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5. Portfolios and Hedging.

Portfolios.

We consider an investor with capital to invest. The simplest model is that in which he has two (or more) choices: to invest in

(i) a *bank account* – assumed *riskless*, and yielding interest. For simplicity, we assume the interest rate is a constant $r > 0$ (usually called the *short rate* of interest: interest rates may be different outside $[0, T]$); thus \$$B$ invested at time t grows to \$$Be^{r(T-t)}$ by time t;

(ii) one (or more) *risky assets* or stocks, whose value (or price) at time t is S_t (a vector of prices if there is more than one type of stock).

A *portfolio* is a division (B_t, S_t) of the investor's capital between bank account and stock holdings at time t.

A *trading strategy* is a rule (suitably restricted – see Chapters IV and VI) chosen by the investor to update his portfolio over time as new price information on the risky stock(s) comes in.

Hedging. Hedging is an attempt to reduce exposure to risk by adopting opposite positions – e.g., in holding both call and put options in the same underlying, or by adjusting a portfolio as above to cover possible losses on an option.

Shorter Oxford English Dictionary [OED]: "6 *trans.* To cover oneself against loss on (a bet etc.) by betting, etc., on the other side. Also *fig.* 1672."

Why buy options? There are two main uses for options: speculation and hedging. In speculation, available funds ('hot money') are invested opportunistically in the hope of making a profit: the underlying itself is irrelevant to the investor (speculator), who is only interested in the potential for possible profit that trade involving it may present. Hedging, by contrast, is typically engaged in by companies who have to deal habitually in intrinsically risky assets such as foreign exchange next year, wheat/copper/oil next year, etc. They may prefer to forego the chance to make exceptional windfall profits when future uncertainty works to their advantage by protecting themselves against exceptional loss. This would serve to protect their economic base (trade in wheat/copper/oil, or manufacture of products using these as raw materials), and also enable them to focus their effort in their chosen area of trade or manufacture. For speculators, on the other hand, it is the market (forex, commodities or whatever) itself which is their main forum of economic activity.

Because the value of an option at expiry is so sensitive to price – it reflects movements in the price of the underlying in exaggerated form – the holding (or more generally, trading) of options and other derivatives presents greater opportunities for profit (and indeed, for loss) than trade in the underlying (this is why speculators buy options!). They are correspondingly more risky than the underlying. The relationship between risk and reward will be considered in more detail below when we discuss arbitrage.

One of the main insights of the fundamental work of Black and Scholes was that one can (at least in the most basic model) *hedge* against meeting a contingent claim by *replicating* it: constructing a portfolio, adjusted or re-balanced as time unfolds and new price information comes in, whose pay-off *is* the amount of the contingent claim.

6. Arbitrage.

Economic agents go to the market for various reasons. One the one hand, companies may wish to insure, or *hedge*, against adverse price movements that might affect their core business. On the other hand, *speculators* may be uninterested in the specific economic background, but only interested in making a profit from some financial transaction. The relation between hedging ('good') and speculation ('bad') is to some extent symbiotic (one cannot lay off a risk unless someone else is prepared to take it on, and why should he unless he expects to make money by doing so). Nevertheless, one feels that it should not be possible to extract money from the market without genuinely engaging in it, by taking *risk*: all business activity is risky. Indeed, were it possible to do so, people would do so – in unlimited quantities, thus sucking money parasitically out of the market, using it as a 'money-pump'. This would undermine the stability and viability of the market in the long run – and in particular, make it impossible for the market to be in *equilibrium*.
Aside: Wholesale v. retail. Wholesale merchants buy from manufacturers, and sell at a higher price to retailers, who sell to the public. The mark-up from wholesale to retail is large, and this suggests an arbitrage opportunity. However, one must bear in mind the overheads of the wholesaler, and the service he offers – bridging the gap between factory and high street, distributing etc. So there is a whole range of costs – of distribution, storage, deterioration of perishable gods in storage, etc. – that must be born in mind here.

The usual theoretical view of modelling markets as NA is not so much that arbitrage opportunities do not exist, but rather that if they do exist in any sizeable quantity, people will rush to exploit them, and by doing so will dissipate them – 'arbitrage them away'.

