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A Multifractal Model of Asset Returns

by

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Abstract

This paper presents the *multifractal model of asset returns* ("MMAR"), based upon the pioneering research into multifractal measures by Mandelbrot (1972, 1974). The multifractal model incorporates two elements of Mandelbrot's past research that are now well-known in finance. First, the MMAR contains long-tails, as in Mandelbrot (1963), which focused on Lévy-stable distributions. In contrast to Mandelbrot (1963), this model does not necessarily imply infinite variance. Second, the model contains long-dependence, the characteristic feature of fractional Brownian Motion (FBM), introduced by Mandelbrot and van Ness (1968). In contrast to FBM, the multifractal model displays long dependence in the absolute value of price increments, while price increments themselves can be uncorrelated. As such, the MMAR is an alternative to ARCH-type representations that have been the focus of empirical research on the distribution of prices for the past fifteen years. The distinguishing feature of the multifractal model is multiscaling of the return distribution's moments under time-rescalings. We define multiscaling, show how to generate processes with this property, and discuss how these processes differ from the standard processes of continuous-time finance. The multifractal model implies certain empirical regularities, which are investigated in a companion paper.

1. Introduction

The probabilistic description of financial prices, pioneered by Bachelier (1900), initially focused on independent and Gaussian distributed price changes. Financial economists have long recognized two major discrepancies between the Bachelier model and actual financial data. First, financial data commonly display temporal dependence in the alternation of periods of large price changes with periods of smaller changes. Secondly, the tails of the histogram of observed data are typically much fatter than predicted by the Gaussian distribution.

Inclusive of its several extensions, the ARCH/GARCH line of research, beginning with Engle (1982) and Bollerslev (1986), has become the predominant mode of thinking about the statistical representation of financial prices. In the original formulation of the GARCH(q, p) model, innovations in returns are specified as

$$\varepsilon_t = u_t h_t^{1/2},$$

where u_t are i.i.d., and

$$h_t = \alpha_0 + \sum_{j=1}^{q} \beta_j h_{t-j} + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2.$$

Among the many extensions to this specification, Nelson (1991) generates asymmetric responses to positive and negative shocks via more general functional dependence between h_t and the past values $\{(\varepsilon_{t-n}, h_{t-n})\}$. Other work adds an independent stochastic component to volatility itself. Also, most of the recent literature weakens the i.i.d. assumption for $\{u_t\}$ to an assumption of stationarity. Bollerslev, Engle and Nelson (1994) survey the literature thoroughly.

The common strand in GARCH-type representations is a conditional distribution of returns that has a finite, time-varying second moment. This directly addresses volatility clustering in the data, and mitigates the problem of fat tails.[1]

In our view, the most important topics in the recent GARCH literature include first, long memory (Baillie, Bollerslev and Mikkelsen, 1996), and second, the relationship between statistical representations at different time scales (Drost and Werker, 1996). These topics are central in our multifractal model, and although

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[1]Many studies find that additional weight in the tails is needed. This leads to the use of Student's t-distributions, Poisson jump components, or nonparametric representations of the conditional distribution.

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the multifractal framework is substantially different than GARCH, it is useful to briefly discuss the treatment of these problems in the familiar GARCH setting.

Long-memory is, intuitively, the idea that the longest apparent cycle in a sample will be proportional to the total number of observations. This idea has been formalized in several ways, most commonly by a slower than exponential decay rate in the auto-correlation function. Hyperbolic decay rates for the absolute value of asset returns were first reported by Taylor (1986) and are now a well documented stylized fact of financial time-series.[2]

Figure 1 presents a visual display of the consequences of long-memory (or more accurately the lack thereof). Figure 1a shows the first-differences of a simulated GARCH(1,1) process with 500 simulation periods. The parameters used in the model, $\{\alpha_1, \beta_1\} = \{.05, .921\}$, come from estimated parameter values in a study of US-UK exchange rates.[3] This graph exhibits the type of mild conditional non-stationarity characteristic of GARCH representations with low values of α and high values of β.[4] There are perceptible changes in volatility over relatively short (in terms of number of lags) periods.

Figure 1b simulates many (100,000) periods. The previously noticeable short cycles disappear between the peaks of the individual cycles. To the eye, this graph is indistinguishable from white noise. This provides an example for a general rule: short-memory processes appear like white noise from a distance.[5]

In contrast to Figure 1b, observed financial data contains noticeable fluctuations in the size of price changes at all time scales. Figure 2 shows first-daily-differences in the logarithm of the DM/US$ exchange rate from 1973 to the

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[2]See Ding. Granger, and Engle (1993) and Dacorogna et al. (1993) for evidence, or Baillie (1996) for further discussion.

[3]See Mills (1993). pp 112-113. The study uses weekly data from 1980-1988, giving 470 observations.

[4]The sum $\alpha + \beta$ is frequently referred to as the persistence in a GARCH(1,1) model. High α coefficients tend to produce sharper changes in volatility, while β contributes to more moderately evolving dependence in volatility. There is a strong trade-off, because of the stability requirements of GARCH representations (e.g. $\sum_{i=1}^{p} \alpha_p + \sum_{j=1}^{q} \beta_j < 1$), between these two types of coefficients. For this reason, the standard GARCH representation is unable to capture large, immediate changes in volatility simultaneously with long cycles. Thus, in later literature, we see the application of jump-processes or other independent stochastic shocks directly to the variance. This allows one to introduce sharp changes in volatility without further restricting the permissible range of β. Nonetheless, even high values of $\beta < 1$ are subject to exponential decline in the ACF, which is at odds with the observed hyperbolic decline in some data. Hence the recent interest in long memory processes.

[5]This is a generalization of Donsker's theorem to weakly dependent increments.

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present. The most compelling aspect of this data is the presence of temporal dependence of varying frequencies – a phenomenon which non-integrated (weak-memory) GARCH-type representations do not capture.

Prior to the development of long-memory GARCH processes, researchers had limited alternatives in modelling low frequency cycles. Attributing low frequency cycles to structural breaks, effectively capping the sample span. is one option. However, since "structural breaks" potentially pose great risk to investors, deliberate censoring of such events can lead to a serious underestimation of market risks.

In theory, GARCH representations also offer the alternative of adding parameters of higher orders to capture low frequency cycles. In practice, representing long-memory phenomena with a non-integrated process leads to non-robust representations whose number of parameters grows with the number of observations.

The recently developed FIGARCH process of Baillie, Bollerslev and Mikkelsen (1996), achieves long memory parsimoniously.. Like GARCH, FIGARCH has an infinite order ARCH representation in squared returns. The model can be viewed as a set of infinite-dimensional restrictions upon its ARCH parameters. Mathematically, the restrictions are transmitted by the fractional differencing operator. In contrast to the FBM and ARFIMA, fractional differencing affects squared errors rather than the error term itself. Hence, the martingale property of prices can be maintained simultaneously with long memory in the absolute value of returns.

To visually reinforce the consequences of long memory, Figure 3a shows 500 simulated first differences of a FIGARCH$(1, d, 0)$ process. The parameter values are taken from Baillie, Bollerslev, and Mikkelsen's study of DM/USD exchange rates. When we repeat the simulation over 100,000 periods in Figure 3b, we see a variety of long-run variation in volatility. Unlike GARCH, there is no convergence to Brownian behavior over very long sampling intervals.

A currently unexplored area in this new branch of the literature is the relationship between FIGARCH representations at different time scales. Drost and Nijman (1993) and Drost and Werker (1996) have studied this problem for GARCH. For a given class of discrete processes, they consider temporal aggregation of log returns, obtaining processes defined on coarser time scales. If the aggregated processes all belong to the same class as the original processes, they refer to the class as closed under temporal aggregation. Another possible term for this property is *scale-consistency*, since it implies an equivalence between representations of the model at different time scales. In empirical work, lack of scale-consistency im-

plies that the researcher adds an additional restriction to the model when choosing the time-scale of the data.

The original GARCH assumptions, which Drost and Nijman call strong GARCH, are not scale-consistent. The weak-GARCH class, which is scale-consistent, assumes only that the parameters $\{\alpha, \beta\}$ are the best linear predictors in terms of lagged values of h_t and ϵ_t^2, and that $\{u_t\}$ are stationary. The exact distribution of the $\{u_t\}$ will, in general, be quite complicated to calculate after aggregation, and correlation between the $\{u_t\}$ is unmodelled. For a continuous-time diffusion with a weak GARCH$(1,1)$ representation, there is a unique correspondence between values of $\{\alpha, \beta\}$ at different time scales. Drost and Werker suggest that estimates which fail to adhere to this correspondence should be considered jump-diffusions.

The recent interest in long-memory and in scale-consistency within the GARCH literature foreshadows two fundamental concepts in the Multifractal Model of Asset Returns ("MMAR"). Multifractal processes will be defined by a restriction on the behavior in their moments as the time-scale of observation changes. Like Drost and Werker, we will argue that information contained in the data at different time scales can identify a model. Reliance upon a single time scale leads to inefficiency, or worse, forecasts that vary with the time-scale of the chosen data.

Section 2 discusses previous financial models with scaling properties. Section 3 briefly introduces the mathematics of multifractal measures and processes. Section 4 applies the idea of multiscaling to financial time series, and presents the Multifractal Model of Asset Returns. Section 5 concludes.

This paper is the first in a three paper series that introduces the concept of multifractality to economics. Like many ideas, multifractality can be understood at several different levels of abstraction, and corresponding mathematical elegance. In this paper, we focus on a very concrete aspect of multifractality - a scaling property in moments of the process or measure. In fact, we define multifractals via this property in Section 3. This has two advantages. First, this definition leads directly to an empirical test. Second, this definition avoids several mathematical technicalities that are not necessary in an expository paper. Thus, we give a simple definition of multifractality, present some examples, extend multifractality from measures to processes (which has not yet been addressed in the mathematics literature), develop a model of financial price changes (the MMAR), and discuss interesting properties of the MMAR from an economic perspective.

The second paper in the series, Calvet, Fisher, and Mandelbrot (1997), approaches the theory of multifractals from an entirely different perspective. It focuses on the local properties of multifractal processes, which substantially differ

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from the standard assumptions in continuous-time finance. In particular, most diffusions are characterized by increments that grow locally at the rate $(\triangle t)^{1/2}$ throughout their sample paths. The exceptions, such as fractional Brownian motion, have local growth rates of order $(\triangle t)^H$, where H is invariant over time. Multifractals, on the other hand, have a multiplicity of local growth rates for increments, which leads to a quite elegant mathematical theory. We explore this aspect of multifractals in the second paper, and develop concepts such as the multifractal spectrum, which characterizes the distribution of local growth rates in a multifractal process.

The third paper in the series, Fisher, Calvet and Mandelbrot (1997), is an empirical study, which tests for multifractality in Deutschemark - US Dollar data. We find strong evidence of a multifractal scaling law, and estimate the multifractal spectrum of the Deutschemark - US Dollar time series. This allows us to recover MMAR components, and simulate a multifractal data generating process. Further, we show that alternatives such as GARCH and FIGARCH are distinguishable from multifractals under simulation, and that the behavior of the exchange-rate data is more consistent with the MMAR hypothesis.

2. Roots of the MMAR

2.1. Three Earlier Themes

While the MMAR is entirely new to economics, it combines three elements of Mandelbrot's previous research that are now well-known. First, the MMAR incorporates long tails, although in a substantially different form than Mandelbrot (1963), which focused on L-stable distributions.[6] At the time of its publication, some researchers reacted against the L-stable model, principally on the grounds that it implied an infinite variance.[7] Most economists now agree that there is no *a priori* justification for rejecting infinite second moments. Indeed, the L-stable model has recently been applied and tested on foreign exchange and stock prices. Contributions to this literature include Koedijk and Kool (1992), Belkacem, Lévy-Véhel and Walter (1995), Phillips, McFarland and McMahon (1996). While the MMAR accounts for long tails in financial data, it does not necessarily imply an infinite variance of returns over discrete sampling intervals.

