M2PM3 EXAMINATION SOLUTIONS 2011

Q1. (i) Write c, s for $\cos\theta$, $\sin\theta$. By de Moivre's theorem, $\cos\ n\theta + i \sin\ n\theta = e^{in\theta} = (e^{i\theta})^n = (c+is)^n$. Taking real parts gives

$$\cos n\theta = c^n - \binom{n}{2} c^{n-2} s^2 + \binom{n}{4} c^{n-4} s^4 \ldots = c^n - \binom{n}{2} c^{n-2}(1-c^2) + \binom{n}{4} c^{n-4}(1-c^2)^2 \ldots = T_n(c),$$

so $\cos\ n\theta$ is a polynomial T_n in $c = \cos\theta$.
(ii) The leading coefficient in T_n is

$$1 + \binom{n}{2} + \binom{n}{4} + \ldots = \sum_{k\ even} \binom{n}{k}.$$

By the Binomial Theorem,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Write $\sum_e := \sum_{k\ even} \binom{n}{k}$, $\sum_o := \sum_{k\ odd} \binom{n}{k}$. Taking $a = b = 1$ and $a = 1, b = -1$ in the Binomial Theorem,

$$2^n = \sum_e + \sum_o, \qquad 0 = \sum_e - \sum_o.$$

Add and halve: $2^{n-1} = \sum_e$: T_n has leading coefficient 2^{n-1}.
(iii) As $\cos\ 2\theta = 2\cos^2\theta - 1$ ($T_2(x) = 2x^2 - 1$),

$$T_2(\cos(\theta/2)) = 2\cos^2(\theta/2) - 1 = \cos\ \theta : \qquad \cos(\theta/2) = \sqrt{\frac{1+\cos\ \theta}{2}}. \qquad (*)$$

Taking $\theta = \pi/2$ in $(*)$: as $\cos(\pi/2) = 0$, (a) $\cos(\pi/4) = 1/\sqrt{2}$; (b) $\cos(\pi/4)$ is a zero of the polynomial $P_1 := T_2$ of degree 2 with integer coefficients: $P_1(\cos(\pi/4)) = T_2(\cos(\pi/4)) = 0$. Taking $\theta = \pi/4$ in $(*)$: (a) $\cos(\pi/8) = \sqrt{(1+1/\sqrt{2})/2}$; (b) $\cos(\pi/8)$ is a zero of the polynomial $P_2 := P_1(T_2) = T_2(T_2)$ of degree 4: $P_2(\cos(\pi/8)) = T_2(T_2(\cos(\pi/8))) = T_2(\cos(\pi/4)) = 0$.

Continuing in this way, we write $P_n := P_{n-1}(T_2) = P_{n-2}(T_2(T_2)) = \ldots$, the nth functional iterate of T_2. Then

$$\begin{aligned}
P_n(\cos(\pi/2^{n+1})) &= P_{n-1}(T_2(\cos(\pi/2^{n+1}))) \quad \text{(definition of } P_n\text{)} \\
&= P_{n-1}(\cos(\pi/2^n)) \quad \text{(by } (*) \text{ with } \theta = \pi/2^n\text{)} \\
&= P_{n-2}(\cos(\pi/2^{n-1})) \quad \text{(by above with } n-1 \text{ for } n\text{)} \\
&= \ldots = P_1(\cos(\pi/4)) = 0. \qquad (**)
\end{aligned}$$

The polynomial P_n, being the n-fold functional iterate of the quadratic T_2 (where $T_2(x) = 2x^2 - 1$), has integer coefficients and degree 2^n.

By n successive applications of $(*)$, $\cos(\pi/2^{n+1})$ is obtained from the integers 1 and 2, divisions and n iterated square roots.

By $(**)$, $\cos(\pi/2^{n+1})$ is a zero of P_n, and so is an algebraic number of the required type.
((i) and (ii): seen; (iii) unseen.)

Q2. (i) *Cantor's theorem for nested compact sets.* If K_n is a sequence of nested (decreasing – $K_{n+1} \subset K_n$) compact sets in the complex plane, their intersection is non-empty.

(ii) *Theorem (Cauchy's Theorem for Triangles).* If f is holomorphic in a domain D containing a triangle γ and its interior $I(\gamma)$ – then $\int_\gamma f = 0$.

