ull5b.tex **Week 5: pm, 26.10.2016**

Theorem (Risk-Neutral Valuation Formula, RNVF). The no-arbitrage price of the claim $h(S_T)$ is given by

$$F(t,x) = e^{-r(T-t)} E^*_{t,x}[h(S_T)|\mathcal{F}_t],$$

where $S_t = x$ is the asset price at time t and P^* is the measure under which the asset price dynamics are given by

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Proof (Step (e) in the above: (a) – (d) are already done). Change measure from P, corresponding to $GBM(\mu, \sigma)$, to P^*, corresponding to $GBM(r, \sigma)$, by Girsanov's Theorem. Then as above, $d\tilde{S}_t = \sigma \tilde{S}_t dW_t$. So by VI.2, $d\tilde{V}_t = H_t d\tilde{S}_t = H_t.\sigma \tilde{S}_t dW_t$, where V is the value process following strategy H to replicate payoff h. Integrating, $\tilde{V}_t$ is a P^*-martingale, as it is an Itô integral. So it has constant expectation. So if $S_t = x$ is the asset price at time t,

$$E^*_{t,x}[\tilde{V}_t(H)|\mathcal{F}_t] = E^*_{t,x}\tilde{V}_T(H) = e^{-rT} E^*_{t,x} h(S_T) :$$

$$F(t,x) = E^*_{t,x} V_t(H) = e^{-r(T-t)} E^*_{t,x} h(S_T). \qquad //$$

Theorem ((Continuous) Black-Scholes Formula, BS).

$$F(t,S) = S\Phi(d_+) - e^{-r(T-t)} K\Phi(d_-), \qquad d_\pm := [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t}.$$

Proof (Step (f) in the above). After the change of measure $P \mapsto P^*$, $\mu \mapsto r$ by Girsanov's Theorem, S_t has P^*-dynamics as in $GBM(r, \sigma)$:

$$dS_t = rS_t dt + \sigma S_t dW_t, \qquad S_t = s, \qquad (*)$$

with W a P^*-Brownian motion. So (VI.1) we can solve this explicitly:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(W_T - W_t)\}.$$

Now $W_T - W_t$ is normal $N(0, T-t)$, so $(W_T - W_t)/\sqrt{T-t} =: Z \sim N(0,1)$:

$$S_T = s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma Z\sqrt{T-t}\}, \qquad Z \sim N(0,1).$$

So by the Risk-Neutral Valuation Formula, the pricing formula is

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} h(s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\}).\frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx.$$

For a general payoff function h, there is no explicit formula for the integral, which has to be evaluated numerically. But we can evaluate the integral for the basic case of a European call option with strike-price K:

$$h(s) = (s-K)^+.$$

Then

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} [s \exp\{(r - \frac{1}{2}\sigma^2)(T-t) + \sigma(T-t)^{\frac{1}{2}} x\} - K]_+ dx.$$

We have already evaluated exactly this integral in Chapter IV, where we obtained the Black-Scholes formula from the binomial model by a passage to the limit. Completing the square in the exponential as before gives the result, as in IV.6 Week 3b. //

Comments.
1. *Risk-neutral measure.* We call P^* the *risk-neutral* probability measure. It is equivalent to P (by Girsanov's Theorem, which gives the Radon-Nikodym derivative showing equivalence), and is a martingale measure (as the discounted asset prices are P^*-martingales, by above), i.e. P^* (or Q) is the *equivalent martingale measure (EMM).*
2. *Fundamental Theorem of Asset Pricing (FTAP).* The above continuous-time result may be summarised just as the FTAP in discrete time: to get the no-arbitrage price of a contingent claim, take the discounted expected value under the equivalent mg (risk-neutral) measure.
3. *Completeness.* In discrete time, we saw that absence of arbitrage corresponded to *existence* of risk-neutral measures, completeness to *uniqueness*. We have obtained existence and uniqueness here (and so completeness), by appealing to Girsanov's Theorem, which we have not proved in full. Completeness questions are linked to the Representation Theorem for Brownian Martingales, below.