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[6] These distributions have alternately been called Lévy-stable, stable, Pareto-Lévy, and stable-Paretian.

[7] See Cootner (1964).

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Second, the multifractal model contains long-dependence, the characteristic feature of fractional Brownian motion (FBM), which was formally introduced by Mandelbrot and van Ness (1968). FBM has received wide applications in the natural sciences, particularly in hydrology. Its use in economics was advanced by the work of Granger and Joyeux (1980) and Hosking (1981), and led to popular models such as ARFIMA. Baillie (1996) provides a good review of this literature. The present model builds on the FBM by obtaining long-memory in the absolute value of returns[8], but allows the possibility that returns themselves are white.

The remaining essential component of the multifractal market model is the concept of trading time, introduced by Mandelbrot and Taylor (1967). The salient feature of trading time is explicit modelling of the relationship between unobserved natural time-scale of the returns process, and clock time, which is what we observe. Trading time models have been extensively used in the literature, including Clark (1973), Dacorogna et al. (1993), Müller et al. (1995), Ghysels, Gouriéroux and Jasiak (1995, 1996).

All three of these components are contained in the MMAR. This model accounts for the most significant empirical regularities of financial time-series, which are long tails relative to the Gaussian and long memory in the absolute value of returns. At the same time, the model incorporates scale-consistency, in the sense that a well-defined scaling rule relates returns over different sampling intervals.

2.2. Self-Affine Processes

Returning to the question of how time should be discretized, Mandelbrot (1963, 1967), followed by Fama (1963), suggested that the shape of the distribution of returns should be the same when the time scale is changed. The property of invariance (up to a scale parameter) under aggregation of independent elements has been studied by Paul Lévy (1925, 1937), and is called L-stability. The class of L-stable random variables contains Gaussians and a continuum of distributions that have scaling (Paretian) tails: $\mathbf{P}(X > x) \sim Cx^{-\alpha}$, with $0 < \alpha < 2$. These random variables can be used to generate continuous-time random motions. Thus the L-stable processes, used in Mandelbrot (1963), have stationary and independent stable increments. They include the Brownian Motion as the only member with continuous sample paths. We can in turn generalize these processes to allow for dependence in the increments.

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[8]Taqqu (1975) establishes that a FBM $B_H(t)$ has long memory in the absolute value of its increments when $H > 1/2$.

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Definition 2.1. *Granted* $X(0) = 0$, *a random process* $\{X(t)\}$ *that satisfies:*

$$X(ct) \stackrel{d}{=} c^H X(t).$$

for some $H > 0$ *and all* $c > 0$, *is called self-affine.*

We call H the self-affinity index, or scaling exponent, of $X(t)$.

Authors such as Samorodnitsky and Taqqu (1994), refer to a process satisfying the above definition as self-similar. We use the term self-similarity in a stricter sense, reserving it for geometric objects which are invariant under isotropic contraction. Self-affinity is a more general term, which allows for different rescalings along the directions of an orthonormal basis. Self-similarity applies when the rescaling operators are the same in each direction, so that the object is not only invariant under dilations, but also rotations.

The distinction between self-similarity and self-affinity was drawn in Mandelbrot (1977). The class of strictly (isotropically) self-similar stochastic processes is degenerate. However, among the broader class of geometric objects and measures, the distinction between self-similarity and self-affinity is essential.

Two main types of self-affine processes have been used in finance. The L-stable motions discussed above assume independent and stable increments. They contrast with *Fractional Brownian Motions (FBM)*, a class of self-affine processes $B_H(t)$ with continuous sample paths and Gaussian increments. The exponent H, called self-affinity exponent, satisfies[9] $0 < H < 1$. The FBM is a Brownian Motion in the special case $H = 1/2$. For other values of H, the FBM has dependent increments. Autocorrelation is negative (anti-persistence) when $0 < H < 1/2$, and positive (persistence) when $1/2 < H < 1$. Persistent FBM have long memory.

3. Multifractal Measures and Processes

L-stable processes miss one the main features of financial markets – the alternation of periods of large price changes with periods of smaller changes.[10] In contrast, the FBM is useful for modelling the tendency of price changes to be followed by changes in the same (or opposite) direction. The fractional Brownian Motions

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[9]Authors sometimes consider the degenerate case $H = 1$. The process is then of the form tZ, where Z is a normal random variable (Samorodnitsky and Taqqu, 1994.)

[10]This phenomenon is typically described as the fluctuation of "volatility" over time. In situations where variance may be infinite, "volatility" should be taken to mean fluctuations in the expected absolute value of price changes.

however, capture neither fat tails nor fluctuations in volatility that are unrelated to the predictability of future returns.

Both models have in common a very strong form of scale-invariance, in which the distribution of returns over different sampling intervals are identical except for a single, non-random contraction. This property is at odds with empirical observations. In particular, many financial data sets become less peaked in the bells and have thinner tails as the sampling interval increases. This does not necessarily imply that the distribution eventually becomes Gaussian at long enough sampling intervals. In particular, the multifractal processes introduced in this paper never become Gaussian.

We build up the theory of multifractals in the following sections. Section 3.1 is a general introduction to multifractality. Section 3.2 presents the binomial measure. which is the simplest example of a multifractal. Section 3.3 generalizes to the broader case of multiplicative measures. Section 3.4 extends multifractality from measures to stochastic processes. Section 3.5 presents a wider definition of multifractal measures based on statistical self-similarity.

3.1. Multifractality

This section provides a general introduction to multifractality. It is aimed at sketching the main ideas of the theory, while detailed explanations are given in later sections. Multifractal measures were introduced in Mandelbrot (1972) and have since been applied in the physical sciences to describe the distribution of energy and matter, e.g. turbulent dissipation, stellar matter, and minerals. They are new to economics.

This paper uses multifractal measures to model temporal heterogeneity in financial time series. It also extends multifractality from measures to stochastic processes. For this reason, our presentation will at times refer to multifractals as either measures or processes. We hope this provides the reader with the greatest exposure to the mathematical generality of multifractals, without being unnecessarily confusing.

We previously discussed self-affine processes, which satisfy the simple scaling rule:

$$X(ct) \stackrel{d}{=} c^H X(t).$$

The theory of multifractals examines the more general relationships:

$$X(ct) \stackrel{d}{=} M(c)X(t), \tag{3.1}$$

where X and M are independent random functions. Under strict stationarity, arbitrary translations along the time axis allow extension of (3.1) to local scaling rules:

$$X(t + c\Delta t) - X(t) \stackrel{d}{=} M(c)\left[X(t + \Delta t) - X(t)\right] \tag{3.2}$$

for all positive c. The scaling factor $M(c)$ is a random variable, whose distribution does not depend on the particular instant t. Self-similar processes satisfy (3.2), with $M(c) = c^H$. To pursue this analogy, we define the generalized index $H(c) = log_c M(c)$, and rewrite the above relation: $X(ct) \stackrel{d}{=} c^{H(c)} X(t)$. In contrast to self-similar processes, the index $H(c)$ is a random function of c.

Multifractality thus permits a richer variety of behaviors than is possible under self-affinity. It also places strong restrictions on the process's distribution. For instance if $c_2/c_1 = c_3/c_2$ and condition (3.1) holds, then

$$\frac{X(c_2 t)}{X(c_1 t)} \stackrel{d}{=} \frac{X(c_3 t)}{X(c_2 t)},$$

since both ratios are distributed like $M(c_2/c_1)$.

We also require that the random scaling factor satisfies the property: $M(ab) \stackrel{d}{=} M_1(a)M_2(b)$, where M_1 and M_2 are independent copies of M. This condition, which is motivated in Section 3.5, implies the scaling rule:

$$\mathbf{E}\left(|X(t)|^q\right) = c(q)t^{\tau(q)+1}, \tag{3.3}$$

where $\tau(q)$ and $c(q)$ are both deterministic functions of q.

This paper presents the scaling rule (3.3) as the defining property of multifractal processes. In this setting, condition (3.1) only characterizes a particular class of multifractals. Multifractality is thus defined as a *global* property of the process's moments. Our approach could also build on the *local* scaling properties of the process's sample paths[11], in the spirit of equation (3.2). This alternative viewpoint is further discussed in the companion paper Calvet, Fisher and Mandelbrot (1997).

We now examine more closely the scaling rule (3.3). Most of our work concentrates on properties of the function $\tau(q)$, which is called the *scaling function*. Setting $q = 0$ in condition (3.3), we see that all scaling functions have intercept $\tau(0) = -1$. In addition, $\tau(q)$ is always concave, as shown in Section 3.4.

A self-affine process with index H is multifractal, with scaling function $\tau(q) = Hq - 1$. Because of its linearity, the scaling function is fully determined by a

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[11]Local scaling builds on the concept of local Hölder exponent.

single coefficient, its slope. It is thus called *uniscaling* or *unifractal*. Multifractal (or *multiscaling*) processes allow more general concave scaling functions.

Section 4.5 shows how to construct and simulate multifractal processes. We compound a Fractional Brownian Motion $B_H(t)$ by the cumulative distribution function $\theta(t)$ of a multifractal measure. The resulting process:

$$X(t) = B_H[\theta(t)]$$

satisfies multiscaling. This construction is the main building block of the MMAR. It requires a good understanding of multifractal measures, which we now present.

3.2. The Binomial Measure is the Simplest Example of a Multifractal

This section introduces the simplest multifractal, the binomial measure[12] on the compact interval $[0,1]$. This is the limit of an elementary iterative procedure called a *multiplicative cascade*.

Let m_0 and m_1 be two positive numbers adding up to 1. At stage $k = 0$, we start the construction with the uniform probability measure μ_0 on $[0,1]$. In the step $k = 1$, the measure μ_1 uniformly spreads mass equal to m_0 on the subinterval $[0,1/2]$ and mass equal to m_1 on $[1/2,1]$. The density of μ_1 is drawn in Figure 4a for $m_0 = 0.6$.

In step $k = 2$, the set $[0,1/2]$ is split into two subintervals, $[0,1/4]$ and $[1/4,1/2]$, which respectively receive a fraction m_0 and m_1 of the total mass $\mu_1[0,1/2]$. We apply the same procedure to the dyadic set $[1/2,1]$ and obtain:

$$\mu_2[0,1/4] = m_0 m_0, \qquad \mu_2[1/4,1/2] = m_0 m_1,$$
$$\mu_2[1/2,3/4] = m_1 m_0, \quad \mu_2[3/4,1] = m_1 m_1.$$

Iteration of this procedure generates an infinite sequence of measures. In step $k + 1$, we assume that the measure μ_k has been defined and construct μ_{k+1} as follows. Consider an interval $[t, t + 2^{-k}]$, where the dyadic number t is of the form:

$$t = 0.\eta_1\eta_2..\eta_k = \sum_{i=1}^{k} \eta_i 2^{-i} \tag{3.4}$$

in the counting base $b = 2$. We uniformly spread a fraction m_0 and m_1 of the mass $\mu_k[t, t + 2^{-k}]$ on the subintervals $[t, t + 2^{-k-1}]$ and $[t + 2^{-k-1}, t + 2^{-k}]$. A

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[12]The binomial measure is sometimes called the Bernoulli or Besicovitch measure.

repetition of this scheme to all subintervals determines μ_{k+1}. The measure μ_{k+1} is now well-defined. Figure 4b represents the measure μ_4 obtained after $k = 4$ steps of the recursion.

