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ARCH and GARCH Models

Advanced Information on the Bank of Sweden Prize in Economic Sciences to Robert F. Engle in Memory of Alfred Nobel, October 8, 2003. Available at: *http://www.nobel.se/economics/laureates/2003/ecoadv.pdf*

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Bollerslev, T., R.F. Engle, and D.B. Nelson (1994), "ARCH Models," in *Handbook of Econometrics, Vol.IV*, (eds. R.F. Engle and D. McFadden). Amsterdam: North-Holland.

Engle, R.F. (2004), "Risk and Volatility: Econometric Models and Financial Practice," *American Economic Review*, Vol.94, p.405-420.

ARCH and GARCH Models

- Basic structures and properties
  - The ARCH(q) and GARCH(p,q) models
  - GARCH(1,1) variance prediction
  - ARMA representation in squares
  - Maximum likelihood estimation and testing

- Variations on the basic GARCH model
  - Asymmetric response and the leverage effect
  - ARCH-M and time-varying risk premia
  - Fat-tailed conditional densities
  - Integrated volatility
  - Long-memory
  - Component GARCH
  - Regime switching
  - Other univariate GARCH models

- Multivariate GARCH models
  - Vech and Diagonal GARCH
  - Factor GARCH models
  - Constant conditional correlation models
  - Dynamic conditional correlation models
  - Asymmetries in correlations
  - Copulas
  - Structural GARCH

Basic Structure and Properties

- Standard time series models:

$$y_t = E(y_t \mid \Omega_{t-1}) + \varepsilon_t$$

$$E(y_t \mid \Omega_{t-1}) = \mu_t(\theta)$$

$$Var(y_t \mid \Omega_{t-1}) = E(\varepsilon_t^2 \mid \Omega_{t-1}) = \sigma^2$$

  - ARMA(p,q) model:

$$\mu_t(\theta) = \varphi_0 + \varphi_1 y_{t-1} + \ldots + \varphi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}$$

  - Conditional mean: varies with Ω_{t-1}
  - Conditional variance: constant

  - k-step-ahead forecasts: generally depends non-trivially on Ω_{t-1}
  - k-step-ahead forecast error variance: depends only on k, not Ω_{t-1}

  - Unconditional mean: constant
  - Unconditional variance: constant

- AutoRegressive Conditional Heteroskedasticity - ARCH

Engle (1982, *Econometrica*)

$$y_t = E(y_t \mid \Omega_{t-1}) + \varepsilon_t$$

$$E(y_t \mid \Omega_{t-1}) \equiv \mu_t(\theta)$$

$$Var(y_t \mid \Omega_{t-1}) = E(\varepsilon_t^2 \mid \Omega_{t-1}) \equiv h_t(\theta)$$

– Conditional mean: varies with Ω_{t-1}

– Conditional variance: varies with Ω_{t-1}

– k-step-ahead forecasts: generally depends on Ω_{t-1}

– k-step-ahead forecast error variance: generally depends on Ω_{t-1}

– Unconditional mean: constant

– Unconditional variance: constant

- How to parameterize $E(\varepsilon_t^2 \mid \Omega_{t-1}) \equiv h_t(\theta)$?

- ARCH(q) process:

$$h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \ldots + \alpha_q \varepsilon_{t-q}^2$$

- AR(1)-ARCH(1) process:

$$y_t = \varphi y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, h_t)$$

$$h_t = \omega + \alpha \varepsilon_{t-1}^2$$

  – Conditional mean: $E(y_t \mid \Omega_{t-1}) =$

  – Conditional variance: $E([y_t - E(y_t \mid \Omega_{t-1})]^2 \mid \Omega_{t-1}) =$

  – Unconditional mean: $E(y_t) =$

  – Unconditional variance: $E(y_t - E(y_t))^2 =$

- Why ARCH(q)?
  - Past residuals: $\varepsilon_{t-1}\ ,\ \varepsilon_{t-2}\ ,\ \ldots\ ,\ \varepsilon_1$
  - Historical sample variance as of time t:

$$\frac{1}{t-1}\left[\,\varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \ldots + \varepsilon_1^2\,\right]$$

  - Only q most recent observations:

$$\frac{1}{q}\left[\,\varepsilon_{t-1}^2 + \varepsilon_{t-2}^2 + \ldots + \varepsilon_{t-q}^2\,\right]$$

  - More weight to most recent observations:

$$\alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \ldots + \alpha_q\varepsilon_{t-q}^2$$

  - ARCH(q):

$$\omega + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \ldots + \alpha_q\varepsilon_{t-q}^2$$

  - Large q and too many “alpha’s”. What to do?

- Generalized ARCH, or GARCH

Bollerslev (1986, *J. of Econometrics*)

- GARCH(p,q) process:

$$h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \dots + \alpha_q \varepsilon_{t-q}^2 + \beta_1 h_{t-1} + \dots + \beta_p h_{t-p}$$

- The simple GARCH(1,1) model often works very well:

$$\varepsilon_t \mid \Omega_{t-1} \sim N(0, h_t)$$

$$h_t = \omega + \alpha\, \varepsilon_{t-1}^2 + \beta\, h_{t-1}$$

  - $\omega > 0,\ \alpha > 0,\ \beta > 0$
  - Conditional variance positively serially correlated
  - Volatility clustering in financial markets

• Homoskedastic and normal GARCH(1,1) confidence bands for AR(4) quarterly U.S. inflation rate forecasts:

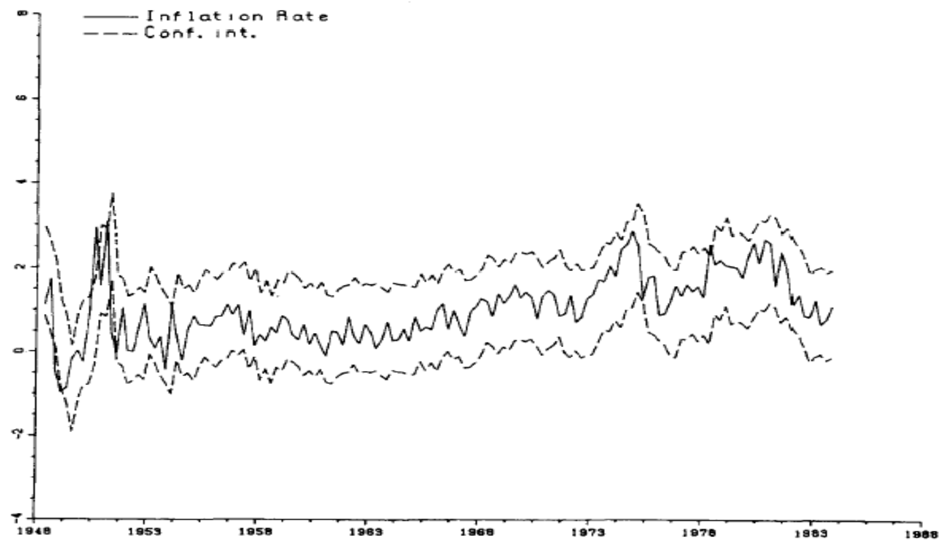


Fig. 3. 95% confidence intervals for OLS.

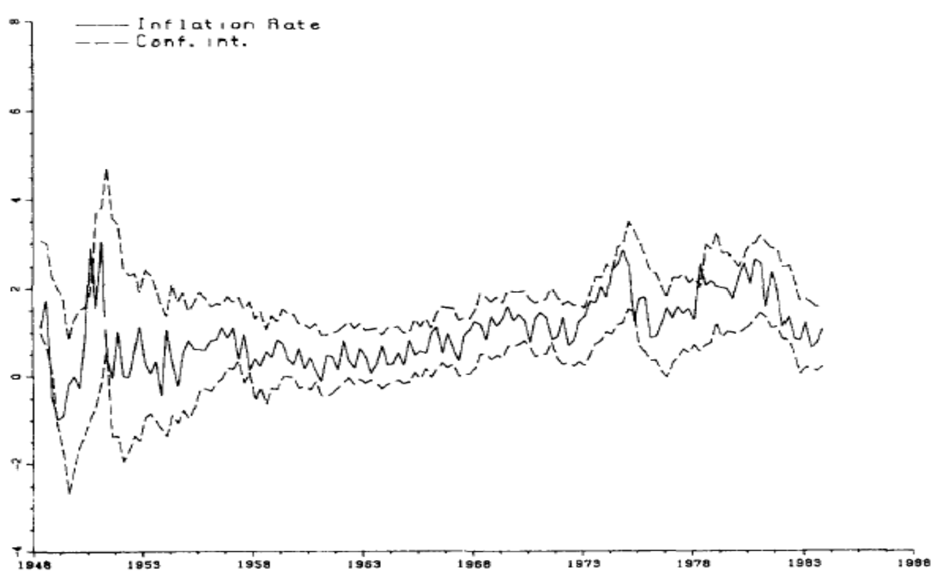


Fig. 4. 95% confidence intervals for GARCH(1,1).

Bollerslev (1986, *Journal of Econometrics*)

Variance Prediction

- Future (predicted) variances depend (non-trivially) on Ω_t

  - GARCH(1,1):

$$h_t = \omega + \alpha \, \varepsilon_{t-1}^2 + \beta \, h_{t-1}$$

  - 1-step-ahead predictions:

$$h_{t+1|t} = \omega + \alpha \, \varepsilon_t^2 + \beta \, h_t$$

  - k-step-ahead predictions:

$$h_{t+k|t} = \omega + \alpha \, h_{t+k-1|t} + \beta \, h_{t+k-1|t}$$

  - Long-run predictions:

$$\lim_{k\to\infty} h_{t+k|t} = \frac{\omega}{1 - \alpha - \beta} = \bar{\omega}$$

- GARCH(1,1) forecasts of Dow Jones Industrial Average:

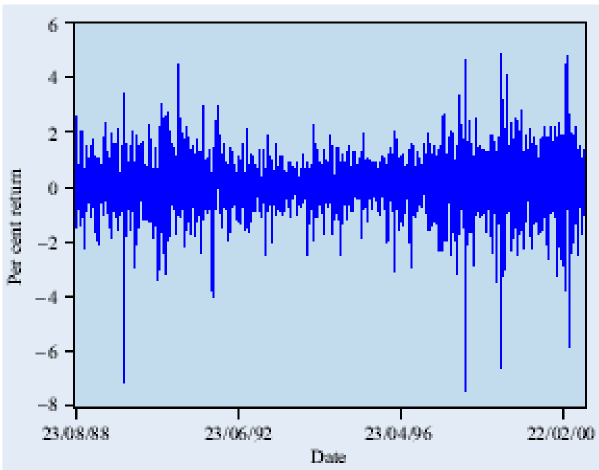


Figure 2. Returns on the Dow Jones Industrial Index.

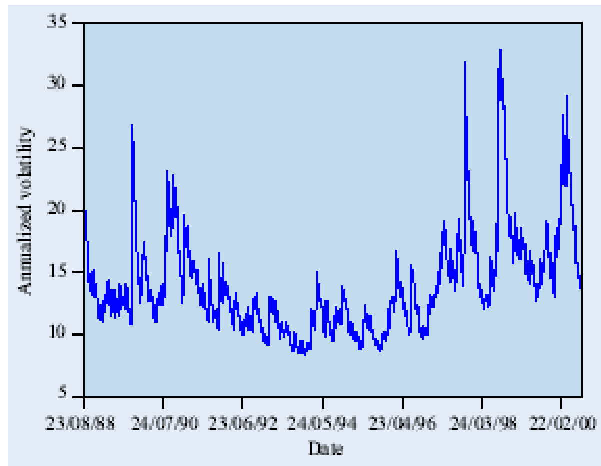


Figure 4. Estimated conditional volatility using a GARCH(1,1) model.