Theorem (Representation Theorem for Brownian Martingales). Let $(M_t : 0 \leq t \leq T)$ be a square-integrable martingale with respect to the

Brownian filtration $(\mathcal{F}_t)$. Then there exists an adapted process $H = (H_t : 0 \leq t \leq T)$ with $E \int H_s^2 ds < \infty$ such that

$$M_t = M_0 + \int_0^t H_s dW_s, \qquad 0 \leq t \leq T.$$

That is, all Brownian martingales may be represented as stochastic integrals with respect to Brownian motion.

We refer to, e.g., [KS], [RY] for proof.

The economic relevance of the Representation Theorem is that it shows (see e.g. [KS, I.6], and below) that the Black-Scholes model is *complete* – that is, that EMMs are unique, and so that *Black-Scholes prices are unique* (we know this already, from FTAP/RNVF above). Mathematically, the result is purely a consequence of properties of the Brownian filtration. The desirable mathematical properties of BM are thus seen to have hidden within them desirable economic and financial consequences of real practical value.

Hedging.

To find a hedging strategy $H = (H_t^0, H_t)$ (H_t^0 for cash, H_t for stock) that replicates the value process $V = (V_t)$, itself given by RNVF (VI.3 Week 5a):

$$V_t = H_t^0 + H_t S_t = E^*[e^{-r(T-t)} h | \mathcal{F}_t].$$

Now

$$M_t := E^*[e^{-rT} h | \mathcal{F}_t]$$

is a martingale (indeed, a uniformly integrable mg: IV.4, V.2) under the filtration $\mathcal{F}_t$, that of the driving BM in (GBM) (VI.1, VI.2), and the filtration is unchanged by the Girsanov change of measure (we quote this). So by the Representation Theorem for Brownian Martingales, there is some adapted process $K = (K_t)$ with

$$M_t = M_0 + \int_0^t K_s dW_s \qquad (t \in [0, T]).$$

Take

$$H_t := K_t / (\sigma \tilde{S}_t), \qquad H_t^0 := M_t - H_t \tilde{S}_t.$$

Then

$$dM_t = K_t dW_t = \frac{K_t}{\sigma \tilde{S}_t} . \sigma \tilde{S}_t dW_t = H_t d\tilde{S}_t,$$

and the strategy given by K is self-financing, by VI.2. This is of limited practical value:
(a) the Representation Theorem does not give $K = (K_t)$ explicitly – it is merely an existence proof;
(b) we already know that, as Brownian paths have infinite variation, exact hedging in the Black-Scholes model is too rough to be practically possible.

Comments on the Black-Scholes formula.
1. The Black-Scholes formula transformed the financial world. Before it (see Ch. I), the expert view was that asking what an option is worth was (in effect) a silly question: the answer would necessarily depend on the attitude to risk of the individual considering buying the option. It turned out that – at least approximately (i.e., subject to the restrictions to perfect – frictionless – markets, including No Arbitrage – an over-simplification of reality) there *is* an option value. One can see this in one's head, without doing any mathematics, if one knows that the Black-Scholes market is *complete* (above). So, every contingent claim (option, etc.) can be *replicated*, by a suitable combination of cash and stock. Anyone can price this: (i) count the cash, and count the stock; (ii) look up the current stock price; (iii) do the arithmetic.
2. The programmable pocket calculator was becoming available around this time. Every trader immediately got one, and programmed it, so that he could price an option (using the Black-Scholes model!) in real time, from market data.
3. The missing quantity in the Black-Scholes formula is the *volatility*, σ. But, the price is continuous and strictly increasing in σ (options like volatility!). So there is *exactly one* value of σ that gives the price at which options are being currently traded. This – the *implied volatility* – is the value that the market currently judges σ to be, and the one that traders use.
4. Because the Black-Scholes model is the benchmark model of mathematical finance, and gives a value for σ at the push of a button, it is widely used.
5. This is *despite* the fact that no one actually believes the Black-Scholes model! It is an over-simplified approximation to reality. Indeed, Fischer Black himself famously once wrote a paper called *The holes in Black-Scholes*.
6. This is an interesting example of theory and practice interacting!
7. Black and Scholes had considerable difficulty in getting their paper published! It was ahead of its time. When published, and its importance understood, it changed its times.
8. Black-Scholes theory and its developments, plus the internet (a global

network of fibre-optic cables – using *photons* rather than *electrons*), were important contributory factors to *globalization*. Enormous sums of money can be transported round the world at the push of a button, and are every day. This has led to *financial contagion* – "one country's economic problem becomes the world's economic problem". (The Ebola virus comes to mind here.) The resulting problems of *systemic stability* are very important, and still largely unsolved; they dominate the agenda at international meetings.