The binomial measure μ is defined as the limit of the sequence (μ_k). We now examine some of its properties. Consider the dyadic interval $[t, t + 2^{-k}]$, where $t = 0.\eta_1\eta_2..\eta_k$ in the counting base $b = 2$. Let φ_0 and φ_1 denote the relative frequencies of 0's and 1's in the binary development of t. The measure of the dyadic interval simplifies to:

$$\mu[t, t + 2^{-k}] = m_0^{k\varphi_0} m_1^{k\varphi_1}.$$

The binomial measure has important characteristics common to many multifractals. It is a continuous but singular probability measure; it thus has no density and no point mass. We also observe that since $m_0 + m_1 = 1$, each stage of the construction preserves the mass of split dyadic intervals. For this reason, the procedure is called *conservative* or *microcanonical.*

This construction can receive several extensions. For instance at each stage of the cascade, intervals can be split not in 2 but in $b > 2$ intervals of equal size. Subintervals, indexed from left to right by β ($0 \leq \beta \leq b - 1$), receive fractions of the total mass equal to $m_0, .., m_{b-1}$. By the conservation of mass, these fractions, also called *multipliers*, add up to one: $\sum m_\beta = 1$. This defines the class of *multinomial* measures, which are discussed in Mandelbrot (1989a) and Evertsz and Mandelbrot (1992).

Another extension randomizes the allocation of mass between subintervals at each step of the iteration. The multiplier of each subinterval is a discrete random variable M_β that takes values $m_0, m_1, ..., m_{b-1}$ with probabilities $p_0, .., p_{b-1}$. The preservation of mass imposes the additivity constraint: $\sum M_\beta = 1$. Figure 4c shows the random density obtained after $k = 10$ iterations with parameters $b = 2$, $p = p_0 = 0.5$ and $m_0 = 0.6$.

3.3. Multifractal Measures Generated as Multiplicative Cascades

An extension of the binomial and multinomial measures allows non-negative multipliers M_β ($0 \leq \beta \leq b - 1$) that are not necessarily discrete, but can be more general random variables. This procedure, usually called a *multiplicative cascade,* helps construct the broader class of *multiplicative measures*. To simplify the presentation, we assume that the multipliers are identically distributed, and denote by M the multiplier M_0.

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We first impose that mass be preserved at every stage of the construction: $\sum M_\beta = 1$. The resulting measure is then called *conservative* or *microcanonical*. In the first stage of the construction, the unit interval $[0, 1]$ receives an initial mass equal to 1 and is subdivided into b-adic cells of length $1/b$. We index these cells from left to right and allocate for every β the random mass M_β to the βth cell.

By a repetition of this scheme, the b-adic cell of length $\Delta t = b^{-k}$, starting at $t = 0.\eta_1...\eta_k = \sum \eta_i b^{-i}$, has measure

$$\mu(\Delta t) = M(\eta_1)M(\eta_1, \eta_2)...M(\eta_1, ..., \eta_k),$$

and thus $[\mu(\Delta t)]^q = M(\eta_1)^q M(\eta_1, \eta_2)^q...M(\eta_1, ..., \eta_k)^q$ for all $q \geq 0$. We take the expectation of this expression and obtain the scaling rule:

$$\mathbb{E}\left[\mu(\Delta t)^q\right] = \left[\mathbb{E}(M^q)\right]^k, \tag{3.5}$$

since the multipliers are independent.

Modifying the previous construction, we now choose that the multipliers M_β be statistically independent. Each iteration only conserves mass "on average" in the sense that $\mathbb{E}(\sum M_\beta) = 1$ or $\mathbb{E}M = 1/b$. The corresponding measure is then called *canonical*. Its total mass, denoted Ω, is generally random[13], and the mass of a b-adic cell takes the form:

$$\mu(\Delta t) = \Omega(\eta_1, ..., \eta_k)M(\eta_1)M(\eta_1, \eta_2)...M(\eta_1, ..., \eta_k).$$

We note that $\Omega(\eta_1, ..., \eta_k)$ has the same distribution as Ω. The measure μ thus satisfies the scaling relationship:

$$\mathbb{E}\left[\mu(\Delta t)^q\right] = \mathbb{E}(\Omega^q)\left[\mathbb{E}(M^q)\right]^k, \tag{3.6}$$

which characterizes multifractals. In the companion empirical paper, we argue that this condition on population expectations can be extended to sample sums of data, allowing us to test 1) for scaling behavior of the moments, and 2) to distinguish between unifractal and multifractal scaling behavior.

This section presented examples of multifractal measures that were constructed as the limit of multiplicative cascades. We did not give a general definition of multifractality. A broader approach, based on the geometric concept of self-similarity, provides better intuition of multifractal measures and is presented in section 3.5. This larger setting is slightly more complicated, and can be skipped in a first reading of the paper.

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[13]The random variable Ω has interesting distributional and tail properties that are discussed in Mandelbrot (1989a).

3.4. Multifractal Processes

We now extend multifractality from measures to stochastic processes. This extension is new to this paper and Mandelbrot (1997). We find it convenient to define multifractal processes in terms of moments, because this has direct graphical and testable implications. In Calvet, Fisher and Mandelbrot (1997), we focus on the local scaling properties of multifractal processes. This alternative leads to a more elegant mathematical presentation, and for some, perhaps a more intuitive understanding of multifractality. For now, we concentrate on the simpler idea of scaling in the moments of the process's increments.

Definition 3.1. *A stochastic process* $\{X(t)\}$ *is called multifractal if it is stationary and satisfies:*

$$\mathbb{E}\left(|X(t)|^q\right) = c(q)t^{\tau(q)+1}, \quad \text{for all } t \in \mathcal{T}, \ q \in \mathcal{Q}, \tag{3.7}$$

where $\mathcal{T}$ *and* $\mathcal{Q}$ *are intervals on the real line,* $\tau(q)$ *and* $c(q)$ *are functions with domain* $\mathcal{Q}$. *Moreover, we assume that* $\mathcal{T}$ *and* $\mathcal{Q}$ *have positive lengths, and that* $0 \in \mathcal{T}$, $[0,1] \subseteq \mathcal{Q}$.

A multifractal process is thus globally scaling, in the sense that its moments satisfy the scaling relationship (3.7). The function $\tau(q)$ is called the *scaling function* of the multifractal process. Setting $q = 0$ in condition (3.7), we see that all scaling functions have the same intercept $\tau(0) = -1$.

Self-affine processes are multifractal, as is now shown. A self-affine process $\{X(t), t \geq 0\}$, with self-affinity index H, satisfies $X(t) \stackrel{d}{=} t^H X(1)$, and therefore $\mathbb{E}\left(|X(t)|^q\right) = t^{Hq}\mathbb{E}\left(|X(1)|^q\right)$. Condition (3.7) thus holds, with:

$$\tau(q) = Hq - 1 \text{ and } c(q) = \mathbb{E}\left(|X(1)|^q\right).$$

In the special case of self-affine processes, the scaling function $\tau(q)$ is linear and fully determined by its index H. More generally, linear scaling functions $\tau(q)$ are determined by a unique parameter, their slope. For this reason, multifractal processes with linear $\tau(q)$ are called *uniscaling* or *unifractal*. In this paper, we focus instead on multifractal processes with non-linear functions $\tau(q)$. Such processes are called *multiscaling*.

The concavity of $\tau(q)$ is easy to derive from condition (3.7). Consider two exponents q_1, q_2, and two positive weights w_1, w_2 adding up to one. Hölder's inequality implies that:

$$\mathbb{E}\left(|X(t)|^q\right) \leq \left[\mathbb{E}\left(|X(t)|^{q_1}\right)\right]^{w_1}\left[\mathbb{E}\left(|X(t)|^{q_2}\right)\right]^{w_2},$$

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where $q = w_1 q_1 + w_2 q_2$. Taking logarithms and using (3.7), we obtain:

$$\ln c(q) + \tau(q) \ln t \leq [w_1 \tau(q_1) + w_2 \tau(q_2)] \ln t + [w_1 \ln c(q_1) + w_2 \ln c(q_2)]. \quad (3.8)$$

We divide by $\ln t < 0$, and let t go to zero:

$$\tau(q) \geq w_1 \tau(q_1) + w_2 \tau(q_2), \quad (3.9)$$

which establishes the concavity of τ.

In fact this proof contains additional information on multifractal processes. Assuming that relation (3.7) holds for $t \in [0, \infty)$, we divide inequality (3.8) by $\ln t > 0$ and let t go to infinity. We obtain the reverse of inequality (3.9), and conclude that $\tau(q)$ is linear. Thus multiscaling can only hold for *bounded* time intervals $\mathcal{T}$. We can reinterpret this results as follows. Processes defined on *unbounded* intervals can only be multifractal over *bounded* ranges of time. They must contain what physicists call *crossovers*, i.e. transitions in their scaling properties. Mandelbrot (1997) also discusses this result. This technical difficulty has little consequence for financial modeling, since multifractal processes can be defined on arbitrarily large time intervals.

We now describe a large class of multiscaling processes, which is inspired by the previous discussion on self-similar random measures. Consider a process $\{X(t)\}$, and assume the existence of an independent process $\{M(c)\}$ that satisfies:

$$X(ct) \stackrel{d}{=} M(c) X(t), \text{ for all } t, 0 < c \leq 1.$$

and

Property 1. *If $0 < a, b \leq 1$, the process M takes positive values and satisfies:*

$$M(ab) \stackrel{d}{=} M_1(a) M_2(b), \quad (3.10)$$

where M_1 and M_2 are two independent copies of M.

Property 1 implies that $\mathbf{E}[M(ab)^q] = \mathbf{E}[M(a)^q]\mathbf{E}[M(b)^q]$ for all $q \geq 0$. When these moments are finite, the process M satisfies the scaling relationship:

$$\mathbf{E}[M(c)^q] = c^{\tau(q)+1},$$

and the process $\{X(t)\}$ is multifractal.

This section has defined multifractality as a scaling property of the process's moments. This presentation has the advantage of having directly testable implications, but somewhat lacks intuitive content. For this reason, we now present an alternative interpretation of multifractality based on the concept of self-similarity.

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3.5. Self-Similar Random Measures

[Please note: This section is not necessary to understand the Multifractal Model of Asset Returns, but generalizes some of the previous presentation.]

We now present a more general approach to multifractality based on the statistical self-similarity of random measures. To simplify the exposition, we only consider the case of random measures defined on an interval X of the real line. An extension of self-similarity to higher dimensions can be found in Mandelbrot (1989a.)

A random measure μ on the interval X is analogous to a random variable. It is a mapping defined on a probability space, and valued in the class of all the measures on X. Moreover given a fixed interval $I \subseteq X$, the mass $\mu(I)$ is a random variable.

We now present the conditions on the random measure that define statistical self-similarity. This will help our discussion of the multifractal market model, in which trading time is viewed as the cumulative distribution function (c.d.f.) of a self-similar random measure. In a general Euclidean space, we call similitude the compound of a translation, a homothetic transformation, and a rotation. The class of similitudes on the real line reduces to the class of affine transformations, and is denoted by $\mathcal{S}$. We first impose that conditional measures be statistically invariant under similitudes.

Assumption 1. *For any $S \in \mathcal{S}$, for any intervals $I_1 \subseteq I_2$, the ratios*

$$\frac{\mu(SI_1)}{\mu(SI_2)} \quad and \quad \frac{\mu(I_1)}{\mu(I_2)}$$

are identically distributed whenever I_1, I_2, SI_1, $SI_2 \subseteq X$.

Successive iterations of a multiplicative cascade are statistically independent. This property generalizes as follows.

Assumption 2. *For all non-decreasing sequence of compact intervals $I_1 \subseteq ... \subseteq I_n$ contained in X, the random variables*

$$\frac{\mu(I_1)}{\mu(I_2)}, ..., \frac{\mu(I_{n-1})}{\mu(I_n)}$$

are statistically independent.

