MA414 STOCHASTIC ANALYSIS: EXAMINATION SOLUTIONS, 2011

Q1.(i) *First Borel-Cantelli Lemma*). $A = \limsup A_n = \cap_n \cup_{m=n}^{\infty} A_m$, so $A \subset \cup_{m=n}^{\infty} A_m$ for each n. So $P(A) \le P(\cup_{m=n}^{\infty} A_m) \le \sum_{m=n}^{\infty} P(A_m) \to 0$ $(n \to \infty)$ (tail of a convergent series): $P(A) = 0$. **[5]**

(ii) (*Second Borel-Cantelli lemma).* If A_n are events, $A := \limsup A_n = \{A_n \ i.o.\}$: if $\sum P(A_n) = \infty$ and the A_n are independent, then $P(A) = 1$.

Proof. By the De Morgan laws, $A^c = \cup_n \cap_{m=n}^{\infty} A_m^c$. But for each n

$$\begin{aligned} P(\cap_{m=n}^{\infty} A_m^c) &= \lim_N P(\cap_{m=n}^{N} A_m^c) \qquad (\sigma\text{-additivity}) \\ &= \prod_{m=n}^{N} (1 - P(A_m)) \qquad \text{(independence)} \\ &\le \prod_{m=n}^{N} \exp\{-P(A_m)\} \qquad (1 - x \le e^{-x} \text{ for } x \ge 0) \\ &= \exp\{-\sum_{m=n}^{N} P(A_m)\} \to 0 \qquad (N \to \infty), \end{aligned}$$

as $\sum P(A_n)$ diverges. So $\cap_{m=n}^{\infty} A_m^c$ is null, so $A^c = \cup_n \cap_{m=n}^{\infty} A_m^c$ is null. // **[8]**

(iii) *Second Borel-Cantelli Lemma for Pairwise Independence.* For A_n pairwise independent, if $\sum P(A_n)$ diverges then $P(\limsup A_n) = P(A_n \ i.o.) = 1$.

Proof. Put $S_n := \sum_1^n I(A_i)$, $S := \sum_1^{\infty} I(A_i)$, $m_n := E[S_n] = \sum_1^n P(A_i)$.
$var(S_n) = E[(S_n - m_n)^2] = E[(\sum_{i=1}^n (I(A_i) - EI(A_i)))(\sum_{j=1}^n (I(A_j) - EI(A_j)))] = E[\sum_i \sum_j (\ldots)(\ldots)] = \sum_i E[(\ldots)^2] + \sum_{i \ne j} E(\ldots)(\ldots)] = \sum_i E[(\ldots)^2]$ (the sum over $i \ne j$ is 0, as there by pairwise independence and the Multiplication Theorem $E[(\ldots)(\ldots)] = E[(\ldots)]E[(\ldots)] = 0.0 = 0$ – variance of sum = sum of variances under pairwise independence). As $I(A_i)$ is Bernoulli with parameter $P(A_i)$, its variance is $P(A_i)[1 - P(A_i)] \le P(A_i)$. So $E[(S_n - m_n)^2] \le \sum_1^n P(A_i) = m_n$, which increases to $+\infty$ as $\sum P(A_n)$ diverges. But

$$\begin{aligned} P(S \le m_n/2) &\le P(S_n \le m_n/2) \qquad (S_n \le S) = P(S_n - m_n \le -m_n/2) \\ &\le P(|S_n - m_n| \ge m_n/2) \le \frac{4}{m_n^2} var(S_n) \qquad \text{(by Tchebycheff's Inequality)} \\ &\le 4/m_n \qquad \text{(by above)} \to 0 \qquad (n \to \infty). \end{aligned}$$

But the LHS increases to $P(S < \infty)$, by continuity (= σ-additivity) of $P(.)$. So $P(S < \infty) = 0$: $P(\sum I(A_n) < \infty) = 0$, i.e. $P(\sum I(A_n) = \infty) = 1$. This says that $P(A_n \ i.o.) = 1$: $P(\limsup A_n) = 1$. // **[12]**

(Standard bookwork: (i), (ii) done in lectures, (iii) done on a problem sheet.)

