pfsprob7.tex

PROBLEMS 7 20.11.2015

Q1 (*Hypergeometric distribution*). Recall that $\binom{n}{k}$ is the number of subsets of size k of a st of size n, so

$$\sum_k \binom{n}{k} = 2^n$$

decomposes the total number 2^n of subsets of a set of size n by size k. Recall also that $\binom{n}{k}$ counts the number of (downward) paths from the vertex (the '1' at the top) to the entry $\binom{n}{k}$ in Pascal's triangle. Show that

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n},$$

(i) by decomposing the number of subsets of size n of a set of $2n$ balls, n white and n black, according to how many white balls they contain;
(ii) by equating coefficients of x^n in the identity $(1+x)^{2n} \equiv (1+x).(1+x)^n$;
(iii) by counting routes from the vertex to the central entry $\binom{2n}{n}$ in row $2n$, according to where they cross row n.

Q2 (*Bernoulli-Laplace urn model*). By Q1 or otherwise, show that the Bernoulli-Laplace urn is
(i) reversible/has detailed balance;
(ii) has invariant distribution the *hypergeometric distribution*

$$\pi_i = \binom{d}{i}^2 / \binom{2d}{d}$$

(this is a probability distribution, by Q1).

Q3 (*Bernoulli-Laplace urn* continued). Show that the mean recurrence time of state 0 is

$$\mu_0 = \binom{2d}{d} \sim 4^d/\sqrt{\pi d} \qquad (d \to \infty).$$

Interpret this in the context of Statistical Mechanics, where d is of the order of Avogadro's number, 6.02×10^{23}.

Q4 *Branching processes.* In a population model, one starts with a single ancestor (the 0th generation). On death, he is replaced by a random number Z of offspring, with PGF $P(s)$ (the 1st generation), and mean $\mu = E[Z]$. They reproduce independently and in the same way, their offspring forming the second generation, with PGF P_2, and so on. Show that:
(i) $P_2(s) = P(P(s))$, the second (functional) iterate of P;
(ii) the nth generation, of size Z_n say, has PGF P_n, the nth functional iterate of P (defined inductively by $P_n = P_{n-1}(P) = P(P_{n-1})$);
(iii) the mean generation size is $E[Z_n] = \mu^n$.

NHB