This leads to the following

Definition 3.2. *A random measure satisfying Assumptions [1] and [2] is called self-similar.*

When the interval X is of the form $[0, T]$, $0 < T \leq \infty$, Assumption [1] implies the existence of a positive random process $M(c)$ independent of μ that satisfies:

$$\mu[0, ct] \stackrel{d}{=} M(c)\mu[0, t] \text{ whenever } 0 < t \leq T, \ 0 < c \leq 1, \tag{3.11}$$

We note in particular that $M(1) = 1$. Given two coefficients $a, b \leq 1$, we can write:

$$\frac{\mu[0, abt]}{\mu[0, t]} = \frac{\mu[0, abt]}{\mu[0, at]} \frac{\mu[0, at]}{\mu[0, t]}.$$

By Assumption [2], the two ratios on the right-hand side are statistically independent, and the process M satisfies Property 1.[14]

The multiplicative measures of the previous sections do not exactly satisfy self-similarity. In these examples, the multiscaling relation (3.11) only holds for certain values of c and t. For instance in the case of the binomial measure, relation (3.11) only applies when c and t are dyadic numbers. When the relation only holds on a dense set of t's, we say that μ is *grid-bound self-similar*. Such is the case of the binomial measure. By contrast, the stronger definition given in this section characterize *grid-free self-similar* measures.

We can now discuss the moments of the measure. Assume without loss of generality that $X = [0, 1]$, and consider the random variables $\mu(a) \equiv \mu[0, a]$. As in the previous section, Property 1 implies that:

$$\mathbf{E}M(a)^q = a^{\tau(q)+1},$$

and therefore:

$$\mathbf{E}[\mu(a)^q] = E[\mu(1)^q]a^{\tau(q)+1}. \tag{3.12}$$

The scaling relation is thus a direct consequence of statistical self-similarity. This provides some justification for the definition of multifractal processes in terms of moments.

We briefly examine the properties of $\tau(q)$. As in previous sections, setting $q = 0$ in (3.12) yields $\tau(0) = -1$. For $q = 1$, we partition the unit interval $[0, 1]$ into n

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[14]Property 1 was defined in section 3.4.

subintervals of equal length, and write that the mass of the subintervals add up to $\mu(1)$. Taking expectations yields $\mathbb{E}[\mu(1/n)] = \mathbb{E}[\mu(1)]/n$, and the scaling function thus satisfies $\tau(1) = 0$. By Hölder's inequality, we also know that $\tau(q)$ is concave.

This framework helps model temporal heterogeneity in financial time series. Large price changes tend to be concentrated in time, and it is natural to assume that the distribution of volatility across time be statistically self-similar. We formalize this intuition in the MMAR.

4. The Multifractal Model of Asset Returns

This section uses the multifractal processes introduced in Section 3 to build a new financial model, the Multifractal Model of Asset Returns (MMAR). In this framework, the price of a financial asset is viewed as a multiscaling process with long memory and long tails. Fluctuations in "volatility" are introduced in the MMAR by a random trading time, generated as the c.d.f. of a random multifractal measure. This construction is new to both mathematics and finance.

We present the multifractal model in the following sections. Section 4.1 introduces trading time and compounding. Section 4.2 defines the MMAR. Section 4.3 compares the MMAR to earlier models. Section 4.4 presents simulation results.

4.1. Trading Time and Compound Processes

Trading time is the key concept facilitating the application of multifractals to financial markets. We introduce the following

Definition 4.1. *Let $\{B(t)\}$ be a stochastic process, and $\theta(t)$ an increasing function of t. The process*

$$X(t) \equiv B[\theta(t)]$$

is called a compound process. The index t denotes clock time, and $\theta(t)$ is called trading time or the time deformation process.

A special form of compound process is subordination, as developed by Bochner (1955), and applied to financial markets by Mandelbrot and Taylor (1968). Subordination originally developed as part of the theory of Markov processes, and requires that $\theta(t)$ has independent increments.[15]

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[15]See Feller (1968) p. 355. More current references to subordination in the mathematics and statistics literature include Rogers and Williams (1987) and Bertoin (1996).

In economics, the concept of subordination has evolved differently, and now encompasses any generic time deformation process.[16] For example. Dacorogna et al. (1992, 1993), use trading time to model time-of-day and day-of-week seasonality. Ghysels, Gouriéroux, and Jasiak (1996), model a stochastic trading time with volatility conditioned on measures of current market activity.

The MMAR posits a trading time that is the c.d.f. of a multifractal measure. Thus, trading time will be both highly variable and contain long memory. Both of these characteristics will be passed on to the price process through compounding. The main significance of compounding is that it allows direct modelling of a processes' variability without affecting the direction of increments or their correlations.

4.2. The Multifractal Model of Asset Returns

This section presents a new model for the price of a financial asset $\{P(t); 0 \leq t \leq T\}$. We introduce the notation:

$$X(t) = \ln P(t) - \ln P(0),$$

and list the main hypotheses of the theory.

Assumption 1. $X(t)$ *is a compound process:*

$$X(t) \equiv B_H[\theta(t)]$$

where $B_H(t)$ is a fractional Brownian Motion with self-affinity index H, and $\theta(t)$ is a stochastic trading time.

Assumption 2. *The trading time $\theta(t)$ is the c.d.f. of a multifractal measure defined on $[0, T]$. That is, $\theta(t)$ is a multifractal process with continuous, non-decreasing paths, and stationary increments.*

Assumption 3. $\{B_H(t)\}$ *and $\{\theta(t)\}$ are independent.*

The trading time $\theta(t)$ plays a crucial role in the MMAR. We first note that $\theta(0) = 0$ almost surely since by definition $X(0) = 0$. Assumption 2 imposes that

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[16]The different usage of subordination in economics can be traced back at least to Clark (1973). This paper develops all of its theory using Markov assumptions, and in empirical work proposes trading volume as a directing process.

$\theta(t)$ be the cumulative distribution function of a self-similar random measure, such as a binomial or a multiplicative measure. Trading time $\theta(t)$ presumably causes the price $X(t)$ to be multifractal, and we expect the scaling functions $\tau_\theta(q)$ and $\tau_X(q)$ to be closely related. This intuition leads to the following

Theorem 4.2. *Under Assumptions* [1] − [3], *the process* $X(t)$ *is multifractal, with scaling function* $\tau_X(q) \equiv \tau_\theta(Hq)$ *and stationary increments.*

Proof: See Appendix. ■

The above construction generates a large class of multifractal processes. Although a complete theory is not currently available, we now discuss some of the most important properties of the MMAR.

4.3. Properties of the MMAR

We first examine tail properties. The multiscaling relation (3.7) imposes that if $\mathbb{E}\,|X(t)|^q$ is finite for some instant t, then it is finite for all t. This justifies dropping the time index when discussing the moments of multifractal processes. By Theorem 4.2, the q-th moment of X exists if (and only if) the process θ has a moment of order Hq. The trading time thus controls the moments of the price $X(t)$.

By assumption, the trading time is generated by a self-similar measure μ. Its moments have very different properties depending upon whether μ is microcanonical or canonical. By definition, microcanonical random measures have a fixed mass on $[0, T]$ in the construction. The corresponding $\theta(t)$ are therefore bounded, and the compound process $X(t)$ has finite moments of all (non-negative) order. Microcanonical measures thus generate "mild" processes with relatively thin tails. On the other hand, canonical measures permit financial models that have diverging moments. Section 3.3 shows that the total mass of a canonical measure is a random variable Ω, which fully determines the tail behavior of prices. Mandelbrot (1972) conjectures and Guivarc'h (1987) proves that Ω generally has Paretian tails and infinite moments.[17] The corresponding process $X(t)$ is then "wild". Overall, the MMAR has enough flexibility to accommodate a wide variety of tail behaviors.

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[17]As a consequence, there exists a critical exponent $q_{crit}(\theta) > 1$. The moment $\mathbb{E}\,\theta^q$ is finite when $0 \leq q < q_{crit}(\theta)$, and infinite when $q \geq q_{crit}(\theta)$. Moreover, the scaling function $\tau_\theta(q)$ is negative when $0 < q < 1$ and positive when $1 < q < q_{crit}(\theta)$. Mandelbrot (1990) provides additional discussion of this topic.

We now study correlation in the process's increments. For a fixed $\Delta t > 0$ and any process Z, we define:

$$Z(t, \Delta t) = Z(t + \Delta t) - Z(t),$$

and the covariance function:

$$\gamma_Z(t) = Cov\left[Z(a, \Delta t), Z(a + t, \Delta t)\right].$$

Since $Z(t)$ has stationary increments, we know that $\gamma_Z(t)$ does not depend on the choice of a. It is natural to first consider the special case $H = 1/2$, in which $B_H(t)$ is a Brownian Motion. We prove the following

Theorem 4.3. *If $B_H(t)$ is a Brownian motion without drift, the following properties hold:*

1. *If $\mathbb{E}\left(\theta^{1/2}\right)$ is finite, then $\{X(t)\}$ is a martingale with respect to its natural filtration.*

2. *If $\mathbb{E}\theta$ is finite, the increments of $X(t)$ are uncorrelated, that is $\gamma_X(t) = 0$ for all $t \geq \Delta t$.*

Proof: See Appendix. ■

Theorem 4.3 shows that when $H = 1/2$, the MMAR generates a price process that has a white spectrum.[18] This result builds on the martingale property of the Brownian motion, and does not extend to the case $H \neq 1/2$. We can in fact prove the following

Theorem 4.4. *If $\mathbb{E}\left(\theta^{2H}\right)$ is finite, the autocovariance function of the price process $X(t)$ satisfies for all $t \geq \Delta t$:*

$$\gamma_X(t) = K\left\{(t + \Delta t)^m + (t - \Delta t)^m - 2t^m\right\} \tag{4.1}$$

where $m = \tau_\theta(2H) + 1$ and $K = c_\theta(2H)Var[B_H(1)]/2$. It is positive when $H > 1/2$, and negative when $H < 1/2$.

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[18]Theorem 4.3 only shows that $\ln P(t)$ is a martingale. By Jensen's inequality, the price $P(t)$ is then a submartingale, but *not* a martingale. A very unstable concept, the martingale property is only discussed to illustrate the flexibility of the model. Moreover, a detailed study of the relation between the MMAR and market efficiency is beyond the scope of this paper.

Proof: See Appendix. ■

When $H > 1/2$, the process $B_H(t)$ has long memory, and price increments are positively correlated. The definition of long memory for the price process, which may only be defined on a bounded time range, is more delicate. This introductory paper informally defines long memory by the following geometric property: the longest apparent cycle has approximately the same length as the interval of definition. In this sense, the c.d.f. of a multiplicative cascade has long memory. By Theorem 4.4, it seems safe to conjecture that the price process has long memory when $H > 1/2$. The MMAR thus allows for a wide variety of autocorrelation structures.

We finally examine dependence in the absolute values of returns, which will indicate whether the MMAR displays persistence in volatility. For any stochastic process Z with stationary increments, it is convenient to define:

$$\delta_Z(t, q) = Cov(|Z(a, \Delta t)|^q, |Z(a + t, \Delta t)|^q).$$

We now prove the following

Theorem 4.5. *If $H \geq 1/2$ and $\mathbb{E}\left(\theta^{Hq}\right)$ is finite, the compound process satisfies:*

$$\delta_X(t, q) \geq \delta_\theta(t, Hq)\left[\mathbb{E}|B_H(1)|^q\right]^2 \tag{4.2}$$

for all non-negative q and $t \geq \Delta t$. Moreover, this result holds as an equality when $H = 1/2$.

Proof: See Appendix. ■

Theorem 4.5 indicates that the price process has long memory in the absolute value of its increments. In particular when $H = 1/2$, the price process displays both uncorrelated increments and persistence in volatility. Thus while the MMAR construction is certainly not familiar in finance, it has quite natural and appealing implications for empirical applications. The MMAR allows for long tails, correlated "volatilities", and either unpredictability or long memory in returns, and thus combines the properties of many earlier models.

4.4. Comparison with Earlier Models

This section discusses similarities and differences between the MMAR and earlier models. Conclusion 1 of Theorem 4.3 guarantees that, when $\{X(t)\}$ is a Brownian Motion without drift, the direction of future returns is not predictable from knowledge of past prices. At the same time, Theorem 4.5 guarantees long memory in the absolute value of price changes. That is, there is not only temporal heterogeneity in the size of price increments, but this temporal heterogeneity is present on any scale at which we choose to look at the data - whether daily, weekly, monthly, or yearly returns.