Q2. Take the Lebesgue probability space $([0,1], \lambda, \mathcal{L})$ modelling the uniform distribution $U[0,1]$ on the unit interval (probability = length). For a random variable $X \sim U[0,1]$, take its dyadic expansion $X = \sum_1^\infty \epsilon_n/2^n$. Thus $\epsilon_1 = 0$ iff $X \in [0,1/2)$, 1 iff $X \in [1/2,1)$ (or $[1/2,1]$: we can omit 1, as it carries 0 probability). If $\epsilon_1, \ldots, \epsilon_{n-1}$ are already defined, on the dyadic intervals $[k/2^{n-1}, (k+1)/2^{n-1})$, and independent fair coin-tosses (Bernoulli $B(\frac{1}{2})$), split each interval into two halves: $\epsilon_n = 0$ on the left half, 1 on the right half. Then ϵ_n is again $B(\frac{1}{2})$, and is independent of $\epsilon_1, \ldots, \epsilon_{n-1}$. By induction, ϵ_n $(n = 1, 2, \ldots)$ are independent $B(\frac{1}{2})$. Conversely, given ϵ_n independent coin tosses, form $X := \sum_1^\infty \epsilon_n/2^n$. Then $X_n := \sum_1^n \epsilon_k/2^k \to X$ a.s. The distribution function F_n of X_n has jumps $1/2^n$ at $k/2^n$, $k = 0, 1, \ldots, 2^n - 1$. This 'saw-tooth jump function' converges to x on $[0,1]$, the distribution function of $U[0,1]$ ($\sup|F_n(x) - x| = 2^{-n} \to 0$ as $n \to \infty$). So $X \sim U[0,1]$. So if $X = \sum_1^\infty \epsilon_n/2^n$, $X \sim U[0,1]$ iff ϵ_n are independent coin tosses – the Lebesgue probability space models *both* (a) length on the unit interval *and* (b) infinitely many independent coin tosses.

(i) From the given $U[0,1]$, we get by dyadic expansion as above a sequence of independent coin-tosses ϵ_n. Rearrange these into a two-suffix array ϵ_{jk} (as with Cantor's proof of 1873 that the rationals are countable). The ϵ_{jk} are all independent, so the $X_j := \sum \epsilon_{jk}/2^k$ are independent, and $U[0,1]$ by above. So from *one* $U(0,1)$, we get in this way *infinitely many copies.*

(ii) If F is a distribution function (right-continuous; increasing from 0 at $-\infty$ to 1 at ∞), define its (left-continuous) inverse function by $F^{-1}(t) := \inf\{F(x) \geq t\}$ for $0 < t < 1$. Then if $U \sim U[0,1]$, $X := F^{-1}(U) \sim F$. For, $\{X \leq x\} = \{F^{-1}(U) \leq x\} = \{U \leq F(x)\}$, which has probability $F(x)$ as U is uniform. By this *probability integral transformation* we can pass from generating copies from the uniform distribution (say by Monte Carlo simulation) to generating copies from the distribution F, in particular, standard normals. Hence by (i) above we can then generate infinitely many independent standard normals.

(iii) We can hence simulate a Brownian motion $B = (B_t)$ from $B_t = \sum_0^\infty \lambda_n Z_n \Delta_n(t)$, with Z_n independent standard normals, $\Delta_n(t)$ the Schauder functions and λ_n suitable normalising constants.

(iv) Similarly, using (ii) rather than (i), we may simulate infinitely many independent Brownian motions.

(Largely standard book work – all covered, in lectures or problem sheets.)

Q3. A distribution is *infinitely divisible* (id) iff, for each $n = 1, 2, \ldots$, it is the n-fold convolution of a probability distribution – equivalently, if its CF is the nth power of the CF of a probability distribution.

The *Lévy-Khintchine formula* states that a probability distribution is id iff its CF has the form $\exp\{-\Psi(u)\}$, where

$$\Psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int (e^{iux} - 1 - iuxI(|x| < 1)\mu(dx),$$

(a is real, $\sigma \geq 0$ and the Lévy measure μ satisfies $\int \min(1, |x|^2)\mu(dx) < \infty$).
(i) $\phi(t) = \int_{-\infty}^{\infty} e^{itx}/(\pi(1 + x^2))dx$. Take γ the semicircle in the upper half-plane on base $[-R, R]$, $t > 0$, and consider $f(z) := e^{itz}/(\pi(1+z^2))$. The only singularity inside γ is at $y = i$, a simple pole.

