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Lecture 13 10.11.2014

Optional Stopping Theorem (continued).

The OST is important in many areas, such as sequential analysis in statistics. We turn in the next section to related ideas specific to the gambling/financial context.

Write $X_n^T := X_{n \wedge T}$ for the sequence (X_n) *stopped* at time T.

Proposition. (i) If (X_n) is adapted and T is a stopping-time, the stopped sequence $(X_{n \wedge T})$ is adapted.
(ii) If (X_n) is a martingale [supermartingale] and T is a stopping time, (X_n^T) is a martingale [supermartingale].

Proof. If $\phi_j := I\{j \le T\}$,

$$X_{T \wedge n} = X_0 + \sum_1^n \phi_j (X_j - X_{j-1}).$$

Since $\{j \le T\}$ is the complement of $\{T < j\} = \{T \le j-1\} \in \mathcal{F}_{j-1}$, $\phi_j = I\{j \le T\} \in \mathcal{F}_{j-1}$, so (ϕ_n) is previsible. So (X_n^T) is adapted.

If (X_n) is a martingale, so is (X_n^T) as it is the martingale transform of (X_n) by (ϕ_n). Since by previsibility of (ϕ_n)

$$E[X_{T \wedge n} | \mathcal{F}_{n-1}] = X_0 + \sum_1^{n-1} \phi_j (X_j - X_{j-1}) + \phi_n (E[X_n | \mathcal{F}_{n-1}] - X_{n-1}),$$

i.e.

$$E[X_{T \wedge n} | \mathcal{F}_{n-1}] - X_{T \wedge n} = \phi_n (E[X_n | \mathcal{F}_{n-1}] - X_{n-1}),$$

$\phi_n \ge 0$ shows that if (X_n) is a supermg [submg], so is $(X_{T \wedge n})$. //

§7. The Snell Envelope and Optimal Stopping.

Definition. If $Z = (Z_n)_{n=0}^N$ is a sequence adapted to a filtration $(\mathcal{F}_n)$, the sequence $U = (U_n)_{n=0}^N$ defined by

$$\begin{cases} U_N := Z_N, \\ U_n := \max(Z_n, E(U_{n+1} | \mathcal{F}_n)) \qquad (n \le N-1) \end{cases}$$

is called the *Snell envelope* of Z (J. L. Snell in 1952; [N] Ch. 6). U is adapted, i.e. $U_n \in \mathcal{F}_n$ for all n. For, Z is adapted, so $Z_n \in \mathcal{F}_n$. Also $E[U_{n+1}|\mathcal{F}_n] \in \mathcal{F}_n$ (definition of conditional expectation). Combining, $U_n \in \mathcal{F}_n$, as required.

We shall see in IV.8 [L20] that the Snell envelope is exactly the tool needed in pricing American options. It is the *least supermg majorant* (also called the *réduite* or *reduced function* – crucial in the mathematics of gambling):

Theorem. The Snell envelope (U_n) of (Z_n) is a supermartingale, and is the smallest supermartingale dominating (Z_n) (that is, with $U_n \geq Z_n$ for all n).

Proof. First, $U_n \geq E(U_{n+1}|\mathcal{F}_n)$, so U is a supermartingale, and $U_n \geq Z_n$, so U dominates Z.

Next, let $T = (T_n)$ be any other supermartingale dominating Z; we must show T dominates U also. First, since $U_N = Z_N$ and T dominates Z, $T_N \geq U_N$. Assume inductively that $T_n \geq U_n$. Then

$$\begin{aligned} T_{n-1} &\geq E(T_n|\mathcal{F}_{n-1}) \qquad \text{(as } T \text{ is a supermartingale)} \\ &\geq E(U_n|\mathcal{F}_{n-1}) \qquad \text{(by the induction hypothesis)} \end{aligned}$$

and

$$T_{n-1} \geq Z_{n-1} \qquad \text{(as } T \text{ dominates } Z\text{)}.$$

Combining,

$$T_{n-1} \geq \max(Z_{n-1}, E(U_n|\mathcal{F}_{n-1})) = U_{n-1}.$$

By backward induction, $T_n \geq U_n$ for all n, as required. //

Note. It is no accident that we are using induction here *backwards in time*. We will use the same method – also known as *dynamic programming (DP)* – in Ch. IV below when we come to pricing American options.

Proposition. $T_0 := \min\{n \geq 0 : U_n = Z_n\}$ is a stopping time, and the stopped sequence $(U_n^{T_0})$ is a martingale.

Proof (not examinable). Since $U_N = Z_N$, $T_0 \in \{0, 1, \cdots, N\}$ is well-defined (and we can use minimum rather than infimum). For $k = 0$, $\{T_0 = 0\} = \{U_0 = Z_0\} \in \mathcal{F}_0$; for $k \geq 1$,

$$\{T_0 = k\} = \{U_0 > Z_0\} \cap \cdots \cap \{U_{k-1} > Z_{k-1}\} \cap \{U_k = Z_k\} \in \mathcal{F}_k.$$

So T_0 is a stopping-time.

