ull6a.tex **Week 6: am, 2.11.2016**

6. Real options (Investment options)

The options considered above concern financial *derivatives* (so called because they derive from the underlying fundamentals such as stock). We turn now to options of another kind, concerned with business decision-making. Typically, we shall be concerned with the decision of whether or not to make a particular investment, and if so, when. Because these options concern the real economy (of manufacturing, etc.) rather than financial markets such as the stock market, such options are often called *real options*. But because they typically concern investment decisions, they are also often called *investment options*. There is a good introductory treatment in [D&P].

The key features are as follows. We are contemplating making some major investment – buying or building a factory, drilling an oil well, etc. While a speculator may consider buying a firm which he thinks is undervalued, or whose assets he thinks are under-utilized, breaking it up, and selling off the parts at a profit ('asset stripping'), we confine attention here to a more constructive and conventional situation – the management of a firm is considering some major investment to further their core economic activity. While if the decision goes wrong it may be possible to recoup some of the cost, much or most of it will usually be irrecoverable (a *sunk cost* – as with an oil well). So the investment is it irreversible – at least in part. Just as stock prices are uncertain – so we model them as random, using some stochastic process – here too, the future profitability of the proposed investment is *uncertain*. Finally, we do not have to act now, or at any specified time. So we have an open-ended – or infinite – time-horizon, in contrast to the options considered earlier where the time-horizon is the expiry time T of the option.

We may choose to delay investment,
(a) to gather more information, to help us assess the project, or
(b) to continue to generate interest on the capital we propose to invest.
So we must recognize, and feed into the decision process, the value of *waiting for further information*. When we commit ourselves and make the decision to invest, it is not just the sunk cost that we lose – we lose the valuable option to wait for new information.

This situation is very reminiscent of the American options of IV.9 with an infinite time-horizon. With such an American *call*, we have the right to buy at a specified price at a time of our choosing (or indeed, not to buy). There, we carried out a full analysis. We formulated an optimal stopping problem, and solved it as a free boundary problem, using the principle of smooth fit. We can apply the same method here (as is done in detail in [D&P], Ch. 5).

We suppose the cost of the investment is I. We suppose that the value of the project is given by a geometric Brownian motion, $X = (X_t) \sim GBM(\mu, \sigma)$ (the value of a project is uncertain for the same reasons that stock prices are uncertain; we model them both as stochastic processes; GBM is the default option here, just as in the Black-Scholes theory of Ch. IV). If τ is the investment time we choose, we want to maximize

$$V(X) := \max_{\tau} \; E[(X_\tau - I)e^{-r\tau}],$$

with r the riskless rate (discount rate) as before. Now if $\mu \leq 0$ the value of the project will fall, so we should invest immediately if $X_0 > I$ and not invest if not. If $\mu > r$, the growth of X will swamp the investment cost I and more than offset the discounting, so we should invest and there is no point in waiting. So we take $\mu \in (0, r]$. The analogues here of (i)-(v) above are

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - rV, \tag{i}$$

$$V(0) = 0, \tag{ii}$$

$$V(x^*) = x^* - I, \tag{iii}$$

$$V'(x^*) = 1 \qquad \text{(smooth pasting)}. \tag{iv}$$

(for (ii), the GBM does not hit 0, but if it approaches 0, so will the value of the project, so (ii) follows from this by continuity). For (iii), this is the value-matching condition: on investment, the firm receives the net pay-off $X^* - I$. As before, we use a trial solution $V(x) = Cx^p$. Substituting in (i), this is a solution if p satisfies the quadratic

$$Q(p) := \frac{1}{2}\sigma^2 p(p-1) + \mu p - r = 0.$$

The product of the roots is negative, and $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$. So one root $p_1 > 1$ and the other $p_2 < 0$. The general solution is $V(x) = C_1 x^{p_1} + C_2 x^{p_2}$, but from $V(0) = 0$ we get $C_2 = 0$, so $V(x) = C_1 x^{p_1}$, or $V(x) = Cx^{p_1}$. If x^* is the critical value at which it is optimal to invest,