4. BS via the Black-Scholes PDE and the Feynman-Kac formula

Theorem (Black-Scholes PDE, 1973). In a market with one riskless asset B_t and one risky asset S_t, with short interest-rate r and dynamics

$$\begin{aligned} dB_t &= rB_t dt, \\ dS_t &= \mu(t, S_t)S_t dt + \sigma(t, S_t)S_t dW_t, \end{aligned}$$

let a contingent claim be tradable, with price $h(S_T)$ at expiry T and price process $\Pi_t := F(t, S_t)$ for some smooth function F. Then the only pricing function F which does not admit arbitrage is the solution to the Black-Scholes PDE with boundary condition:

$$F_1(t,x) + rxF_2(t,x) + \frac{1}{2}x^2\sigma^2(t,x)F_{22}(t,x) - rF(t,x) = 0, \qquad (BS)$$

$$F(T,x) = h(x). \qquad (BC)$$

Proof. By Itô's Lemma ($\Pi = F$, $d\Pi = dF$),

$$d\Pi_t = F_1 dt + F_2 dS_t + \frac{1}{2}F_{22}(dS_t)^2$$

(since t has finite variation, the F_{11}- and F_{12}-terms are absent as $(dt)^2$ and $dtdS_t$ are negligible with respect to the terms retained)

$$= F_1 dt + F_2(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}F_{22}(\mu S_t dt + \sigma S_t dW_t)^2$$

$$= F_1 dt + F_2(\mu S_t dt + \sigma S_t dW_t) + \frac{1}{2}F_{22}(\sigma S_t dW_t)^2$$

(the contribution of the FV terms in dt are negligible, as above)

$$= (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 {S_t}^2 F_{22})dt + \sigma S_t F_2 dW_t$$

(as $(dW_t)^2 = dt$). Now $\Pi = F$, so (multiplying by Π, dividing by F)

$$d\Pi_t = \Pi_t(\mu_\Pi(t)dt + \sigma_\Pi(t)dW_t),$$

where

$$\mu_\Pi(t) := (F_1 + \mu S_t F_2 + \frac{1}{2}\sigma^2 S_t^2 F_{22})/F, \qquad \sigma_\Pi(t) := \sigma S_t F_2/F.$$

Now form a portfolio based on two assets: the underlying stock and the option (recall that options are also assets in their own right – they have a value (Black-Scholes formula), and are traded (in large quantities)). Let the *relative portfolio* in stock S and derivative Π be (U_t^S, U_t^Π). Then the dynamics for the value V of the portfolio are given by

$$\begin{aligned} dV_t/V_t &= U_t^S dS_t/S_t + U_t^\Pi d\Pi_t/\Pi_t \\ &= U_t^S(\mu dt + \sigma dW_t) + U_t^\Pi(\mu_\Pi dt + \sigma_\Pi dW_t) \\ &= (U_t^S \mu + U_t^\Pi \mu_\Pi)dt + (U_t^S \sigma + U_t^\Pi \sigma_\Pi)dW_t, \end{aligned}$$

by above. Now both brackets are linear in U^S, U^Π, and $U^S + U^\Pi = 1$ as proportions sum to 1. This is one linear equation in the two unknowns U^S, U^Π, and we can obtain a second one by eliminating the driving Wiener term in the dynamics of V – for then, the portfolio is *riskless*. So it must have return r, the riskless interest rate, to avoid arbitrage. We thus solve the two equations

$$\begin{aligned} U^S + U^\Pi &= 1 \\ U^S\sigma + U^\Pi\sigma_\Pi &= 0. \end{aligned}$$

The solution of the two equations above is

$$U^\Pi = \frac{\sigma}{\sigma - \sigma_\Pi}, \qquad U^S = \frac{-\sigma_\Pi}{\sigma - \sigma_\Pi},$$

which as $\sigma_\Pi = \sigma S F_2/F$ gives the portfolio explicitly as

$$U^\Pi = \frac{F}{F - SF_2}, \qquad U^S = \frac{-SF_2}{F - SF_2}.$$

With this choice of relative portfolio, the dynamics of V are given by

$$dV_t/V = (U_t^S \mu + U_t^\Pi \mu_\Pi)dt,$$

which has no driving Wiener term. So, no arbitrage as above implies that the return rate is the short interest rate r:

$$U_t^S \mu + U_t^\Pi \mu_\Pi = r.$$

Now substitute the values (obtained above)

$$\mu_\Pi = (F+\mu S F_2+\frac{1}{2}\sigma^2 S^2 F_{22})/F, \quad U^S = (-SF_2)/(F-SF_2), \quad U^\Pi = F/(F-SF_2).$$

