M2PM3 COMPLEX ANALYSIS PROBLEMS 3, 5.2.2009

Q1 *The Gamma function (real case).*
Define
$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \qquad (x > 0).$$
(i) Show that the integral converges for $x > 0$, but not for $x \leq 0$.
(ii) Show that
$$\Gamma(x+1) = x\Gamma(x) \qquad (x > 0).$$
(iii) Show that
$$\Gamma(n+1) = n! \qquad (n = 0, 1, 2, \ldots).$$
Interpret Γ as providing a continuous extension of the factorial.
(iv) Show that
$$\Gamma(1/2) = \sqrt{\pi}.$$

Q2 *The Gamma function (complex case).*
For $z \in \mathbf{C}$, define
$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt \qquad (x = \Re z > 0).$$
(i) Show that the integral converges for $\Re z > 0$.
(ii) Show that
$$\Gamma(z+1) = z\Gamma(z) \qquad (x = \Re z > 0).$$
(iii) Use (ii) to extend the domain of definition of $\Gamma(z)$ from $\Re z > 0$ to $\Re z > -1$.
(iv) By repeated use of (iii) (or by induction), extend the domain of definition of $\Gamma(z)$ to $\Re z > -n$ ($n = 1, 2, 3, \ldots$), and so to all of the complex plane $\mathbf{C}$. Show also that Γ so extended has complex values (i.e., in the complex plane $\mathbf{C}$ rather than the extended complex plane $\mathbf{C}^*$) except for $z = 0, -1, -2, \ldots$.
(v) Find $\Gamma(-2.5)$ and $\Gamma(3.5)$.

Q3 *The Riemann zeta function* ($\sigma > 1$).
(i) For real σ, define (where convergent)
$$\zeta(\sigma) := \sum_{n=1}^{\infty} 1/n^\sigma.$$
Find the set of σ for which the series is convergent.
(ii) For $s = \sigma + i\tau \in \mathbf{C}$, define (where convergent)
$$\zeta(s) := \sum_{n=1}^{\infty} 1/n^s.$$
Find the half-plane of convergence of this Dirichlet series, and its half-plane of absolute convergence.

Q4 *The Riemann zeta function* ($\sigma > 0$).
(i) By using the Alternating Series Test, or otherwise, show that

$$\sum_{n=1}^{\infty} (-1)^{n-1}/n^{\sigma}$$

converges for $\sigma > 0$ but not for $\sigma \leq 0$. Find the half-planes of convergence and of absolute convergence of the Dirichlet series

$$\sum_{n=1}^{\infty} (-1)^{n-1}/n^{s}.$$

(ii) By considering the sums over n odd and even separately, or otherwise, show that

$$\sum_{n=1}^{\infty} (-1)^{n-1}/n^{s} = (1 - 2^{1-s})\zeta(s) \qquad (\sigma = \Re s > 1).$$

(iii) Deduce that defining

$$\zeta(s) := \frac{1}{(1 - 2^{1-s})} \sum_{n=1}^{\infty} (-1)^{n-1}/n^{s}$$

defines a function ζ for $\sigma = \Re s > 0$, $s \neq 1$.
(iv) Show that as $s \to 1$,

$$\zeta(s) \sim \frac{1}{s-1}.$$

(You may assume that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} \ldots = \log 2$. Hint: use L'Hospital's Rule.)

NHB