$$V(x^*) = x^* - I,$$

and 'smooth pasting' gives

$$V'(x^*) = 1.$$

From these two equations, we can find C and x^*. The second is

$$V'(x^*) = Cp_1(x^*)^{p_1-1} = 1, \qquad C = (x^*)^{1-p_1}/p_1.$$

Then the first gives

$$C(x^*)^{p_1} = x^* - I, \qquad x^*/p_1 = x^* - I, \qquad x^* = \frac{p_1}{(p_1-1)}I.$$

The main feature here is the factor

$$q := p_1/(p_1 - 1) > 1$$

by which the value must exceed the investment cost I before investment should be made (q is used because this is related to "Tobin's q" in Economics). One can check that q increases with σ (the riskier the project, the more reluctant we are to invest), and also q increases with r (as then investing our capital risklessly becomes more attractive). Then the critical threshold above which it is optimal to invest is

$$x^* = qI.$$

Also

$$C = (qI)^{1-p_1}/p_1, \qquad V(x) = (qI)^{1-p_1}x^{p_1}/p_1.$$

The results above show that the traditional *net present value* (NPV – accountancy-based) approach to valuing real options is misleading – see [DP].

§7. Further results

1. Exotic options.

The options considered so far (put/call, European/American) are so standard now as to be commonly called *vanilla options.* More complicated types of option are called *exotic options.* We turn to some of the commoner types below.

Asian options.

Here the payoff is a function of the *average price* of the underlying between contract time and expiry time. Asian options are widely used in practice - for instance, for oil and foreign currencies. The averaging complicates the mathematics, but, e.g., protects the holder against speculative attempts to manipulate the asset price near expiry. For details and references, see e.g. [RS] ROGERS, L. C. G. & SHI, Z. (1995): The value of an Asian option. *J. Applied Probability* **32**, 1077-1088.

Asian options are *pathwise* options, as their payoff depends on the whole sample path of the price process, not just the terminal value at expiry.

Lookback options.

It is natural to look back with the benefit of hindsight, and wish that we had acted optimally throughout – bought at the low, sold at the high, etc. How nice it would be to have a piece of paper that entitled us to the benefits that would have resulted if we had ... Such things exist, and are called *lookback options*. These are again exotics, and – like the Asian options above – are pathwise options. Their theory involves the powerful *reflection principle* (below) for Brownian motion (loosely: if we reflect a Brownian path in a mirror, we get another Brownian path).

Barrier options.

Another common type of pathwise exotic is that of *barrier options*, where whether or not an option ends in the money depends on whether or not some price level is crossed up to expiry. They may 'knock in' or 'knock out' (become in or out of the money), and the barrier may be crossed from above or below. So there are four types: 'up and in', 'up and out', 'down and in', 'down and out'. Again, their theory involves the reflection principle.

Sometimes there are *two* barriers, one above and one below. One can use the reflection principle at each barrier. As one might expect (from sitting in a barber's shop, where one has mirrors in front and behind, and sees an infinite sequence of reflections), this involves infinite summations.

The Reflection Principle.

There is a brief account of this in [BK] 6.3.3, in connection with barrier options. As mentioned there, the method goes back to Kelvin's *method of images* in electrostatics in 1848, although it is often known by the name of Désiré André (1887).

The idea of reflecting a Brownian path in a mirror can be formalised, by using the Markov property to re-start the process from the time when it hits the mirror (at level $b > 0$, say). But the time, τ_b say, that BM takes to reach level b is random. So one actually needs the *strong Markov property* – the Markov property applied at a stopping time. This can be justified: BM does have the strong Markov property (and so, more generally, do Lévy processes).

