liv6.tex Week 6: 4.3.2013

§3. Complete Markets: Uniqueness of EMMs.

A contingent claim (option, etc.) can be defined by its *payoff* function, h say, which should be non-negative (options confer rights, not obligations, so negative values are avoided by not exercising the option), and $\mathcal{F}_N$-measurable (so that we know how to evaluate h at the terminal time N).

Definition. A contingent claim defined by the payoff function h is *attainable* if there is an admissible strategy worth (i.e., replicating) h at time N. A market is *complete* if every contingent claim is attainable.

THEOREM (complete iff EMM unique). A viable market is complete iff there exists a unique probability measure P^* equivalent to P under which discounted asset prices are martingales - that is, iff equivalent martingale measures are unique.

Proof. $\Rightarrow$: Assume viability and completeness. Then for any $\mathcal{F}_N$-measurable random variable $h \geq 0$, there exists an admissible (so self-financing) strategy H replicating h: $h = V_N(H)$. As H is self-financing, by §1

$$h/S_N^0 = \tilde{V}_N(H) = V_0(H) + \Sigma_1^N H_j.\Delta\tilde{S}_j.$$

We know by the Theorem of §2 that an equivalent martingale measure P^* exists; we have to prove uniqueness. So, let P_1, P_2 be two such equivalent martingale measures. For $i = 1, 2$, $(\tilde{V}_n(H))_{n=0}^N$ is a P_i-martingale. So,

$$E_i(\tilde{V}_N(H)) = E_i(V_0(H)) = V_0(H),$$

since the value at time zero is non-random ($\mathcal{F}_0 = \{\emptyset, \Omega\}$). So

$$E_1(h/S_N^0) = E_2(h/S_N^0).$$

Since h is arbitrary, E_1, E_2 have to agree on integrating all non-negative integrands. Taking negatives and using linearity: they have to agree on non-positive integrands also. Splitting an arbitrary integrand into its positive and negative parts: they have to agree on all integrands. Now E_i is expectation (i.e., integration) with respect to the measure P_i, and measures that agree on integrating all integrands must coincide. So $P_1 = P_2$. //

Before proving the converse, we prove a lemma. Recall that an admissible strategy is a self-financing strategy with all values non-negative. The Lemma shows that the non-negativity of contingent claims extends to all values of any self-financing strategy replicating it – in other words, this gives equivalence of admissible and self-financing replicating strategies.

LEMMA. In a viable market, any attainable h (i.e., any h that can be replicated by an admissible strategy H) can also be replicated by a self-financing strategy H.

Proof. If H is self-financing and P^* is an equivalent martingale measure under which discounted prices $\tilde{S}$ are P^*-martingales (such P^* exist by viability and the Theorem of §2), $\tilde{V}_n(H)$ is also a P^*-martingale, being the martingale transform of § by H (see §1). So

$$\tilde{V}_n(H) = E^*(\tilde{V}_N(H)|\mathcal{F}_n) \qquad (n = 0, 1, \cdots, N).$$

If H replicates h, $V_N(H) = h \geq 0$, so discounting, $\tilde{V}_N(H) \geq 0$, so the above equation gives $\tilde{V}_n(H) \geq 0$ for each n. Thus *all* the values at each time n are non-negative – not just the final value at time N – so H is admissible. //

Proof of the Theorem (continued). $\Leftarrow$: Assume the market is viable but incomplete: then there exists a non-attainable $h \geq 0$. By the Lemma, we may confine attention to self-financing strategies H (which will then automatically be admissible). By the Proposition of §1, we may confine attention to the risky assets $S^1, \cdots, S^d$, as these suffice to tell us how to handle the bank account S^0.