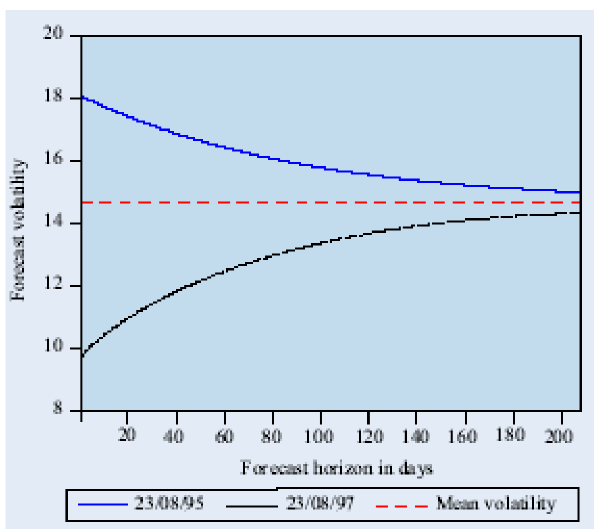


Figure 6. Forecasts of daily return volatility using the GARCH(1,1) model.

Engle and Patton (2001, *Quantitative Finance*)

- k-period returns:

$$r_{t+k}(k) = r_{t+k} + r_{t+k-1} + \ldots + r_{t+1}$$

- k-period return variance:

$$\text{Var}(r_{t+k}(k) \mid \Omega_t) \approx h_{t+k|t} + h_{t+k-1|t} + \ldots + h_{t+1|t}$$

- Dow Jones Industrial Average:

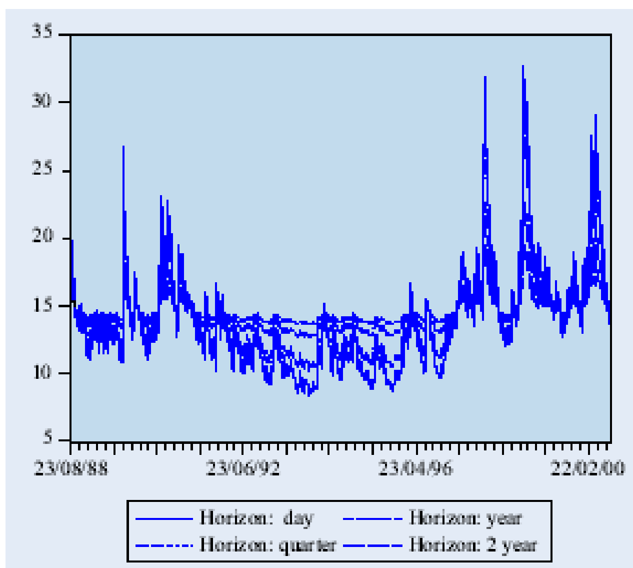


Figure 7. Volatilities at different horizons from GARCH(1,1).

Engle and Patton (2001, *Quantitative Finance*)

- Scaling:

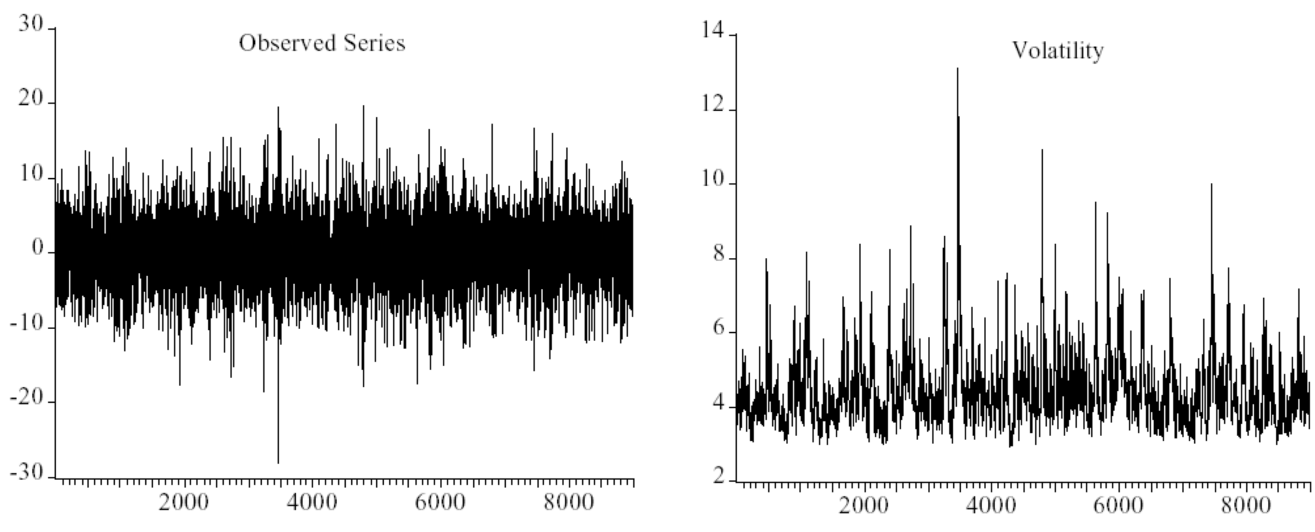
$$\mathrm{Var}(r_{t+k}(k)) \approx k \times \mathrm{Var}(r_{t+1} \mid \Omega_t) = k \times h_{t+1|t}$$

  – Easy to calculate

  – Correct on average

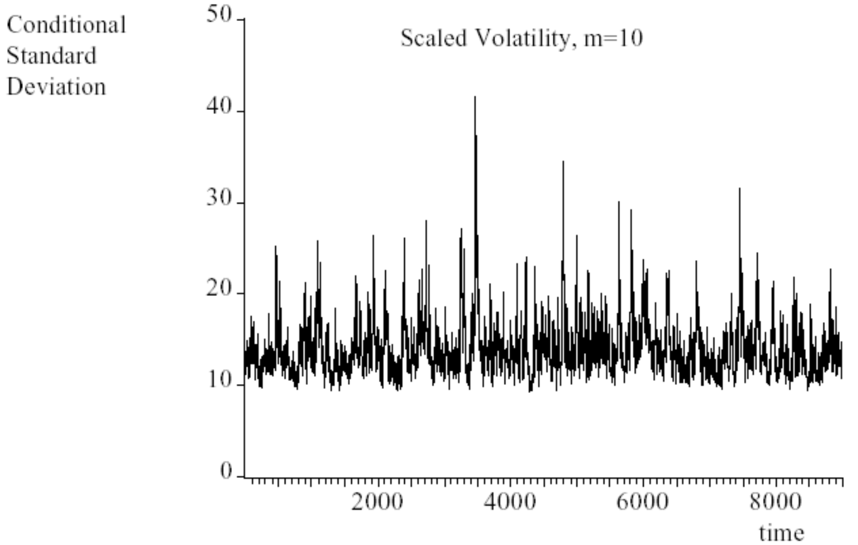
  – Exaggerates volatility-of-volatility

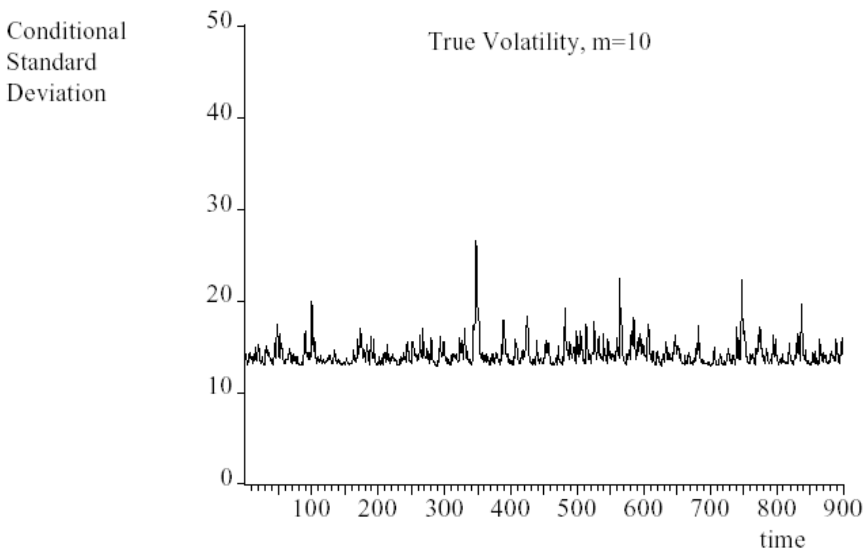
- Simulated GARCH(1,1):



Diebold, Schuerman, Hickman and Inoue (1998, *Risk*)

10-Day Volatility, Scaled and Actual





Diebold, Schuerman, Hickman and Inoue (1998, *Risk*)

ARMA Representation in Squares

- ARCH(1) implies an AR(1) representation for y_t^2:

$$y_t^2 = \omega + \alpha y_{t-1}^2 + \nu_t$$

$$\nu_t = y_t^2 - h_t$$

- GARCH(1,1) implies an ARMA(1,1) representation for r_t^2:

$$y_t^2 = \omega + (\alpha + \beta) y_{t-1}^2 - \beta \nu_{t-1} + \nu_t$$

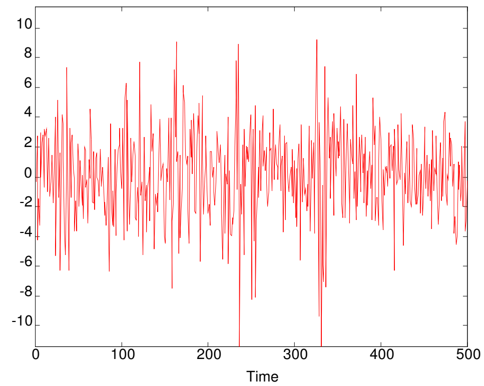
$$\nu_t = y_t^2 - h_t$$

- Important result - y_t^2 is a *noisy* indicator for h_t:

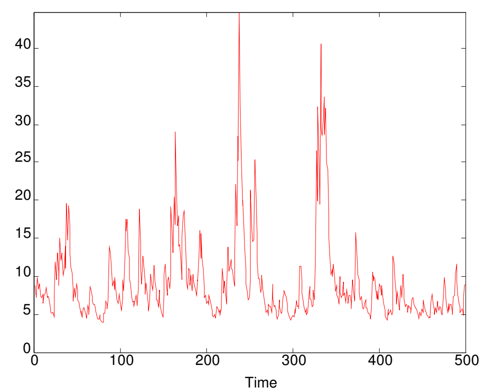
$$y_t^2 = \left(\omega + (\alpha + \beta) y_{t-1}^2 - \beta \nu_{t-1}\right) + \nu_t$$

$$= h_t + \nu_t$$

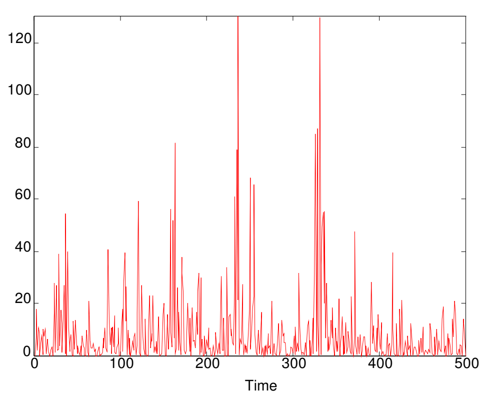
GARCH(1,1) Realization



Conditional Variance



Squared GARCH(1,1) Realization



Maximum Likelihood Estimation and Testing

- Conditional normal GARCH process:

$$y_t \mid \Omega_{t-1} \sim N(\,\mu_t(\theta)\,,\, h_t(\theta)\,)$$

- Conditional densities:

$$f\left(y_t \mid \Omega_{t-1};\theta\right) = \frac{1}{\sqrt{2\pi}}\, h_t(\theta)^{-1/2} \exp\left(-\frac{1}{2}\frac{(y_t - \mu_t(\theta))^2}{h_t(\theta)} \right)$$

- Prediction error decomposition:

$$f(y_T, y_{T-1}, \ldots, y_1;\theta) =$$

$$f(y_T \mid \Omega_{T-1};\theta) \times f(y_{T-1} \mid \Omega_{T-2};\theta) \times \ldots \times f(y_1 \mid \Omega_0;\theta)$$

- Log-likelihood function:

$$\log L(y_T,\ \ldots,\ y_1;\theta) = \log f(y_T,\ \ldots,\ y_1;\theta)$$

$$= -\frac{T}{2}\log(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\log h_t(\theta) - \frac{1}{2}\sum_{t=1}^{T}\frac{(y_t - \mu_t(\theta))^2}{h_t(\theta)}$$

- Non-linear function in θ
  - Numerical optimization techniques

- Testing and model diagnostics
  - Likelihood ratio test:

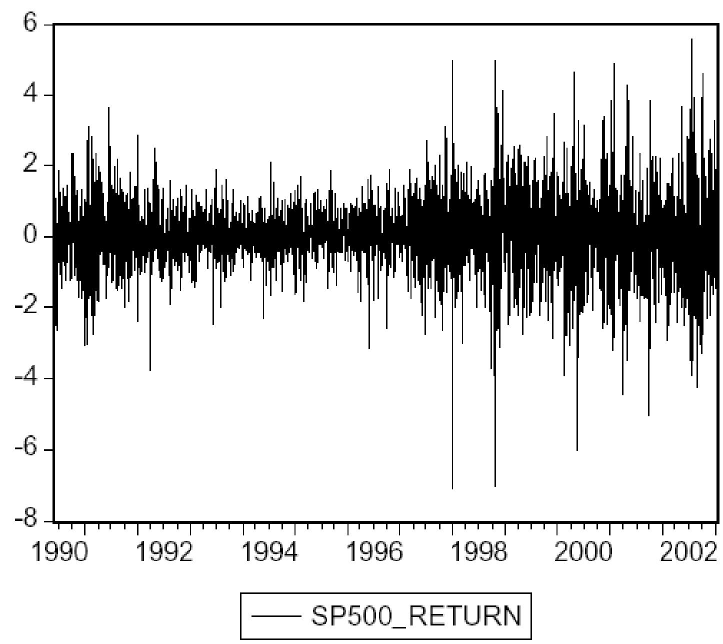
$$2 \times \Big(\log L(y_T, \ldots, y_1; \hat{\theta}_U) \; - \; \log L(y_T, \ldots, y_1; \hat{\theta}_R) \Big) \sim \chi^2$$

  - Autocorrelations of standardized residuals:

$$\frac{\varepsilon_t(\hat{\theta})}{h_t(\hat{\theta})^{1/2}} \qquad \frac{\varepsilon_t^2(\hat{\theta})}{h_t(\hat{\theta})} \qquad \frac{|\,\varepsilon_t(\hat{\theta})\,|}{h_t(\hat{\theta})^{1/2}}$$

- Many different software packages available
  - EViews

- Daily S&P500 returns:



- Autocorrelations:

Raw Returns

Sample: 6/05/1990 1/03/2003
Included observations: 3284

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.008	0.008	0.2075	0.649
		2	-0.026	-0.026	2.3678	0.306
		3	-0.034	-0.034	6.2702	0.099
		4	-0.002	-0.002	6.2829	0.179
		5	-0.027	-0.029	8.7126	0.121
		6	-0.029	-0.030	11.467	0.075
		7	-0.029	-0.030	14.229	0.047
		8	-0.009	-0.012	14.482	0.070
		9	0.016	0.013	15.367	0.081
		10	0.014	0.011	16.041	0.098

Squared Returns

Sample: 6/05/1990 1/03/2003
Included observations: 3284

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.205	0.205	137.54	0.000
		2	0.197	0.161	264.52	0.000
		3	0.190	0.132	383.04	0.000
		4	0.140	0.063	447.62	0.000
		5	0.189	0.118	565.24	0.000
		6	0.139	0.049	629.02	0.000
		7	0.150	0.063	702.68	0.000
		8	0.146	0.054	773.00	0.000
		9	0.112	0.019	814.13	0.000
		10	0.140	0.051	878.60	0.000

- GARCH(1,1) estimates:

Dependent Variable: SP500_RETURN
Method: ML - ARCH (Marquardt)

Sample(adjusted): 6/05/1990 1/03/2003
Included observations: 3284 after adjusting endpoints
Convergence achieved after 15 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.050394	0.013766	3.660814	0.0003
Variance Equation				
C	0.005686	0.002064	2.754869	0.0059
ARCH(1)	0.061016	0.009813	6.217920	0.0000
GARCH(1)	0.935759	0.009149	102.2819	0.0000

R-squared	-0.000418	Mean dependent var	0.028847
Adjusted R-squared	-0.001333	S.D. dependent var	1.053664
S.E. of regression	1.054366	Akaike info criterion	2.670108
Sum squared resid	3646.337	Schwarz criterion	2.677534
Log likelihood	-4380.318	Durbin-Watson stat	1.983258

- Residual Diagnostics:

Correlogram of GARCH Standardized Residuals

Sample: 6/05/1990 1/03/2003
Included observations: 3284

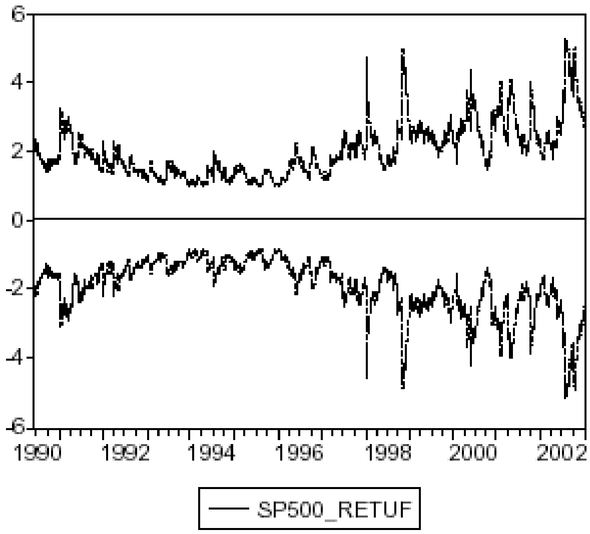
Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.040	0.040	5.2606	0.022
		2	-0.009	-0.011	5.5249	0.063
		3	-0.045	-0.044	12.152	0.007
		4	-0.017	-0.013	13.072	0.011
		5	-0.029	-0.029	15.901	0.007
		6	-0.012	-0.012	16.338	0.012
		7	-0.016	-0.017	17.153	0.016
		8	-0.011	-0.013	17.536	0.025
		9	0.002	0.000	17.543	0.041
		10	0.016	0.013	18.403	0.049

Correlogram of GARCH Standardized Residuals Squared

Sample: 6/05/1990 1/03/2003
Included observations: 3284

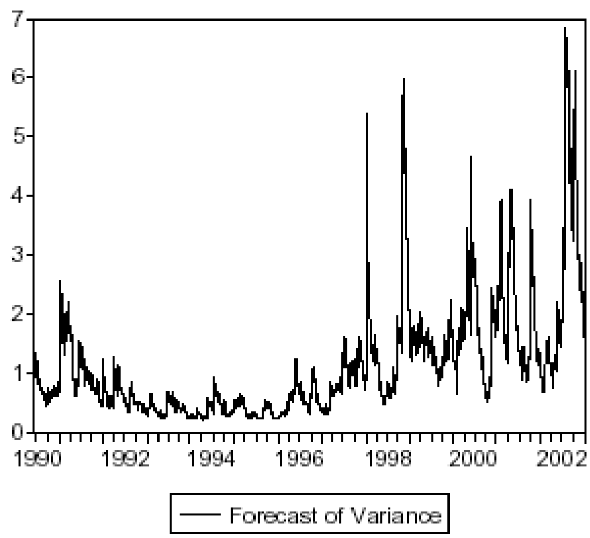
Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.004	0.004	0.0597	0.807
		2	0.034	0.034	3.8383	0.147
		3	0.002	0.002	3.8502	0.278
		4	-0.001	-0.002	3.8543	0.426
		5	0.005	0.004	3.9215	0.561
		6	0.002	0.002	3.9386	0.685
		7	-0.015	-0.015	4.6452	0.703
		8	-0.001	-0.001	4.6483	0.794
		9	-0.007	-0.006	4.8291	0.849
		10	0.002	0.002	4.8459	0.901

- GARCH(1,1) forecasts:



Forecast: SP500_RETUF
Actual: SP500_RETURN
Forecast sample: 6/01/1990 1/09/2003
Adjusted sample: 6/05/1990 1/09/2003
Included observations: 3284

Root Mean Squared Error	1.053724
Mean Absolute Error	0.748226
Mean Abs. Percent Error	122.6389
Theil Inequality Coefficient	0.954207
Bias Proportion	0.000418
Variance Proportion	0.999582
Covariance Proportion	0.000000



• Higher order GARCH models:

Dependent Variable: SP500_RETURN
Method: ML - ARCH (Marquardt)

Sample(adjusted): 6/05/1990 1/03/2003
Included observations: 3284 after adjusting endpoints
Convergence achieved after 17 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.050118	0.013833	3.622982	0.0003
Variance Equation				
C	0.005973	0.002192	2.724882	0.0064
ARCH(1)	0.052668	0.025417	2.072178	0.0382
ARCH(2)	0.010486	0.027597	0.379985	0.7040
GARCH(1)	0.933460	0.010370	90.01703	0.0000

R-squared	-0.000408	Mean dependent var	0.028847
Adjusted R-squared	-0.001628	S.D. dependent var	1.053664
S.E. of regression	1.054521	Akaike info criterion	2.670621
Sum squared resid	3646.299	Schwarz criterion	2.679903
Log likelihood	-4380.159	Durbin-Watson stat	1.983279

Dependent Variable: SP500_RETURN
Method: ML - ARCH (Marquardt)

Sample(adjusted): 6/05/1990 1/03/2003
Included observations: 3284 after adjusting endpoints
Convergence achieved after 70 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.050300	0.013791	3.647391	0.0003
Variance Equation				
C	0.005459	0.002801	1.948877	0.0513
ARCH(1)	0.058316	0.025484	2.288355	0.0221
GARCH(1)	0.990786	0.431960	2.293697	0.0218
GARCH(2)	-0.052196	0.407329	-0.128143	0.8980

R-squared	-0.000415	Mean dependent var	0.028847
Adjusted R-squared	-0.001635	S.D. dependent var	1.053664
S.E. of regression	1.054525	Akaike info criterion	2.670686
Sum squared resid	3646.324	Schwarz criterion	2.679969
Log likelihood	-4380.267	Durbin-Watson stat	1.983265

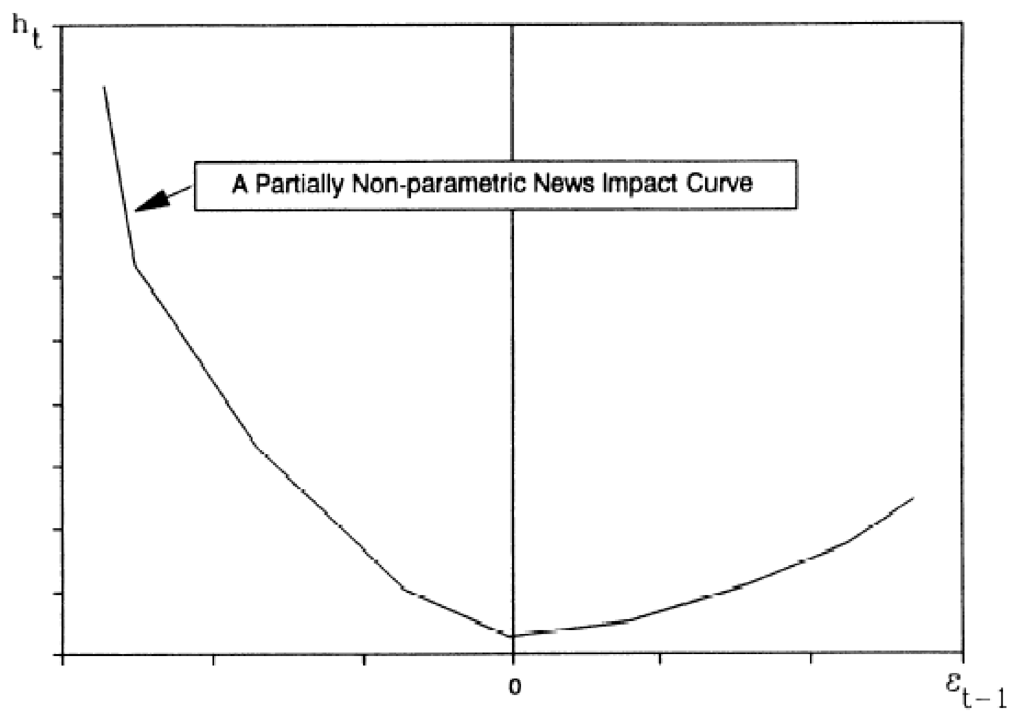
Asymmetric Response and the Leverage Effect