This result may surprise readers who have learned to associate (from the FBM) long memory with the predictability of market returns. In fact, the MMAR is not the first to combine long memory with a martingale property. The FIGARCH model of Baillie, Bollerslev and Mikkelsen (1996) also has this feature. The most important distinguishing feature of the multifractal model is that returns are scale-consistent, while FIGARCH is not.

To place the multifractal market model more concretely within the existing literature on the distribution of financial returns, we present the following tables.

Table 1: Typical Characteristics for Models of Financial Returns

Volatility Clustering Implies Predictable Price	Volatility Clustering Consistent with Martingale Price	PROPERTIES
ARMA*	GARCH	no long memory scale-inconsistent
ARFIMA*	FIGARCH	long memory scale-inconsistent
FBM	MMAR	long memory scale-consistent

* Asymptotically, ARMA scales like Brownian Motion, and ARFIMA like FBM.

Table 2: Typical Covariance Characteristics of Models of Returns

	$Cov(|X_{t+s+\Delta t} - X_{t+s}|, |X_{t+\Delta t} - X_t|)$	$Cov(X_{t+s+\Delta t} - X_{t+s}, X_{t+\Delta t} - X_t)$
ARMA	weak	weak
GARCH	weak	zero
ARFIMA	strong	strong
FIGARCH	strong	zero
FBM	strong	strong
MMAR	strong	zero or strong

The distinguishing features of the multifractal market model are thus:

1. Long memory in volatility

2. Compatibility with the martingale property of returns

3. Scale-consistency

4. Multiscaling

Properties 1 and 2 are given by Theorems 4.5 and 4.3. Properties 3 and 4 are consequences of the definition of multifractal processes. They respectively correspond to the time-invariance and nonlinearity of the scaling function $\tau(q)$.

4.5. Simulation of Multifractal Processes

We can generate a wide range of continuous multifractal processes with sequences of increments that are indistinguishable from real financial data *at all time scales*.

Example 4.6. *Figures 5a and 5b show two simulated sample paths of multifractal processes. Figures 6a and 6b show their respective first differences. In these simulations, $\{B(t)\}$ is a standardized Brownian Motion. $\{\theta(t)\}$ is the c.d.f. of a randomized binomial measure with multiplier $m_0 = .6$.*

The most striking aspect of these simulations is their strong resemblance in character to the exchange rate data in Figure 2. In contrast to the GARCH simulation in Figure 1, these simulations display marked temporal heterogeneity at all time scales. In addition, we observe crashes, upcrashes, slow but persistent growth and decline, and a variety of behavior interesting to technical analysts. This is quite atypical of the Brownian Motion and GARCH processes.

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Other factors are of far greater importance to financial economists. Foremost among these are the testable implications of multifractality. Moreover, the clearly non-Gaussian behavior of multifractal processes has tremendous implications for the behavior of risk-averse investors and the pricing of financial derivatives.

5. Conclusion

The Multifractal Model of Asset Returns, which is new to this paper and Mandelbrot (1997), incorporates important regularities observed in financial time series including long tails and long memory. Multifractality is defined by a set of restrictions on the process's moments as the time scale of observations changes. It is integrated in the model through *trading time*, a random distortion of clock time that accounts for changes in volatility.

The properties of the MMAR include multifractality, scale-consistency and long memory in volatility. The predictability of (log) prices is not theoretically specified, since the model has enough flexibility to satisfy the martingale property in some cases and long memory in its increments otherwise. The MMAR is thus a promising alternative to ARCH-type models. Like FIGARCH, the MMAR incorporates long memory in volatility. In addition, the MMAR allows the possibility that returns are uncorrelated, but does not require it. This is an important property for researchers interested in issues of market efficiency. The main advantage of the MMAR over alternatives like FIGARCH is the property of scale-consistency. Because of this property, aggregation characteristics of the data (otherwise thought of as the information contained at different sampling frequencies) can be used to test and identify the model.

The main disadvantage of the MMAR is the dearth of applicable statistical methods. We propose that new econometric methods are needed for models which are both time-invariant and scale-invariant. A demonstration of this type of method appears in the companion paper, Fisher, Calvet, and Mandelbrot (1997).

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6. Appendix

6.1. Proof of Theorem 4.2

The argument builds on the iterated expectation:

$$\mathbb{E}\left(|X(t)|^q\right) = \mathbb{E}\left[\mathbb{E}\left(|X(t)|^q \,|\theta(t) = u\right)\right]$$

Since the trading time and the self-affine process $\{B_H(t)\}$ are independent, conditioning on $\theta(t)$ yields:

$$\begin{aligned}
\mathbb{E}\left(|X(t)|^q \,|\theta(t) = u\right) &= \mathbb{E}\left[|B_H(u)|^q \,|\theta(t) = u\right] \\
&= \theta(t)^{Hq}\,\mathbb{E}\left[|B_H(1)|^q\right],
\end{aligned}$$

and thus

$$\mathbb{E}\left[|X(t)|^q\right] = \mathbb{E}\left[\theta(t)^{Hq}\right]\,\mathbb{E}\left[|B_H(1)|^q\right]$$

The process $X(t)$ therefore satisfies multiscaling relation (3.7), with $\tau_X(q) \equiv \tau_\theta(Hq)$ and $c_X(q) \equiv c_\theta(Hq)\,\mathbb{E}\left[|B_H(1)|^q\right]$.

6.2. Proof of Theorem 4.3

1. Let $\mathcal{F}_t$ and $\mathcal{F}_t'$ respectively denote the natural filtrations of $\{X(t)\}$ and $\{X(t), \theta(t)\}$. We compute $\mathbb{E}\{X(t+T)|\mathcal{F}_t\}$ as the iterated expectation:

$$\mathbb{E}\left\{\,\mathbb{E}\left\{B_H[\theta(t+T)]|\,\mathcal{F}_t', \theta(t+T) = u\right\}\,|\,\mathcal{F}_t\right\}$$

For any t, T and $u \geq t$, the independence of B_H and θ implies that:

$$\begin{aligned}
\mathbb{E}\left\{B_H[\theta(t+T)]|\,\mathcal{F}_t', \theta(t+T) = u\right\} &= \mathbb{E}\left[B_H(u)|\mathcal{F}_t'\right] \\
&= B_H[\theta(t)]
\end{aligned}$$

since $\{B_H(t)\}$ is a martingale in this case. We now infer that

$$\mathbb{E}\left[X(t+T)|\mathcal{F}_t\right] = X(t).$$

2. This property, which holds for all square integrable martingales, is a direct consequence of the fact that $\mathbb{E}\left[X(t'+\Delta t) - X(t')|\mathcal{F}_t\right] = 0$ when $t + \Delta t \leq t'$.

6.3. Proof of Theorem 4.4

Consider the conditional expectation:

$$\mathbb{E}\left\{ X(0,\Delta t)X(t,\Delta t)\,|\,\theta(\Delta t) = u_1, \theta(t) = u_2, \theta(t+\Delta t) = u_3\right\}. \qquad (6.1)$$

Since $B_H(t)$ and $\theta(t)$ are independent processes, this expression simplifies to:

$$\mathbb{E}\left\{ B_H(u_1)[B_H(u_3) - B_H(u_2)]\right\}.$$

or[19]

$$\frac{1}{2}\left\{ |u_3|^{2H} - |u_2|^{2H} + |u_2 - u_1|^{2H} - |u_3 - u_1|^{2H}\right\} Var[B_H(1)],$$

which we rewrite in terms of trading time:

$$\frac{Var[B_H(1)]}{2}\left\{ |\theta(t+\Delta t)|^{2H} - |\theta(t)|^{2H} + |\theta(t) - \theta(\Delta t)|^{2H} - |\theta(t+\Delta t) - \theta(\Delta t)|^{2H}\right\}.$$

Taking the expectation yields (4.1).

The sign of $\gamma_X(t)$ is determined by the concavity of the function $f(x) = x^{\tau_\theta(2H)+1}$, i.e. by the sign of $\tau_\theta(2H)$. We know that $\tau_\theta(1) = 0$, so that for all q satisfying $0 \le q < q_{crit}(\theta)$, the scaling function $\tau_\theta(q)$ has the same sign as $q - 1$. When $H = 1/2$, the autocovariance equals zero since $\tau_\theta(2H) = 0$. When $H > 1/2$, $f(x)$ is convex and the function $\gamma_X(t)$ is positive. Conversely $\gamma_X(t)$ is negative for $H < 1/2$.

6.4. Proof of Theorem 4.5

The argument builds on the properties of the conditional expectation:

$$\mathbb{E}\left\{ |X(0,\Delta t)X(t,\Delta t)|^q\,|\,\theta(\Delta t) = u_1, \theta(t) = u_2, \theta(t+\Delta t) = u_3\right\}. \qquad (6.2)$$

We start by rewriting this quantity:

$$\mathbb{E}\left\{ |B_H(u_1)[B_H(u_3) - B_H(u_2)]|^q\right\}$$

---

[19]For FBM, we know that:

$$\mathbb{E}[B_H(u_1)B_H(u_2)] = \left(|u_1|^{2H} + |u_2|^{2H} - |u_1 - u_2|^{2H}\right) Var[B_H(1)]/2.$$

since $B_H(t)$ and $\theta(t)$ are independent processes.

We assumed that the FBM has either independent $(H = 1/2)$ or positively correlated $(H > 1/2)$ increments. Therefore expression (6.2) is equal to. or bounded below by:

$$\mathbb{E}\left[|B_H(u_1)|^q\right] \mathbb{E}\left[|B_H(u_3) - B_H(u_2)|^q\right] = |u_1|^{Hq} |u_3 - u_2|^{Hq} \left[\mathbb{E}|B_H(1)|^q\right]^2 .$$

In terms of trading time, this lower bound can be rewritten:

$$|\theta(\Delta t)|^{Hq} |\theta(t, \Delta t)|^{Hq} \left[\mathbb{E}|B_H(1)|^q\right]^2 . \tag{6.3}$$

Taking the expectation of (6.2) and (6.3), we infer that:

$$\mathbb{E}\left[|X(0, \Delta t)X(t, \Delta t)|^q\right] \geq \mathbb{E}\left[|\theta(0, \Delta t)\theta(t, \Delta t)|^q\right] \left[\mathbb{E}|B_H(1)|^q\right]^2 .$$

The rest of the proof is straightforward.

