

M3PM16/M4PM16 SOLUTIONS 7. 8.3.2012

Q2 (J 93-4, Prop. 2.6.5). Write $q := p^n$ for a generic prime power, and for primes p with $p^2 \leq x$, let r_p be the largest ‘relevant power’ (largest r with $p^r \leq x$). Then

$$\Delta := \sum_{q \leq x} 1/q - \sum_{p \leq x} 1/p = \sum_{p \leq \sqrt{x}} \sum_{r=2}^{r_p} 1/p^r.$$

But $\sum_2^\infty 1/p^r = 1/(p(p-1))$, summing the GP, so

$$\Delta \leq \sum_p \frac{1}{p(p-1)} = S$$

(above). Write

$$S_0 := \sum_{p \leq \sqrt{x}} \frac{1}{p(p-1)};$$

then

$$\begin{aligned} S - S_0 &\leq \sum_{p > \sqrt{x}} < \sum_{n > \sqrt{x}} \frac{1}{n(n-1)} \\ &= \frac{1}{\sqrt{[x]}} \quad \left(\frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}, \text{ sum telescopes} \right) \\ &\leq 2/\sqrt{x}. \end{aligned}$$

As $p^{r_p+1} \geq x$:

$$\sum_{r > r_p} \frac{1}{p^r} < \frac{1}{x} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots \right) = \frac{1}{x(1-1/p)} \leq 2/\sqrt{x} \quad (p \geq 2).$$

So

$$S_0 - \Delta = \sum_{p \leq \sqrt{x}} \sum_{r > r_p} 1/p^r < \pi(\sqrt{x}) \cdot 2/x \leq 2/\sqrt{x}$$

($\pi(x) := \sum_{p \leq x} 1 \leq \sum_{n \leq x} 1 \leq x$). Combining, $S - \Delta \leq 4/\sqrt{x} = O(1/\log x)$. So the difference Δ in the sums here and in Mertens’ Second Theorem is $S + O(1/\log x)$, and the result follows from Mertens’ Second Theorem. //

Q2 (Tom M. Apostol: A proof that Euler missed: Evaluating $\zeta(2)$ the easy way. *Mathematical Intelligencer* **8** no. 1 (1983), 59-60;
W. J. LeVeque, *Topics in number theory*, Vol. 1, Addison-Wesley, 1956, p.122 Ex.6).

$$\begin{aligned} I &:= \int_0^1 \int_0^1 dx dy / (1 - xy) = \int_0^1 \int_0^1 \sum_0^\infty x^n y^n dx dy = \sum_0^\infty \int_0^1 x^n dx \int_0^1 y^n dy \\ &= \sum_0^\infty 1/(n+1)^2 = \sum_1^\infty 1/n^2 = \zeta(2). \end{aligned}$$

The change of variable has Jacobian 1, and takes the bounding lines of the unit square to those of S , and $1 - xy = 1 - (u^2 - v^2)/2 = (2 - u^2 + v^2)/2$. So symmetry between $\pm u$ reduces I to

$$I = 4 \int_0^{1/\sqrt{2}} \left(\int_0^u \frac{dv}{2 - u^2 + v^2} \right) du + 4 \int_{1/\sqrt{2}}^{\sqrt{2}} \left(\frac{dv}{2 - u^2 + v^2} \right) du = I_1 + I_2,$$

say. Evaluating the inner integrals,

$$I_1 = 4 \int_0^{1/\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2 - u^2}} \right) \frac{du}{\sqrt{2 - u^2}}, \quad I_2 = 4 \int_{1/\sqrt{2}}^{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}} \right) \frac{du}{\sqrt{2 - u^2}}.$$

In I_1 , $u = \sqrt{2} \sin \theta$, $du = \sqrt{2} \cos \theta d\theta = \sqrt{2 - u^2} d\theta$, $\tan \theta = u/\sqrt{2 - u^2}$, $\tan^{-1}(u/\sqrt{2 - u^2}) = \theta$; the limits are $u = 0$, $\theta = 0$ and $u = 1/\sqrt{2}$, $\sin \theta = 1/2$, $\theta = \pi/6$. So $I_1 = 4 \int_0^{\pi/6} \theta d\theta = 2(\pi/6)^2$.

In I_2 , $u = \sqrt{2} \cos 2\theta$, $du = -2\sqrt{2} \sin 2\theta d\theta = -2\sqrt{2} \sqrt{1 - \cos^2 2\theta} d\theta$
 $= -2\sqrt{2} \sqrt{1 - u^2/2} d\theta = -2\sqrt{2 - u^2} d\theta$: $du/\sqrt{2 - u^2} = -2d\theta$.

$$\frac{\sqrt{2} - u}{\sqrt{2 - u^2}} = \frac{\sqrt{2}(1 - \cos 2\theta)}{\sqrt{2 - 2\cos^2 2\theta}} = \frac{(1 - \cos 2\theta)}{\sqrt{1 - \cos^2 2\theta}} = \frac{1 - \cos 2\theta}{\sqrt{(1 - \cos 2\theta)(1 + \cos 2\theta)}}$$

$$= \sqrt{\frac{(1 - \cos 2\theta)}{1 + \cos 2\theta}} = \sqrt{\frac{2\sin^2 \theta}{2\cos^2 \theta}} = \tan \theta : \quad \tan^{-1} \left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}} \right) = \theta.$$

The limits are $u = 1/\sqrt{2}$, $\cos 2\theta = 1/2$, $2\theta = \pi/3$, $\theta = \pi/6$ and $u = \sqrt{2}$, $\cos 2\theta = 1$, $\theta = 0$. So $I_2 = 4 \int_{\pi/6}^0 \theta (-2d\theta) = 8 \int_0^{\pi/6} \theta d\theta = 4(\pi/6)^2$. So $I = I_1 + I_2 = (2 + 4)(\pi/6)^2 = \pi^2/6$. //

NHB