$$Res_i \frac{e^{itz}}{\pi(z-1)(z+1)} = \frac{e^{-t}}{\pi.2i} = \frac{-ie^{-t}}{2\pi}.$$

By Cauchy's Residue Theorem:

$$\int_\gamma f = 2\pi i.\left(\frac{-ie^{-t}}{2\pi}\right) = e^{-t}.$$

But

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \to \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{\pi(1+x^2)} + 0 \quad \text{(Jordan's Lemma)}.$$

This gives the result for $t > 0$. For $t = 0$, it is an arctan (or $\tan^{-1}$) integral. For $t < 0$: replace t by $-t$. //

Thus the CF of the symmetric Cauchy density $1/(\pi(1 + x^2))$ is $e^{-|t|}$.
(ii) This is id, as $e^{-|t|} = [e^{-|t|/n}]^n$ for each n, and each [.] is a CF.
(iii) Substituting $\mu(dx) = 1/(\pi|x|^2)dx$ above gives $\Psi(u)$ as the sum of two integrals, I_1 over $(-1, 1)$ and I_2 over its complement. In I_1, the $\pm iux$ terms over $(-1, 0)$ and $(0, 1)$ cancel; we can then combine I_1 and I_2 to get

$$\Psi(u) = \frac{2}{\pi}\int_0^{\infty} (\cos\ ux - 1)dx/x^2.$$

This gives $\Psi'(u) = -(2/\pi)\int_0^\infty \sin ux\, dx/x = -(2/\pi)\int_0^\infty \sin\ t\, dt/t = -(2/\pi).\pi/2 = -1$. So $\Psi(u) = -u$ for $u > 0$. So $\Psi(u) = -|u|$. //

For X_i independent Cauchy, $(X_1 + \ldots + X_n)/n$ has CF $[e^{-|t|/n}]^n = e^{-|t|}$, the CF of X_1. So $(X_1 + \ldots + X_n)/n =_d X_1$. This does not contradict the SLLN: it does not apply, as the mean of X_i is undefined.
((i), (ii): standard bookwork, covered in lectures; (iii): unseen (but obvious) example; last part: similar seen.)

Q4. (i) For $t \neq 0$, X is Gaussian with zero mean (as B is), and continuous (again, as B is). The covariance of B is $\min(s,t)$. The covariance of X is

$$\begin{aligned} cov(X_s, X_t) &= cov(sB(1/s), tB(1/t)) \\ &= E[sB(1/s).tB(1/t)] \\ &= st.E[B(1/s)B(1/t)] \\ &= st.cov(B(1/s), B(1/t)) \\ &= st.\min(1/s, 1/t) = \min(t,s) = \min(s,t). \end{aligned}$$

This is the same covariance as Brownian motion. So, away from the origin, X *is* Brownian motion, as a Gaussian process is uniquely characterized by its mean and covariance (from the properties of the multivariate normal distribution). So X is continuous. So we can define it at the origin by continuity. So X is Brownian motion everywhere – X is BM.
(ii) Since Brownian motion is 0 at the origin, $X(0) = 0$. Since Brownian motion is continuous at the origin, $X(t) \to 0$ as $t \to 0$. This says that

$$tB(1/t) \to 0 \qquad (t \to 0),$$

which is

$$B(t)/t \to 0 \qquad (t \to \infty),$$

as required.

By construction, Brownian motion is given by its expansion

$$B_t = \sum_{n=0}^{\infty} \lambda_n Z_n \Delta_n(t),$$

where the Z_n are independent standard normal random variables, the $\Delta_n(t)$ are the Schauder functions and the λ_n are normalising constants. Now $\Delta_n(0) = 0$ for $n \geq 1$, while $\Delta_0(t) = t$, so $\Delta_0(1) = 1$. Also $\lambda_0 = 1$. Putting $t = 1$, $B_1 = Z_0$. So Brownian bridge is

$$B_0(t) := B(t) - tB(1) = B(t) - tZ_0 :$$

the expansion of Brownian bridge in the Schauder functions is

$$B_0(t) = \sum_{n=1}^{\infty} \lambda_n Z_n \Delta_n(t).$$

(Standard bookwork: covered in lectures or problem sheets.)

