m2pm3l4.tex
**Lecture 4. 17.1.2011.**

**Euler's Formula (L. Euler (1707-1783)).**

$$e^z = 1 + z + \frac{z^2}{2!} + ... + \frac{z^n}{n!}...,$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...,$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

Take $z = i\theta$, $\theta$ real:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} \\ &= (1 - \frac{z^2}{2!} + \frac{z^4}{4!} - ...) + i(z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...) \\ &= \cos\theta + i\sin\theta. \end{aligned}$$

This is implicit in the *Argand representation*:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta};$$

$r = |z| := \sqrt{x^2 + y^2}$ is the *modulus* of $z$, $\theta$, where $\tan\theta = y/x$ (so $\theta = \tan^{-1} z$) is the *argument* of $z$.
Note. 1. The argument (or $\tan^{-1}$, like $\sin^{-1}$ and $\cos^{-1}$) is many-valued; see later.
2. Take $\theta = \pi$; then
$$e^{i\pi} = -1.$$

**The extended complex plane $\mathbf{C}^*$.**

$\mathbf{R}$ is totally ordered, so there are two directions in which to "go off to infinity", right to $+\infty$, and left to $-\infty$. We write $\mathbf{R}^* := \mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$. What about $\mathbf{C}$?

On $\mathbf{R}$, recall graphs with asymptotes, e.g. $g(x) = 1/x$ or $g(x) = \tan(x)$. This suggests that, in some sense, '$+\infty$ and $-\infty$ are the same place'.

**Stereographic Projection** (G.F. RIEMANN (1826-66) in 1851), PTOLEMY, c. 160 AD).

Draw a picture of the unit sphere ('Earth'), showing the following:

| | |
|---|---|
| $\Sigma$ | Unit sphere |
| C | Unit circle in $Oxy$ ("Equator") |
| N,S | North and South poles |
| GM | "Greenwich Meridian" |

Line NP cuts $Oxy$-plane in $P'$.

$$P \longrightarrow P'$$

is called *stereographic projection*.

| | | |
|---|---|---|
| $P$ Northern Hemisphere | $\longleftrightarrow$ | $P'$ outside unit circle C |
| $P$ Southern Hemisphere | $\longleftrightarrow$ | $P'$ inside unit circle C |
| $P$ on equator | $\longleftrightarrow$ | $P' = P$ on unit circle |
| $P$ South Pole S | $\longleftrightarrow$ | $P'$ origin 0 |
| $P$ North Pole N | $\longleftrightarrow$ | $P'$ ? |

Stereographic projection gives a 1-1 correspondence between the complex plane $\mathbf{C}$ and $\Sigma \setminus \{N\}$. Call this the *punctured sphere*

$$\Sigma' := \Sigma \setminus \{N\}.$$

We now complete $\Sigma'$ to get $\Sigma$, by including the North Pole N. This corresponds (under stereographic projection) to completing $\mathbf{C}$ to get $\mathbf{C}^*$:

$$\mathbf{C}^* := \mathbf{C} \cup \{\infty\},$$

where $\infty$, the "point at infinity in $\mathbf{C}$", corresponds to "going off to infinity in all directions".

*Note.* 1. This is a special case of a general procedure, called *Alexandrov (one-point) compactification.*

2. See also the subject of *Projection Geometry* (Girard DESARGUES (1591-1661) in 1631) - the mathematics of perspective and computer graphics.