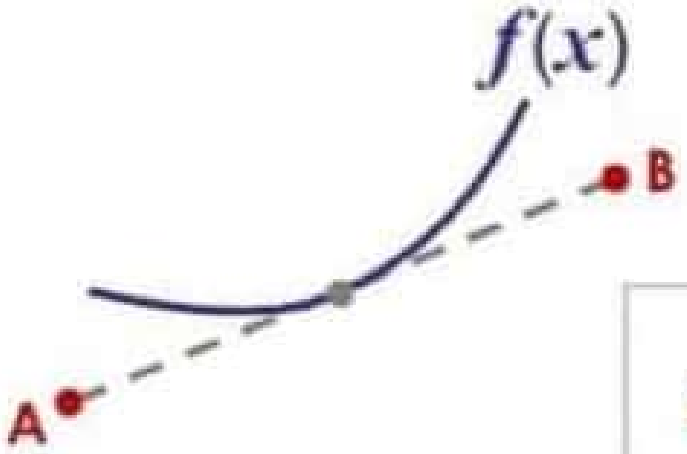
The word 'derived' means 'produced from', for example orange juice is derived from oranges.

The derived function $f'(x)$ is the rate of change of the function $f(x)$ with respect to x.

NOTE

Tangents to Functions

A tangent to a function is a straight line which intersects the function in only one place, with the same gradient as the function.



$$m_{AB} = f'(x)$$

The gradient of any tangent to the function $f(x)$ can be found by substituting the x-coordinate of intersection into $f'(x)$.

NOTE

Equations of Tangents

REMEMBER

$$y - b = m(x - a)$$

Straight Line Equation

To find the equation of a tangent:

- differentiate
- substitute x-coordinate to find gradient at point of intersection
- substitute gradient and point of intersection into $y - b = m(x - a)$

Example

Find the equation of the tangent to the function

$$f(x) = \frac{1}{2}x^3$$

at the point (2,4).

$$f'(x) = \frac{3}{2}x^2$$

$$\therefore \quad m = f'(2) = \frac{3}{2} \times (2)^2 = 6$$

substitute: $\quad y - 4 = 6(x - 2)$

$$\therefore \quad \underline{\underline{6x - y - 8 = 0}}$$

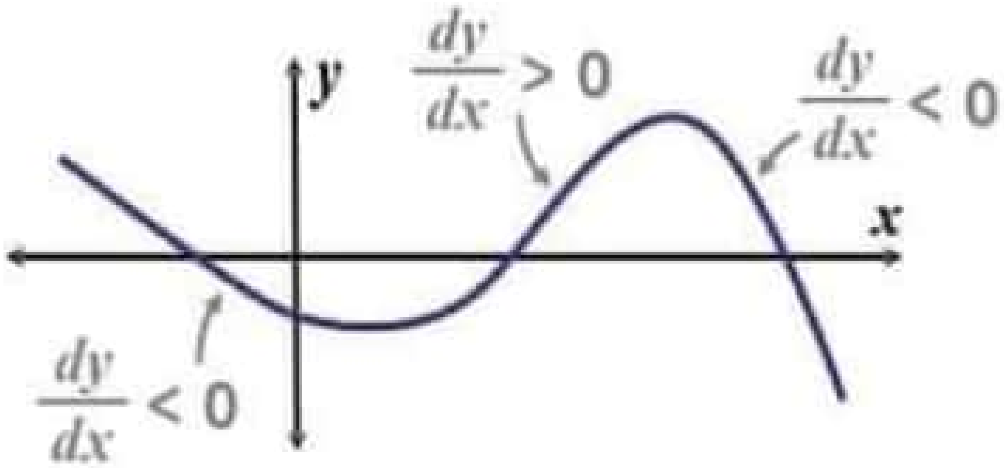
NOTE

Increasing and Decreasing Curves

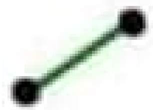
The gradient at any point on a curve can be found by differentiating.

If $\dfrac{dy}{dx} > 0$ then y is increasing.

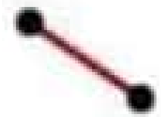
If $\dfrac{dy}{dx} < 0$ then y is decreasing.