Call $\tilde{\mathcal{V}}$ the set of random variables of the form

$$U_0 + \Sigma_1^N H_n.\Delta\tilde{S}_n$$

with U_0 $\mathcal{F}_0$-measurable (i.e. deterministic) and $((H_n^1, \cdots, H_n^d))_{n=0}^N$ predictable; this is a vector space. Then by above, the discounted value h/S_N^0 does not belong to $\tilde{\mathcal{V}}$, so $\tilde{\mathcal{V}}$ is a *proper* subspace of the vector space $\mathbf{R}^\Omega$ of all random variables on Ω. Let P^* be a probability measure equivalent to P under which discounted prices are martingales (such P^* exist by viability, by the Theorem of §2). Define the scalar product

$$(X, Y) \to E^*(XY)$$

on random variables on Ω. Since $\tilde{\mathcal{V}}$ is a proper subspace, by Gram-Schmidt orthogonalisation there exists a non-zero random variable X orthogonal to $\tilde{\mathcal{V}}$. That is,

$$E^*(X) = 0.$$

Write $\|X\|_\infty := \max\{|X(\omega)| : \omega \in \Omega\}$, and define P^{**} by

$$P^{**}(\{\omega\}) = \Big(1 + \frac{X(\omega)}{2\|X\|_\infty}\Big) P^*(\{\omega\}).$$

By construction, P^{**} is equivalent to P^* (same null-sets - actually, as $P^* \sim P$ and P has no non-empty null-sets, neither do P^*, P^{**}). As X is non-zero, P^{**} and P^* are *different*. Now

$$\begin{aligned} E^{**}(\Sigma_1^N H_n.\Delta\tilde{S}_n) &= \Sigma_\omega P^{**}(\omega)\Big(\Sigma_1^N H_n.\Delta\tilde{S}_n\Big)(\omega) \\ &= \Sigma_\omega \Big(1 + \frac{X(\omega)}{2\|X\|_\infty}\Big) P^*(\omega)\Big(\Sigma_1^N H_n.\Delta\tilde{S}_n\Big)(\omega). \end{aligned}$$

The '1' term on the right gives $E^*(\Sigma_1^N H_n.\Delta\tilde{S}_n)$, which is zero since this is a martingale transform of the E^*-martingale $\tilde{S}_n$. The 'X' term gives a multiple of the inner product

$$(X, \Sigma_1^N H_n.\Delta\tilde{S}_n),$$

which is zero as X is orthogonal to $\tilde{\mathcal{V}}$ and $\Sigma_1^N H_n.\Delta\tilde{S}_n \in \tilde{\mathcal{V}}$. By the Martingale Transform Lemma, $\tilde{S}_n$ is a P^{**}-martingale since H (previsible) is arbitrary. Thus P^{**} is a second equivalent martingale measure, different from P^*. So incompleteness implies non-uniqueness of equivalent martingale measures, as required. //

Martingale Representation. To say that every contingent claim can be replicated means that every P^*-martingale (where P^* is the risk-neutral measure, which is unique) can be written, or *represented*, as a martingale transform (of the discounted prices) by the replicating (perfect-hedge) trading strategy H. In stochastic-process language, this says that all P^*-martingales can be *represented* as martingale transforms of discounted prices. Such Martingale Representation Theorems hold much more generally, and are very important. For the Brownian case, see VI and [RY], Ch. V.
Note. In the example of Chapter I, we saw that the simple option there could

be replicated. More generally, in our market set-up, *all* options can be replicated – our market is *complete*. Similarly for the Black-Scholes theory below.

§4. The Fundamental Theorem of Asset Pricing.

We summarise what we have learned so far. We call a measure P^* under which discounted prices $\tilde{S}_n$ are P^*-martingales a *martingale measure*. Such a P^* equivalent to the true probability measure P is called an *equivalent martingale measure*. Then
1 (**No-Arbitrage Theorem**: §2). If the market is *viable* (arbitrage-free), equivalent martingale measures P^* *exist*.
2 (**Completeness Theorem**: §3). If the market is *complete* (all contingent claims can be replicated), equivalent martingale measures are *unique*. Combining:

THEOREM (Fundamental Theorem of Asset Pricing). In a complete viable market, there exists a unique equivalent martingale measure P^*.

