pfsl28(14).tex
Lecture 28. 15.12.2014
Poisson point processes (continued).

This counts the Poisson points in B – and is a Poisson process with rate (parameter) $\nu(B)$. All this reverses: starting with an $e = (e(t) : t \geq 0)$ whose counting processes over Borel sets B are Poisson $P(\nu(B))$, then – as no point can contribute to more than one count over disjoint sets – disjoint counting processes never jump together, so are independent by above, and $\phi := \sum_{t \geq 0} \delta_{(e(t),t)}$ is a Poisson measure with intensity $\mu = \nu \times dt$.

Lévy Processes; Lévy-Khintchine Formula; Lévy-Itô decomposition.

We can now sketch the close link between the general Lévy process on the one hand and the general infinitely-divisible law given by the Lévy-Khintchine formula (LK) on the other.

First, if $X = (X_t)$ is Lévy, the law of each X_1 is infinitely divisible, so

$$E \exp\{iuX_1\} = \exp\{-\Psi(u)\} \qquad (u \in \mathbb{R})$$

with Ψ a Lévy exponent as in (LK). Similarly,

$$E \exp\{iuX_t\} = \exp\{-t\Psi(u)\} \qquad (u \in \mathbb{R}),$$

for rational t at first and general t by approximation and càdlàg paths. Then Ψ is called the *Lévy exponent*, or *characteristic exponent*, of the Lévy process X. Conversely, given a Lévy exponent $\Psi(u)$ as in (LK), construct a Brownian motion, and an independent Ppp $\Delta = (\Delta_t : t \geq 0)$ with characteristic measure μ, the Lévy measure in (LK). Then $X_1(t) := at + \sigma B_t$ has CF

$$E \exp\{iuX_1(t)\} = \exp\{-t\Psi_1(t)\} = \exp\left\{-t(iau + \frac{1}{2}\sigma^2 u^2)\right\},$$

giving the non-integral terms in (LK). For the 'large' jumps of Δ, write

$$\Delta_t^{(2)} := \Delta_t \text{ if } \quad |\Delta_t| \geq 1, \quad 0 \text{ else.}$$

Then $\Delta^{(2)}$ is a Poisson point process with characteristic measure $\mu^{(2)}(dx) := I(|x| \geq 1)\mu(dx)$. Since $\int \min(1, |x|^2)\mu(dx) < \infty$, $\mu^{(2)}$ has finite mass, so $\Delta^{(2)}$, a $Ppp(\mu^{(2)})$, is discrete and its counting process

$$X_t^{(2)} := \sum_{s \leq t} \Delta_s^{(2)} \qquad (t \geq 0)$$

is compound Poisson, with Lévy exponent

$$\Psi^{(2)}(u) = \int (1 - e^{iux}) I(|x| \geq 1)\mu(dx) = \int (1 - e^{iux})\mu^{(2)}(dx).$$

There remain the 'small jumps',

$$\Delta_t^{(3)} := \Delta_t \ \text{ if } \quad |\Delta_t| < 1, \quad 0 \text{ else},$$

a $Ppp(\mu^{(3)})$, where $\mu^{(3)}(dx) = I(|x| < 1)\mu(dx)$, and independent of $\Delta^{(2)}$ because $\Delta^{(2)}$, $\Delta^{(3)}$ are Poisson point processes that never jump together. For each $\epsilon > 0$, the 'compensated sum of jumps'

$$X_t^{(\epsilon,3)} := \sum_{s \leq t} I(\epsilon < |\Delta_s| < 1)\Delta_s - t \int x I(\epsilon < |x| < 1)\mu(dx) \qquad (t \geq 0)$$

is a Lévy process with Lévy exponent

$$\Psi^{(\epsilon,3)}(u) = \int (1 - e^{iux} + iux) I(\epsilon < |x| < 1)\mu(dx).$$

Use of a suitable maximal inequality allows passage to the limit $\epsilon \downarrow 0$ (going from finite to possibly countably infinite sums of jumps): $X_t^{(\epsilon,3)} \to X_t^{(3)}$, a Lévy process with Lévy exponent

$$\Psi^{(3)}(u) = \int (1 - e^{iux} + iux) I(|x| < 1)\mu(dx),$$

independent of $X^{(2)}$ and with càdlàg paths. Combining:

Theorem (Lévy-Itô decomposition). For a Lévy exponent

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (1 - e^{iux} + iuxI(|x| < 1)\mu(dx),$$

the construction above yields a Lévy process

$$X = X^{(1)} + X^{(2)} + X^{(3)}$$

with Lévy exponent $\Psi = \Psi^{(1)} + \Psi^{(2)} + \Psi^{(3)}$. Here the $X^{(i)}$ are independent Lévy processes, with Lévy exponents $\Psi^{(i)}$; $X^{(1)}$ is Gaussian, $X^{(2)}$ is a compound Poisson process with jumps of modulus ≥ 1; $X^{(3)}$ is a compensated sum of jumps of modulus < 1. The jump process $\Delta X = (\Delta X_t : t \geq 0)$ is a $Ppp(\mu)$, and similarly $\Delta X^{(i)}$ is a $Ppp(\mu^{(i)})$ for $i = 2, 3$.