- Standard GARCH model:

$$h_t = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}$$

  - Volatility respond symmetrically to past returns

- News impact curve:



Engle and Ng (1993, *Journal of Finance*)

- Asymmetric response I - GJR or TGARCH models:

Glosten, Jagannathan and Runkle (1993, *Journal of Finance*)
Zakoian (1994, *Journal of Economic Dynamics and Control*)

$$h_t = \omega + \alpha r_{t-1}^2 + \gamma r_{t-1}^2 D_{t-1} + \beta h_{t-1}$$

$$D_t = \begin{cases} 1 & \text{if } r_t < 0 \\ 0 & \text{otherwise} \end{cases}$$

  - Positive return (good news): α
  - Negative return (bad news): $\alpha + \gamma$

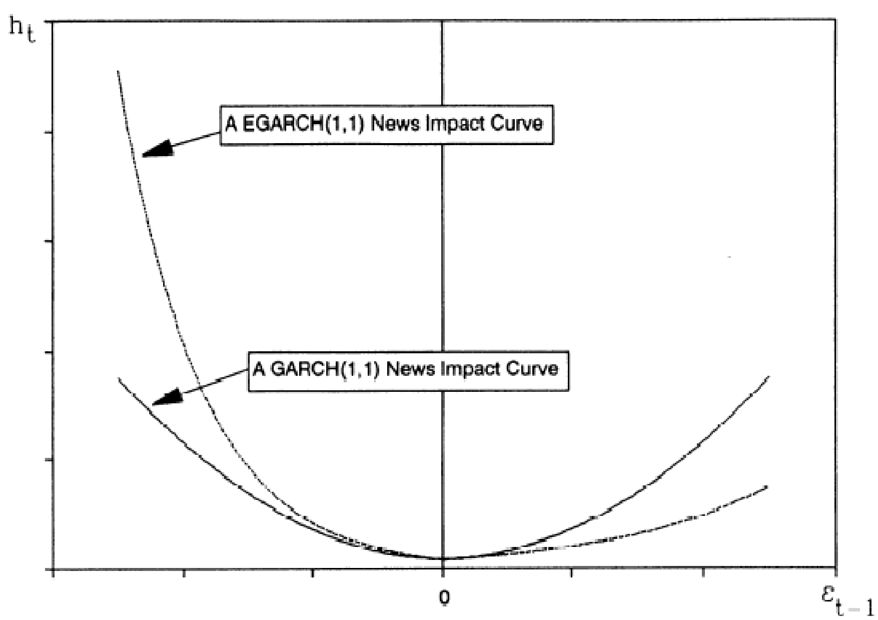
- Empirically: $\gamma > 0$
  - "Leverage" effect

- Asymmetric response II - EGARCH model:

Nelson (1991, *Econometrica*)

$$\ln(h_t) = \omega + \alpha \left| \frac{r_{t-1}}{h_{t-1}^{1/2}} \right| + \gamma \frac{r_{t-1}}{h_{t-1}^{1/2}} + \beta \ln(h_{t-1})$$

  - Like a GARCH model in logs
  - Log specification ensures positivity of the variance
  - Complicates forecasts of h_{t+k}
  - Volatility driven by both size and sign of shocks
  - "Leverage" effect when $\gamma < 0$



- Daily SP500 returns:

Dependent Variable: SP500_RETURN
Method: ML - ARCH (Marquardt)

Sample(adjusted): 6/05/1990 1/03/2003
Included observations: 3284 after adjusting endpoints
Convergence achieved after 18 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.027738	0.013508	2.053491	0.0400
Variance Equation				
C	0.011914	0.002736	4.354824	0.0000
ARCH(1)	0.007538	0.010296	0.732140	0.4641
(RESID<0)*ARCH(1)	0.113922	0.019805	5.752311	0.0000
GARCH(1)	0.924656	0.010713	86.31432	0.0000

R-squared	-0.000001	Mean dependent var	0.028847
Adjusted R-squared	-0.001221	S.D. dependent var	1.053664
S.E. of regression	1.054307	Akaike info criterion	2.644816
Sum squared resid	3644.817	Schwarz criterion	2.654098
Log likelihood	-4337.787	Durbin-Watson stat	1.984085

Dependent Variable: SP500_RETURN
Method: ML - ARCH (Marquardt)

Sample(adjusted): 6/05/1990 1/03/2003
Included observations: 3284 after adjusting endpoints
Convergence achieved after 21 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.023302	0.013840	1.683632	0.0923
Variance Equation				
C	-0.096127	0.015431	-6.229354	0.0000
|RES|/SQR[GARCH](1	0.121627	0.019525	6.229333	0.0000
RES/SQR[GARCH](1)	-0.093223	0.013756	-6.776698	0.0000
EGARCH(1)	0.981289	0.003676	266.9692	0.0000

R-squared	-0.000028	Mean dependent var	0.028847
Adjusted R-squared	-0.001248	S.D. dependent var	1.053664
S.E. of regression	1.054321	Akaike info criterion	2.636527
Sum squared resid	3644.914	Schwarz criterion	2.645810
Log likelihood	-4324.177	Durbin-Watson stat	1.984032

ARCH-M and Time-Varying Risk Premia

Engle, Lilien and Robins (1997, *Econometrica*)

- GARCH-in-Mean, or GARCH-M, model:

$$r_t = x_t' b + \delta h_t + \varepsilon_t$$

$$\varepsilon_t \mid \Omega_{t-1} \sim N(0, h_t)$$

  - Information matrix not block-diagonal
  - Risk-return tradeoff
  - Time-varying risk premium
  - Other functional forms: $\delta \cdot f(h_t)$

- Daily SP500 returns:

Dependent Variable: SP500_RETURN
Method: ML - ARCH (Marquardt)

Sample(adjusted): 6/05/1990 1/03/2003
Included observations: 3284 after adjusting endpoints
Convergence achieved after 14 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
GARCH	0.037345	0.024269	1.538758	0.1239
C	0.025856	0.021404	1.207990	0.2271
Variance Equation				
C	0.006035	0.002100	2.874151	0.0041
ARCH(1)	0.062866	0.010102	6.222943	0.0000
GARCH(1)	0.933578	0.009333	100.0313	0.0000

Dependent Variable: SP500_RETURN
Method: ML - ARCH (Marquardt)

Sample(adjusted): 6/05/1990 1/03/2003
Included observations: 3284 after adjusting endpoints
Convergence achieved after 15 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
SQR(GARCH)	0.055272	0.050604	1.092245	0.2747
C	0.008235	0.041430	0.198781	0.8424
Variance Equation				
C	0.005946	0.002079	2.860298	0.0042
ARCH(1)	0.062183	0.010015	6.209229	0.0000
GARCH(1)	0.934301	0.009264	100.8485	0.0000

Fat-tailed Conditional Densities

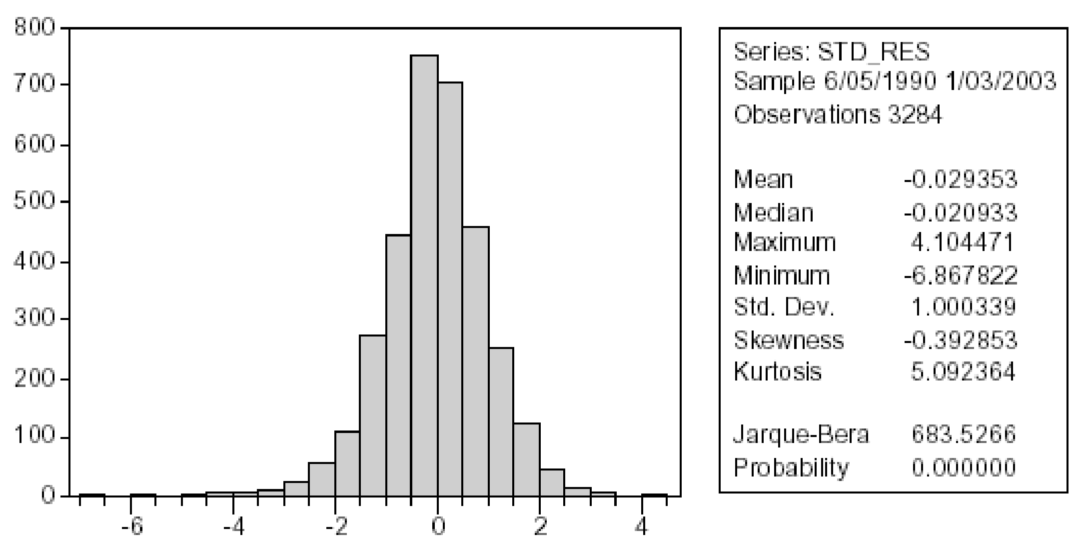
- Standard GARCH models assume:

$$r_t \mid \Omega_{t-1} \sim N(0, h_t)$$

  - If the model is correctly specified: $\dfrac{r_t}{\sqrt{\hat{h}_t}} \sim N(0,1)$

  - In practice: $\dfrac{r_t}{\sqrt{\hat{h}_t}} \sim \text{fat tailed}$

- GARCH(1,1) standardized residuals for daily SP500 returns:



- Two approaches for dealing with this problem

  - Robust inference

  - Fat tailed conditional distributions
    Parametric
    Non-Parametric

- Robust inference

Bollerslev and Wooldridge (1992, *Econometric Reviews*)

  - "Sandwich form" of the covariance matrix

- Daily SP500 GARCH(1,1) estimates and standard errors:

Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.050394	0.013766	3.660814	0.0003
Variance Equation				
C	0.005686	0.002064	2.754869	0.0059
ARCH(1)	0.061016	0.009813	6.217920	0.0000
GARCH(1)	0.935759	0.009149	102.2819	0.0000

Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.050394	0.014246	3.537407	0.0004
Variance Equation				
C	0.005686	0.001240	4.586609	0.0000
ARCH(1)	0.061016	0.004967	12.28500	0.0000
GARCH(1)	0.935759	0.005223	179.1672	0.0000

- Fat tailed conditional distributions
  - Important for VaR (quantile) predictions

- Parametric
  - GARCH-t

Bollerslev (1987, *Review of Economics and Statistics*)

$$r_t h_t^{-1/2} \overset{iid}{\sim} t_\nu \quad \text{instead of} \quad r_t h_t^{-1/2} \overset{iid}{\sim} N(0,1)$$

  - GARCH-GED

Nelson (1991, *Econometrica*)

$$r_t h_t^{-1/2} \overset{iid}{\sim} GED \quad \text{instead of} \quad r_t h_t^{-1/2} \overset{iid}{\sim} N(0,1)$$

- Non Parametric
  - Semiparametric GARCH

Engle and Gonzales-Rivera (1991, *J. of Business and Economic Statistics*)

  - Semi Non Parametric (SNP) Density Estimation

Gallant and Tauchen (1989, *Econometrica*)
Gallant, Hsieh and Tauchen (1991, *Proceedings*)

- Volatility clustering produces unconditional fat tails

$$r_t \mid \Omega_{t-1} \sim N(0, h_t) \quad \Rightarrow \quad r_t \sim \text{fat tailed distribution}$$

- Convergence to normality under temporal aggregation

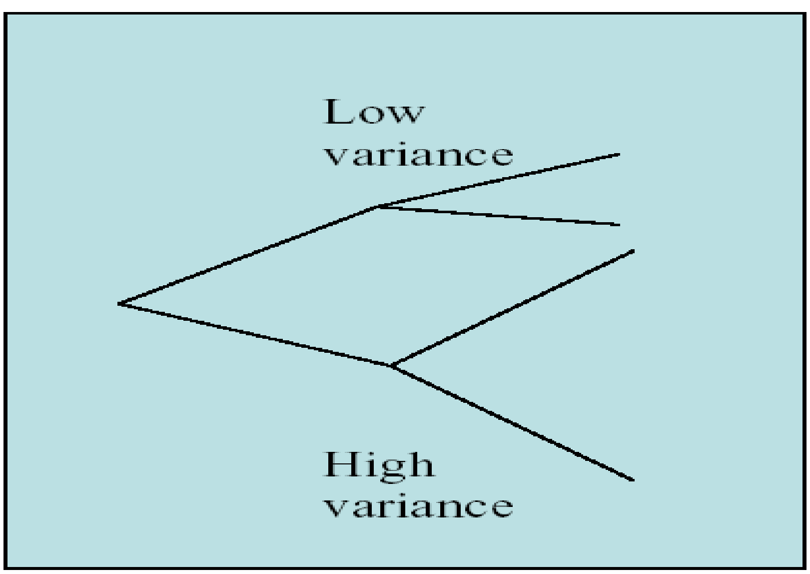
$$r_t \mid \Omega_{t-1} \sim N(0, h_t) \quad \Rightarrow$$

$$r_t(k) = r_t + r_{t-1} + .. + r_{t-k+1} \sim N(0, Var_k) \quad \text{for large } k$$