Another approach is to work discretely, and use simple random walk as a discrete alternative (or approximation) to BM. Here things are simpler and more elementary: one can reduce the problem to *combinatorics* (counting paths). See e.g. [GS] §3.10. Incidentally, this book (though not the 1st or 2nd editions) contains a brief account of Black-Scholes theory (§13.10).
American options

With finite time-horizon T, the expiry as above, the results are basically the same as in discrete time.

With infinite time-horizon, see above.

2. What time-horizon?

There is an old saying (which I love: I was brought up in a farming village): live as if you would die tomorrow, farm as if you would live forever.

For us as individuals, conscious of our own mortality: there is a sense in which death cancels all debts and releases all obligations. But, we continue, in at least three ways:
(i) in our genes, for those of us who have children;
(ii) in the memories of those we leave behind who knew us;
(iii) in our life's work (visible in tangible form for writers, scientists, scholars etc., even for cinema actors, but more ephemeral for stage actors, say).

The clear evidence of global warming, as a result of mankind's abuse or over-exploitation of the environment (denied only by the wilfully blinkered, or by modern-day Flat Earthists), has led to environmental issues becoming ever more prominent in our lives, public and private, and in our politics. Anyone with any social conscience wants to do their best by the environment, not through self-interest but for posterity.

The great economist J. M. Keynes (Ch. I) was once asked what would happen in the long run, and famously replied "In the long run, we are all dead". This was not meant to be flippant, but to focus attention on what time-horizon we are to judge our actions and intentions by.

One of the great problems of contemporary politics is that a politician is necessarily constrained by a time-frame centred on the next election, or perhaps also the one after that. Rhetoric is cheap, but taking effective action to preserve the environment is expensive, and likely to be punished rather than rewarded as the polls. Many people are too poor, hungry or insecure to worry much about posterity and the environment, and take a short-term view centred on themselves and their families.

Sustainability is clearly crucial here, and clearly needs international, indeed global, cooperation. Finding effective ways to do this is one of the challenges of our time. It is hard enough to do this with a scientifically literate population. How hard this is may be judged by the frighteningly large percentage of the US population (including prominent politicians) who don't even believe in Darwinian evolution, let alone global warming.

There is lots of work to be done here, in areas such as *cost-benefit analysis* – informed by a sensible choice of *time-horizon.*

3. Discontinuities in stock price.

The Black-Scholes model relies on stock-price movements being continuous. If stock prices jump – for instance, in response to abrupt events such as outbreaks of war/devaluations/natural disasters such as major earthquakes/the oil-price crisis of 1973, etc. – the Black-Scholes analysis fails. In particular, the market will no longer be complete, and it will no longer be possible to hedge against a contingent claim by replicating it. Because of incompleteness, there will be many equivalent martingale measures, so many prices (which fill an interval – familiar as the *bid-ask spread*). One should seek the optimal measure, which minimises risk or maximises payoff (minimal equivalent martingale measures – this involves the Föllmer-Schweizer decomposition).

One model for price discontinuities is a *Poisson* model, in which 'shocks' occur, but prices move in a Black-Scholes way between shocks.

Recent work by Barndorff-Nielsen and Shephard, by Eberlein, by Bingham & Kiesel and others, has focussed on *Lévy* models (stationary independent increments - generalising Brownian motion to include jumps). This is sensible because:
(i) when prices are examined in sufficient detail, they are in fact seen to be discontinuous, the jumps resulting from the individual transactions by which the assets are traded;
(ii) the extra flexibility provided by the larger class of Lévy models gives more scope for model fitting to observed data. The subclass of *hyperbolic* models seems particularly well-suited here.

There is a whole field of such *Lévy finance.* For background and details, see e.g. [BK] §5.5.

