M4A22/M5A22 MASTERY QUESTION SOLUTION 2014

Ornstein-Uhlenbeck (OU) process. The OU SDE $dV = -\beta V dt + \sigma dW$ (OU) models the velocity of a diffusing particle. The $-\beta V dt$ term is *frictional drag*; the σdW term is *noise*. **[2]**

(ii) $e^{-\beta t}$ solves the corresponding homogeneous DE $dV = -\beta V dt$. So by variation of parameters, take a trial solution $V = Ce^{-\beta t}$. Then

$$dV = -\beta C e^{-\beta t} dt + e^{-\beta t} dC = -\beta V dt + e^{-\beta t} dC,$$

so V is a solution of (OU) if $e^{-\beta t} dC = \sigma dW$, $dC = \sigma e^{\beta t} dW$, $C = c + \sigma \int_0^t e^{\beta u} dW$. So with initial velocity v_0, $V = e^{-\beta t} C$ is

$$V = v_0 e^{-\beta t} + \sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u. \qquad [4]$$

(iii) V comes from W, Gaussian, by linear operations, so is Gaussian.
V_t has mean $v_0 e^{-\beta t}$, as $E[e^{\beta u} dW_u] = \int_0^t e^{\beta u} E[dW_u] = 0$.
By the Itô isometry, V_t has variance

$$E[(\sigma e^{-\beta t} \int_0^t e^{\beta u} dW_u)^2] = \sigma^2 e^{-2\beta t} \int_0^t (e^{\beta u})^2 du$$
$$= \sigma^2 e^{-2\beta t} [e^{2\beta t} - 1]/(2\beta) = \sigma^2 [1 - e^{-2\beta t}]/(2\beta).$$

So the limit distribution as $t \to \infty$ is $N(0, \sigma^2/(2\beta))$. **[4]**

(iv) For $u \geq 0$, the covariance is $cov(V_t, V_{t+u})$, which is

$$\sigma^2 E[e^{-\beta t} \int_0^t e^{\beta v} dW_v . e^{-\beta(t+u)} (\int_0^t + \int_t^{t+u}) e^{\beta w} dW_w].$$

By independence of Brownian increments, $\int_t^{t+u}$ contributes 0, so by above

$$cov(V_t, V_{t+u}) = e^{-\beta u} var(V_t) = \sigma^2 e^{-\beta u} [1 - e^{-2\beta t}]/(2\beta) \to \sigma^2 e^{-\beta u}/(2\beta) \quad (t \to \infty).$$

[4]

(v) V is Markov (a diffusion), being the solution of the SDE (OU). **[3]**

(vi) The process shows *mean reversion* – a strong push towards the central value. This is characteristic of interest rates (under normal conditions). The financial relevance is to the *Vasicek model* of interest-rate theory. **[3]**

Seen, lectures.

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