Proof. Join the three midpoints of the sides of the triangle γ. This *quadrisects* γ into 4 *similar* triangles. Call these Γ_1 to Γ_4: then $\int_\gamma f = \sum_1^4 \int_{\Gamma_i} f$. For, there are 12 terms, 3 for each of the 4 triangles. The 'outer 6' add to $\int_\gamma f$; the 'inner 6' cancel in pairs. So, for at least one i, $\left|\int_{\Gamma_i} f\right| \geq \frac{1}{4}|I|$, where $I = \int_\gamma f$. For if not, each $\left|\int_{\Gamma_i} f\right| < \frac{1}{4}|I|$, so $|I| = \left|\int_\Gamma f\right| = \left|\sum_1^4 \int_{\Gamma_i} f\right| \leq \sum_1^4 |\int_{\Gamma_i} f| < \sum_1^4 \frac{1}{4}|I| = |I|$, a contradiction. Call this Γ_i γ_1. So: $\left|\int_{\gamma_1} f\right| \geq \frac{1}{4}|I|$.

Now quadrisect γ_1. Repeating the argument above, at least one of the 4 resulting triangles, γ_2 say, has $\left|\int_{\gamma_2} f\right| \geq \frac{1}{4}\left|\int_{\gamma_1} f\right| \geq \frac{1}{4^2}|I|$. Continue (or use induction): we obtain a sequence of triangles $\gamma_1, \gamma_2, ..., \gamma_n, ...$ s.t. if Δ denotes the union of γ and its interior $I(\gamma)$ and similarly for Δ_n, γ_n, $\Delta_{n+1} \subset \Delta_n \subset ... \subset \Delta_2 \subset \Delta_1 \subset \Delta$; lengths: $L(\gamma_n) = 2^{-n}L$ ($L = L(\gamma)$, length of γ); and $4^{-n}|I| \leq \left|\int_{\gamma_n} f\right|$. (i)

The sets Δ_n are decreasing, closed and bounded (so *compact*), and non-empty. So by *Cantor's Theorem*, $\bigcap_{n=1}^\infty \triangle_n \neq \emptyset$. Take $z_0 \in \cap_{n=1}^\infty \triangle_n$. As Δ is compact, $z_0 \in \Delta$. Since *by assumption* f is holomorphic in D, f is holomorphic at z_0. So $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall z$ with $0 < |z - z_0| < \delta$, $\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$:

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| < \epsilon|z - z_0|. \qquad (*)$$

As $diam(\gamma_n) = 2^{-n} \downarrow 0$ as $n \to \infty$, Δ_n tends to $\{z_0\}$ as $n \to \infty$. So $\Delta_N \subset N(z_0, \delta)$ for all large enough n. For this n, and all $z \in \Delta_n$, $|z - z_0| \leq L(\gamma_n) = 2^{-n}L$ (Triangle Lemma, Problems 4). Now $f(z_0) + f'(z_0)(z - z_0) = F'(z)$, with $F(z) = f(z_0)(z - z_0) + \frac{1}{2}f'(z_0)(z - z_0)^2$.

$$\begin{aligned}
\int_{\gamma_n}[f(z_0) + f'(z_0)(z - z_0)]\,dz &= \int_{\gamma_n} F'(z)\,dz \\
&= \int_a^b F'(z(t))\dot{z}(t)\,dt \quad (\text{if } \gamma_n \text{ is parametrised by } [a,b]) \\
&= [F(z(t))]_{t=a}^b \quad (\text{Fundamental Th. of Calculus}) \\
&= F(z(b)) - F(z(a)) \\
&= F(z(a)) - F(z(a)) = 0 \quad (z(b) = z(a) \text{ as } \gamma_n \text{ is closed}).
\end{aligned}$$

This and $(*)$ give $\left|\int_{\gamma_n} f\right| < \epsilon \int_{\gamma_n} |z - z_0|\,dz$. But $\int_{\gamma_n} |z - z_0|\,dz \leq \max_{\gamma_n} |z - z_0| \cdot L(\gamma_n)$ (ML) $\leq L(\gamma_n) \cdot L(\gamma_n)$ (Triangle Lemma) $\leq 4^{-n}L^2$ ($L(\gamma_n) \leq L \cdot 2^{-n}$). So

$$|\int_{\gamma_n} f| < \epsilon.4^{-n}L^2. \qquad (ii)$$

By (i) and (ii): $4^{-n}|I| \leq \left|\int_{\gamma_n} f\right| \leq \epsilon \cdot 4^{-n} \cdot L^2 : \quad 4^{-n}|I| \leq \epsilon 4^{-n}L^2 : \quad |I| \leq \epsilon \cdot L^2$. But $\epsilon > 0$ is arbitrarily small. So $|I| = 0$: $I = 0$: $\int_\gamma f = 0$. //

(Covered in full in lectures.)