As in the proof of the Proposition in §6,

$$U_n^{T_0} = U_{n \wedge T_0} = U_o + \sum_1^n \phi_j \Delta U_j,$$

where $\phi_j = I\{T_0 \geq j\}$ is adapted. For $n \leq N-1$,

$$U_{n+1}^{T_0} - U_n^{T_0} = \phi_{n+1}(U_{n+1} - U_n) = I\{n+1 \leq T_0\}(U_{n+1} - U_n).$$

Now $U_n := \max(Z_n, E(U_{n+1}|\mathcal{F}_n))$, and

$$U_n > Z_n \qquad \text{on } \{n+1 \leq T_0\}.$$

So from the definition of U_n,

$$U_n = E(U_{n+1}|\mathcal{F}_n) \qquad \text{on } \{n+1 \leq T_0\}.$$

We next prove

$$U_{n+1}^{T_0} - U_n^{T_0} = I\{n+1 \leq T_0\}(U_{n+1} - E(U_{n+1}|\mathcal{F}_n)). \tag{1}$$

For, suppose first that $T_0 \geq n+1$. Then the left of (1) is $U_{n+1} - U_n$, the right is $U_{n+1} - E(U_{n+1}|\mathcal{F}_n)$, and these agree on $\{n+1 \leq T_0\}$ by above. The other possibility is that $T_0 < n+1$, i.e. $T_0 \leq n$. Then the left of (1) is $U_{T_0} - U_{T_0} = 0$, while the right is zero because the indicator is zero. Combining, this proves (1) as required. Apply $E(.|\mathcal{F}_n)$ to (1): since $\{n+1 \leq T_0\} = \{T_0 \leq n\}^c \in \mathcal{F}_n$,

$$E[(U_{n+1}^{T_0} - U_n^{T_0})|\mathcal{F}_n] = I\{n+1 \leq T_0\}E([U_{n+1} - E(U_{n+1}|\mathcal{F}_n)]|\mathcal{F}_n)$$

$$= I\{n+1 \leq T_0)\}[E(U_{n+1}|\mathcal{F}_n) - E(U_{n+1}|\mathcal{F}_n)] = 0.$$

So $E(U_{n+1}^{T_0}|\mathcal{F}_n) = U_n^{T_0}$. This says that $U_n^{T_0}$ is a martingale, as required. //

Note. Just because U is a supermartingale, we knew that stopping it would give a supermartingale, by the Proposition of §6. The point is that, using the special properties of the Snell envelope, we actually get a *martingale*.

Write $\mathcal{T}_{n,N}$ for the set of stopping times taking values in $\{n, n+1, \cdots, N\}$ (a finite set, as Ω is finite). We next see that the Snell envelope solves the *optimal stopping problem*: it *maximises* the expectation of our final value of Z – the value when we choose to quit – conditional on our present (publicly available) information. This is the best we can hope to do in practice (without cheating – insider trading, etc.)

Theorem. T_0 solves the optimal stopping problem for Z:

$$U_0 = E(Z_{T_0}|\mathcal{F}_0) = \max\{E(Z_T|\mathcal{F}_0) : T \in \mathcal{T}_{0,N}\}.$$

Proof. As $(U_n^{T_0})$ is a martingale (above),

$$\begin{aligned} U_0 &= U_0^{T_0} \qquad (\text{since } 0 = 0 \wedge T_0) \\ &= E(U_N^{T_0}|\mathcal{F}_0) \qquad (\text{by the martingale property}) \\ &= E(U_{T_0}|\mathcal{F}_0) \qquad (\text{since } T_0 = T_0 \wedge N) \\ &= E(Z_{T_0}|\mathcal{F}_0) \qquad (\text{since } U_{T_0} = Z_{T_0}), \end{aligned}$$

proving the first statement. Now for any stopping time $T \in \mathcal{T}_{0,N}$, since U is a supermartingale (above), so is the stopped process (U_n^T) (§6). So

$$\begin{aligned} U_0 &= U_0^T \qquad (0 = 0 \wedge T, \text{ as above}) \\ &\geq E(U_N^T|\mathcal{F}_0) \qquad ((U_n^T) \text{ a supermartingale}) \\ &= E(U_T|\mathcal{F}_0) \qquad (T = T \wedge N) \\ &\geq E(Z_T|\mathcal{F}_0) \qquad ((U_n) \text{ dominates } (Z_n)), \end{aligned}$$

and this completes the proof. //

The same argument, starting at time n rather than time 0, gives an apparently more general version:

Theorem. If $T_n := \min\{j \geq n : U_j = Z_j\}$,

$$U_n = E(Z_{T_n}|\mathcal{F}_n) = \sup\{E(Z_T|\mathcal{F}_n) : T \in \mathcal{T}_{n,N}\}.$$

To recapitulate: as we are attempting to maximise our payoff by stopping $Z = (Z_n)$ at the most advantageous time, the Theorem shows that T_n gives the best stopping-time that is realistic: it maximises our *expected payoff* given only information *currently available* (it is easy, but irrelevant, to maximise things with hindsight!). We thus call T_0 (or T_n, starting from time n) the *optimal* stopping time for the problem.