Substituting the values above in the no-arbitrage relation gives

$$\frac{-SF_2}{F-SF_2}.\mu + \frac{F}{F-SF_2}.\frac{F_1+\mu S F_2+\frac{1}{2}\sigma^2 F_{22}}{F} = r.$$

So

$$-SF_2\mu + F_1 + \mu S F_2 + \frac{1}{2}\sigma^2 S^2 F_{22} = rF - rSF_2,$$

giving the Black-Scholes PDE as required:

$$F_1 + rSF_2 + \frac{1}{2}\sigma^2 S^2 F_{22} - rF = 0. \qquad (BS) \quad //$$

Black and Scholes were classically trained applied mathematicians. When they derived their PDE, they recognised it as parabolic, and so a relative of the *heat equation*. After some months' work, they were able to transform it into the heat equation. The solution to this is known classically.[1] On transforming back, they obtained the Black-Scholes formula.

Theorem (Feynman-Kac Formula). The solution $F(t,x)$ to the PDE

$$F_1(t,x) + \mu(t,x)F_2(t,x) + \frac{1}{2}\sigma^2(t,x)F_{22}(t,x) = g(t,x) \qquad (PDE)$$

with final condition $F(T,x) = h(x)$ has the stochastic representation

$$F(t,x) = E_{t,x}h(X_T) - E_{t,x}\int_t^T g(s,X_s)ds, \qquad (FK)$$

[1]See e.g. the link to MPC2 (Mathematics and Physics for Chemists, Year 2) on my website, Weeks 4, 9. The solution is in terms of *Green functions*. The Green function for (fundamental solution of) the heat equation has the form of a *normal density* (*heat kernel*). This reflects the close link between the mathematics of the heat equation (Fourier in 1807) and the mathematics of Brownian motion (Wiener in 1923) noted earlier (Kakutani, 1944 – Potential Theory).

where X satisfies the SDE

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \leq s \leq T) \qquad (SDE)$$

with initial condition $X_t = x$.

Proof. Consider a SDE, with initial condition (IC), of the form

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s \qquad (t \leq s \leq T), \qquad (SDE)$$

$$X_t = x. \qquad (IC)$$

For suitably well-behaved functions μ, σ, this SDE has a unique solution $X = (X_s : t \leq s \leq T)$, a diffusion. We refer for details on solutions of SDEs and diffusions to an advanced text such as [RW2], [RY], [KS §5.7]. Uniqueness of solutions of the SDE is related to *completeness*, and uniqueness of prices (Representation Theorem for Brownian Martingales, above). (This is as in the FTAP of Ch. IV, but the continuous-time case is harder – here we have to quote uniqueness rather than prove it.)

Taking existence of a unique solution for granted for the moment, consider a smooth function $F(s, X_s)$ of it. By Itô's Lemma, as above,

$$dF = F_1 ds + F_2 dX + \frac{1}{2}F_{22}(dX)^2,$$

and as $(dX)^2 = (\mu ds + \sigma dW_s)^2 = \sigma^2(dW_s)^2 = \sigma^2 ds$, this is

$$dF = F_1 ds + F_2(\mu ds + \sigma dW_s) + \frac{1}{2}\sigma^2 F_{22} ds = (F_1 + \mu F_2 + \frac{1}{2}\sigma^2 F_{22})ds + \sigma F_2 dW_s. \qquad (**)$$

Now suppose that F satisfies the PDE, with boundary condition (BC),

$$F_1(t, x) + \mu(t, x)F_2(t, x) + \frac{1}{2}\sigma^2 F_{22}(t, x) = g(t, x) \qquad (PDE)$$

$$F(T, x) = h(x). \qquad (BC)$$

Then $(**)$ gives

$$dF = g ds + \sigma F_2 dW_s,$$

which can be written in stochastic-integral form as

$$F(T, X_T) = F(t, X_t) + \int_t^T g(s, X_s)ds + \int_t^T \sigma(s, X_s)F_2(s, X_s)dW_s.$$

