

**Lemma.** If  $A(x) := \sum_{n \leq x} a_n$ ,  $B(x) := \sum_{n \leq x} b_n$ ,

$$\sum_{n \leq x} (a * b)(n) = \sum_{j, k \leq x} a_j b_k = \sum_{j \leq x} a_j B(x/j) = \sum_{k \leq x} b_k A(x/k).$$

*Proof.*

$$\sum_{n \leq x} (a * b)(n) = \sum_{n \leq x} \sum_{jk=n} a_j b_k = \sum_{jk \leq x} a_j b_k = \sum_{j \leq x} a_j \sum_{k \leq x/j} b_k = \sum_{j \leq x} a_j B(x/j),$$

and symmetrically. //

*Defn.* Call  $a$  *multiplicative* if  $a(\cdot)$  is not  $\equiv 0$  and

$$a(mn) = a(m)a(n) \quad \text{for } (m, n) = 1$$

(( $m, n$ ) = gcd of  $m$  and  $n$ : ( $m, n$ ) = 1 means  $m, n$  are *coprime* – have no common factors.

Call  $a$  *completely multiplicative* if it is not  $\equiv 0$  and

$$a(mn) = a(m)a(n) \quad \forall m, n.$$

**Propn.** (i) If  $a$  is multiplicative,  $a(1) = 1$ .

(ii) If  $a, b$  are multiplicative, so is  $a * b$ .

*Proof.* (i) As  $(n, 1) = 1$  for all  $n$ ,  $a(n)a(1) = a(n)$ . As  $a$  is not  $\equiv 0$ ,  $a(n) \neq 0$  for some  $n$ . Then cancelling gives  $a(1) = 1$ .

(ii) Take  $m, n$  with  $(m, n) = 1$ . As  $m, n$  have no common factors, every divisor  $r$  of  $mn$  is uniquely expressible as  $r = jk$  with  $j|m$  and  $k|n$ . Then also  $j, k$  have no common factors, so  $(j, k) = 1$ . Similarly,  $(m/j, n/k) = 1$ . So

$$(a * b)(n) = \sum_{r|mn} a(r)b(mn/r) = \sum_{j|m} \sum_{k|n} a(jk)b\left(\frac{m}{j}, \frac{n}{k}\right) = \sum_{j|m} \sum_{k|n} a(j)a(k)b\left(\frac{m}{j}\right)b\left(\frac{n}{k}\right)$$

(as both  $a$  and  $b$  are multiplicative)

$$= (a * b)(m)(a * b)(n). \quad //$$

**Cor.** If  $f$  is multiplicative, so is  $F := f * u$ :  $F(n) = \sum_{d|n} f(d)$ .

There is a converse: if  $F(n) = \sum_{d|n} f(d)$  is multiplicative, so is  $f$  (Problems 5, Q3).

#### §4. EULER PRODUCTS

Throughout, write  $p$  for a prime,  $P$  for the set of primes,

**Theorem (Euler).** If  $a$  is completely multiplicative with  $\sum_1^\infty |a_n| < \infty$ , then

- (i)  $\sum_1^\infty a_n \neq 0$ ;
- (ii)  $\sum_1^\infty a_n = \prod_p 1/(1 - a_p)$ .

*Proof.* By I.5,  $\prod_p (1 - a_p)$  converges to a non-zero value (as  $\sum |a_n| < \infty$ ); thus so does  $\prod_p 1/(1 - a_p)$ .

Fix  $N$ ; write  $P[N]$  for the set of primes  $p \leq N$ ,  $E_N$  for the set of integers with all prime factors in  $P[N]$ ,  $E_N^*$  for the remaining natural numbers,

$$T_N := \prod_{p \in P[N]} 1/(1 - a_p) = \prod_{p \in P[N]} (1 + a_p + a_p^2 + \dots).$$

Multiply out. Each  $n \in E_N$  appears on RHS exactly once, by FTA (I.1). So

$$T_N = \sum_{n \in E_N} a_n.$$

As  $\{1, 2, \dots, N\} \subset E_N$ ,  $E_N^* \subset \{N+1, N+2, \dots\}$ , so

$$\left| \sum_1^\infty a_n - T_N \right| = \left| \sum_{n \in E_N^*} a_n \right| \leq \sum_{n > N} |a_n| \rightarrow 0 \quad (N \rightarrow \infty). \quad //$$

The special case  $a_n \equiv 1/n^s$  gives

**Theorem (Euler).**  $\zeta(s) = \prod_p 1/(1 - 1/p^s)$  ( $\text{Res} > 1$ ).

*Proof.*

$$RHS = \prod_p (1 + p^{-s} + p^{-2s} + \dots) = \sum_{k, p_1, \dots, p_k} p_1^{-n_1 s} p_2^{-n_2 s} \dots p_k^{-n_k s} = \sum_n n^{-s} = \zeta(s)$$

by FTA, as each  $n = p_1^{n_1} \dots p_k^{n_k}$ , uniquely. //