

M3H PROBLEMS 6. 24.2.2017

Q1 (*Viète's infinite product for π* : Francois Viète (1540-1603) in 1593).

(i) Show that if A_n is the area of the regular inscribed n -gon to a unit circle, then

$$A_n/A_{2n} = \cos(\pi/n).$$

(ii) Deduce Viète's infinite product for π :

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

(Viète, *Variorum liber VIII*, 1593: the first exact expression for π).

Q2 (*Girard's formula for spherical excess*: Albert Girard (1595-1632), *Invention nouvelle en algèbre*, 1629).

Show that on sphere of radius r , a spherical triangle with angles A, B, C has area

$$S = r^2(A + B + C - \pi).$$

Q3 (*Wallis' product for π* : John Wallis (1616-1703), *Arithmetica infinitorum*, 1656).

By integrating by parts, or otherwise, show that if $I_n := \int \sin^n x \, dx$,

$$nI_n = -\sin^{n-1}x \cos x + (n-1)I_{n-2}.$$

Deduce that if $J_n := \int_0^{\pi/2} \sin^n x \, dx$, one has the *reduction formula*

$$J_n = \frac{n-1}{n} \cdot J_{n-2}.$$

Hence show that

$$J_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2};$$

$$J_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{2}{3}.$$

Show that $J_{2m+1} \leq J_{2m} \leq J_{2m-1}$, and hence that (Wallis' product)

$$\frac{2^2}{3^2} \cdot \frac{4^2}{5^2} \cdots \frac{(2m-2)^2}{(2m-1)^2} \rightarrow \frac{\pi}{2} \quad (m \rightarrow \infty). \quad \text{NHB}$$