Gradient REMEMBER



Positive
uphill slope



Negative
downhill slope

Alternatively,

If $f'(x) > 0$ then $f(x)$ is increasing.

If $f'(x) < 0$ then $f(x)$ is decreasing.

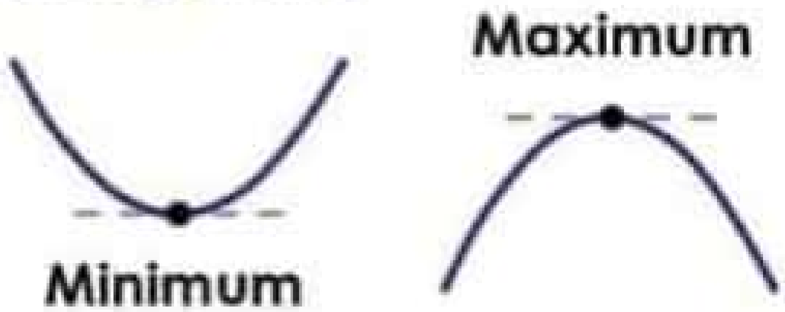
NOTE

Stationary Points

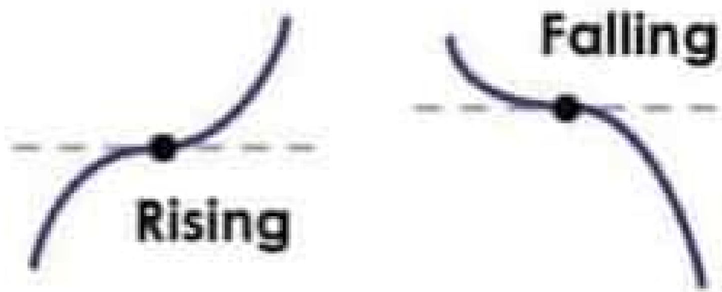
If a function is neither increasing or decreasing, the gradient is zero and the function can be described as stationary.

There are two main types of stationary point.

Turning Points



Points of Inflection



At any stationary point, $\frac{dy}{dx} = 0$ or alternatively $f'(x) = 0$

NOTE

Investigating Stationary Points

Example

Use a **nature table** to reduce the amount of working.

Find the stationary point of

$$f(x) = x^2 - 8x + 3$$

and determine its nature.

Stationary point given by

$$f'(x) = 0$$

$$f'(x) = 2x - 8$$

$$\therefore 2x - 8 = 0$$

$$\underline{x = 4}$$

'slightly less than four'

'slightly more'

x	4^-	4	4^+
$f'(x)$	$-$	0	$+$
slope	\	—	/

gradient is positive

$\therefore$ The stationary point at $x = 4$ is a minimum turning point.

NOTE

Investigating Stationary Points

Example 2

Investigate the stationary points of

$$y = 4x^3 - x^4$$

$$\therefore \frac{dy}{dx} = 12x^2 - 4x^3 = 0$$

$$4x^2(3 - x) = 0$$

$4x^2 = 0$ or $3 - x = 0$

$\underline{x = 0}$ $\qquad$ $\underline{x = 3}$

$\therefore y = 0$ $\qquad$ $\therefore y = 27$

stationary point at $(0,0)$:

x	0^-	0	0^+
$\frac{dy}{dx}$	$+$	0	$+$
slope	/	—	/

$\therefore$ rising point of inflection

stationary point at $(3, 27)$:

x	3^-	3	3^+
$\frac{dy}{dx}$	$+$	0	$-$
slope	/	—	\

$\therefore$ maximum turning point

NOTE

Positive and Negative Infinity

The symbol ∞ is used for infinity.

$+\infty$ 'positive infinity'

$-\infty$ 'negative infinity'

REMEMBER $\infty + 1 = \infty$

Example

$$y = 5x^3 + 7x^2$$

For very large $\pm x$, the value of $7x^2$ becomes insignificant compared with the value of $5x^3$.