Let h (≥ 0, $\mathcal{F}_N$-measurable) be any contingent claim, H an admissible strategy replicating it:

$$V_N(H) = h.$$

As $\tilde{V}_n$ is the martingale transform of the P^*-martingale $\tilde{S}_n$ (by H_n), $\tilde{V}_n$ is a P^*-martingale. So $V_0(H)(= \tilde{V}_0(H)) = E^*(\tilde{V}_N(H))$. Writing this out in full:

$$V_0(H) = E^*(h/S_N^0).$$

More generally, the same argument gives $\tilde{V}_n(H) = V_n(H)/S_n^0 = E^*[(h/S_N^0)|\mathcal{F}_n]$:

$$V_n(H) = S_n^0 E^*(\frac{h}{S_N^0}|\mathcal{F}_n) \qquad (n = 0, 1, \cdots, N).$$

It is natural to call $V_0(H)$ above the *value* of the contingent claim h at time 0, and $V_n(H)$ above the value of h at time n. For, if an investor *sells* the claim h at time n for $V_n(H)$, he can follow strategy H to replicate h at time N and clear the claim. To sell the claim for *any other amount* would provide an arbitrage opportunity (as with the argument for put-call parity). So this value $V_n(H)$ is the *arbitrage price* (or more exactly, *arbitrage-free price* or *no-arbitrage price*); an investor selling for this value is *perfectly hedged*.

We note that, to calculate prices as above, we need to know only
(i) Ω, the set of all possible states,
(ii) the σ-field $\mathcal{F}$ and the filtration (or information flow) $(\mathcal{F}_n)$,
(iii) P^*.
We do **NOT** need to know the underlying probability measure P - only its null sets, to know what 'equivalent to P' means (actually, in this model, only the empty set is null).

Now option pricing is our central task, and for pricing purposes P^* is vital and P itself irrelevant. We thus may – and shall – focus attention on P^*, which is called the *risk-neutral* probability measure. *Risk-neutrality* is the central concept of the subject. The concept of risk-neutrality is due in its modern form to Harrison and Pliska [HP] in 1981 – though the idea can be traced back to actuarial practice much earlier. Harrison and Pliska call P^* the *reference measure*; other names are *risk-adjusted* or *martingale measure*. The term 'risk-neutral' reflects the P^*-martingale property of the risky assets, since martingales model fair games.

To summarise, we have the

THEOREM (Risk-Neutral Pricing Formula). In a complete viable market, arbitrage-free prices of assets are their discounted expected values under the risk-neutral (equivalent martingale) measure P^*. With payoff h,

$$V_n(H) = (1+r)^{-(N-n)} E^*[V_N(H)|\mathcal{F}_n] = (1+r)^{-(N-n)} E^*[h|\mathcal{F}_n].$$

§5. European Options. The Discrete Black-Scholes Formula.

We consider the simplest case, the Cox-Ross-Rubinstein *binomial model* of 1979; see [CR], [BK].

We take $d = 1$ for simplicity (one risky asset, one riskless asset or bank account); the price vector is (S_n^0, S_n^1), or $((1+r)^n, S_n)$, where

$$S_{n+1} = \begin{cases} S_n(1+a) & \text{with probability } p, \\ S_n(1+b) & \text{with probability } 1-p \end{cases}$$

with $-1 < a < b$, $S_0 > 0$. So writing N for the expiry time,

$$\Omega = \{1+a, 1+b\}^N,$$

each $\omega \in \Omega$ representing the successive values of $T_{n+1} := S_{n+1}/S_n$, $n = 0, 1, \cdots, N-1$. The filtration is $\mathcal{F}_0 = \{\emptyset, \Omega\}$ (trivial σ-field), $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ (power-set of Ω: class of all subsets of Ω), $\mathcal{F}_n = \sigma(S_1, \cdots, S_n) = \sigma(T_1, \cdots, T_n)$. For $\omega = (\omega_1, \cdots, \omega_N) \in \Omega$, $P(\{\omega_1, \cdots, \omega_N\}) = P(T_1 = \omega_1, \cdots, T_N = \omega_N)$, so knowing the probability measure P (equivalently, knowing p) means we know the distribution of $(T_1, \cdots, T_N)$.

