7 Your Daily Dose of Vitamin i

1. We will use complex numbers to find identities for $\cot$. Use Pascal's triangle to expand the following:

   (a) $(a+b)^3$ (b) $(a+b)^4$ (c) $(a+b)^5$

   Then substitute $b = i = \sqrt{-1}$ and expand:

   (d) $(a+i)^3$ (e) $(a+i)^4$ (f) $(a+i)^5$

   Finally, substitute $a = \cot\theta$ and expand:

   (g) $(\cot\theta + i)^3$ (h) $(\cot\theta + i)^4$ (i) $(\cot\theta + i)^5$

   Consider $z = i + \cot\theta$.

   (j) Use the above results to find identities for (i) $\cot 3\theta$, (ii) $\cot 4\theta$, and (iii) $\cot 5\theta$.

   (k) Graph z, z^2, z^3, z^4, and z^5, with $\theta \approx 75°$. What method did you use?

2. Compute $(1+i)^n$ for $n = 3, 4, 5, \ldots$. Can you find a general pattern?

3. Expand and graph $\operatorname{cis}^n \theta$ for $n = 2, 3, 4, \ldots$ and $\theta \approx 50°$.

   (a) Why is the real part $\cos n\theta$ and the imaginary part $\sin n\theta$?

   (b) Use your results to write identities for $\cos n\theta$ and $\sin n\theta$ for $n = 2, 3, 4, 5$.

4. Compute $\cos 7° + \cos 79° + \cos 151° + \cos 223° + \cos 295°$ without a calculator. (Hint: what does this have to do with complex numbers?)

5. Factor the following:

   (a) $x^6 - 1$ as a difference of squares

   (b) $x^6 - 1$ as a difference of cubes

   (c) $x^4 + x^2 + 1$ over the real numbers

   (d) $x^6 - 1$ completely

   (e) $x^4 + x^2 + 1$ completely

6. Let $f(z) = \frac{z+1}{z-1}$.

   (a) Without a calculator, compute $f^{2020}(z)$.

   (b) What if you replace 2020 with the current year?

7. Find $\operatorname{Im}\left((\operatorname{cis} 12° + \operatorname{cis} 48°)^6\right)$.

8. Let x satisfy the equation $x + \frac{1}{x} = 2\cos\theta$.

   (a) Compute $x^2 + \frac{1}{x^2}$ in terms of θ.

   (b) Compute $x^n + \frac{1}{x^n}$ in terms of n and θ.