ma414prob2.tex

MA414 PROBLEMS 2. 19.1.2012

Q1. *Central moments.* Write $\mu_k := E[X^k]$, $\mu_{k,0} := E[(X - EX)^k]$ for the kth moment and central moment of a random variable X. Define the sample central moments, and show that they converge to the (population) central moments (as the sample size n increases).

Q2. *Cumulants.* The *moment-generating function* (MGF) and *cumulant-generating function* (CGF) are

$$M(t) := E[e^{tX}] = \sum_0^\infty \mu_k t^k/k!, \qquad K(t) := \log\, M(t) = \sum_1^\infty \kappa_k t^k/k! = \kappa_1 t + \ldots$$

Their centred versions have X replaced by $X - EX$, μ_k, κ_k by $\mu_{k,0}$, $\kappa_{k,0}$. Show that (i) $\kappa_1 = \mu$, $\kappa_k = \kappa_{k,0}$ for $k = 2, 3, \ldots$, (ii) $\kappa_2 = \sigma^2$, (iii) $\kappa_3 = \mu_{3,0}$, (iv) $\kappa_{4,0} = \mu_{4,0} - 3\sigma^4$. Show also that X is normal iff all cumulants above the second vanish.

Q3. (i) Show that the standard normal distribution $N(0, 1)$ has CF $e^{-t^2/2}$.
(ii) Deduce that the general normal distribution $N(\mu, \sigma)$ has CF
$\exp\{i\mu t - \sigma^2 t^2/2\}$.

Q4. (i) Show that the symmetric exponential distribution SE with density $f(x) := e^{-|x|}/2$ has CF $\phi(t) = 1/(1 + t^2)$. (One can do this by Real Analysis – integrate by parts twice.)
(ii) Show that the Cauchy distribution with density

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

has CF

$$\phi(t) = e^{-|t|}.$$

(This uses Complex Analysis, and Jordan's Lemma.)

Q5. Comment on the similarity between density and CF in Q3, and between Q4 (i) and (ii).

NHB