For $p^* \in (0,1)$ to be determined, let P^* correspond to p^* as P does to p. Then the discounted price $(\tilde{S}_n)$ is a P^*-martingale iff

$$E^*[\tilde{S}_{n+1}|\mathcal{F}_n] = \tilde{S}_n, \qquad E^*[(\tilde{S}_{n+1}/\tilde{S}_n)|\mathcal{F}_n] = 1,$$

$$E^*[T_{n+1}|\mathcal{F}_n] = 1 + r \qquad (n = 0, 1, \cdots, N-1),$$

since $S_n = \tilde{S}_n(1+r)^n$, $T_{n+1} = S_{n+1}/S_n = (\tilde{S}_{n+1}/\tilde{S}_n)(1+r)$. But

$$E^*(T_{n+1}|\mathcal{F}_n) = (1+a).p^* + (1+b).(1-p^*)$$

is a weighted average of $1+a$ and $1+b$; this can be $1+r$ iff $r \in [a,b]$. As P^* is to be *equivalent* to P and P has no non-empty null-sets, $r = a, b$ are excluded. Thus by §2:

LEMMA. The market is viable (arbitrage-free) iff $r \in (a,b)$.

Next, $1+r = (1+a)p^* + (1+b)(1-p^*)$, $r = ap^* + b(1-p^*)$: $r - b = p^*(a-b)$:

LEMMA. The equivalent martingale measure exists, is unique, and is given by

$$p^* = (b-r)/(b-a).$$

COROLLARY. The market is complete.

Now $S_N = S_n \Pi_{n+1}^N T_i$. By the Fundamental Theorem of Asset Pricing, the price C_n of a call option with strike-price K at time n is

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} E^*[(S_N - K)_+|\mathcal{F}_n] \\ &= (1+r)^{-(N-n)} E^*[(S_n \Pi_{n+1}^N T_i - K)_+|\mathcal{F}_n]. \end{aligned}$$

Now the conditioning on $\mathcal{F}_n$ has no effect – on S_n as this is $\mathcal{F}_n$-measurable (known at time n), and on the T_i as these are independent of $\mathcal{F}_n$. So

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} E^*[(S_n \Pi_{n+1}^N T_i - K)_+] \\ &= (1+r)^{-(N-n)} \Sigma_{j=0}^{N-n} \binom{N-n}{j} p^{*j}(1-p^*)^{N-n-j}(S_n(1+a)^j(1+b)^{N-n-j} - K)_+; \end{aligned}$$

here j, $N-n-j$ are the numbers of times T_i takes the two possible values $1+a, 1+b$. This is the *discrete Black-Scholes formula* of Cox, Ross & Rubinstein (1979) for pricing a European call option in the binomial model. For a European put option, we can either argue similarly or use put-call parity (I.3).

We can find the (perfect-hedge) strategy for replicating this explicitly. Write

$$c(n,x) := \Sigma_{j=0}^{N-n} \binom{N-n}{j} p^{*j}(1-p^*)^{N-n-j}(x(1+a)^j(1+b)^{N-n-j} - K)_+.$$

Then $c(n,x)$ is the undiscounted P^*-expectation of the call at time n given that $S_n = x$. This must be the value of the portfolio at time n if the strategy $H = (H_n)$ replicates the claim:

$$H_n^0(1+r)^n + H_n S_n = c(n, S_n)$$

(here by previsibility H_n^0 and H_n are both functions of $S_0, \cdots, S_{n-1}$ only). Now $S_n = S_{n-1}T_n = S_{n-1}(1+a)$ or $S_{n-1}(1+b)$, so:

$$\begin{aligned} H_n^0(1+r)^n + H_n S_{n-1}(1+a) &= c(n, S_{n-1}(1+a)) \\ H_n^0(1+r)^n + H_n S_{n-1}(1+b) &= c(n, S_{n-1}(1+b)). \end{aligned}$$