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Figure 1a. DM/USD Exchange Rate: June 4, 1973 – Dec 31, 1996
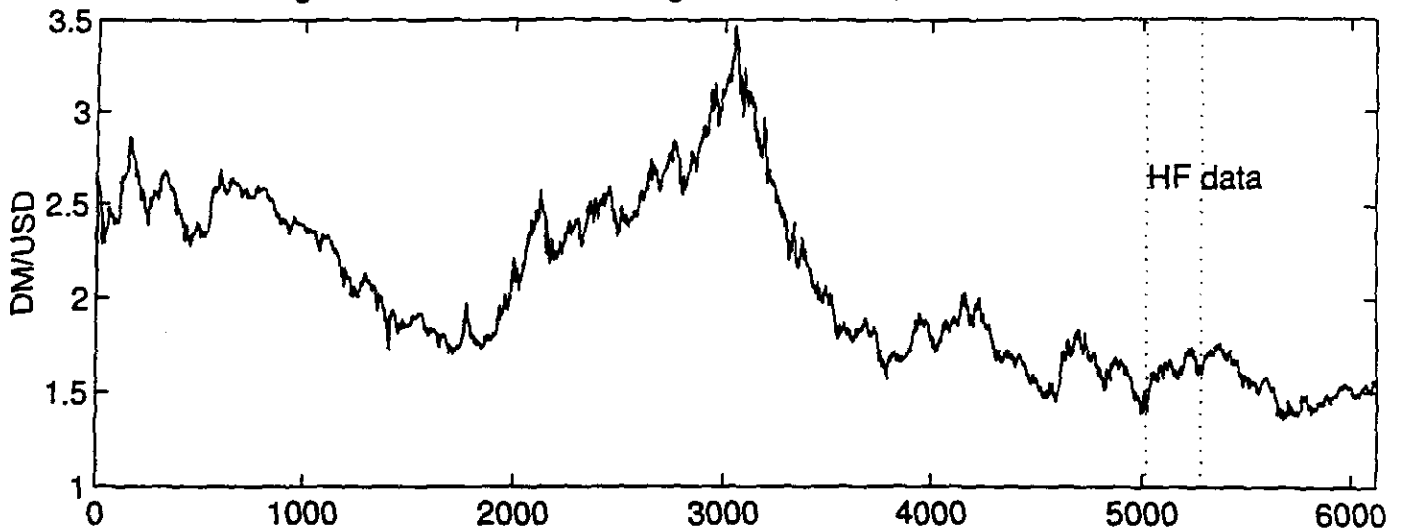
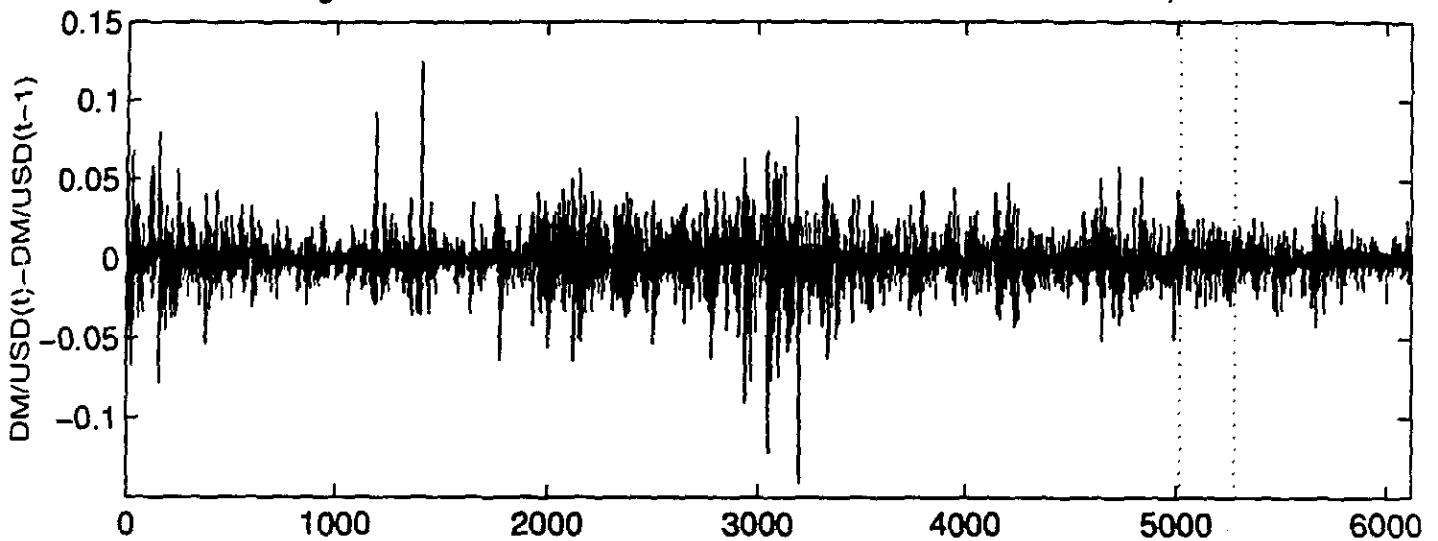Figure 1b. DM/USD First Differences: June 4, 1973 – Dec 31, 1996

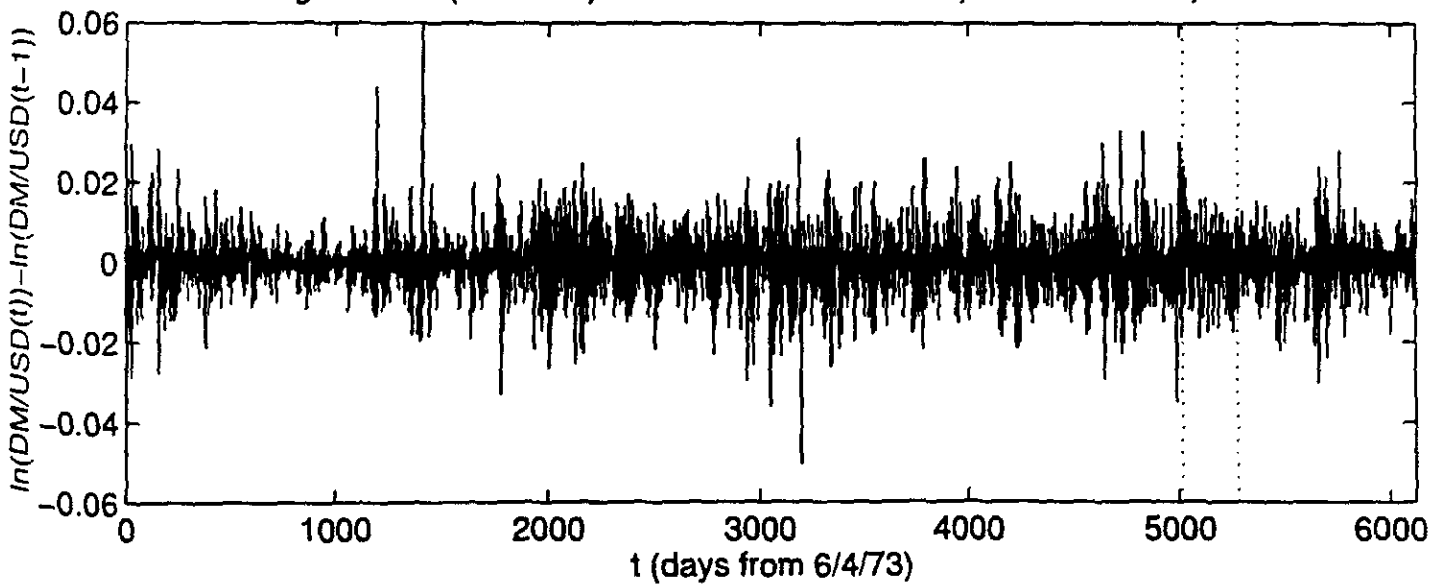Figure 1c. ln(DM/USD) First Differences: June 4, 1973 – Dec 31, 1996
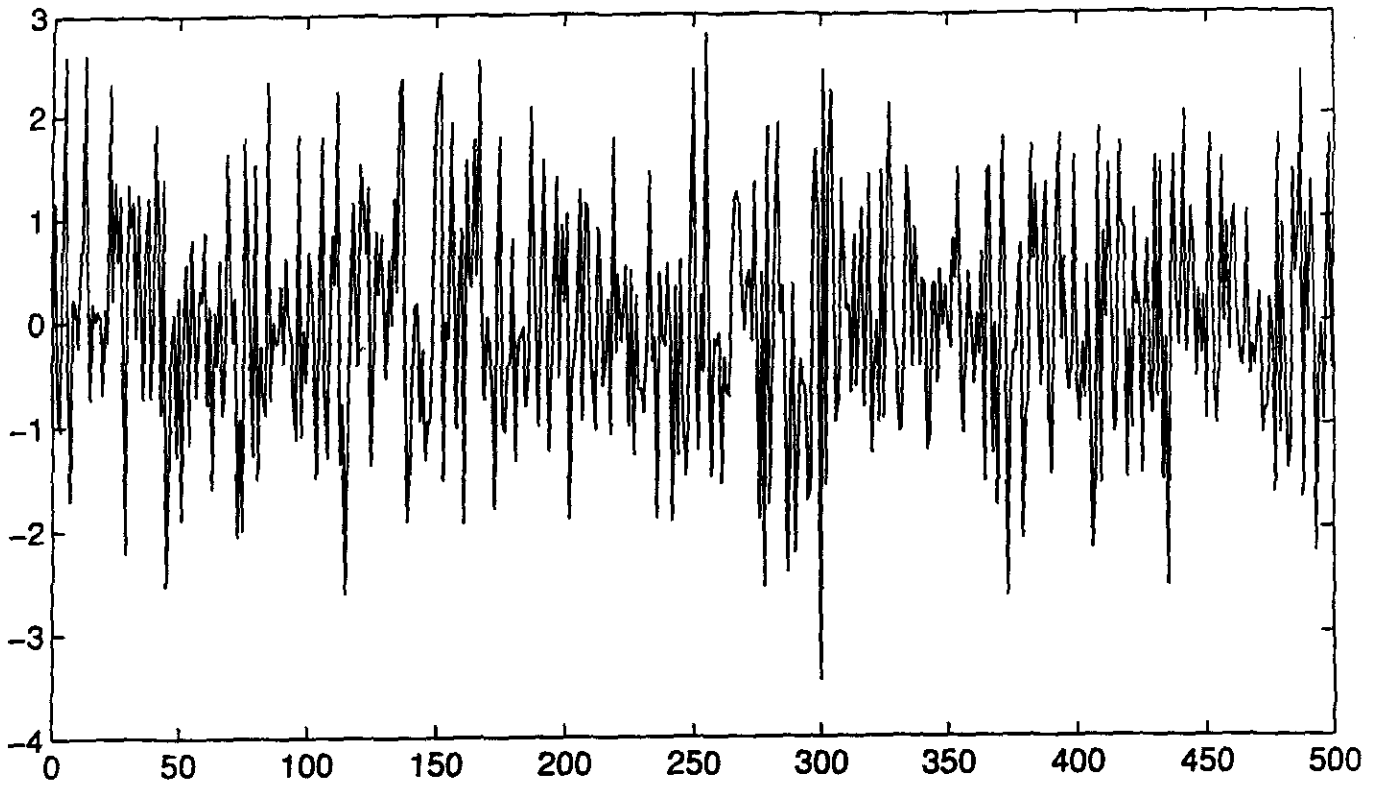
Fig 2a. GARCH Simulation, n=500

