MATL480 RESIT EXAMINATION SOLUTIONS 2016-17

Q1. (i) *Types of risk.* Institutions encounter risks of various types. Perhaps the biggest one starts at the top: how good is the board? If the board of directors, and particularly the chairman and CEO, do not have a good overview and good judgement, this alone can bring the institution down.

Other specific types of risk include:

Market risk. This is the risk that one's current market position (the aggregate of risky assets one holds) goes down in value (things one is long on get cheaper, and/or things one is short on get dearer). **[3]**

Credit risk. This is the risk that counter-parties to one's financial transactions may default on their obligations. When this happens, debts cannot be (or are not) paid in full. Usually, payment is made in part, by negotiation between the parties (it may be cheaper to agree a partial repayment than to force the other party into bankruptcy), or by the administrators or liquidators in the case of companies. [**3**]

Operational risk. This is risk arising from the internal procedures of an institution: failure of computer systems for implementing transactions; fraudulent or unauthorised trading made possible by inadequate supervision; etc. [**3**]

Liquidity risk. This is the risk that one will be unable to implement a planned or agreed transaction because of lack of cash-in-hand to trade with, and/or willingness to trade. The Credit Crunch of 2007/8 on was caused by banks realising they had piles of toxic debt on their hands (see below), and so did not know what their balance sheets were worth; that other banks were similarly placed; hence that banks no longer trusted themselves or each other, and so refused to lend to each other. So the financial system froze up; so the real economy froze up. [**3**]

Model risk. To handle real-world phenomena of any complexity, one needs to model them mathematically. To quote Box's Dictum: All models are wrong; some models are useful. Use of an inappropriate model to set the prices at which one buys and sells exposes the institution to open-ended losses, to competitors with better models. [**3**]

(ii) *Stress testing.* Financial regulators test the adequacy of the performance of a financial institution by subjecting it to *stress testing*: seeing how well its operations would perform under hypothetical but unfavourable market scenarios. This tests various aspects: their models, systems (how management and trading teams would react under pressure), capital reserves, etc. [**5**]

[Mainly seen – lectures]

Q2 *Doubling strategy.* (i) With N the number of losses before the first win:

$$P(N = k) = P(L, L, \cdots, L(k \text{ times}), W) = (\frac{1}{2})^k.\frac{1}{2} = (\frac{1}{2})^{k+1}.$$

That is, N is geometrically distributed with parameter 1/2. As

$$\sum_{k=0}^{\infty} P(N = k) = \sum_{0}^{\infty} (\frac{1}{2})^{k+1} = \frac{1}{2}/(1 - \frac{1}{2}) = 1,$$

$P(N < \infty) = 1$: $N < \infty$ a.s. So one is certain to win eventually. **[4]**
(ii) Let S_n be one's fortune at time n. When $N = k$, one has losses at trials $1, 2, 3, \ldots, k$, with losses $1, 2, 4, \ldots, 2^{k-1}$, followed by a win at trial $k + 1$ (of 2^k). So one's fortune then is

$$2^k - (1 + 2 + 2^2 + \ldots + 2^{k-1}) = 2^k - (2^k - 1) = 1,$$

summing the finite geometric progression. So one's eventual fortune is $+1$ (which, by (i), one is certain to win eventually). **[4]**
(iii) N has PGF

$$P(s) := E[s^N] = \sum_{n=0}^{\infty} s^k P(N = k) = \sum_{0}^{\infty} s^k.(\frac{1}{2})^{k+1}$$

$$= \frac{1}{2} \sum_{0}^{\infty} (\frac{1}{2}s)^k = \frac{1}{2}/(1 - \frac{1}{2}s) = 1/(2 - s):$$

$$P'(s) = E[Ns^{N-1}] = (2 - s)^{-2}; \qquad P'(1) = E[N] = 1.$$

So the mean number of losses is 1, and the mean time the game lasts is 2. **[4]**
(iv) This is an impossible strategy to use in reality, for two reasons:
(a) It depends on one's opponent's cooperation. What is to stop him trying this on you? If he does, the game degenerates into a simple coin toss, with the winner walking away with a profit of 1 (pound, or million pounds, say) – suicidally risky. **[4]**
(b) Even with a cooperative opponent, it relies on the gambler having an unlimited amount of cash to bet with, or an unlimited line of credit – both hopelessly unrealistic in practice. **[4]**
[Seen – Problems]