OED: "3 [Comm.]. The traffic in Bills of Exchange drawn on sundry places, and bought or sold in sight of the daily quotations of rates in the several markets. Also, the similar traffic in Stocks. 1881."

Used in this broad sense, the term covers financial activity of many kinds, including trade in options, futures and foreign exchange. However, the term arbitrage is nowadays also used in a narrower and more technical sense. Financial markets involve both riskless (bank account) and risky (stocks, etc.) assets. To the investor, the only point of exposing oneself to risk is the opportunity, or possibility, of realising a greater profit than the riskless procedure of putting all one's money in the bank (the mathematics of which – compound interest – does not require a degree or MSc course!). Generally speaking, the greater the risk, the greater the return required to make investment an attractive enough prospect to attract funds. Thus, for instance, a clearing bank lends to companies at higher rates than it pays to its account holders. The companies' trading activities involve risk; the bank tries to spread the risk over a range of different loans, and makes its money on the difference between high/risky and low/riskless interest rates.

It is usually better to work, not in face-value or nominal terms, but in *discounted* terms, allowing for the exponential growth-rate e^{rt} of risklessly invested money. So, profit and loss are generally reckoned against this discounted benchmark.

The above makes it clear that a market with arbitrage opportunities would be a disorderly market – too disorderly to model. The remarkable thing is the converse. It turns out that the minimal requirement of absence of arbitrage opportunities is enough to allow one to build a model of a financial market which – while admittedly idealised (frictionless market – no transaction costs, etc.) – is realistic enough both to provide real insight and to handle the mathematics necessary to price standard options (Black-Scholes theory). We shall see that arbitrage arguments suffice to determine prices – the *arbitrage pricing technique* (APT).

Short-selling. First, consider a riskless asset (bank account), with interest-rate $r > 0$. If our bank deposit is positive, we *lend* money and *earn* interest at rate r. If our bank deposit is negative (overdraft), we *borrow* money and *pay* interest. [We assume for simplicity that we pay interest also at rate r, though in practice of course it will be at some higher rate $r' > r$. Models taking these different interest-rates into account are topical at research level; we defer consideration of them till VI.7.]

In many markets, risky assets such as stocks may be treated in the same

way. We may have a positive or zero holding – or a *negative holding* (notionally borrowing stock, which we will be obliged to repay – or repay its current value). In particular, we may be allowed to *sell stock we do not own.* This is called *short-selling*, and is perfectly legal (subject to appropriate regulation) in many markets. Think of short-selling as borrowing. Not only is short-selling both routine and necessary in some contexts, such as foreign exchange and commodities futures, it simplifies the mathematics. So we assume, unless otherwise specified, no restriction on short-selling. By extension, we call a portfolio, or position, *short* in an asset if the holding of the asset is negative, *long* if the holding of the asset is positive.
Note. It turns out that in some important contexts – such as the Black-Scholes theory of European and American calls – short-selling can be avoided. In such cases, it is natural and sensible to do so: see Ch. VI.

7. Put-Call Parity.

Just as long and short positions are diametrical opposites, so are call and put options. We now show how an arbitrage argument demonstrates that these are linked.

Suppose there is a risky asset, value S (or S_t at time t), with European call and put options on it, value C, P (or C_t, P_t), with expiry time T and strike-price K. Consider a portfolio which is long one asset, long one put and short one call; write Π (or Π_t) for the value of this portfolio. So

$$\Pi = S + P - C \qquad \text{(S: long asset; P: long put; -C: short call).}$$

Recall that the payoffs at expiry are:

$$\begin{cases} \max(S-K,0) & \text{or} \quad (S-K)_+ \quad \text{for a call,} \\ \max(K-S,0) & \text{or} \quad (K-S)_+ \quad \text{for a put.} \end{cases}$$

So the value of the above portfolio at expiry is

$$\begin{cases} S + 0 - (S-K) = K & \text{if } S \geq K \\ S + (K-S) - 0 = K & \text{if } K \geq S, \end{cases}$$

namely K.

This portfolio thus guarantees a payoff K at time T. How much is it worth at time t?