Fig 2b. GARCH Simulation, n=100000
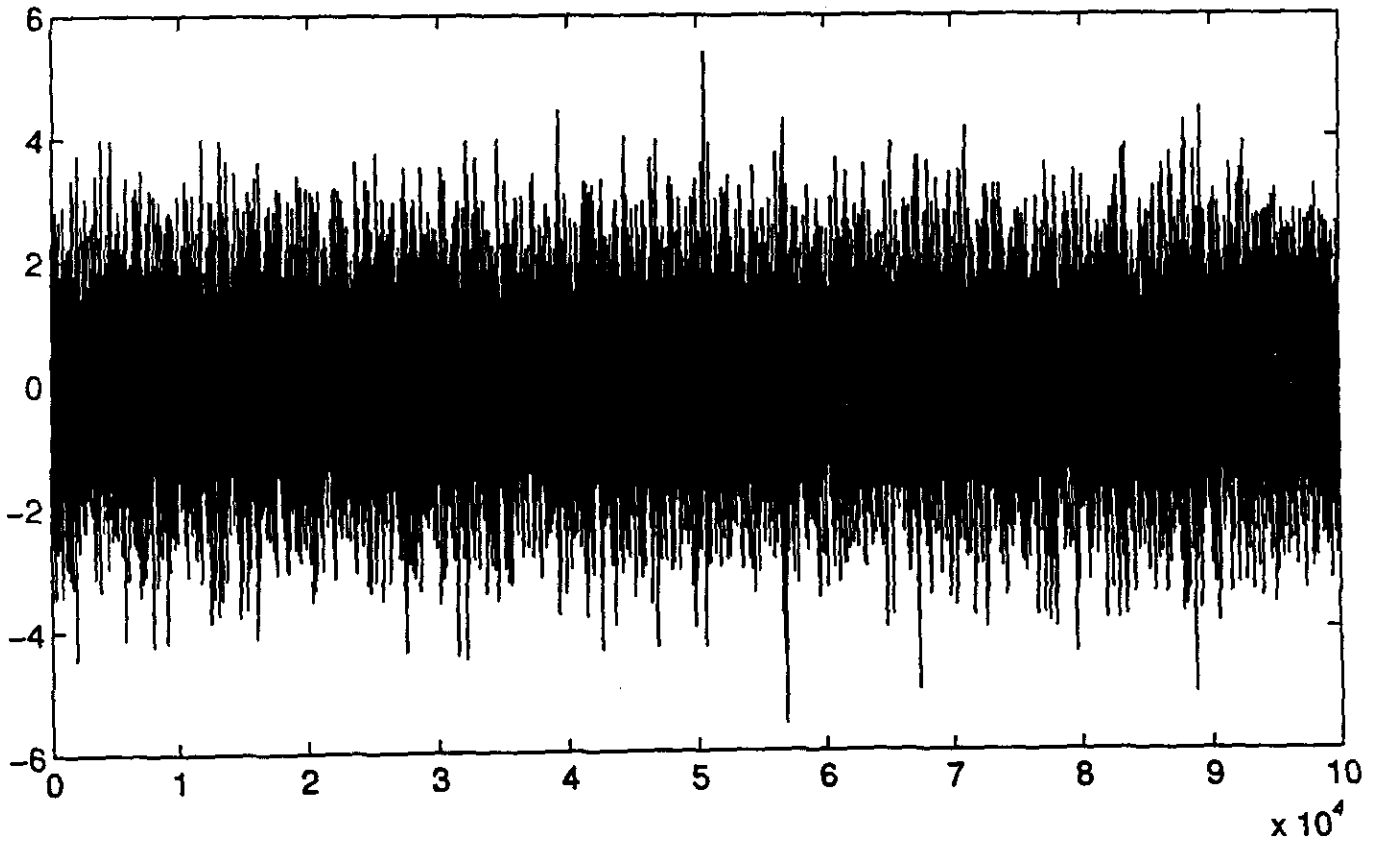
Figure 3a. FIGARCH Increments, n=500
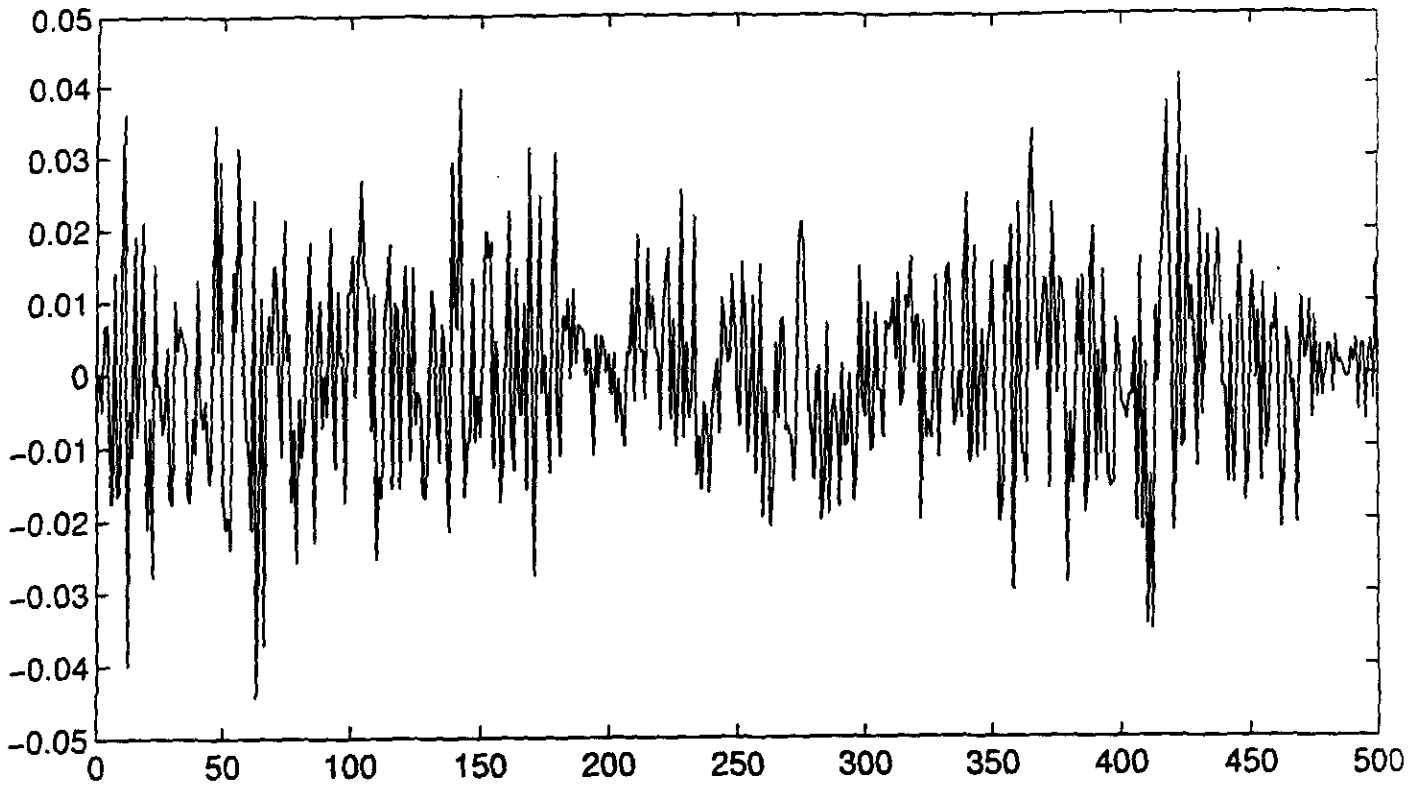
Figure 3b. FIGARCH Increments, n=10^5
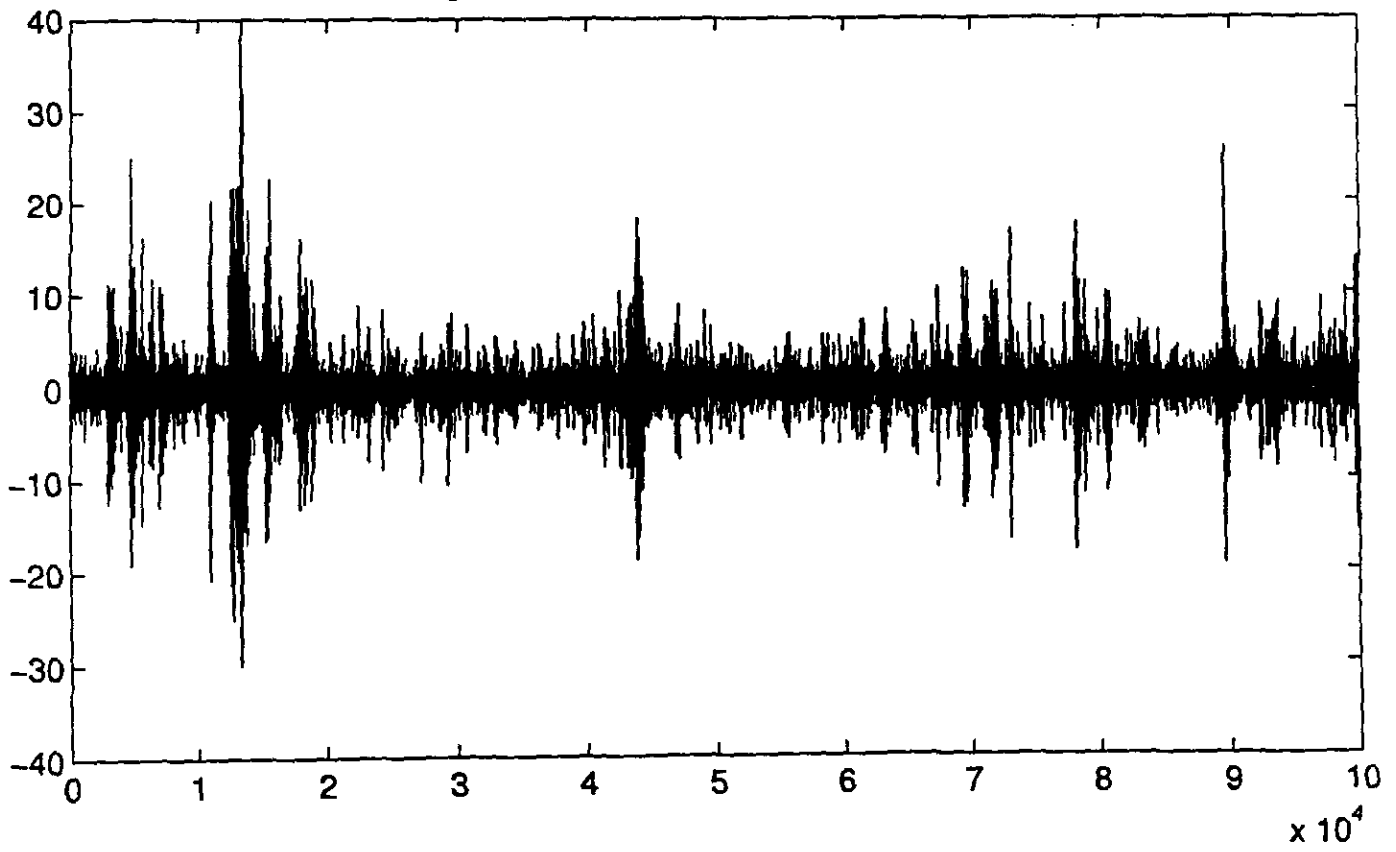
Fig 4.a   Density Function (Iteration k = 1)
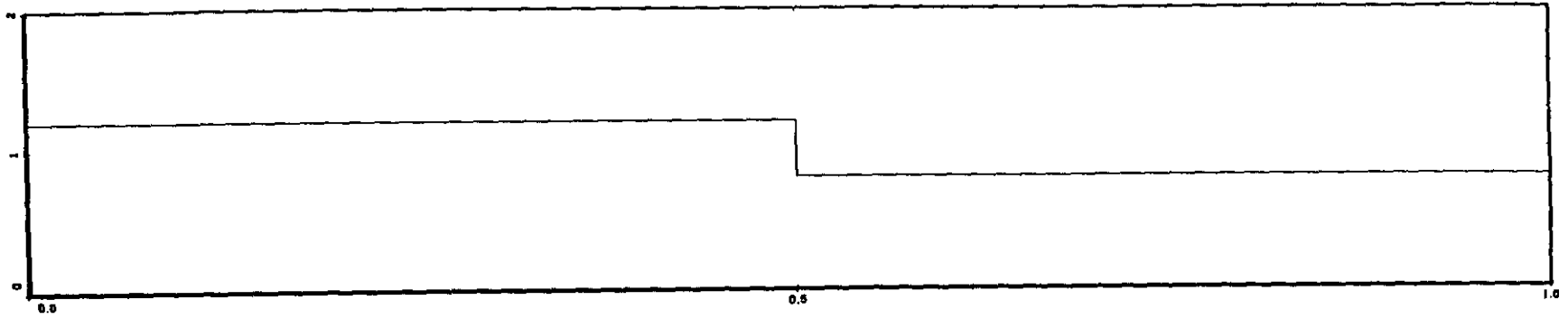
Fig 4.b   Density Function (Iteration k = 4)
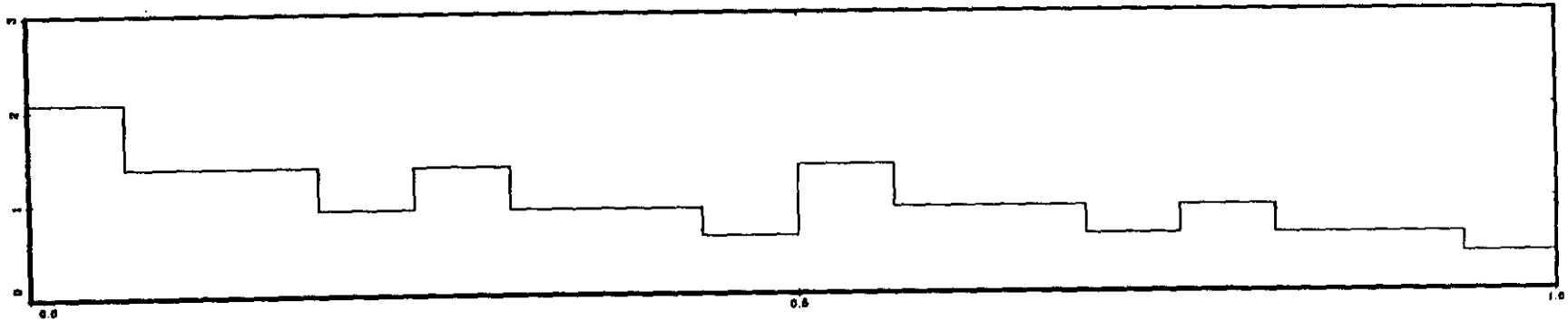
Fig 4.c   Density Function (Iteration k = 10)
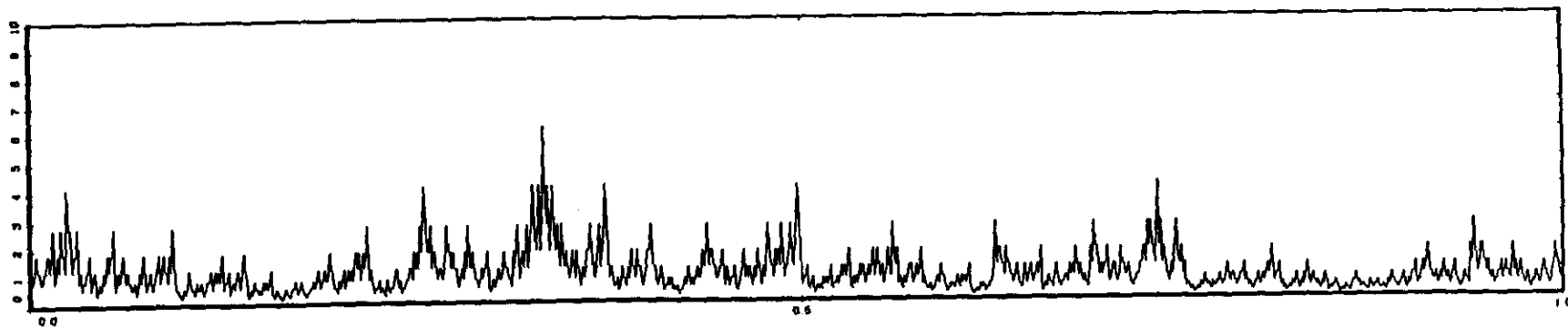
Fig 5.a— Price P(t) — Simulation 1
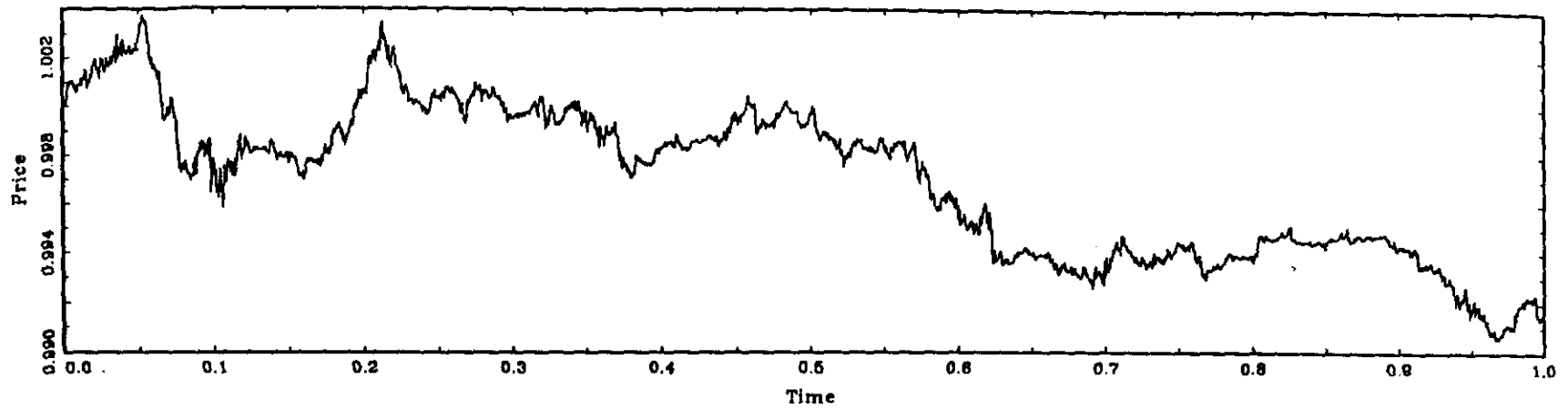
Fig 5.b— Price P(t) — Simulation 2