Stable processes.

A stable process has (to within location and scale) a Lévy exponent involving two parameters, $\alpha \in (0, 2]$, called the *index*, and $\beta \in [-1, 1]$, called the *skewness parameter*:

$$\Psi(u) = |u|^\alpha(1 - i\beta(sgn\ u)\tan(\frac{1}{2}\pi\alpha)) \quad (\alpha \neq 1), \quad |u|(1 + i\beta(sgn\ u)\frac{2}{\pi}\log|u|) \quad (\alpha = 1)$$

(for $\alpha = 2$, β drops out as tan $\pi = 0$, so $\Psi(u) = u^2$, giving the normal (Gaussian) distribution). The case $\alpha = 1$ is the *Cauchy* case; the asymmetric Cauchy case $\alpha = 1, \beta \neq 0$ is awkward, and we do not consider it further.

The Lévy measure μ in the stable case is absolutely continuous, with density ν, $\mu(dx) = \nu(x)dx$, where

$$\nu(x) = c_+/x^{1+\alpha} \quad (x > 0), \quad c_-/|x|^{1+\alpha} \quad (x < 0) \quad (c_\pm \geq 0,\ c_+ + c_- > 0).$$

Here

$$\beta = (c_= - c_-)/(c_+ + c_-).$$

The calculations are simpler in the symmetric case, $c_+ = c_-, = c$ say. Then

$$\Psi'(u) = 2cu^{\alpha-1}I \qquad (u > 0), \qquad I := \int_0^\infty v^{-\alpha}\sin v dv.$$

So $\Psi(u) = 2cIu^\alpha/\alpha$ for $u > 0$, and similarly for $u < 0$: $\Psi(u) = |u|^\alpha.2cI/a$ But (see e.g. M2PM3 L30 on my website: there $t = 1 - \alpha \in (0, 1)$, but we can extend by analytic continuation to $-1 < t < 1$, $\alpha \in (0, 2)$) $I = \Gamma(1 - \alpha)\cos(\frac{1}{2}\pi\alpha)$ (here $\alpha \neq 1$: $\Gamma(z)$ has a pole at $z = 0$; for the Cauchy case $\alpha = 1$ see above). Choose $c := \sigma/(2I)$; then $\Psi(u) = |u|^\alpha$.

Example: The Holtsmark distribution.

The symmetric stable law with $\alpha = 3/2$ is called the *Holtsmark distribution*, proposed by the Danish physicist J. Holtsmark in 1919 as a model for the distribution of galaxies in space (here 3/2 comes from the 3 dimensions of space and the 2 in Newton's Inverse Square Law of Gravity). Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(1 + x) = x\Gamma(x)$, the constant $\Gamma(1 - \alpha)\cos(\frac{1}{2}\pi\alpha)$ here is

$$\int_0^\infty v^{3/2}\sin v dv = \Gamma(-\frac{1}{2})\cos(3\pi/4) = (-2\sqrt{\pi}).(-1/\sqrt{2}) = \sqrt{2\pi}.$$

Subordinators.

We resort to complex numbers in the CF $\phi(u) = E(e^{iuX})$ because this always exists – for all real u – unlike the ostensibly simpler moment-generating function (MGF) $M(u) := E(e^{uX})$, which may well diverge for some real u. However, if the random variable X is *non-negative*, then for $s \geq 0$ the *Laplace-Stieltjes transform* (LST)

$$\psi(s) := E[e^{-sX}] \leq E(1) = 1$$

always exists. For $X \geq 0$ we have both the CF and the LST to hand, but the LST is usually simpler to handle. We can pass from CF to LST formally by taking $u = is$, and this can be justified by analytic continuation.

Some Lévy processes X have increasing (i.e. non-decreasing) sample paths; these are called *subordinators.* From the construction above, subordinators can have no negative jumps, so μ has support in $(0, \infty)$ and no mass on $(-\infty, 0)$. Because increasing functions have FV, one must have paths of (locally) finite variation, the condition for which can be shown to be

$$\int \min(1, |x|)\mu(dx) < \infty.$$

Thus the Lévy exponent must be of the form

$$\Psi(u) = -idu + \int_0^\infty (1 - e^{iux})\mu(dx),$$

with $d \geq 0$. It is more convenient to use the Laplace exponent $\Phi(s) = \Psi(is)$:

$$E\left(\exp\{-sX_t\}\right) = \exp\{-t\Phi(s)\} \qquad (s \geq 0), \qquad \Phi(s) = ds + \int_0^\infty (1 - e^{-sx})\mu(dx).$$

Random time-change.

Because of the *arrow of time*, the fact that subordinator paths increase, as time elapsed does, makes them suitable for *random changes of time.* It may be useful to pass from *real time* to *operational time*, speeding things up when nothing much is happening and slowing things down when too much is happening. We have evolved to experience time this way ourselves in a crisis!