- Asymmetry in distributions

$$r_t \mid \Omega_{t-1} \sim N(0, h_t) \quad \Rightarrow$$

$$r_t(k) \sim \text{not necessarily symmetric}$$



Engle (2004, *Am.Eco.Rev.*)

Integrated Volatility

Engle and Bollerslev (1986, *Econometric Reviews*)
Bollerslev and Engle (1993, *Econometrica*)

- "Integration in variance"
  - Like a "unit root"

- For the GARCH(1,1) model this occurs when: $\alpha + \beta = 1$
  - Empirically: $\hat{\alpha} + \hat{\beta} \approx 1$

- Daily SP500 GARCH(1,1) estimates:

Convergence achieved after 15 iterations
Bollerslev-Wooldrige robust standard errors & covariance
Variance backcast: ON

	Coefficient	Std. Error	z-Statistic	Prob.
C	0.050394	0.013766	3.660814	0.0003
Variance Equation				
C	0.005686	0.002064	2.754869	0.0059
ARCH(1)	0.061016	0.009813	6.217920	0.0000
GARCH(1)	0.935759	0.009149	102.2819	0.0000

- The IGARCH model imposes: $\alpha + \beta = 1$

- Features of the IGARCH model:

  – Corresponds to EWMA (RiskMetric) with $\omega = 0$

  – Squared process is ARIMA but strictly stationary

    Nelson (1990, *Econometric Theory*)

  – Likelihood-based inference may proceed in standard fashion

    Lumsdaine (1996, *Econometrica*)
    Lee and Hansen (1994, *Econometric Theory*)

  – Continuous-record, or fill-in, asymptotics "justifies" IGARCH

    Nelson (1990, 1992, *J. of Econometrics*)
    Nelson and Foster (1994, *Econometrica*)

- Problems with IGARCH as a model:

  – Infinite dependence on initial conditions

  – Unconditional variance doesn't exist

- Maybe dominant root is close to, but less than, unity

  – Maybe long-memory!

Long Memory

- ARFIMA model:

$$(1-L)^d \; \Phi(L) \, y_t \; = \; \Theta(L) \, \varepsilon_t$$

$$(1-L)^d \; = \; 1 - \; dL \; + \; \frac{d(d-1)}{2!} L^2 \; - \; \frac{d(d-1)(d-2)}{3!} L^3 \; + \; \ldots$$

  - d=0: ARMA model
  - d=1: Unit root
  - 0<d<1: Fractionally integrated

- Autocorrelations:

  - d=0: Fast exponential decay
  - d=1: Infinite persistence
  - 0<d<½: Eventual but slow hyperbolic decay

- Can do the same for absolute or squared returns: $|r_t|$, r_t^2

- Daily S&P returns 1928-1991:

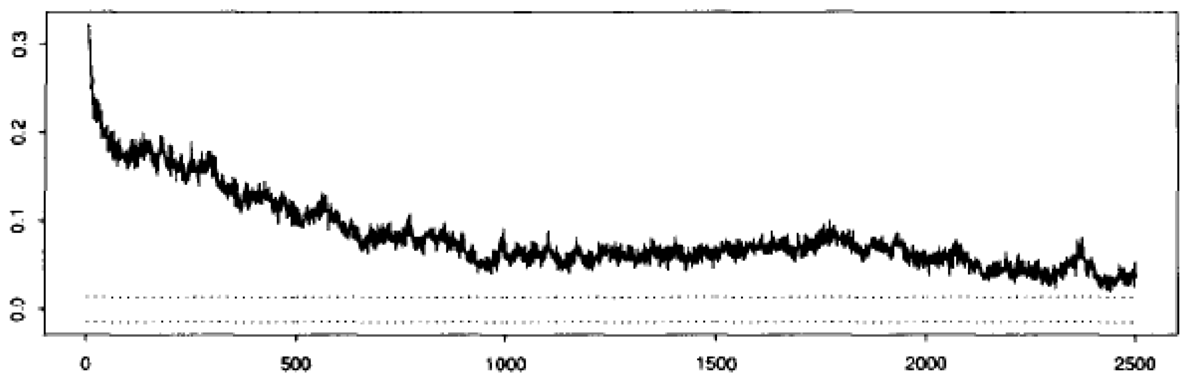


Fig. 3.8. Autocorrelation of $|r|$ up to lag 2500.

Ding, Granger and Engle (1993, *J. of Empirical Finance*)

- FIGARCH and FIEGARCH models

Baillie, Bollerslev and Mikkelsen (1996, *J. of Econometrics*)
Bollerslev and Mikkelsen (1996, *J. of Econometrics*)

  – GARCH(1,1) ~ ARMA(1,1)

$$r_t^2 = \omega + (\alpha + \beta) r_{t-1}^2 - \beta \nu_{t-1} + \nu_t$$

  – FIGARCH(0,d,1)

$$(1 - L)^d r_t^2 = \omega - \beta \nu_{t-1} + \nu_t$$

- FIGARCH(0,d,1):

$$(1 - L)^d r_t^2 = \omega - \beta \nu_{t-1} + \nu_t$$

$$h_t = \omega(1 - \beta)^{-1} + [1 - (1 - \beta L)^{-1}(1 - L)^d] r_t^2$$
$$= \omega(1 - \beta)^{-1} + \sum_{k=1}^{\infty} \lambda_k r_{t-k}^2$$

$$\lambda_k = (1 - \beta - (1 - d) k^{-1}) \Gamma(k + d - 1) \Gamma(k)^{-1} \Gamma(d)^{-1}$$

  – By Stirling's Formula for large k

$$\lambda_k \sim k^{d-1}$$

  – Typically $\hat{d} \approx 0.35 - 0.45$

- FIEGARCH:

$$(1 - L)^d \log(h_t) = \omega + \alpha \left| \frac{r_{t-1}}{h_{t-1}^{1/2}} \right| + \gamma \frac{r_{t-1}}{h_{t-1}^{1/2}}$$

Component GARCH

Engle and Lee (1999, *Festschrift for C.W.J. Granger*)

- Standard GARCH(1,1) model:

$$(h_t - \bar{\omega}) = \alpha (r_{t-1}^2 - \bar{\omega}) + \beta (h_{t-1} - \bar{\omega})$$

  – long-run volatility: $\bar{\omega} = \dfrac{\omega}{1 - \alpha - \beta}$

- Component GARCH:

$$(h_t - q_t) = \alpha (r_{t-1}^2 - q_{t-1}) + \beta (h_{t-1} - q_{t-1})$$

  – long-run volatility:

$$q_t = \omega + \rho (q_{t-1} - \omega) + \varphi (r_{t-1}^2 - h_{t-1})$$

  – Transitory dynamics governed by: $\alpha + \beta$

  – Persistent dynamics governed by: ρ

  – Equivalent to non-linearly restricted GARCH(2,2)

  – Looks "like" long-memory

Andersen and Bollerslev (1997, *J. of Finance*)
Gallant, Hsu and Tauchen (1999, *Rev. Econ. Stat.*)

Regime Switching

Hamilton (1989, *Econometrica*)

- Markov-switching model:
  - $\{s_t\}_{t=1}^{T}$ a two-state first-order Markov process with transition probabilities:

$$M = \begin{pmatrix} p_{HH} & p_{HL} \\ p_{LH} & p_{LL} \end{pmatrix} = \begin{pmatrix} p_{HH} & 1-p_{HH} \\ 1-p_{LL} & p_{LL} \end{pmatrix}$$

  - Conditional distribution:

$$f(r_t \,|\, s_t;\ \theta) = \frac{1}{\sqrt{2\pi}\ \sigma_{s_t}} \exp\left(\frac{-r_t^2}{2\sigma_{s_t}^2} \right)$$

  - Like GARCH with only two states: "High" (H) and "Low" (L) volatility
  - GARCH and regime switching

Cai (1994, *J. of Business and Economic Statistics*)
Hamilton and Susmel (1994, *J. of Econometrics*)
Gray (1996, *J. of Financial Economics*)

  - Intimately related to long-memory

Liu (2000, *J. of Econometrics*)
Park (2000, *Rev. Econ. Stat.*)

Other Univariate GARCH Models

- An *incomplete* list:

Bollerslev (2010, *Engle Festschrift*)

ARCH	Engle (1982)
GARCH	Bollerslev (1986)
IGARCH	Bollerslev and Engle (1986)
Log-GARCH	Geweke (1986), Milhøj (1987), Pantula (1986)
TS-GARCH	Taylor (1986), Schwert (1989)
GARCH-t	Bollerslev (1987)
ARCH-M	Engle, Lilien and Robins (1987)
MGARCH	Bollerslev, Engle and Wooldridge (1998)
CCC GARCH	Bollerslev (1990)
AGARCH	Engle (1990)
CGARCH	Engle and Lee (1990)
EGARCH	Nelson (1991)
SPARCH	Engle and Gonzalez-Rivera (1991)
LARCH	Robinson (1991)
AARCH	Bera, Higgins and Lee (1992)
NGARCH	Higgins and Bera (1992)
QARCH	Sentana (1992)
STARCH	Harvey, Ruiz and Sentana (1992)
QTARCH	Gourieroux and Monfort (1992)
GARCH-EAR	LeBaron (1992)
GJR-GARCH	Glosten, Jagannathan and Runkle (1993)
Weak GARCH	Drost and Nijman (1993)
VGARCH	Engle and Lee (1993)
APARCH	Ding, Granger and Engle (1993)
SWARCH	Hamilton and Susmel (1994)
β-ARCH	Guegan and Diebolt (1994)
TGARCH	Zakoian (1994)
GQARCH	Sentana (1995)
HGARCH	Hentschel (1995)
SGARCH	Liu and Brorsen (1995)
PGARCH	Bollerslev and Ghysels (1996)
GARCH-X	Brenner, Harjes and Kroner (1996)
GRS-GARCH	Gray (1996)
VS-GARCH	Fornari and Mele (1996)
FIGARCH	Baillie, Bollerslev and Mikkelsen (1996)
FIEGARCH	Bollerslev and Mikkelsen (1996)
ANN-ARCH	Donaldson and Kamstra (1997)
HARCH	Müller, Dacorogna, Davé, Olsen, Pictet and Weizsäcker (1997)
ATGARCH	Crouchy and Rockinger (1997)
AUG-GARCH	Duan (1997)
STGARCH	Gonzalez-Rivera (1998)
FIAPARCH	Tse (1998)
SQR-GARCH	Heston and Nandi (2000)
OGARCH	Alexander (2001)
DCC GARCH	Engle (2002)
ANST-GARCH	Nam, Pyum and Arize (2002)
RGARCH	Park (2002)
Flex-GARCH	Ledoit, Santa-Clara and Wolf (2003)
GARJI	Maheu and McCurdy (2004)
HYGARCH	Davidson (2004)
COGARCH	Klüppelberg, Lindner and Maller (2004)
LMGARCH	Conrad and Karanasos (2006)
REGARCH	Brandt and Jones (2006)
FCGARCH	Medeiros and Veiga (2009)
....	