$\therefore$ as $x \to \infty$, $y \to 5x^3$

The symbol $\rightarrow$ means approaches'.

$$5\times(+\infty)^3 = +\infty$$

$$5\times(-\infty)^3 = -\infty$$

$\therefore$ as $x \to +\infty$ and as $x \to -\infty$

$y \to +\infty$ $\qquad$ $y \to -\infty$

NOTE

Curve Sketching

To sketch the graph of any function, the following basic information is required:

- the stationary points and their nature

  solve for $\frac{dy}{dx} = 0$ and use nature table

- the x-intercept(s) and y-intercept

  solve for $y = 0$ and $x = 0$

- the value of y as x approaches positive and negative infinity

Example

$$y = x^3 - 2x^4$$

as $x \to \infty$

$y \to -2x^4$

$\therefore$ as $x \to +\infty$

$y \to -\infty$

and as $x \to -\infty$

$y \to +\infty$

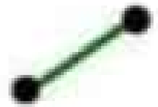
NOTE

Graph of the Derived Function

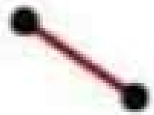
The graph of $f'(x)$ can be thought of as the graph of the gradient of $f(x)$.

REMEMBER

Gradient

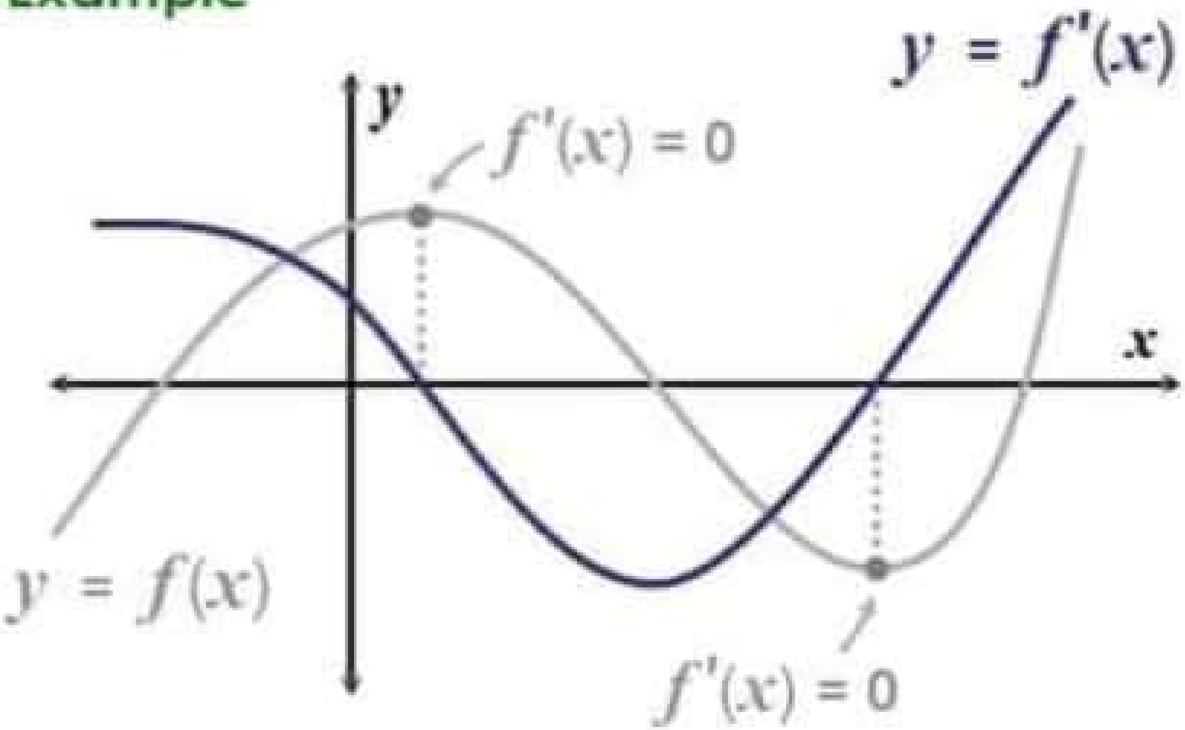


Positive $f'(x) > 0$



Negative $f'(x) < 0$

Example



The roots of $f'(x)$ are given by the stationary points of $f(x)$.