We mentioned the *jitter* in stock prices: these jump, when looked at closely enough. Also, big trades move prices, and so do economic shocks. The Black-Scholes model based on BM, which is continuous, cannot handle this. More general processes (Lévy processes – stationary independent increments) are needed here. But these model *incomplete* markets – so prices are no longer unique (one has a bid-ask spread). We stress: *real markets are incomplete*. Real prices jump. The completeness of the Black-Scholes model, and the Brownian Martingale Representation Theorem, reflects the *continuity* of BM.

4. Varying or random interest rates.

We have assumed that the interest rate r is a positive constant. It is more realistic, though more complicated, to let $r = r_t$ vary with time. More generally still, r may be random, i.e. $r = (r_t) = (r_t(\omega))$ may be a stochastic process.

A number of possible models for such interest-rate processes r have been proposed and studied. Interest-rate theory is a subject in its own right, and we must leave it here (but see the Postscript below).

5. Transaction costs.

Real markets suffer from friction: there are actual costs in trading and making transactions, which complicate the theory. One classic is by M. H. A. Davis and A. R. Norman (1990).

As we have seen, there are strong arguments for *introducing* transaction costs to reduce the enormous volume of speculative trading, and so its potentially destabilising effect on the world economy.

6. Higher interest rates for borrowing than lending.

Real financial markets have higher interest rates for borrowing than for lending (which is how banking works), and this introduces another kind of friction into the market. For further detail, see e.g.
[CK] CVITANOVIC, J. & KARATZAS, I. (1993): Hedging contingent claims with constrained portfolios, *Ann. Appl. Prob.* **3**, 652-681, §9.

7. Stochastic volatility (SV).

The Black-Scholes theory above – in discrete or continuous time – has involved the volatility – the parameter that describes the sensitivity of the stock price to new information, to the market's assessment of new information. Volatility is so important that it has been subjected to intensive scrutiny, in the light of much real market data. Alas, such detailed scrutiny reveals that volatility is not really constant at all – the Black-Scholes theory over-simplifies reality. (This is hardly surprising: real financial markets are more complicated than the contents of this course, as they involve *investor psychology*, rather than straight mathematics!) One way out is to admit that volatility is *random* (stochastic), and then try to model the stochastic process generating it. Volatility exhibits *clustering*, linked to *mean reversion*, so *Ornstein-Uhlenbeck* models are useful here. Such *stochastic volatility models* are topical today.

8. Stochastic Volatility (SV) continued; ARCH and GARCH

There are a number of *stylised facts* in mathematical finance. E.g.:

(i). Financial data show *skewness*. This is a result of the asymmetry between profit and loss (large losses are lethal!; large profits are just nice to have).

(ii). Financial data have much *fatter tails* than the normal (Gaussian). We have discussed this in I.5.

(iii) Financial data show *volatility clustering*. This is a result of the economic and financial environment, which is extremely complex, and which moves between good times/booms/upswings and bad times/slumps/downswings. Typically, the market 'gets stuck', staying in its current state for longer than is objectively justified, and then over-correcting. As investors are highly sensitive to losses (see (i) above), downturns cause widespread nervousness, which is reflected in higher volatility. The upshot is that good times are associated with periods of growth but low volatility; downturns spark extended periods of high volatility (as well as stagnation, or shrinkage, of the economy).

$ARCH$ and $GARCH$. We turn to models that can incorporate such features.

The model equations are (with Z_t ind. $N(0,1)$)

$$X_t = \sigma_t Z_t, \qquad \sigma_t^2 = \alpha_0 + \sum_1^p \alpha_i X_{i-1}^2, \qquad (ARCH(p))$$

while in $GARCH(p,q)$ the σ_t^2 term becomes

$$\sigma_t^2 = \alpha_0 + \sum_1^p \alpha_i X_{i-1}^2 + \sum_1^q \beta_j \sigma_{t-j}^2. \qquad (GARCH(p,q))$$

The names stand for (generalised) autoregressive conditionally heteroscedastic (= variable variance). These are widely used in Econometrics, to model *volatility clustering* – the common tendency for periods of high volatility, or variability, to cluster together in time. They were introduced in 1987 by