ARCH:	$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2$
GARCH:	$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$
IGARCH	$\sigma_t^2 = \omega + \varepsilon_{t-1}^2 + \sum_{i=2}^{p} \alpha_i (\varepsilon_{t-i}^2 - \varepsilon_{t-1}^2) + \sum_{j=1}^{q} \beta_j (\sigma_{t-j}^2 - \varepsilon_{t-1}^2)$
Taylor/Schwert:	$\sigma_t = \omega + \sum_{i=1}^{p} \alpha_i \lvert\varepsilon_{t-i}\rvert + \sum_{j=1}^{q} \beta_j \sigma_{t-j}$
A-GARCH:	$\sigma_t^2 = \omega + \sum_{i=1}^{p} \left[\alpha_i \varepsilon_{t-i}^2 + \gamma_i \varepsilon_{t-i}\right] + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$
NA-GARCH:	$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \left(\varepsilon_{t-i} + \gamma_i \sigma_{t-i}\right)^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$
V-GARCH:	$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \left(e_{t-i} + \gamma_i\right)^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$
Thr.-GARCH:	$\sigma_t = \omega + \sum_{i=1}^{p} \alpha_i \left[(1-\gamma_i)\varepsilon_{t-i}^{+} - (1+\gamma_i)\varepsilon_{t-i}^{-}\right] + \sum_{j=1}^{q} \beta_j \sigma_{t-j}$
GJR-GARCH:	$\sigma_t^2 = \omega + \sum_{i=1}^{p_1} \left[\alpha_i + \gamma_i I_{\{\varepsilon_{t-i}>0\}}\right] \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$
log-GARCH:	$\log(\sigma_t) = \omega + \sum_{i=1}^{p} \alpha_i \lvert e_{t-i}\rvert + \sum_{j=1}^{q} \beta_j \log(\sigma_{t-j})$
EGARCH:	$\log(\sigma_t^2) = \omega + \sum_{i=1}^{p} \left[\alpha_i e_{t-i} + \gamma_i \left(\lvert e_{t-i}\rvert - E\lvert e_{t-i}\rvert\right)\right] + \sum_{j=1}^{q} \beta_j \log(\sigma_{t-j}^2),$
NGARCH[a]:	$\sigma_t^\delta = \omega + \sum_{i=1}^{p} \alpha_i \lvert\varepsilon_{t-i}\rvert^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta$
A-PARCH:	$\sigma^\delta = \omega + \sum_{i=1}^{p} \alpha_i \left[\lvert\varepsilon_{t-i}\rvert - \gamma_i \varepsilon_{t-i}\right]^\delta + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta$
GQ-ARCH:	$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i} + \sum_{i=1}^{p} \alpha_{ii} \varepsilon_{t-i}^2 + \sum_{i<j}^{p} \alpha_{ij} \varepsilon_{t-i}\varepsilon_{t-j} + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$
H-GARCH:	$\sigma_t^\delta = \omega + \sum_{i=1}^{p} \alpha_i \delta \sigma_{t-i}^\delta \left[\lvert e_t - \kappa\rvert - \tau\,(e_t - \kappa)\right]^\nu + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^\delta$
Aug-GARCH[b]:	$\sigma_t^2 = \begin{cases} \lvert\delta\phi_t - \delta + 1\rvert^{1/\delta} & \text{if } \delta \neq 0 \\ \exp(\phi_t - 1) & \text{if } \delta = 0 \end{cases}$
	$\phi_t = \omega + \sum_{i=1}^{p} \left[\alpha_{1i} \lvert\varepsilon_{t-i} - \kappa\rvert^\nu + \alpha_{2i} \max(0, \kappa - \varepsilon_{t-i})^\nu\right] \phi_{t-j}$
	$+ \sum_{i=1}^{p} \left[\alpha_{3i} f(\lvert\varepsilon_{t-i} - \kappa\rvert, \nu) + \alpha_{4i} f(\max(0, \kappa - \varepsilon_{t-i}), \nu)\right] \phi_{t-j}$
	$+ \sum_{j=1}^{q} \beta_j \phi_{t-j}^2$

[a] This is A-PARCH without the leverage effect.
[b] Here $f(x, \nu) = (x^\nu - 1)/\nu$.

Hansen and Lunde (2005, *J. of Applied Econometrics*)

Multivariate GARCH Models

- Univariate GARCH models, y_t scalar:
  - $\mathrm{Var}(y_t \,|\, \Omega_{t-1}) = h_t$

- Multivariate GARCH models, r_t $N \times 1$ vector:
  - $\mathrm{Var}(y_t \,|\, \Omega_{t-1}) = H_t$
  - Positive definite, $N \times N$ matrix:

$$\mathrm{Var}(\lambda' y_t \,|\, \Omega_{t-1}) = h_t = \lambda' H_t \lambda > 0$$

- Maximum likelihood estimation and testing:

$$y_t \,|\, \Omega_{t-1} \sim N(\mu_t, H_t)$$

$$\mathrm{Log}L = -\frac{TN}{2}\ln(2\pi) - \frac{1}{2}\sum \ln\left|H_t\right| - \frac{1}{2}\sum \varepsilon_t' H_t^{-1} \varepsilon_t$$

- How to parameterize H_t ?

Vech and Diagonal GARCH

Bollerslev, Engle and Wooldridge (1988, *J. of Political Economy*)

- General vech(·) model:

$$\mathrm{vech}\left(H_t\right) = W + \sum_{i=1}^{q} A_i \,\mathrm{vech}\left(\varepsilon_{t-i}\,\varepsilon_{t-i}'\right) + \sum_{j=1}^{p} B_j \,\mathrm{vech}\left(H_{t-j}\right)$$

  – Vech(·) ensures symmetry

  – Not necessarily positive definite

  – Large number of parameters, $O(N^4)$

- Diagonal model

  – A_i and B_i matrices diagonal

  – BEKK representation ensures positive definiteness

  Engle and Kroner (1995, *Econometric Theory*)

  – Still $O(N^2)$ parameters

- EWMA - RiskMetric

  – $W = 0$, $A_1 = \mathrm{diag}\{\gamma\}$ and $B_1 = \mathrm{diag}\{1-\gamma\}$

  – Positive definite, but too simplistic

Factor ARCH Models

Diebold and Nerlove (1989, *J. of Applied Econometrics*)
Engle, Ng and Rothschild (1990, *J. of Econometrics*)

- Commonalities in volatility

- One-Factor GARCH(1,1) model:

$$r_t = bF_t + \nu_t$$

$$F_t \mid \Omega_{t-1} \sim N(0, h_t)$$

$$h_t = \omega + \alpha F_{t-1}^2 + \beta h_{t-1}$$

$$\nu_t \mid \Omega_{t-1} \sim N(0, \Gamma)$$

  – Conditional distribution:

$$r_t \mid \Omega_{t-1} \sim N\left(0, H_t\right)$$

$$H_t = h_t b b' + \Gamma$$

– H_t guaranteed to be positive definite

– j^{th} time-t conditional variance:

$$H_{jj,t} = b_j^2\, h_t + \gamma_j = b_j^2 \left(\omega + \alpha F_{t-1}^2 + \beta h_{t-1}\right) + \gamma_j$$

– jk^{th} time-t conditional covariance (for Γ diagonal):

$$H_{jk,t} = b_j b_k\, h_t = b_j b_k \left(\omega + \alpha F_{t-1}^2 + \beta h_{t-1}\right)$$

– Parameters in one-factor GARCH(1,1) model, 2N + 3

– Parameters in k-factor GARCH(p,q) model, O(N)

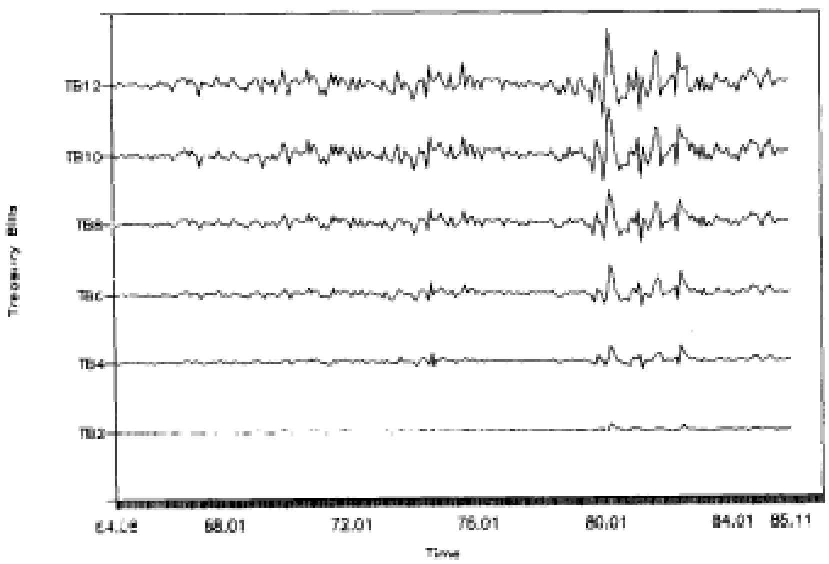


Fig. 1. Monthly excess returns.

Engle, Ng and Rothschild (1990, *J. of Econometrics*)

- Drawbacks to the factor GARCH model
  - What are the factor(s) ?
  - Latent factor(s) complicates estimation
  - Portfolios with no ARCH, $\lambda' b = 0$:

$$r_{p,t} = \lambda' r_t = \lambda' b F_t + \lambda' \nu_t = \lambda' \nu_t$$

Constant Conditional Correlation (CCC) Model

Bollerslev (1990, *Review of Economics and Statistics*)

- Unconditional correlation:

$$\rho_{ij} = \frac{E(r_{i,t} r_{j,t})}{\sqrt{E(r_{i,t}^2) E(r_{j,t}^2)}}$$

- *Conditional* correlation:

$$\rho_{ij,t} = \frac{E(r_{i,t} r_{j,t} \mid \Omega_{t-1})}{\sqrt{E(r_{i,t}^2 \mid \Omega_{t-1}) E(r_{j,t}^2 \mid \Omega_{t-1})}}$$

- *Constant* Conditional Correlations:

$$\rho_{ij,t} = \rho_{ij}$$

- Time-Varying Conditional Covariances:

$$E(r_{i,t} r_{j,t} \mid \Omega_{t-1}) = \rho_{ij} \sqrt{E(r_{i,t}^2 \mid \Omega_{t-1})} \sqrt{E(r_{j,t}^2 \mid \Omega_{t-1})}$$

- Constant conditional correlation GARCH model:

$$H_t = D_t R D_t$$

– D_t^2 diagonal matrix of conditional variances:

$$D_t^2 = \text{diag}(h_{ii,t}) = \text{diag}(E(r_{i,t}^2 \mid \Omega_{t-1}))$$

– R matrix of constant *conditional* correlations for $\boldsymbol{\varepsilon}_t = D_t^{-1} r_t$:

$$E(\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t^{\prime} \mid \Omega_{t-1}) = R$$

- Easy to use and estimate
  - N univariate GARCH models
  - R estimated by sample cross correlations of $\hat{\boldsymbol{\varepsilon}}_t = \hat{D}_t^{-1} r_t$
  - Guaranteed to be positive definite

- So are the correlations constant ?
  - Maybe in the short-run
  - But probably not over longer horizons

- Rolling sample international equity market correlations:

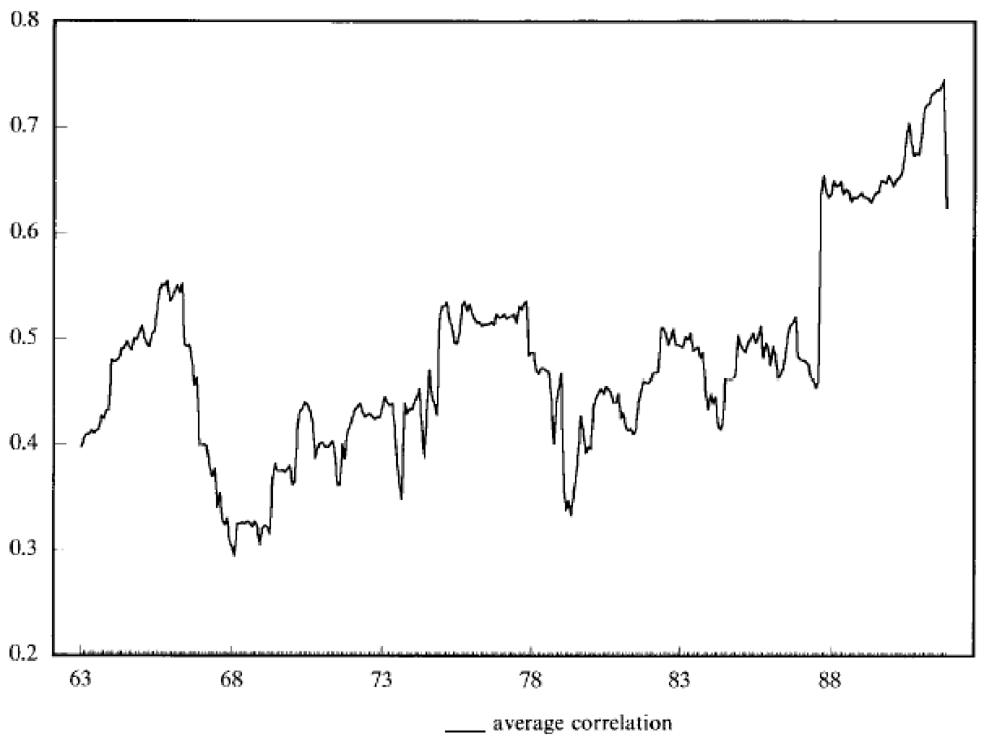


FIGURE 1. Correlation of the US stock market. This figure reports the (unweighted) average correlation of the US stock market with the other seven stock markets. The correlation is computed over sliding windows of four years, using local currency monthly total returns. The period is December 1959–91.