The riskless way to guarantee a payoff K at time T is to deposit $Ke^{-r(T-t)}$

in the bank at time t and do nothing. If the portfolio is offered for sale at time t too cheaply – at a price $\Pi < Ke^{-r(T-t)}$ – I can *buy* it, *borrow* $Ke^{-r(T-t)}$ from the bank, and pocket a positive profit $Ke^{-r(T-t)} - \Pi > 0$. At time T my portfolio yields K (above), while my bank debt has grown to K. I clear my cash account – use the one to pay off the other – thus locking in my earlier profit, which is *riskless*. If on the other hand the portfolio is offered for sale at time t at too high a price – at price $\Pi > Ke^{-r(T-t)}$ – I can do the exact opposite. I *sell the portfolio short* – that is, I *buy its negative*, long one call, short one put, short one asset, for $-\Pi$, and *invest* $Ke^{-r(T-t)}$ in the bank, pocketing a positive profit $-(-\Pi) - Ke^{-r(T-t)} = \Pi - Ke^{-r(T-t)} > 0$. At time T, my bank deposit has grown to K, and I again clear my cash account – use this to meet my obligation K on the portfolio I sold short, again locking in my earlier riskless profit.

Thus the rational price for the portfolio at time t is *exactly* $Ke^{-r(T-t)}$. *Any other price* presents arbitrageurs with an arbitrage opportunity (to make and lock in a riskless profit) – which they will take! Thus
(i) The price (or value) of the portfolio at time t is $Ke^{-r(T-t)}$, that is,

$$S + P - C = Ke^{-r(T-t)}.$$

This link between the prices of the underlying asset S and call and put options on it is called *put-call parity.*
(ii) The value of the portfolio $S+P-C$ is the *discounted value of the riskless equivalent.* This is a first glimpse at the central principle, or insight, of the entire subject of option pricing.
(iii) Arbitrage arguments, although apparently qualitative, have quantitative conclusions, and allow one to calculate precisely the rational price – or *arbitrage price* – of a portfolio. The put-call parity argument above is the simplest example – though a typical one – of the *arbitrage pricing technique.*
(iv) The arbitrage pricing technique is due to S. A. Ross in 1976-78 (details in [BK], Preface).
Put-call parity has a long history (footnote from Wikipedia).[1]

[1] Forms of put-call parity appeared in practice as early as medieval ages, and was formally described by a number of authors in the early 20th century.

Michael Knoll, in The Ancient Roots of Modern Financial Innovation: The Early History of Regulatory Arbitrage, describes the important role that put-call parity played in developing the equity of redemption, the defining characteristic of a modern mortgage, in Medieval England.

Note. 1. History shows both that arbitrage opportunities exist (or are sought) in the real world and that the exploiting of them is a delicate matter. The collapse of Baring's Bank in 1995 (the UK's oldest bank, and bankers to HMQ) was triggered by unauthorised dealings by one individual, who aimed to exploit a fine margin between the Singapore and Osaka Stock Exchanges (and failed!).

The leadership of Baring's Bank at that time thought that the trader involved had discovered a clever way to exploit price movements in either direction between Singapore and Osaka. This is obviously impossible on theoretical grounds, to anyone who knows any Physics. See Problems 2 Q1 (key phrases: perpetual motion machine; Maxwell's demon; Second Law of Thermodynamics; entropy).
2. Major finance houses have an *arbitrage desk*, where their *arbs* work.

8. An Example: Single-Period Binary Model.

We consider the following simple example, taken from
[CRR] COX, J. C., ROSS, S. A. & RUBINSTEIN, M. (1979): Option pricing: a simplified approach. *J. Financial Economics* **7**, 229-263 (cf. [SKKM, I, §4.5, 36-37).
For definiteness, we use the language of foreign exchange. Our risky asset will be the current price in Swiss francs (SFR) of (say) 100 US \$, supposed

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In the 19th century, financier Russell Sage used put-call parity to create synthetic loans, which had higher interest rates than the usury laws of the time would have normally allowed.

Nelson, an option arbitrage trader in New York, published a book: "The A.B.C. of Options and Arbitrage" in 1904 that describes the put-call parity in detail. His book was re-discovered by Espen Gaarder Haug in the early 2000s and many references from Nelson's book are given in Haug's book "Derivatives Models on Models".