Fig 5 — PRICE P(t) = exp $W_{u(t)}$,
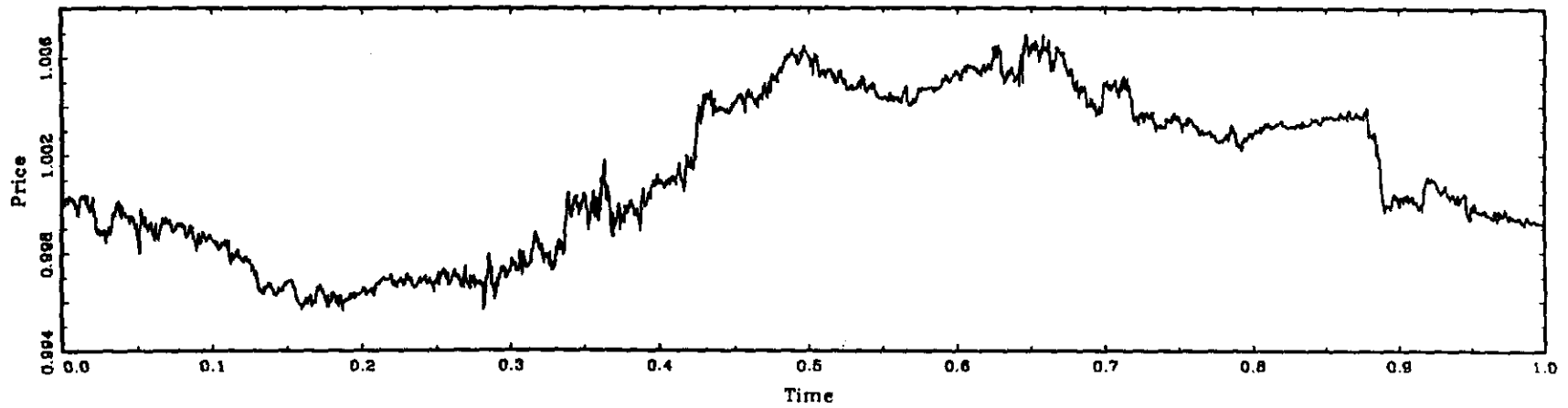
W = Wiener Process, u = Binomial Fractal Time (m_0 = 0.6, p = 0.5)

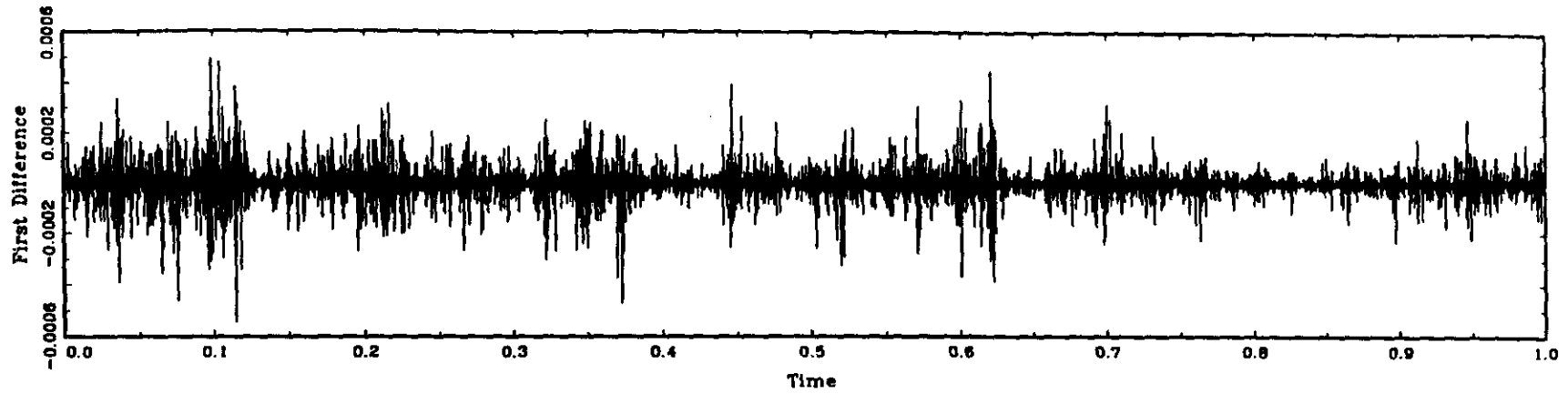## Fig 6.a— First Differences — Simulation 1

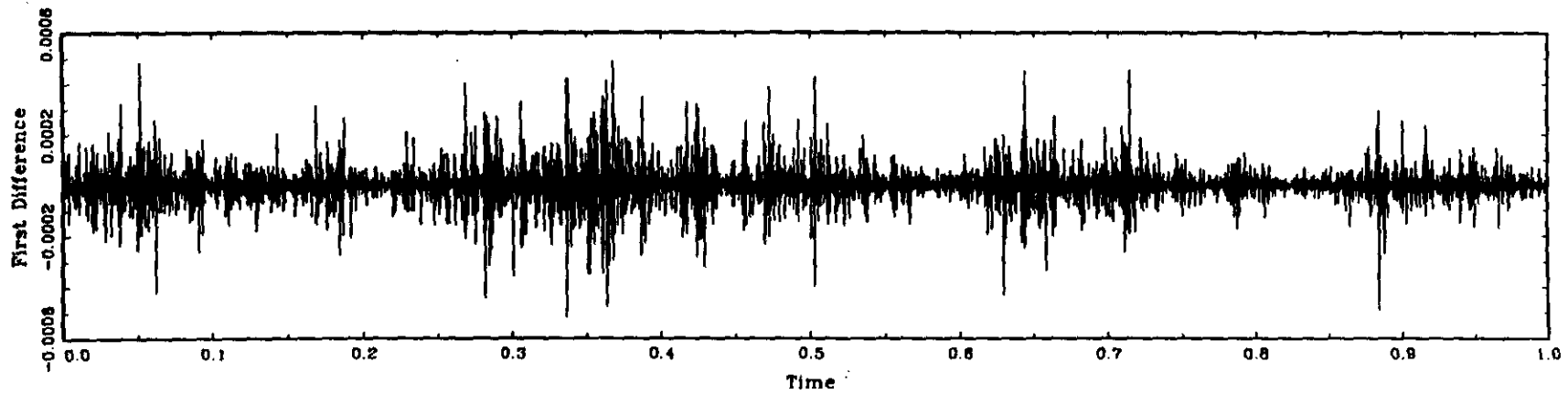
Fig 6.b— First Differences — Simulation 2

Fig 6 — FIRST DIFFERENCES OF $P(t) = \exp W_{u(t)}$,

W = Wiener Process, u = Binomial Fractal Time (m_0 = 0.6, p = 0.5)