m3pm16soln4.tex

**M3PM16/M4PM16 SOLUTIONS 4. 14.2.2014**

Q1 (HW §16.1,2, J, 68-9). (i) Using $|.|$ for cardinality,

$$\sum_{d|n} |a : 1 \leq a \leq n, (a,n) = d| = n,$$

as each integer $a$ from 1 to $n$ has a unique gcd with $n$, $d := (a,n)$, which divides $n$. Also, if $d|n$ then

$$\begin{aligned} \phi(n/d) &= |a : 1 \leq a \leq n/d, (a, n/d) = 1| \qquad (\text{definition of } \phi(n/d) \\ &= |b : 1 \leq b \leq n, (b,n) = d| \qquad (b := da). \end{aligned}$$

Combining,

$$n = \sum_{d|n} \phi(n|d), = \sum_{d|n} \phi(d),$$

since as $d$ runs through the divisors of $n$, so does $n/d$.
(ii) This follows by II.3 Propn. and (i).
(iii) There are $p^c - 1$ positive integers $< p^c$, of which the multiples of $p$ are $p, 2p, \ldots, p^c - p$ (so $p^{c-1} - 1$ of these), and the rest are coprime to $p^c$. So

$$\phi(p^c) = (p^c - 1) - (p^{c-1} - 1) = p^c(1 - \frac{1}{p}).$$

So if $n = \prod p^c$ is the prime-power factorisation of $n$ (FTA), (ii) gives

$$\phi(n) = \prod \phi(p^c) = \prod p^c \prod (1 - \frac{1}{p}) = n \prod_{p|n} (1 - \frac{1}{p}). \qquad //$$

$$\prod_{p|n} (1 - \frac{1}{p}) = 1 - \sum_i 1/p_i + \sum_{ij} 1/p_i p_j - - \ldots = \sum_k \mu(k)/k$$

(the middle sum is over the set of divisors $k$ of $n$ with $\mu(k) \neq 0$, so we can include those with $\mu(k) = 0$). Now (iv) follows from (iii). In convolution form, this says $\phi = \mu * I$, where $I(n) \equiv n$. Taking Dirichlet series, this gives $\sum_1^\infty \phi(n)/n^s = \zeta(s-1)/\zeta(s)$, as $\mu(n)$, $n$ have Dirichlet series $1/\zeta(s) = \sum_1^\infty \mu(n)/n^s$, $\zeta(s-1) = \sum_1^\infty n/n^s = \sum_1^\infty 1/n^{s-1}$.

Q2 (HW Th. 260). (i) If $a \in A$ belongs to exactly $m$ of the sets: if $m = 0$, $a$ is counted in the RHS $S - S_1 + S_2 ...$ just once, in $S$ itself. If $m > 0$, then $a$ is counted:
$1 = \binom{m}{0}$ times in $S$; $\binom{m}{1}$ times in $S_1$, ..., $\binom{m}{r}$ times in $S_r$.
Altogether, $a$ is counted

$$\binom{m}{0} - \binom{m}{1} + \binom{m}{2} \ldots = (1-1)^m = 0$$

times. So the RHS is the cardinality of the set of points in none of the $A_i$.
(ii) (HW Th. 261). The number of integers $\leq n$ and divisible by $a$ is $[n/a]$. If $a$ is coprime to $b$, the number of integers $\leq n$ and divisible by both $a$ and $b$ is $[n/ab]$, etc. So the number of integers $\leq n$ and not divisible by any of a coprime set of integers $a, b, \ldots$ is $[n] - \sum[n/a] + \sum[n/ab] \ldots$.
Taking $a, b, \ldots$ as the prime divisors of $n$,

$$\phi(n) = n - \sum \frac{n}{p} + \sum \frac{n}{pq} \ldots = n \prod_{p|n} \Big(1 - \frac{1}{p}\Big).$$

Q3 (see e.g. R. V. Churchill, *Fourier series and boundary value problems*, McGraw-Hill 1963, Ch. 4). Write $a_n$ for the Fourier cosine coefficients of $|x|$ on $[-\pi, \pi]$ ($|.|$ is even, so we do not need sine terms). Then

$$\frac{1}{2}a_0 = \frac{1}{2}.\frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \frac{1}{\pi}\int_0^{\pi} x dx = \frac{1}{\pi}[\frac{1}{2}x^2]_0^{\pi} = \frac{\pi}{2},$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos\ nx dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx dx = \frac{2}{n\pi}\int_0^{\pi} x d \sin nx \\ &= \frac{2[x \sin nx]_0^{\pi}}{n\pi} - \frac{2}{n\pi}\int_0^{\pi} \sin nx dx = \frac{2}{n^2\pi}[\cos nx]_0^{\pi} = \frac{2(\cos n\pi - 1)}{n^2\pi} \\ &= \frac{2((-1)^n - 1)}{n^2\pi} = -\frac{4}{\pi n^2} \end{aligned}$$

if $n$ is odd, 0 if $n$ is even. So

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos(2m-1)x}{(2m-1)^2} : \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{odd} 1/n^2 : \quad \sum_{odd} = \pi^2/8.$$

But

$$\zeta(2) = \sum_1^{\infty} 1/n^2 = \sum_{odd} + \sum_{even} = \sum_{odd} + \frac{1}{4}\zeta(2) : \qquad \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}, \qquad \zeta(2) = \frac{\pi^2}{6}.$$

NHB