Henry Deutsch describes the put-call parity in 1910 in his book "Arbitrage in Bullion, Coins, Bills, Stocks, Shares and Options, 2nd Edition". London: Engham Wilson but in less detail than Nelson (1904).

Mathematics professor Vinzenz Bronzin also derives the put-call parity in 1908 and uses it as part of his arbitrage argument to develop a series of mathematical option models under a series of different distributions. The work of professor Bronzin was just recently rediscovered by professor Wolfgang Hafner and professor Heinz Zimmermann. The original work of Bronzin is a book written in German and is now translated and published in English in an edited work by Hafner and Zimmermann ("Vinzenz Bronzin's option pricing models", Springer Verlag).

Its first description in the modern academic literature appears to be (Stoll 1969).

$X_0 = 150$ at time 0. Consider a call option with strike price $c = 150$ at time T. The simplest case is the binary model, with two outcomes: suppose the price X_T of 100 \$ at time T is (in SFR)

$$X_T = \begin{cases} 180 & \text{with probability } p \\ 90 & \text{with probability } 1-p. \end{cases}$$

The payoff H of the option will be $30 = 180 - 150$ with probability p, 0 with probability $1-p$, so has expectation $EH = 30p$. This would seem to be the fair price for the option at $t = 0$, or allowing for an interest-rate r and discounting, we get the value

$$V_0 = E(\frac{H}{1+r}) = \frac{30p}{1+r}.$$

Take for simplicity $p = \frac{1}{2}$ and $r = 0$ (no interest): the naive, or expectation, value of the option at time 0 is

$$V_0 = 15.$$

The *Black-Scholes value* of the option, however, is different. To derive it, we follow the Black-Scholes prescription (Ch. IV, VI):
(i) First replace p by p^* so that the price, properly discounted, *behaves like a fair game*:

$$X_0 = E^*(\frac{X_T}{1+r}).$$

That is,

$$150 = \frac{1}{1+r}(p^*.180 + (1-p^*).90);$$

for $r = 0$ this gives $60 = 90p^*$ or $p^* = 2/3$.
(ii) Now compute the fair price of the expected value in this new model:

$$V_0 = E^*(\frac{H}{1+r}) = \frac{30p^*}{1+r};$$

for $r = 0$ this gives the Black-Scholes value as $V_0 = 20$.
Justification: it works! – as the arbitrage constructed below shows. For simplicity, take $r = 0$.
We *sell* the option at time 0, for a price $\pi(H)$, say. We then prepare for the resulting contingent claim on us at time T by the option holder by using the

following strategy:

Sell the option for $\pi(H)$	$+\pi(H)$
Buy \$33.33 at the present exchange rate of 1.50	-50
Borrow SFR 30	$+30$
Balance	$\pi(H) - 20.$

So our balance at time 0 is $\pi(H) - 20$. At time T, two cases are possible:
(i) The dollar has risen:

Option is exercised (against us)	-30
Sell dollars at 1.80	$+60$
Repay loan	-30
Balance	0.

(ii) The dollar has fallen:

Option is worthless	0.00
Sell dollars at 0.90	$+30$
Repay loan	-30
Balance	0.

So the balance at time T is zero in both cases. The balance $\pi(H) - 20$ at time 0 should thus also be zero, giving the Black-Scholes price $\pi(H) = 20$ as above. For, *any other price* gives an arbitrage opportunity. Argue as in put-call parity in §4: if the option is offered too cheaply, buy it; if it is offered too dearly, write it (the equivalent for options to 'sell it short' for stock). Thus any other price would offer an arbitrageur the opportunity to extract a riskless profit, by appropriately buying and selling (Swiss francs, US dollars and options) so as to exploit your mis-pricing.

The same argument with interest-rate r also applies: divide everything through by $1 + r$.

Note. This argument, and result, are **independent** of p, the 'real' probability, and depend instead **only** on this 'fictitious' new probability, p^* (which is called the *risk-neutral* or *risk-adjusted* probability.