Robert Engle (1942) and C. W. J. (Sir Clive) Granger (1934-2009), who received the Nobel Prize for this in 2003. From Granger's obituary (*The Times*, 1.6.2009): "Following Granger's arrival at UCSD in La Jolla, he began the work with his colleague Robert F. Engle for which he is most famous, and for which they received the Bank of Sweden Nobel Memorial Prize in Economic Sciences in 2003. They developed in 1987 the concept of cointegration. Cointegrated series are series that tend to move together, and commonly occur in economics. Engle and Granger gave the example of the price of tomatoes in N. and S. Carolina .... Cointegration may be used to reduce non-stationary situations to stationary ones, which are much easier to handle statistically and so to make predictions for. This is a matter of great economic importance, as most macroeconomic time series are non-stationary, so temporary disturbances in, say, GDP may have a long-lasting effect, and so a permanent economic cost. The Engle-Granger approach helps to separate out short-term effects, which are random and unpredictable, from long-term effects, which reflect the underlying economics. This is invaluable for macroeconomic policy formulation, on matters such as interest rates, exchange rates, and the relationship between incomes and consumption."

9. Volatility Modelling

In the standard Black-Scholes theory we have developed, volatility σ is *constant*. Thus a graph of volatility against strike K (or stock price S) should be flat. But typically it isn't, and displays curvature. Such volatility curves often turn upwards at both ends ('volatility smile'); there may well be asymmetry ('volatility smirk').

As above, it may be useful to model volatility stochastically, and use an SV model. However, the driving noise in this model will have a volatility of its own ('vol of vol'), etc. Practitioners often use *computer graphics* to represent *volatility surfaces* – the three-dimensional equivalents of graphs, where e.g. σ is graphed against K and S. The subject is too big to pursue further here; there is a good account (mixing theory with practice) in

J. GATHERAL: *The volatility surface: A practitioner's guide*. Wiley 2006.

Continuous time: SV and Econometrics.

Stochastic volatility can be studied in *continuous time*, where one can use calculus. The difference equations above are then replaced by differential equations. There is a whole subject of *continuous-time econometrics*, pioneered by Rex Bergstrom (1925-2006):

A. R. BERGSTROM, *Continuous-time econometric modelling*, OUP, 1990.

10. Volatility Index (VIX).

Just as there are indices of stocks (FTSE, S&P, DAX etc.), there is also a volatility index (VIX). Just as options on the Footsie etc. can be traded, so too can options on VIX. It may amuse you to know that VIX has already entered popular fiction (the novel by Robert HARRIS: *The fear index*. Hutchinson, 2011).

11. State-space models.

We cannot measure volatility directly; we can measure option prices, which depend on volatility. This is an example of a *state-space model*. The prototype here is the *Kalman filter* (1960, from control engineering), which can be used in discrete or continuous time.

12. Portfolios and Multivariate Time Series.

By Markowitzian diversification, we should carry a portfolio of risky assets. Its evolution over time involves two areas of Statistics, Time Series and Multivariate Analysis. Note for now that the more Statistics you can learn here, the better.

13. Discrete v. continuous models in mathematical finance: comment on the theorems required.

In Ch. IV we have studied discrete models, obtaining (discrete versions of) the No-Arbitrage and Completeness Theorems and the FTAP, and proving everything. In Ch. VI we study continuous models, obtaining continuous versions of these results and proving as much as possible – but not everything. One mathematical difference between IV and VI is in Measure Theory (which we could do without in IV, but not in VI). Another is seen in the Separating Hyperplane Theorem (SHT), used to prove the NA Theorem in IV.2. In IV, spaces are finite-dimensional, and the SHT is just Euclidean geometry. (We have a cone, with axis L say. The required hyperplane is that through the origin perpendicular to L.) In VI, spaces are infinite-dimensional, and now the SHT needs the Hahn-Banach Theorem, the cornerstone of Functional Analysis. In turn, the Hahn-Banach Theorem needs the Axiom of Choice (AC), or some variant of it such as Zorn's Lemma.