The stochastic integral on the right is a martingale, so has constant expectation, which must be 0 as it starts at 0. Recalling that $X_t = x$, writing $E_{t,x}$ for expectation with value x and starting-time t, and the price at expiry T as $h(X_T)$ as before, taking $E_{t,x}$ gives the Feynman-Kac formula:

$$E_{t,x} h(X_T) = F(t,x) + E_{t,x} \int_t^T g(s, X_s) ds. \qquad //$$

Re-derivation of the Black-Scholes formula via the Black-Scholes PDE and the Feynman-Kac formula.

Now replace $\mu(t,x)$ by rx, $\sigma(t,x)$ by σx, g by rF in the Feynman-Kac formula above. The SDE becomes that for $GBM(r, \sigma)$:

$$dX_s = rX_s ds + \sigma X_s dW_s \qquad (*)$$

– the same as for a risky asset with mean return-rate r (the short interest-rate for a riskless asset) in place of μ (which disappeared in the Black-Scholes result). The PDE becomes

$$F_1 + rxF_2 + \frac{1}{2}\sigma^2 x^2 F_{22} = rF, \qquad (BS)$$

the Black-Scholes PDE. So by the Feynman-Kac formula,

$$dF = rF ds + \sigma F_2 dW_s, \qquad F(T, s) = h(s).$$

We can eliminate the first term on the right by discounting at rate r: write $G(s, X_s) := e^{-rs} F(s, X_s)$ for the discounted price process. Then as before,

$$dG = -re^{-rs} F ds + e^{-rs} dF = e^{-rs}(dF - rF ds) = e^{-rs}.\sigma F_2 dW.$$

Then integrating, G is a stochastic integral, so a martingale: *the discounted price process* $G(s, X_s) = e^{-rs} F(s, X_s)$ *is a martingale*, under the measure P^* giving the dynamics in $(*)$. This is the measure P we started with, *except* that μ has been changed to r. By the martingale property of G:

$$E^*_{t,x}[G(t, X_t)] = E^*_{t,x}[e^{-rt} F(t, X_t)] = e^{-rt} F(t, x)$$

$$= E^*_{T,x}[e^{-rT} F(T, X_T) | \mathcal{F}_t] = e^{-rT} E^*_{T,x} h(X_T) | \mathcal{F}_t].$$

This gives the Black-Scholes formula, as before. //

§5. Infinite time-horizon; American puts

We sketch here the theory of the American option (one can exercise at any time), over an infinite time-horizon; for details see Peskir & Shiryaev [PS, VII, 25.1]. We deal first with a *put* option (see Week 6a, VI.6 under *Real options (Investment options)* for the corresponding 'call option') – giving the right to sell at the strike price K, at any time τ of our choosing. This τ has to be a *stopping time*: we have to take the decision whether or not to stop at τ based on information already available (that is, contained in $\mathcal{F}_\tau$ – no access to the future, no insider trading). As above, we pass to the risk-neutral measure.

Under the risk-neutral measure, the SDE for GBM becomes

$$dX_t = rX_t dt + \sigma X_t dB_t. \qquad (GBM_r)$$

To evaluate the option, we have to solve the *optimal stopping problem*

$$V(x) := \sup_\tau E_x[e^{-r\tau}(K - X_\tau)^+],$$

with the supremum taken over all stopping times τ and $X_0 = x$ under P_x.

The process X satisfying (GBM_r) – a *diffusion* – is specified by a second-order linear differential operator, called its (infinitesimal) *generator*,

$$L_X := rxD + \frac{1}{2}\sigma^2 x^2 D^2, \qquad D := \partial/\partial x.$$

Now the closer X gets to 0, the less likely we are to gain by continuing. This suggests that our best strategy is to stop when X gets too small: to stop at $\tau = \tau_b$, where

$$\tau_b := \inf\{t \geq 0 : X_t \leq b\},$$

for some $b \in (0, K)$. This gives the following *free boundary problem* for the *unknown value function* $V(x)$ and the *unknown point* b:

$$L_X V = rV \qquad \text{for } x > b; \qquad (i)$$

$$V(x) = (K - x)^+ \qquad \text{for } x = b; \qquad (ii)$$

$$V'(x) = -1 \qquad \text{for } x = b \text{ (smooth fit)}; \qquad (iii)$$

$$V(x) > (K - x)^+ \qquad \text{for } x > b; \qquad (iv)$$

$$V(x) = (K - x)^+ \qquad \text{for } 0 < x < b. \tag{v}$$

Writing $d := \sigma^2/2$ ('d for diffusion'), (i) is

$$dx^2 V'' + rxV' - rV = 0. \tag{i^*}$$

Trial solution (the ODE is *homogeneous*!):

$$V(x) = x^p.$$

Substituting gives a quadratic for p:

$$p^2 - (1 - \frac{r}{d})p - \frac{r}{d} = 0.$$

One root is $p = 1$; the other is $p = -r/d$. So the general solution (GS) to the DE (i^*) is

$$V(x) = C_1 x + C_2 x^{-r/d},$$

for some constants C_1 and C_2. But $V(x) \leq K$ for all $x \geq 0$ (an option giving the right to sell at price K cannot be worth more than K!). So V is bounded. Taking x large ($x < b$ is covered by (v)), we must have $C_1 = 0$. This gives

$$C_2 = \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d}, \qquad b = \frac{K}{1 + d/r}.$$

So

$$\begin{aligned} V(x) &= \frac{d}{r}\Big(\frac{K}{1 + d/r}\Big)^{1+r/d} x^{-r/d} \qquad \text{if } x \in [b, \infty) \\ &= K - x \qquad \text{if } x \in (0, b]. \end{aligned}$$

This is in fact the full and correct solution to the problem. For details, see [P&S], §25.1.

The 'smooth fit' in (iii) is characteristic of free boundary problems. For a heuristic analogy: imagine trying to determine the shape of a rope, tied to the ground on one side of a convex body, stretched over the body, then pulled tight and tied to the ground on the other side. We can see on physical grounds that the rope will be:
straight to the left of the convex body;
continuously in contact with the body for a while, then
straight to the right of the body, and

there should be no kink in the rope at the points where it makes and then leaves contact with the body.
This 'no kink' condition corresponds to 'smooth fit' in (iii).

Note.
1. A classic free-boundary problem is the *Stefan problem*, on melting ice:
A. M. MEIRMANOV: *The Stefan problem*. Walter de Gruyter, Berlin, 1992.
This relates to the *phase change*, of passing between water and ice. Phase changes release or absorb *latent heat* ('hidden' heat). Thus evaporation has a cooling effect (which is why we as warm-blooded mammals sweat)[2]. Conversely, this latent heat is released on condensation (which is why steam burns are even worse than scalding by hot water).
2. Phase changes typically involve an 'energy-entropy competition'. Nature tries to *increase entropy* (entropy is a measure of disorder). Overall, the *Law of Conservation of Energy*, or *First Law of Thermodynamics*, applies: energy is conserved – in the large. But in the small, 'Nature tries to mimimise energy': an isolated system will settle in equilibrium in a configuration that uses least energy (it 'settles down to be comfortable' – just as we do!). The ice-water interface (which may be complicated) is what emerges from the balance between these two conflicting tendencies.
3. There are links with the *Calculus of Variations* (classical examples: brachistochrone, the curve of fastest descent; catenary, the curve followed by a chain hanging under gravity). There are links too with *Optimal Stopping*; see Peskir and Shiryaev [PS], esp. Ch. III, for a monograph treatment.
4. We are familiar with the three phases of water – solid (ice), liquid (water) and gas (steam). At normal temperature, freezing point of water is 0^o C (Centigrade) and its boiling point is 100^o C. But at high temperature, one can have a *triple point*, where all three phases can coexist stable.
5. Titan, the largest moon of Saturn, has a great deal of atmospheric and liquid surface methane, and has 'methane weather systems' analogous to the weather systems of the Earth, which are of course driven by water in the atmosphere and oceans.

[2] The molecules of the water differ in their velocity, according to the Maxwell-Boltzmann distribution (which we met in V.6 Week 5a in connection with the Ornstein-Uhlenback process). Because of surface-tension effects, it is the faster-moving water molecules that are energetic enough to break through the surface film and evaporate. Thus evaporation cools liquid by draining it differentially of its faster-moving ('hotter') molecules.