LTCC MEASURE-THEORETIC PROBABILITY
EXAM SOLUTIONS, March 2019

Q1. *Variances add over pairwise independent summands.*

For $X_i \in L_2$ (and so $\in L_1$ also), $i = 1, \cdots, n$, $S_n := var\ \sum_1^n X_i$: to show that $var\ S_n = \sum_1^n var\ X_i$, it suffices to assume that each $E[X_i] = 0$, and then apply this to the $X_i - E[X_i]$. So for means 0,

$$\begin{aligned} var\ S_n = E[S_n^2] &= E[(\sum_1^n X_i)^2] = E[\sum_1^n X_i . \sum_1^n X_j] \\ &= E[\sum_1^n X_i^2] + E[\sum_{i \neq j} X_i X_j] \\ &= E[\sum_1^n X_i^2] + \sum_{i \neq j} E[X_i] E[X_j], \end{aligned}$$

by pairwise independence. As the means are 0, this says

$$var\ S_n = \sum_1^n var\ X_i. \qquad //$$

Q2. *The direct half of the SLLN holds under pairwise independence.*

Examination of the proof shows that we only use the *first* Borel-Cantelli lemma ($\sum P(A_n) < \infty$ implies $P(\limsup A_n)$, or $P(A_n i.o.)$, is 0), and that variances add over independent summands, which by Q1 needs only *pairwise* independence.

Q3. *But not the converse half.*

We cannot expect the converse half (a.s. convergence implies that the mean exists) to hold if independence is weakened. As we say in lectures, the Ergodic Theorem (which generalises the SLLN) can give a.s. convergence without the mean existing.

Q4. *Strongly dependent errors need not cancel: an example.*

As in lectures: the SLLN says that independent errors tend to *cancel*; *weakly* dependent errors also tend to cancel (e.g. under *mixing* conditions – there are whole books on this!) But strongly dependent errors need *not* cancel.

Example.

Consider a school Physics class, of 20 pupils. There is an exam soon; the physics teacher has marking to do from another exam. He divides the class into 10 pairs, sets them a time-consuming practical, and goes into his room to mark his scripts. As soon as the door is closed, 18 of the class gang up on the pair who are good at physics practical, and by threats force them to do the practical for them while they revise.

The ‘good pair’ finish in time, and show their result to the others. Say, they have measured the electrical conductivity of copper, giving their result to 6 significant figures. The ‘bad 18’ realise that if they just copy this, it will be obvious what has happened. So, the ‘bad 9 pairs’ copy the first 4 sig figs, making up two ‘nonsense digits’ at the end, to disguise the copying.

Alas, the ammeter the good pair used was dropped that morning, and the culprit did not dare own up. So it was reading *way too high.* So, *no* sig figs are correct in anyone’s results.

Observe the contrast here. The 9 replicates of the two ‘nonsense digits’ *are* independent, and they *do* tend to cancel. But so what? The first 4 sig figs are as strongly dependent as they could be: they are identical. So there is *no* cancellation possible.

Moral. Be on the alert for such pitfalls, when handling real data! NHB