Q5. A function ϕ is *convex* if

$$\phi(\lambda x + (1-\lambda)y) \leq \lambda\phi(x) + (1-\lambda)\phi(y) \qquad \forall \lambda \in [0,1], x, y.$$

Jensen's inequality states that

$$\phi(E[X]) \leq E[\phi(X)]$$

for convex ϕ and random variables X with X, $\phi(X)$ both integrable. The *conditional Jensen inequality* states that for $\mathcal{C}$ a σ-field, ϕ, X as above,

$$\phi(E[X|\mathcal{C}]) \leq E[\phi(X)|\mathcal{C}].$$

(i) For $s < t$, $M_s = E[M_t|\mathcal{F}_s]$ as M is a martingale. So by the conditional Jensen inequality,

$$\phi(M_s) = \phi(E[M_t|\mathcal{F}_s]) \leq E[\phi(M_t)|\mathcal{F}_s],$$

which says that $\phi(M)$ is a submartingale.
(ii) If M is a submartingale, $M_s \leq E[M_t|\mathcal{F}_s]$. As ϕ is non-decreasing on the range of M,

$$\begin{aligned}\phi(M_s) &\leq \phi(E[M_t|\mathcal{F}_s]),\\ &\leq E[\phi(M_t)|\mathcal{F}_s]\end{aligned}$$

by the conditional Jensen inequality again, and again $\phi(M)$ is a submartingale.
(iii) As Brownian motion B is a martingale (lectures), and x^2 is convex (its second derivative is $1 \geq 0$), B^2 is a submartingale by (i).
(iv) As $B_t^2 - t$ is a martingale (which you may quote here as it is not asked – but is easy to prove, as in lectures)

$$B_t^2 = [B_t^2 - t] + t \qquad (\text{submg} = \text{mg} + \text{incr})$$

is the Doob-Meyer decomposition of B_t^2. The increasing process here is t, which is thus the quadratic variation of Brownian motion B.
(Standard bookwork – covered in lectures or problem sheets.)

Q6. (i) The *Itô isometry* states that for $f \in H^2 := H^2(0,T)$, the class of measurable f with $\{f : E[\int_0^T f^2(\omega,t)dt] < \infty\}$,

$$E[(\int_0^t f^2(\omega,u)dB_u)^2] = E[\int_0^t f^2(\omega,t)dt]. \qquad \textbf{[2]}$$

(ii) *Conditional Itô isometry.* For $0 \le s \le t \le T$,

$$E[(\int_s^t f^2(\omega,u)dB_u)^2|\mathcal{F}_s] = E[\int_s^t f^2(\omega,t)dt|\mathcal{F}_s].$$

Proof. It suffices to show that for all $A \in \mathcal{F}_s$,

$$E[I(A)(\int_s^t f^2(\omega,u)dB_u)^2] = E[I(A)\int_s^t f^2(\omega,t)dt].$$

This follows from the unconditional Itô isometry, applied to the integrand $g(\omega,u) := fI_A(\omega)I_{(s,t]}(u)$. // **[6]**
(iii) For $s \le t$,

$$E[M_t|\mathcal{F}_s] = E[\{(\int_0^s + \int_s^t)f_u dB_u\}^2|\mathcal{F}_s] - \int_0^s f_u^2 du - E[\int_s^t f_u^2 du|\mathcal{F}_s]$$

$$= E[(\int_0^s f_u dB_u)^2] + 2(\int_0^s b_u dB_u)E[\int_s^t b_u dB_u|\mathcal{F}_s] + E[(\int_s^t f_u dB_u)^2|\mathcal{F}_s] - \int_0^s f_u^2 du - E[\int_s^t f_u^2 du|\mathcal{F}_s].$$

The first and fourth terms give M_s. The third and fifth terms cancel, by the conditional Itô isometry (ii). The second factor in the second term involves an Itô integral, which (for an integral $f \in H^2$) is a martingale, so has constant expectation, which is 0 on taking $t = s$, so the second term is 0. Combining, the RHS is M_s, which proves that M is a martingale. **[13]**
(iv) Taking $f \equiv 1$ gives $M_t := B_t^2 - t$ is a martingale. **[4]**
((i), (ii): standard book work, covered in lectures; (iii), (iv): similar problems seen.)

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