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Lecture 18. 20.11.2013

Given an SP X, it is sometimes possible to improve the regularity of its paths without changing its distribution (that is, without changing its finite-dimensional distributions). For background on such results (separability, measurability, versions, regularization etc.) see e.g. [D].

There are several ways to define 'sameness' of two processes X and Y. We say
(i) X and Y have the *same finite-dimensional distributions* if, for any integer n and $\{t_1, \cdots, t_n\}$ a finite set of time points in $[0, \infty)$, the random vectors $(X(t_1), \ldots, X(t_n))$ and $(Y(t_1), \ldots, Y(t_n))$ have the same distribution;
(ii) Y is a *modification* of X if, for every $t \geq 0$, we have $P(X_t = Y_t) = 1$.

A process is called *progressively measurable* if the map $(t, \omega) \mapsto X_t(\omega)$ is measurable, for each $t \geq 0$. Progressive measurability holds for adapted processes with right-continuous (or left-continuous) paths – and so always in the generality in which we work.

A random variable $\tau : \Omega \to [0, \infty]$ is a *stopping time* if $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

For a set $A \subset \mathbf{R}^d$ and a stochastic process X, we can define the *hitting time* of A for X as

$$\tau_A := \inf\{t > 0 : X_t \in A\}.$$

For our usual situation (RCLL processes and Borel sets) hitting times are stopping times.

We will also need the *stopping time σ-algebra* $\mathcal{F}_\tau$ defined as

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t.$$

Intuitively, $\mathcal{F}_\tau$ represents the events known at time τ.

The continuous-time theory is technically much harder than the discrete-time theory, for two reasons:
1. questions of path-regularity arise in continuous time but not in discrete time;
2. uncountable operations (such as taking the supremum over an interval) arise in continuous time. But measure theory is constructed using countable operations: uncountable operations risk losing measurability.
This is why discrete and continuous time are often treated separately.

Conditional expectation.

The central definition of modern probability (Williams' phrase, [W]) is due to Kolmogorov in 1933, and explained in Doob's Lemma above. The *conditional expectation* of a random variable X *given* a σ-field $\mathcal{C}$, $E[X|\mathcal{C}]$, is any $\mathcal{C}$-measurable random variable that 'integrates the right sets the right way', i.e. satisfies

$$\int_C E[X|\mathcal{C}]dP = \int_C XdP \qquad \forall C \in \mathcal{C}, \quad a.s. \tag{CE}$$

This captures the idea of conditioning given known information, which you may have met in elementary Probability, or Statistics (e.g. regression). We will see it in use in V.2 below. For more on this, see e.g. SP, L 15, 16.

2. Martingales: discrete time.

We refer for a fuller account to [W]. The classic exposition is Ch. VII in Doob's book [D] of 1953.

Definition. A process $X = (X_n)$ in discrete time is called a *martingale* (mg) relative to $(\{\mathcal{F}_n\}, P)$ if
(i) X is adapted (to $\{\mathcal{F}_n\}$);
(ii) $E|X_n| < \infty$ for all n;
(iii) $[X_n|\mathcal{F}_{n-1}] = X_{n-1}$ P-a.s.
X is a *supermartingale* (supermg) if in place of (iii)

$$E[X_n|\mathcal{F}_{n-1}] \le X_{n-1} \qquad P-a.s. \qquad (n \ge 1);$$

X is a *submartingale* (submg) if in place of (iii)

$$E[X_n|\mathcal{F}_{n-1}] \ge X_{n-1} \qquad P-a.s. \qquad (n \ge 1).$$

Martingales have a useful interpretation in terms of dynamic games: a mg is 'constant on average', and models a fair game; a supermg is 'decreasing on average', and models an unfavourable game; a submg is 'increasing on average', and models a favourable game.

Note. 1. Mgs have many connections with harmonic functions in probabilistic potential theory. Supermgs correspond to superharmonic functions, submgs to subharmonic functions.
2. X is a submg (supermg) iff $-X$ is a supermg (submg); X is a mg if and only if it is both a submg and a supermg.
3. (X_n) is a mg iff $(X_n - X_0)$ is a mg. So w.l.o.g. take $X_0 = 0$ if convenient.
4. If X is a martingale, then for $m < n$ using the iterated conditional expectation (tower) property and the martingale property repeatedly (all equalities are in the a.s.-sense)

$$E[X_n|\mathcal{F}_m] = E[E(X_n|\mathcal{F}_{n-1})|\mathcal{F}_m] = E[X_{n-1}|\mathcal{F}_m] = \ldots = E[X_m|\mathcal{F}_m] = X_m,$$

and similarly for submgs, supermgs. See e.g. SP, L16, 17.