Q3 (*Euler's reflection formula for the Gamma function*).
(i) For $0 < x < 1$, the integrals for both $\Gamma(x)$ and $\Gamma(1-x)$ converge, and $\Gamma(x)\Gamma(1-x) = \int_0^\infty t^{x-1}e^{-t}dt.\int_0^\infty u^{-x}e^{-u}du$.
Substituting $u = tv$, the second integral on the right is $t^{1-x}\int_0^\infty v^{-x}e^{-tv}dv$. Cancelling powers of t and changing the order of integration, the RHS becomes

$$\int_0^\infty v^{-x}dv.\int_0^\infty e^{-(1+v)t}dt \quad = \int_0^\infty v^{-x}.\frac{1}{1+v}dv.\int_0^\infty e^{-w}dw \quad (w := (1+v)t).$$

The w-integral is 1. Interchanging x and $1-x$ (which preserves the LHS, and so the RHS also) gives

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty \frac{v^{x-1}}{1+v}dv.$$

(ii) *Cut* the plane along the positive real axis. Take $f(z) := z^{a-1}/((1+z)$; this is now holomorphic, except for a simple pole at $z = -1 = e^{i\pi}$, with residue (by the cover-up rule) $e^{i\pi(a-1)}$. Take γ the keyhole contour consisting of:
γ_1, the top of the x-axis between ϵ and R;
γ_2, a large circle radius R, +ve sense [avoiding the cut];
γ_3, the bottom of the x-axis from R to ϵ;
γ_4, a small circle around the origin of radius ϵ, -ve sense [avoiding the cut].
$\int_\gamma f = \sum_1^4 \int_{\gamma_i} f = \sum_1^4 I_i$, say.
$I_1 \to I$ as $R \to \infty$, $\epsilon \to 0$.
By ML, $I_2 = O((R^{a-1}/R).R) = O(R^{a-1}) \to 0$ as $R \to \infty$, as $a < 1$.
On γ_3, $z = xe^{2\pi i}$, $z^{a-1} = x^{a-1}e^{2\pi i(a-1)}$, so $I_3 \to -e^{2\pi i(a-1)}.I = -e^{2\pi ia}I$.
By ML, $I_4 = O(\epsilon^{a-1}.\epsilon) = O(\epsilon^a) \to 0$ as $\epsilon \to 0$, as $a > 0$.
By Cauchy's Residue Theorem, this gives

$$I(1 - e^{2\pi i(a-1)}) = 2\pi i.e^{\pi i(a-1)} = -2\pi ie^{i\pi a}:$$

$$I = -2\pi ie^{\pi ia}/(1-e^{2\pi ia}) = -\pi.2i/(e^{-\pi ia} - e^{\pi ia}) = -\pi/-\sin\pi a = \pi/\sin\pi a.$$

(iii) Changing a, x in (ii) to x, v and combining with (i) gives

$$\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x \qquad (0 < x < 1).$$

(iv) Since $\Gamma(z)\Gamma(1-z) - \pi/\sin\pi z$ vanishes on the interval $(0,1)$, by above, and this set has a limit point in the region (the plane less the integers) in which the function is holomorphic, it vanishes identically by *analytic continuation*. So for all complex z,

$$\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z, \qquad \frac{1}{\Gamma(z)}.\frac{1}{\Gamma(1-z)} = \frac{\sin\pi z}{\pi}.$$

(v) The Gamma function $\Gamma(z)$ has poles at 0 and the negative integers. So $\Gamma(1-z)$ has poles at the positive integers, and $\Gamma(z)\Gamma(1-z)$ has poles at the integers. So too does $\pi/\sin\ \pi z$. But $\Gamma(z)$ has no zeros (or $\sin\pi z$ would have a pole). So: $\Gamma(z)$, $\Gamma(1-z)$, $\pi/\sin\ \pi z$ have poles but no zeros; $1/\Gamma(z)$, $1/\Gamma(1-z)$, $(\sin\ \pi z)/\pi$ are *entire* – they have (integer) zeros but no poles.
All seen – (v) re analytic continuation in Ch. II, (ii) re integration in Ch. III.