The example above is highly instructive. First, it clearly represents the simplest possible non-trivial case: only two time-points (with one time-period between them, hence the 'single-period' of the title), and only two possible outcomes (hence the 'binary' of the title). Secondly, it shows that there is a theory hidden here, which gives us a definite prescription to follow (and some surprises, such as not involving the 'real' probability p above). This prescription is simple to implement, and can be justified by explicitly constructing an arbitrage to exploit doing anything else [if the option is offered for sale too cheaply, buy it, if too dearly, write it]. This theory is the Black-Scholes theory, which we consider in detail in Chapters IV and VI. The technical key to the Black-Scholes prescription is the introduction of p^* and its associated expectation operator E^*. In technical language, this is the *equivalent martingale measure*. Now each of these three terms needs full introduction. We shall talk about measures in II.1 below, about equivalent measures in II.4, and martingales in III.3 and V.2. We stress: the Black-Scholes theory – that is, rational option pricing – cannot be done without all these concepts. This is why we need Chapter II on the necessary background on measure theory, and Chapters III and V on the necessary background on stochastic processes.

There are basically three options open to those teaching, and learning, how to price options etc.

1. One can avoid measure theory altogether (cf. [CR]). This is technically possible rigorously in the discrete-time setting of Ch. III – though at greater length, because the key concepts cannot be addressed explicitly. It is also possible non-rigorously in continuous time; cf. [WHD], who base their approach on partial differential equations (PDE).
2. One can learn measure theory first – say, from the excellent book [W]. This, however, puts the subject beyond the reach of most people who need it and use it in practice – and beyond reach of most of this audience.
3. One can do as we shall do (and as [BK], [E], [BR] do): state what we need from measure theory, and use its language, concepts, viewpoint and results, without proving anything. This makes good sense: the constructions and proofs of measure theory are quite hard (say, final year undergraduate or first-year postgraduate level for good mathematics students with a bent for analysis – quite a select group!). Using measure theory taking its results for granted, however, is quite easy, as we shall see.

9. Complements

1. Types of risk. Institutions encounter risks of various types. These include:
Market risk. This is the risk that one's current market position (the aggregate of risky assets one holds) goes down in value (things one is long on get cheaper, and/or things one is short on get dearer).
Credit risk. This is the risk that counter-parties to one's financial transactions may default on their obligations.

When this happens, debts cannot be (or are not) paid in full. Usually, payment is made in part, by negotiation between the parties (it may be cheaper to agree a partial repayment than to force the other party into bankruptcy), or by the administrators or liquidators in the case of companies. This raises issues of *moral hazard*, below.
Operational risk. This is risk arising from the internal procedures of an institution: failure of computer systems for implementing transactions (the failure of the Taurus clearing system on the London Stock Exchange was one example); fraudulent or unauthorised trading made possible by inadequate supervision; etc.
Liquidity risk. This is the risk that one will be unable to implement a planned or agreed transaction because of lack of cash-in-hand to trade with, and/or willingness to trade. The Credit Crunch of 2007/8 on was caused by banks realising they had piles of toxic debt on their hands (see below), and so did not know what their balance sheets were worth; that other banks were similarly placed; hence that banks no longer trusted themselves or each other, and so refused to lend to each other. So the financial system froze up; so the real economy froze up.
Model risk. To handle real-world phenomena of any complexity, one needs to model them mathematically. To quote Box's Dictum: All models are wrong; some models are useful.[2] Use of an inappropriate model to set the prices at which one buys and sells exposes the institution to open-ended losses, to competitors with better models.
2. Risk management. The problems of 2007/8 on have made the importance of risk management obvious. For an excellent book-length treatment, see e.g. [MFE] A. J. McNEIL, R. FREY & P. EMBRECHTS: *Quantitative risk management: Concepts, techniques, tools*. Princeton UP, 2005.

