

SOLUTIONS 7a. 7.11.2018

Q1. *Gamma distributions and Renewal.*

(i) *Density.*

$$\int f = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-\lambda x} \cdot \lambda^\alpha x^{\alpha-1} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-u} \cdot u^{\alpha-1} du = 1,$$

putting $u := \lambda x$ and using the definition of the Gamma function.

(ii) *Mean.*

$$\begin{aligned} \mu &= \int x f(x) dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty x \cdot e^{-\lambda x} \cdot \lambda^\alpha x^{\alpha-1} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty (u/\lambda) \cdot e^{-u} \cdot u^{\alpha-1} du = \frac{1}{\lambda \Gamma(\alpha)} \int_0^\infty e^{-u} \cdot u^\alpha du \\ &= \frac{\Gamma(\alpha+1)}{\lambda \Gamma(\alpha)} \\ &= \alpha/\lambda. \end{aligned}$$

(iii) *LST: $\hat{f}(s)$.*

$$\begin{aligned} \hat{f}(s) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-sx} \cdot e^{-\lambda x} \cdot \lambda^\alpha x^{\alpha-1} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda+s)x} \cdot \lambda^\alpha x^{\alpha-1} dx \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-u} \left(\frac{\lambda}{\lambda+s} \right)^\alpha u^{\alpha-1} du \quad (u := (\lambda+s)x) \\ &= \left(\frac{\lambda}{\lambda+s} \right)^\alpha. \end{aligned}$$

(iv) *LST: $\hat{U}(s)$.*

$$\begin{aligned} \hat{U}(s) &= \frac{1}{1 - \hat{f}(s)} = \frac{1}{1 - \left(\frac{\lambda}{\lambda+s} \right)^\alpha} \\ &= \frac{(\lambda+s)^\alpha}{(\lambda+s)^\alpha - \lambda^\alpha} = \frac{(\lambda+s)^\alpha}{\lambda^\alpha \left[\left(\left(1 + \frac{s}{\lambda} \right)^\alpha - 1 \right) \right]}. \end{aligned}$$

As $s \downarrow 0$,

$$(\lambda+s)^\alpha / \lambda^\alpha \rightarrow 1, \quad \left[\left(\left(1 + \frac{s}{\lambda} \right)^\alpha - 1 \right) \right] \sim \sigma \cdot s / \lambda = \mu s,$$

by the (generalised) Binomial Theorem (Newton: see M3H). So

$$\hat{U}(s) \sim \frac{1}{\mu s} \quad (s \downarrow 0).$$

So by HLK with $\rho = 1$,

$$U(x) \sim x/\mu \quad (x \rightarrow \infty),$$

giving the Renewal Theorem in this case.

NHB