Longin and Solnik (1995, *J. of International Money and Finance*)

<u>Dynamic Conditional Correlation (DCC) Models</u>

Engle (2002, *J. of Business and Economic Statistics*)
Tse and Tsui (2002, *J. of Business and Economic Statistics*)

- Allow the correlations to be time-varying:

$$H_t = D_t R_t D_t$$

- R_t must be a correlation matrix for ε_t:
  - Positive definite
  - Ones along the diagonal

- Exponential smoothing - DCC_INT model:

$$\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t}\, q_{jj,t}}}$$

$$q_{ij,t} = (1 - \gamma)\varepsilon_{i,t-1}\varepsilon_{j,t-1} + \gamma q_{ij,t-1}$$

- Could parameterize R_t in many other ways

• Equity volatilities and correlations:

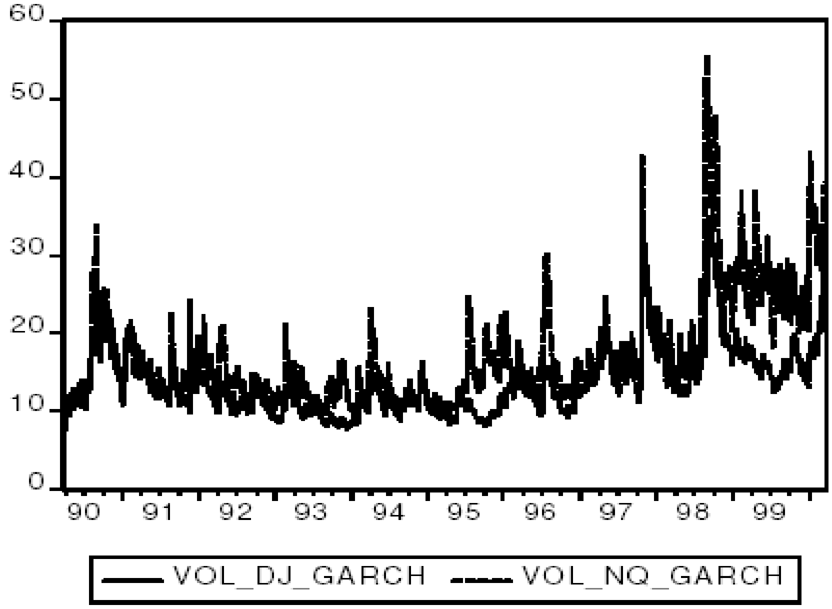


Figure 2. Ten Years of Volatilities.

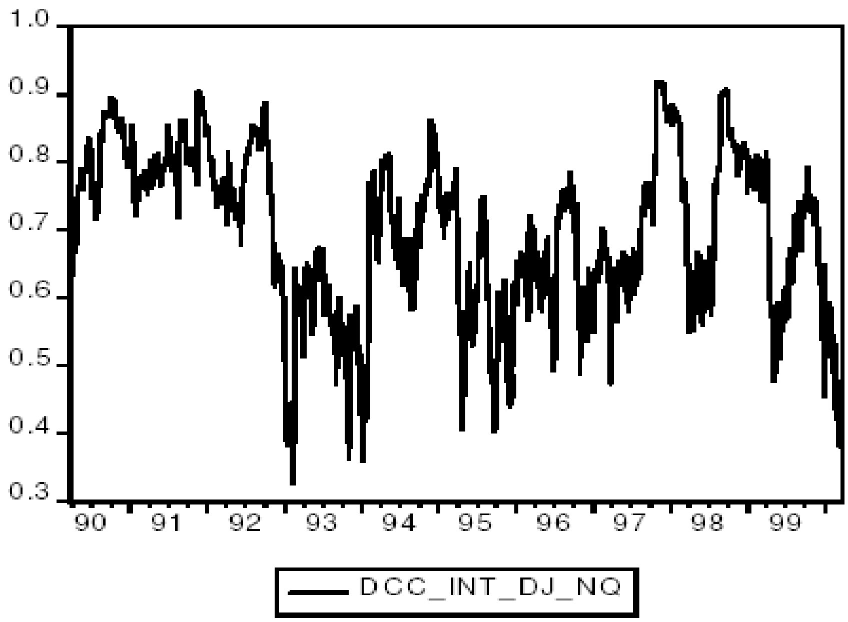
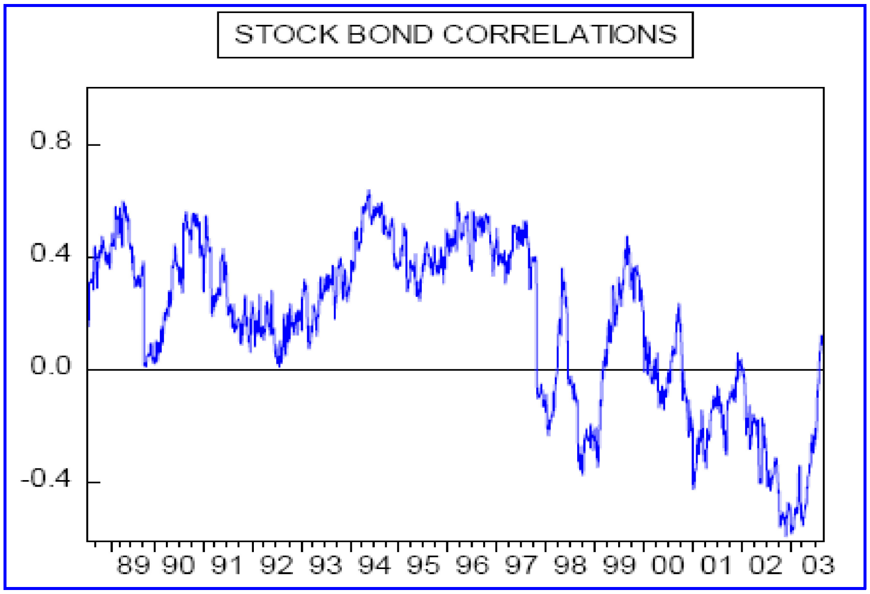


Figure 3. Ten Years of Dow Jones–NASDAQ Correlations.

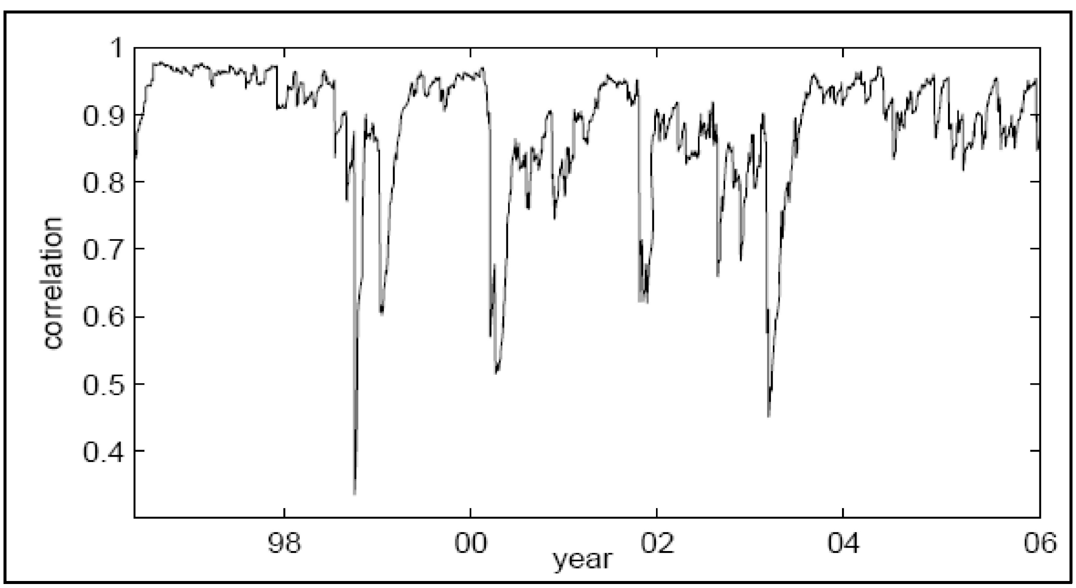
Engle (2002, *J. of Business and Economic Statistics*)

• DCC-based Bond-Equity correlations:



Engle (2009, *Anticipating Correlations*)

• DCC-based T-Bond and AAA Corporate Bond correlations:



Buraschi,Porchia and Trojani (2009, *manuscript, Imperial College, London*)

Asymmetries in Correlations

- Up- and down-markets
  - Domestic
  - International

- High and low volatility

- Recessions and expansions

- Benefits to diversification

  - Least when you want it the most !

- US equity returns:

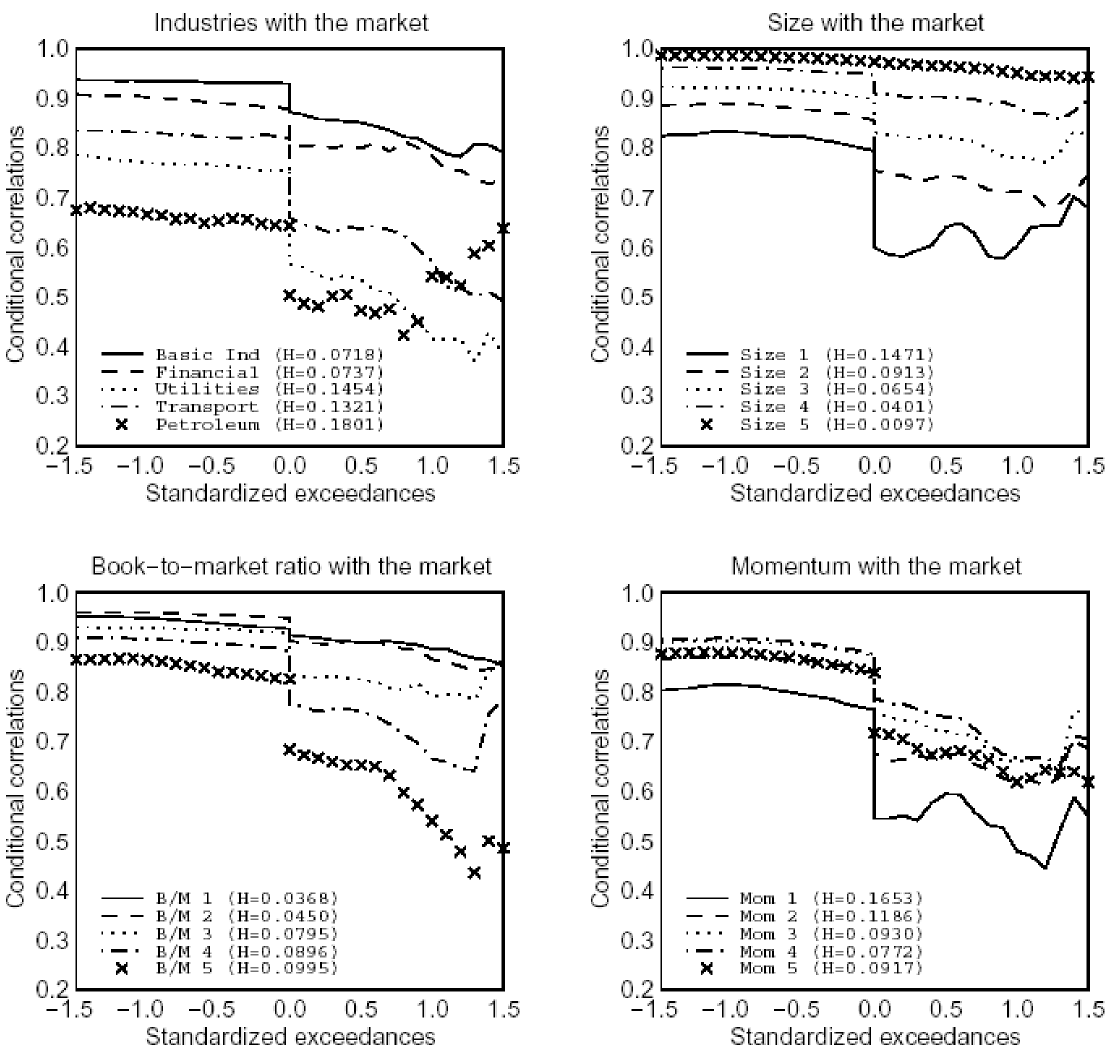


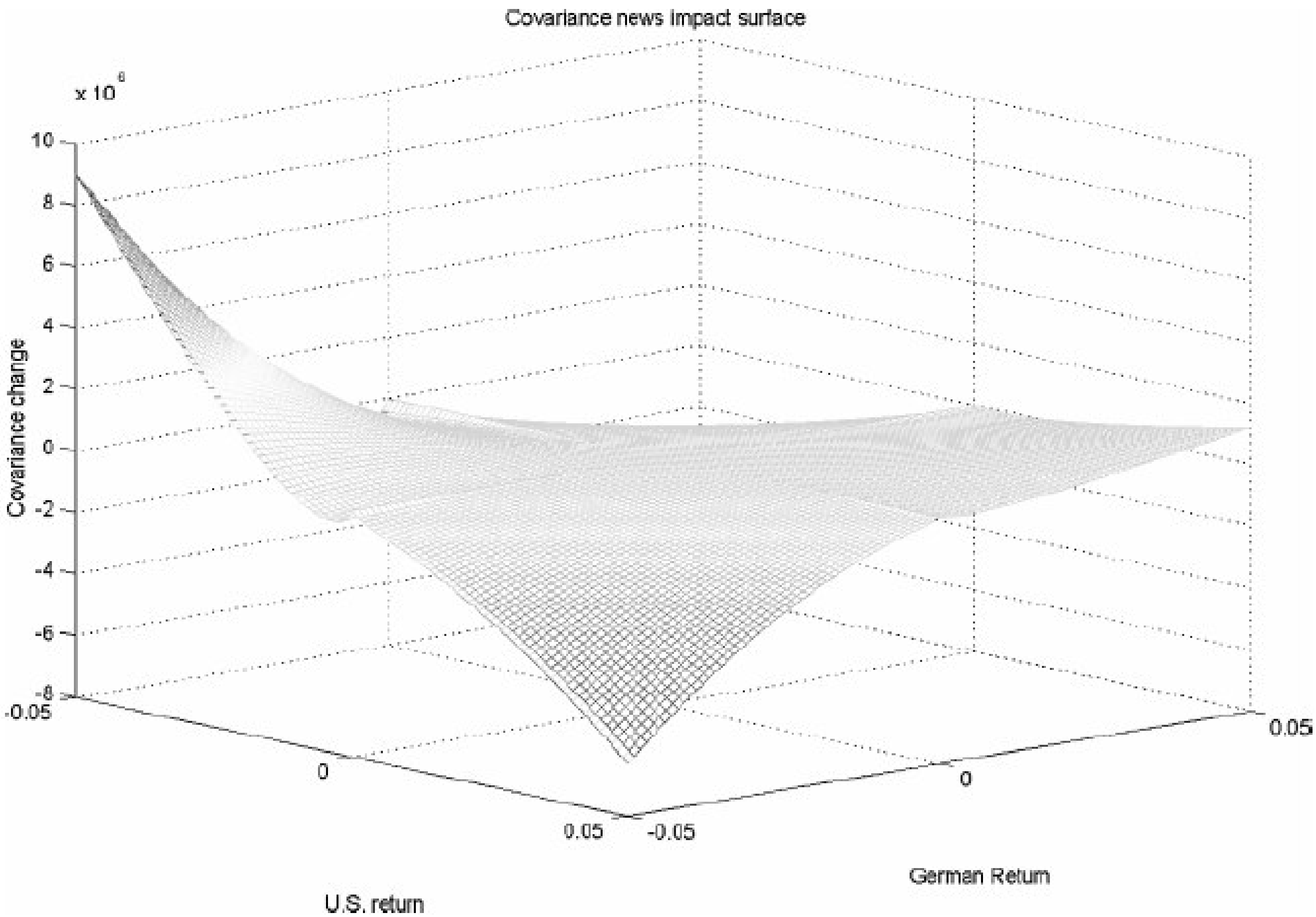
Figure 3: Exceedance correlations of industry, size, book-to-market, and momentum portfolios

We plot exceedance correlations with the market portfolio for selected industry, size, book-to-market, and momentum portfolios. These are the conditional correlations $\text{corr}(\tilde{x}, \tilde{y} | \tilde{x} > \vartheta, \tilde{y} > \vartheta; \rho)$ for exceedance $\vartheta > 0$ for normalized portfolio $\tilde{x}$ and the normalized market portfolio $\tilde{y}$. For $\vartheta < 0$, the exceedance correlation is defined as $\text{corr}(\tilde{x}, \tilde{y} | \tilde{x} < \vartheta, \tilde{y} < \vartheta; \rho)$. Exceedance correlations are calculated at the weekly frequency. The H statistic in the legend is the measure of correlation asymmetry developed in Section 4.

Ang and Chen (2002, *J. of Financial Economics*)

- How to model asymmetries in (covariances) correlations
  - Multivariate vech GJR model
  - Multivariate EGARCH model
  - Regime switching GARCH
  - DCC with asymmetric $q_{ij,t}$
  - Copulas

- Asymmetric DCC covariance estimates:



Cappiello, Engle and Shephard (2007, *Journal of Financial Econometrics*)

Copulas

- "Standard" univariate GARCH:

$$r_{i,t} \,|\, \Omega_{t-1} \sim N(0, h_{ii,t}) \qquad (\text{ or } \sim D(0, h_{ii,t})\ )$$

- "Standard" multivariate GARCH:

$$r_t \,|\, \Omega_{t-1} \sim N(0, H_t) \qquad (\text{ or } \sim D(0, H_t)\ )$$

- Copulas:
  - Marginal distributions (cdf)

    $$F(r_{i,t} \,|\, \Omega_{t-1})$$

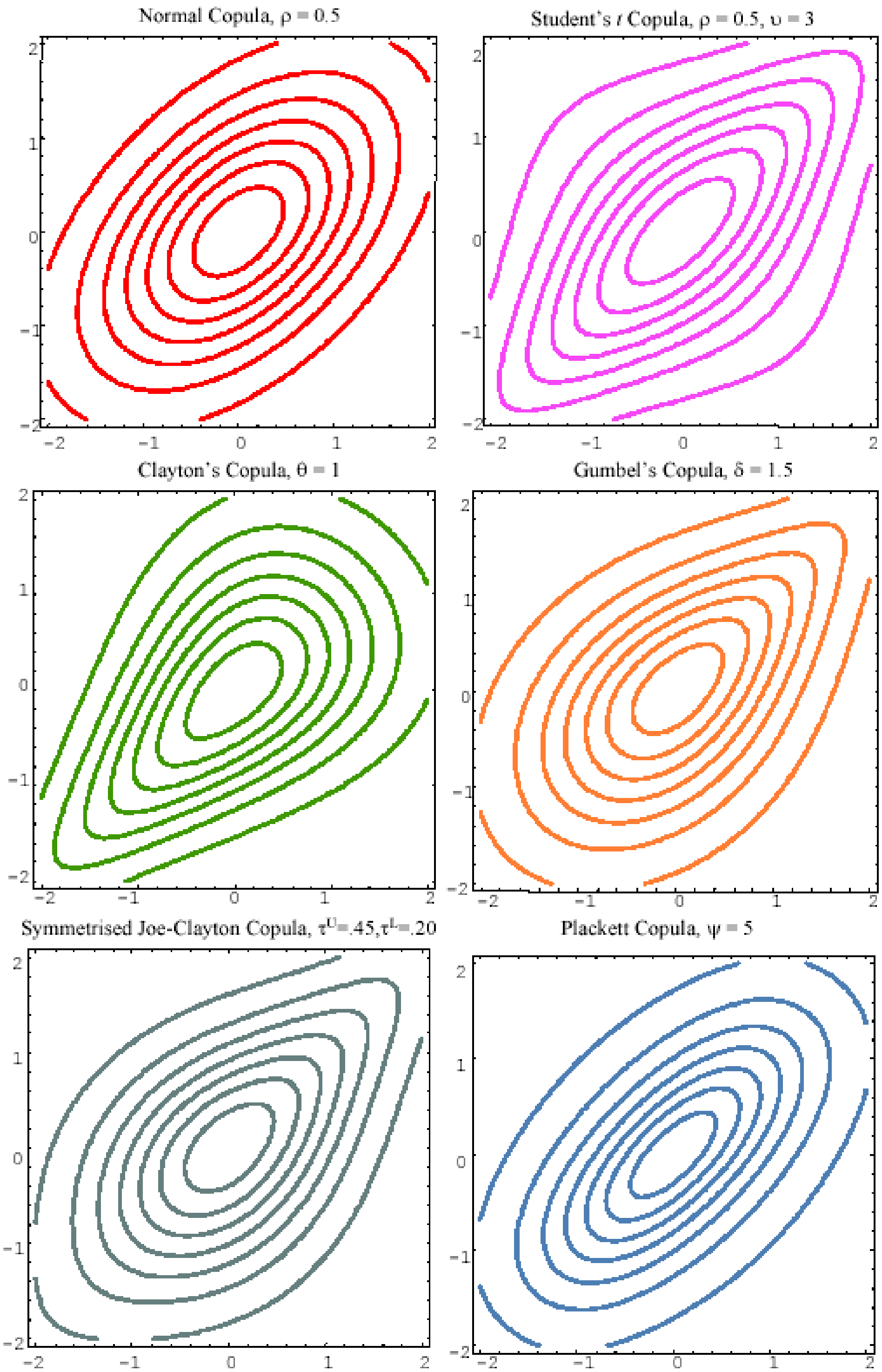
  - Joint distribution (cdf)

    $$F(r_t \,|\, \Omega_{t-1}) = C\Big(F^{-1}(r_{1,t} \,|\, \Omega_{t-1}), \ldots, F^{-1}(r_{N,t} \,|\, \Omega_{t-1})\Big)$$

  - Sklar's Theorem:

    The copula C is unique, although not necessarily time-invariant

• Examples of copulas:



Patton (2005, *International Economic Review*)

Structural GARCH

Sentana and Fiorentini (2001, *J. of Econometrics*)
Rigobon and Sack (2004, *manuscript*)
Andersen, Bollerslev, Diebold and Vega (2007, *J.Int.Eco.*)

- Structural (homoskedastic) VAR

$$\Phi_0 y_t = \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + \varepsilon_t$$

$$Var_{t-1}(\varepsilon_t) = diag\{h_1, h_2, \ldots, h_N\}$$

– Reduced form

$$y_t = A_1 y_{t-1} + \ldots + A_p y_{t-p} + e_t$$

$$A_i \equiv \Phi_0^{-1} \Phi_i \qquad e_t \equiv \Phi_0^{-1} \varepsilon_t$$

– Φ_0 not identified (in general)

- Structural VAR-GARCH

$$Var_{t-1}(\varepsilon_t) = diag\{h_{1,t}, h_{2,t}, \ldots, h_{N,t}\} \equiv D_t$$

– Reduced form errors

$$Var_{t-1}(e_t) = H_t = \Phi_0^{-1} D_t \Phi_0^{-1\prime}$$

– Φ_0 identified through *heteroskedasticity*