Q4 (i). $I_n := \int_0^\infty \frac{1}{1+x^n}\,dx = \frac{\pi}{n\sin\pi/n} \qquad (n = 2, 3, ...)$.
First Proof: sector contour. Let $f(z) := 1/(1+z^n)$, and take the contour as a sector, with $\gamma_1 = [0, R]$, γ_2 on the arc $|z| = R$, $0 \le \arg\ z \le 2\pi/n$, and γ_3 the path back to the origin. By ML, $\int_{\gamma_2} f = O(R^{-n}.R) \to 0$ as $R \to \infty$, and $\int_{\gamma_1} f \to I$. On γ_3, z goes from R to O, $z = xe^{2\pi i/n}$, $dz = e^{2\pi i/n}\,dx$. So $\int_{\gamma_3} f \to -e^{2\pi i/n} I$. So $\int_\gamma f \to I\left(1 - e^{2\pi i/n}\right)$. By CRT: $\int_\gamma f = 2\pi i \sum Res f$, and f singular where $z^n = -1 = e^{i\pi} = e^{(2k+1)\pi}$, $z = e^{i\pi(2k+1)/n}$. Only $k = 0$, $z = e^{i\pi/n}$ is inside γ. This is a simple pole. Write $f(z) = (z - e^{i\pi/n})/((1 + z^n)(z - e^{i\pi/n})$. By the Cover-Up Rule and L'Hospital's Rule,

$$Res_{e^{i\pi/n}} f = \lim_{z\to e^{i\pi/n}} \frac{z - e^{i\pi n}}{z^n + 1} = \lim_{z\to e^{i\pi/n}} \frac{1}{nz^{n-1}} = \frac{1}{ne^{i\pi(n-1)/n}} = \frac{e^{i\pi/n}}{ne^{i\pi}} = -\frac{e^{i\pi/n}}{n}.$$

$$I = \frac{2\pi i\cdot(-1)^n e^{i\pi/n}}{n(1 - e^{2\pi i/n})} = \frac{\pi}{n}\cdot\frac{-2i}{e^{i\pi/n} - e^{i\pi/n}} = \frac{\pi}{n\sin\pi/n}.$$

Second Proof: by the integral of III.6 (keyhole contour). Put $x^n = y$, $x = y^{1/n}$, $dx = (1/n)y^{(1/n)-1}dy$.

$$I = \int_0^\infty \frac{1}{n}.\frac{y^{(1/n)-1}dy}{1+y} = \frac{\pi}{n\sin(\pi/n)}.$$

(ii). To prove $\sum_{n=1}^\infty 1/n^2 = \pi^2/6$ (Euler). We quote the *Squares Lemma*: $\cot\pi z$ is *uniformly bounded* on the squares C_N with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$.
Proof. Take $f(z) = 1/z^2$. Then $f(z)\cot\pi z$ has simple poles at $z = n \ne 0$ residue $f(n)/\pi = 1/(\pi n^2)$, and a triple pole at $z = 0$. Near 0,

$$f(z)\cot\pi z = \frac{\cos\pi z}{z^2\sin\pi z} = \frac{1 - \frac{\pi^2 z^2}{2} + ...}{z^2\left(\pi z - \frac{\pi^3 z^3}{6} + ...\right)}$$

$$= \frac{1}{\pi z^3}.(1 - \frac{\pi^2 z^2}{2} + ...)(1 + \frac{\pi^2 z^2}{6} - ...) = \frac{1}{\pi z^3}.(1 - \frac{\pi^2 z^2}{3} + ...).$$

So $Res_0\ f(z)\cot\pi z = -\pi/3$. Take $f(z) = 1/z^2$. By CRT:

$$\left|\int_{C_N} \frac{\cot\pi z}{z^2}\,dz\right| = 2\pi i \sum Res = 2\pi i\left(-\pi/3 + (\sum_{n=-N}^{-1} + \sum_{n=1}^{N})\frac{1}{\pi n^2}\right) \quad \text{(Cover-Up Rule).}$$

By ML and the Squares Lemma: as $\cot\pi z = O(1)$, $1/z^2 = O(1/N^2)$ on C_N, which has length $O(N)$, $|LHS| = O(1).O(1/N^2).O(N) = O(1/N) \to 0$. So $-\frac{\pi}{3} + \frac{2}{\pi}\sum_{n=1}^N 1/n^2 \to 0$: $\zeta(2) := \sum_{n=1}^\infty 1/n^2 = \pi^2/6$.
Second Proof. We quote the infinite product for sin: $\sin\ z = z\Pi_1^\infty(1 - \frac{z^2}{n^2\pi^2})$. So $\frac{\sin z}{z} = \sum_{k=0}^\infty \frac{(-)^k z^{2k}}{(2k+1)!} = \Pi_1^\infty(1 - \frac{z^2}{n^2\pi^2})$. Equate coefficients of z^2: $-\frac{1}{3!} = -\frac{1}{\pi^2}.\sum_1^\infty \frac{1}{n^2}$: $\zeta(2) = \sum_1^\infty 1/n^2 = \pi^2/6$.
(All seen – lectures, or problem sheets).

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