Q3 *Properties of conditional expectation.*
(i) The *conditional expectation* of a random variable Y with $E[|Y|] < \infty$ *given* a σ-field $\mathcal{C}$, $E[Y|\mathcal{C}]$, is defined by:

$E[Y|\mathcal{C}]$ is $\mathcal{C}$-measurable;

$$\int_C E[Y|\mathcal{C}]dP = \int_C Y dP \qquad \forall C \in \mathcal{C} \quad a.s. \qquad [\mathbf{4}]$$

(ii) If $\mathcal{C}$ is the trivial σ-field $\{\emptyset, \Omega\}$, $E[Y|\mathcal{C}] = E[Y]$: a conditional expectation given no information is just an ordinary unconditional expectation.

If $\mathcal{C} = \mathcal{F}$ is the whole σ-field (in the definition of the probability space), $E[Y|\mathcal{C}] = Y$: a conditional expectation given full information is the random variable itself (no randomness left to average over). **[4]**
(iii) *Iterated conditional expectation.* (a) If $\mathcal{C} \subset \mathcal{B}$, $E[E(Y|\mathcal{B})|\mathcal{C}] = E[Y|\mathcal{C}]$ $a.s.$
Proof. $E_{\mathcal{C}} E_{\mathcal{B}} Y$ is $\mathcal{C}$-measurable, and for $C \in \mathcal{C} \subset \mathcal{B}$,

$$\int_C E_{\mathcal{C}}[E_{\mathcal{B}} Y] dP = \int_C E_{\mathcal{B}} Y dP \qquad (\text{definition of } E_{\mathcal{C}} \text{ as } C \in \mathcal{C})$$

$$= \int_C Y dP \qquad (\text{definition of } E_{\mathcal{B}} \text{ as } C \in \mathcal{B}).$$

So $E_{\mathcal{C}}[E_{\mathcal{B}} Y]$ satisfies the defining relation for $E_{\mathcal{C}} Y$. Being also $\mathcal{C}$-measurable, it *is* $E_{\mathcal{C}} Y$ (a.s.). //
(b) If $\mathcal{C} \subset \mathcal{B}$, $E[E(Y|\mathcal{C})|\mathcal{B}] = E[Y|\mathcal{C}]$ $\quad a.s.$ For, $E[Y|\mathcal{C}]$ is $\mathcal{C}$-measurable, so $\mathcal{B}$-measurable as $\mathcal{C} \subset \mathcal{B}$, so $E[.|\mathcal{B}]$ has no effect on it ('taking out what is known': given $\mathcal{B}$, we know Y, so it counts as a constant and we can take it out through integrals, i.e. expectations). **[4]**
(iv) *Conditional Mean Formula.* $E[E(Y|\mathcal{B})] = EY \quad P - a.s.$
Proof. Take $\mathcal{C} = \{\emptyset, \Omega\}$ in (iii). // **[4]**
Projections. Above, take $\mathcal{B} = \mathcal{C}$:

$$E[E[X|\mathcal{C}]|\mathcal{C}] = E[X|\mathcal{C}].$$

This says that the operation of taking conditional expectation given a sub-σ-field $\mathcal{C}$ is *idempotent* – doing it twice is the same as doing it once. Also, taking conditional expectation is a *linear* operation (it is defined via an integral, and integration is linear). So as in Linear Algebra, being idempotent and linear it is called a *projection* (Example: $(x, y, z) \mapsto (x, y, 0)$ projects from 3-dimensional space onto the (x, y)-plane). **[4]**
[Seen – lectures]

Q4. *Discrete and continuous Black-Scholes models and formulae.*
(i) In the *discrete* Black-Scholes (BS) model, we use the (Cox-Ross-Rubinstein, CRR) *binomial tree* model of 1979. At each step, the price can go up or down; we use a 'recombining' tree, so that 'up' and 'down' paths link. The price at expiry thus depends on the number of up and down steps, and this has a *binomial distribution*. We can calculate the payoff at each terminal node. The tree has a unique *risk-neutral measure*, P^* say, under which discounted asset prices become martingales. The price of the option at any time is thus the P^* expectation of the discounted value of the payoff, and we can find this as a suitable *binomial sum*. This gives the *discrete BS formula*.