Martingale convergence

One reason why martingales (mgs) are so useful is that they have good convergence properties – under suitable conditions. We state some of the key results, without proof; for details, see e.g. SP, L18-19.

Call $X = (X_n)$ L_1-*bounded* if $\sup_n E[|X_n|] < \infty$, i.e.

$$E[|X_n|] \leq K \quad \text{for all } n,$$

for some constant K.

Doob's (Sub-)Martingale Convergence Theorem. An L_1-bounded (sub)martingale is a.s. convergent.

The proof depends on Doob's Upcrossing Inequality (see e.g. SP L18).

Uniform integrability (UI). Call X_n *uniformly integrable* (UI) if

$$\sup_n \int_{\{|X_n|>a\}} |X_n| dP \to 0 \qquad (a \to \infty).$$

If the index set $\{1, 2, \ldots\}$ of the filtration $(\mathcal{F}_n)$ extends to $\{1, 2, \ldots, \infty\}$ so that $\{X_n : n = 1, 2, \ldots, \infty\}$ is a (sub-)mg w.r.t. this filtration, the (sub-)mg is called *closed*, with *closing* (or *last*) element X_∞.

Theorem. Let (X_n) be a UI submg. Then $\sup_n E[X_n^+] < \infty$, and X_n converges to a limit X_∞ a.s. and in L_1, which closes the submg: $X = (X_n)$ is a closed submg, closed by X_∞.

Theorem. X_n is a UI mg iff X_n is a closed mg iff there exists $Y \in L_1$ with

$$X_n = E[Y|\mathcal{F}_n].$$

Then $X_n \to E[Y|\mathcal{F}_\infty]$ a.s. and in L_1.

Corollary (UI Mg Convergence Theorem). For a mg $X = (X_n)$, the following are equivalent:

(i) X is UI;

(ii) X converges a.s. and in L_1 (to X_∞, say);

(iii) X is closed by a random variable Y: $X_n = E[Y|\mathcal{F}_n]$;

(iv) X is closed by its limit X_∞: $X_n = E[X_\infty|\mathcal{F}_n]$.

Note. 1. The UI mgs – equivalently by above the closed mgs – (also called *regular* mgs) are the 'nice' mgs. Note that all the randomness is in the closing rv $Y = X_\infty$. As time progresses, more of Y is revealed as more information becomes available. (Think of progressive revelation, as in – choose your metaphor – a 'striptease', or, 'the Day of Judgement'.)
2. UI (or closed) mgs are also common, and crucially important in Mathematical Finance. There, one does two things: (i) *discount* all asset prices (so as to work with real rather than nominal prices); (ii) change from the real-world probability measure P to an equivalent martingale measure Q (EMM, or *risk-neutral measure*) under which discounted asset prices $\tilde{S}_t$ become (Q)-mgs:

$$\tilde{S}_t = E_Q[\tilde{S}_T|\mathcal{F}_t]$$

(here $T < \infty$ is typically the expiry time of an option). See e.g. [BK], esp. Ch. 4.

Matters are simpler in the L_p case for $p \in (1, \infty)$. Call $X = (X_n)$ L_p-*bounded* if

$$\sup_n \|X_n\|_p < \infty$$

(so in particular each $X_n \in L_p$). We may take $p = 2$ for simplicity, and because of the link with Hilbert-space methods and the important *Kunita-Watanabe Inequalities.* We quote (for proof see e.g. SP L19)

Theorem (L_p-Mg Theorem). If $p > 1$, an L_p-bounded mg X_n is UI, and converges to its limit X_∞ a.s. and in L_p.

3. Martingales in continuous time

A stochastic process $X = (X(t))_{0 \le t < \infty}$ is a *martingale* (mg) relative to $(\{\mathcal{F}_t\}, P)$ if
(i) X is adapted, and $E|X(t)| < \infty$ for all $\le t < \infty$;
(ii) $E[X(t)|\mathcal{F}_s] = X(s)$ P- a.s. $(0 \le s \le t)$,
and similarly for submgs (with $\le$ above) and supermgs (with $\ge$).

In continuous time there are regularization results, under which one can take $X(t)$ RCLL in t (basically $t \to EX(t)$ has to be right-continuous). Then the analogues of most results for discrete-time martingales hold true.