Note carefully the results needing SHT or AC. These tend to be the hard ones, and the crucial ones.

14. High-frequency trading and order books.

For an interesting account of how ultra-fast fibre optic cable networks enable (ultra-)high frequency traders to "game the system", and for some constructive suggestions on market design response, see e.g.
Eric Budish, Peter Cramton, and John Shim, *The High-Frequency Trading Arms Race: Frequent Batch Auctions as a Market Design Response.*

15. Partial Differential Equations (PDEs). The most important PDEs encountered in Mathematics or Physics (or Finance!) are *linear* PDEs of *second order* (involving partial derivatives of first or second order only). These may be classified, in a way analogous to the classification of *conics* or conic sections (whose equations are algebraic of second order), into three broad categories:
Elliptic PDEs – prototype, *Laplace's equation* (*Poisson's*, with $4\pi\rho$ on RHS)

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$$

in electromagnetism or potential theory;
Parabolic PDEs – prototype, the *heat equation*

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = \kappa^{-1}\partial u/\partial t;$$

Hyperbolic PDEs – prototype, the *wave equation*

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = c^{-2}\partial u/\partial t^2.$$

The Black-Scholes PDE is parabolic, and can be transformed into the heat equation, whose solution can be written down in terms of an integral and the *heat kernel.* This is the same as the probabilistic solution obtained above.
Note. 1. Black and Scholes were classically trained applied mathematicians. When they derived their PDE, they recognised it as parabolic. After some months' work, they were able to transform it into the heat equation. The solution to this is known classically (see Week 5b). Transforming back, they obtained the Black-Scholes formula – which transformed the financial world.

Postscript.

1. One recent book on Financial Mathematics describes the subject as being composed of three strands:
arbitrage – the core economic concept, which we have used throughout;
martingales – the key probabilistic concept (Ch. III on);
numerics. Finance houses in the City use *models*, which they need to *calibrate to data* – a task involving both statistical and numerical skills, and in particular an ability to *programme*.

2. You will probably already have experience with at least one general mathematics package (e.g., Matlab and/or Python) (if not: get it, a.s.a.p.!). You may also know some Numerical Analysis, the theory behind computation. You may have encountered *simulation*, also known as *Monte Carlo*, and/or a branch of Probability and Statistics called *Markov Chain Monte Carlo (MCMC)* – computer-intensive methods for numerical solutions to problems too complicated to solve analytically. The leaders of R & D teams in the City need to be expert at both stochastic modelling (e.g., to propose new products), and simulation (to evaluate how these perform). Most of the ones I know use Matlab for this. At a lower level, quantitative analysts (quants) working under them need expertise in a computer language; C++ is the industry standard. If you are thinking of a career in Mathematical Finance, learn C++, as soon as possible, and for academic credit.

3. This course deals with *equity markets* – with *stocks*, and financial derivatives of them – options on stocks, etc. The relevant mathematics is *finite-dimensional*. Lurking in the background are *bond markets* ('money markets': bonds, gilts etc., where *interest rates* dominate), and the relevant options – *interest-rate derivatives*, and *foreign exchange* between different currencies ('forex'). The resulting mathematics (which is highly topical, and so in great demand in the City!) is *infinite-dimensional*, and so harder than the equity-market theory we have done. However, the underlying principles are basically the same. One has to learn to walk before one learns to run, and this course serves as a preparation for the Interest Rate Theory course next term.

4. The aim of this lecture course is simple. It is to familiarize the student with the basics of Black-Scholes theory, as the core of modern finance, and with the mathematics necessary to understand this. The motivation driving the ever-increasing study of this material is the financial services industry and the City. I hope that any of you who seek City careers will find this introduction to the subject useful in later life. NHB, 2016