Now just as the binomial distribution has a histogram approximating a suitable normal density, the binomial sum in the discrete BS formula also has a limit, given by two terms, both involving Φ, one involving the stock price S, the other the strike price K. This limit of the discrete BS formula is the (continuous) *Black-Scholes formula* of 1973.

Not only does the *formula* have a limit, as above, the *model* has a limit. Brownian motion is a suitable limit of random walks. So we can treat the continuous BS formula directly via BM (as Black and Scholes did in 1973), or indirectly via the CRR tree of 1979. [**10**]

(ii) When we pass from the real probability measure P to the risk-neutral probability measure (or equivalent martingale measure, EMM) P^*, the *mean* return μ on the stock is lost, and replaced by the riskless return rate r. What survives is the relevant *variance*, or rather its square root, the *volatility*, σ. So: both discrete and continuous BS formulae involve σ but *not* μ. [**5**]

(iii) For an American put, we have at each node of the tree the option of exercising early. We calculate both the option value and the optimal exercise strategy by working backwards through the tree:
1. Draw the tree, and fill in the stock price at each node.
2. Using the strike price K and the prices at the *terminal nodes*, fill in the payoffs ($f_{N,j} = \max[K - Su^j d^{N-j}, 0]$) at the terminal nodes.
3. Go back one time-step. Fill in the 'European' value at the penultimate nodes as the discounted values of the upper and lower right (terminal) values, under P^* - '$p^* \times$ lower right plus $1 - p^* \times$ upper right'. Fill in the 'intrinsic' (or early-exercise) value. The American put value is the higher of these.
4. Iterate, working back down the tree to the root. The value of the American put at time 0 is the value at the root. The nodes split into the 'early-exercise region' and the 'continuation region'. [**5**]

[Seen, lectures]

Q5. *Geometric Brownian Motion (GBM).* (i) Consider the Black-Scholes model, with dynamics given by the stochastic differential equation (SDE)

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t. \qquad (GBM)$$

The interpretation here is that B_t is our bank account at time t – money invested risklessly at rate r, so growing exponentially. The risky stock S has a similar term, this time with growth-rate μ (which models the systematic part of the price dynamics), plus a second term which models the risky part. The uncertainty in the economic and financial climate is represented by the Brownian motion (BM) $W = (W_t)$; this is coupled to the stock-price dynamics via the paramater σ, the *volatility*, which measures how sensitive this particular risky stock is to changes in the overall economic climate. [**5**]
(ii) Discounting the prices by e^{rt}, the discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= -r\tilde{S}_t dt + \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t. \end{aligned}$$

Thus discounting changes the rate μ on the RHS of (GBM) to $\mu - r$. [**5**]
(iii) Now use Girsanov's Theorem to change from the real probability measure P to an equivalent probability measure P^* under which the μdt in (GBM) is rdt. Then under P^*, the stock-price dynamics become

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t \qquad (\text{under } P^*).$$

Integrating, $\tilde{S}$ on the left is a stochastic integral w.r.t. Brownian motion – which is a martingale. This P^* is the *equivalent martingale measure (EMM)*, or *risk-neutral measure*. The EMM is that in the continuous-time version of the Fundamental Theorem of Asset Pricing: *to price assets, take expectations of discounted prices under the risk-neutral measure.* This leads to the Black-Scholes formula by direct probabilistic means, rather than via the Black-Scholes PDE. [**5**]
(iv) In the Black-Scholes model, markets are complete. So the EMM is unique. This is a result of the representation theorem for Brownian martingales: *any* Brownian martingale can be represented as a stochastic integral w.r.t. BM. Completeness results from the *continuity* of the paths of BM. [**5**]
[Mainly seen – lectures] N. H. Bingham