

# **SOLUTIONS 2a. 4.10.2017**

Q1. (i) In spherical polar coordinates  $(r, \theta, \phi)$  ( $r$ : distance from centre, range 0 to  $\infty$ ;  $\theta$ : colatitude ( $= \frac{1}{2}\pi$  - latitude), range 0 to  $\pi$ ;  $\phi$  longitude, range 0 to  $2\pi$ ): increase  $r$  to  $r + dr$ , etc. The element of volume  $dV$  is a (to first order) cuboid, of sides  $dr$  ("up"),  $r d\theta$  ("South"),  $r \sin \theta d\phi$  ("East") (draw a diagram – or consult a textbook if you need one!) So

$$dV = dr \cdot r d\theta \cdot r \sin \theta d\phi = r^2 \sin \theta dr d\theta d\phi.$$

So

$$V = \int_0^r r^2 dr \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta = \frac{1}{3} r^3 \cdot 2\pi [-\cos \theta]_0^\pi = \frac{2\pi}{3} r^3 [ -(-1) - (-1) ] = 4\pi r^3 / 3.$$

(ii) Holding  $r$  fixed,

$$dS = r^2 \sin \theta \cdot d\theta d\phi.$$

So

$$A = r^2 \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta = r^2 \cdot 2\pi \cdot 2 = 4\pi r^2,$$

by above.

(iii) To first order,

$$dV = S dr : \quad S = dV/dr, \quad V = \int_0^r S dr$$

('flattening out' the spherical shell: volume = area  $\times$  thickness: the curvature effects are second-order). So (i), (ii) are equivalent: ((ii) follows from (i) by differentiating, and (i) from (ii) by integrating.

Q2. This follows by the same method as the area of an ellipse  $A = \pi ab$ : wlog  $a \geq b \geq c$ . Compress [squash] the  $x$ - and  $y$ -axes in the ratios  $a/c$ ,  $b/c$ , to get a sphere of radius  $c$ . This has volume  $4\pi c^3/3$ . Now dilate [unsquash] the  $x$ - and  $y$ -axes in the ratios  $a/c$ ,  $b/c$ , to get volume

$$V = \frac{4\pi c^3}{3} \cdot \frac{a}{c} \cdot \frac{b}{c} = \frac{4\pi abc}{3}.$$

Q3. (i) Choose the vertex  $V$  as origin, and the  $z$ -axis vertical – the perpendicular from  $V$  to the horizontal base (with  $z$  going downwards, if we draw the tetrahedron the usual way). Slice the volume into thin horizontal slices. The area of the slice between  $z$  and  $z + dz$  is  $A(z/h)^2$ , by similarity. So

$$V = \int_0^h A(z/h)^2 dz = Ah^{-2} \int_0^h z^2 dz = Ah^{-2} \cdot h^3/3 : \\ V = Ah/3.$$

(ii) Similarly in the general case: the above does not use that the base is triangular.

Q4. (i) The range between  $x$  and  $x + dx$  generates volume  $dV = \pi y^2 dx = \pi f(x)^2 dx$ . Integrate this from  $a$  to  $b$ .

(ii) The semicircle on base  $[-r, r]$  is  $y = f(x) = \sqrt{r^2 - x^2}$ . This generates the sphere on revolution, giving

$$V = \int_{-r}^r \pi(r^2 - x^2) dx = \pi[r^2 x - \frac{1}{3} x^3]_{-r}^r = \pi r^3 [1 - \frac{1}{3} - (-1) + (-\frac{1}{3})] \\ = \pi r^3 (2 - \frac{2}{3}) = 4\pi r^3/3.$$

Q5 (Georges BOULIGAND, 1935). *First Proof.* For the region  $S_1$  with area  $A_1$  with base the hypotenuse, side 1: use cartesian coordinates to approximate its area, arbitrarily closely, by decomposing it into small squares of area  $dA_1 = dx dy$ .

For each such small square on side 1, construct similar small squares on sides 2 and 3, of areas  $dA_2, dA_3$ .

By Pythagoras' theorem,  $dA_1 = dA_2 + dA_3$ .

Summing, we get  $A_1 = A_2 + A_3$  arbitrarily closely, and so exactly.

*Second Proof.* Drop a perpendicular from the right-angled vertex to the hypotenuse. This splits the 'big figure' into two 'smaller figures', each similar to it. With  $l_1$  the length of the hypotenuse and  $l_2, l_3$  those of the other two sides, by similarity lengths scale by  $l_2/l_1, l_3/l_1$  on going from the big figure to the smaller ones, so areas scale by  $(l_2/l_1)^2, (l_3/l_1)^2$ . So

$$A_2 + A_3 = A_1[(l_2/l_1)^2 + (l_3/l_1)^2] = A_1(l_2^2 + l_3^2)/l_1^2 = A_1,$$

by Pythagoras' theorem. //

NHB