We know from Markowitz that we should have a balanced portfolio, with lots of negative correlation. The danger is *large* losses. These are quantified by the *tails* of the distributions – the joint distribution of our portfolio. The point of diversifying is so that what we lose on the swings we gain on the roundabouts. Two comments:
(a) Whether this works for large losses depends on the tail properties of the joint distribution. It does *not* work if this is normal – as it is in the benchmark Black-Scholes model.
(b) When the whole market is falling – as in a financial crisis – none of the

[2] George E. Box, 1919-. British statistician

risk-management techniques useful under normal market conditions work.
3. Moral hazard. Before the limited liability company, if one defaulted, one was liable to the *whole* of the loss incurred by one's counter-party. This made trading very dangerous (the early traders were called merchant adventurers) – all the more as insurance had not developed by then.[3]

Limited liability was what made ordinary people willing to undertake the risks of trading, and so paved the way for the development of modern business, commerce, capitalism etc.

The moral hazard here is the possibility of gambling with other people's money. If it works, fine. If not, walk away (writing off one's limited liability) and leave them to bear the loss.

Bankruptcy law varies from country to country, and is too complicated to pursue here. But one sees moral hazard where it concerns us in, e.g.:
(a) start-ups of hedge funds (or, dot-com companies);
(b) aggressive traders – who (for the sake of their bonuses) gamble with their careers – but with other people's money;
(c) credit rating – where the credit rating agencies had a financial incentive to pass as AAA some highly questionable financial asset, etc.
4. Securitization. This term covers the drive in recent years to seek out new financial markets by identifying risks that people might want to cover themselves against, and creating new financial derivatives that can be sold to address this perceived need. These derivatives too could be traded, etc. The upshot was an explosion in the volume of trade in increasingly artificial financial products, developed by the R&D departments of the financial institutions (investment banks, US, or merchant banks, UK). It became obvious in 2007/8 that the leaders of these institutions did not understand these products – could not price them, and could not value their holdings of them (above).

One specific trigger of the US crash in 2007 was the explosive growth in sub-prime mortgages.[4] These were granted to people who would not have qualified as financially sound enough to get a mortgage previously, but who wanted to buy their own house. This new and profitable market proved ir-

[3]Lloyds of London predates limited liability. The Lloyds participants – "names" – had unlimited liability. Many were driven into personal bankruptcy in the Lloyds scandals of the 90s. See Google for the ghastly details.

[4]Sub-prime mortgages are to the mortgage market what junk bonds – bonds allowing companies of sub-prime credit-worthiness to raise money – are to bond markets. Again, see Google for the ghastly details.

resistible to US banks – leading to a great house-price bubble, which burst (as bubbles do) in 2007. The knock-on effects hit the UK in 2008 (Northern Rock, etc.). The real damage of this failure of the financial sector has been its devastating and ongoing consequences on the real economy.

5. Macro-prudential issues. As the above illustrates, financial matters are too important to be left to financiers. Proper regulation is vital. These are extremely important (and difficult, and topical) issues; anything important enough becomes political (Couve de Murville); politics is not an exact science (Bismarck). Mathematics is an exact science. So there are limits to which mathematics – including this course, which contains a lot of mathematics – can help. Your job as citizens, as well as students of this course, is to get and keep well-informed enough about these "big-picture" issues to be able to form a view.

Turning from the big picture to specifics:

6. Forwards and futures. Forwards are agreements between buyer and seller made now, but concerning delivery in the future. They are not traded. Futures are options on things that will come to market in the future (next year's grain crop, for example), and these are traded (extensively). There are good accounts in Hull's books, [H1], [H2].

7. OTC and exchange-traded contracts. OTC – "over-the-counter" – denotes a transaction made between an individual buyer and an individual seller. As markets develop, options on standard transactions develop, and these are assets themselves that can be traded in exchanges (e.g., the CBOE, which opened in 1973: I.3).

8. Marking to market. This is a system whereby the exchanges cover themselves and their clients against the risk of large losses. If one party to a trade is, on current market prices, exposed to a potentially heavy loss, a *margin call* will be required by the exchange. It is the need to provide such margin calls that actually triggers many financial failures (but limits the losses of the counter-parties).

9. Forex. Forex is an abbreviation for foreign exchange. International trade involves more than one currency' currencies move against each other. There is a vast market in derivatives to cover the risks involved.

10. Swaps. From Hull [H2] Ch. 5: "Swaps are private agreements betweentwo companies to exchange cash flows in the future ... The first swap contracts were negotiated in 1981. Since then the market has grown very rapidly. ..." There are even options on swaps – *swaptions* – etc.