Subtract:

$$H_n S_{n-1}(b-a) = c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a)).$$

So H_n in fact depends only on $S_{n-1}, H_n = H_n(S_{n-1})$ (by previsibility), and

PROPOSITION. The perfect hedging strategy H_n replicating the European call option above is given by

$$H_n = H_n(S_{n-1}) = \frac{c(n, S_{n-1}(1+b)) - c(n, S_{n-1}(1+a))}{S_{n-1}(b-a)}.$$

Notice that the numerator is the difference of two values of $c(n,x)$ with the larger value of x in the first term (recall $b > a$). When the payoff function $c(n,x)$ is an increasing function of x, as for the European call option considered here, this is non-negative. In this case, the Proposition gives $H_n \geq 0$: the replicating strategy does not involve short-selling. We record this as:

COROLLARY. When the payoff function is a non-decreasing function of the final asset price S_N, the perfect-hedging strategy replicating the claim does not involve short-selling of the risky asset.

§6. Continuous-Time Limit of the Binomial Model.

Suppose now that we wish to price an option in continuous time with initial stock price S_0, strike price K and expiry T. We can use the work above to give a discrete-time approximation, where $N \to \infty$. Given $R \geq 0$, to be thought of as an instantaneous interest rate in continuous time, define r by

$$r := RT/N : \qquad e^{RT} = \lim_{N\to\infty}(1 + \frac{RT}{N})^N = \lim_{N\to\infty}(1+r)^N.$$

Here r, which tends to zero as $N \to \infty$, represents the interest rate in discrete time for the approximating binomial model.
For $\sigma > 0$ fixed (σ^2 is to be a variance in continuous time, which will correspond to the *volatility* of the stock), define a, b by

$$\log((1+a)/(1+r)) = -\sigma/\sqrt{N}, \qquad \log((1+b)/(1+r)) = \sigma/\sqrt{N}$$

(a, b both go to zero as $N \to \infty$). We now have a sequence of binomial models, for each of which we can price options as in §5. We shall show that the pricing formula converges as $N \to \infty$ to a limit (which we shall identify later with the continuous Black-Scholes formula of Ch. VI); see e.g. [BK], 4.6.2.

LEMMA. Let $(X_j^N)_{j=1}^N$ be iid with mean μ_N satisfying

$$N\mu_N \to \mu \qquad (N \to \infty)$$

and variance $\sigma^2(1+o(1))/N$. If $Y_N := \Sigma_1^N X_j^N$, then Y_N converges in distribution to normality:

$$Y_N \to Y = N(\mu, \sigma^2) \qquad (N \to \infty).$$

Proof. Use characteristic functions: since Y_N has mean $\mu_N = \mu(1+o(1))/N$ and variance as given, it also has second moment $\sigma^2(1+o(1))/N$. So it has characteristic function

$$\phi_N(u) := E\exp\{iuY_N\} = \Pi_1^N E\exp\{iuX_j^N\} = [E\exp\{iuX_1^N\}]^N$$

$$= (1 + \frac{iu\mu}{N} - \frac{1}{2}\frac{\sigma^2 u^2}{N} + o(\frac{1}{N}))^N \to \exp\{iu\mu - \frac{1}{2}\sigma^2 u^2\} \qquad (N \to \infty).$$

This is the characteristic function of the normal law $N(\mu, \sigma^2)$. The result follows, since convergence of characteristic functions implies convergence in distribution by Lévy's continuity theorem for characteristic functions ([W], §18.1). //

We can apply this to pricing the call option above:

$$C_0^{(N)} = (1 + \frac{RT}{N})^{-N} E^*[(S_0 \Pi_1^N T_n - K)_+]$$

$$= E^*[(S_0 \exp\{Y_N\} - (1 + \frac{RT}{N})^{-N} K)_+], \tag{1}$$

where

$$Y_N := \sum_1^N \log(T_n/(1+r)).$$

Since $T_n = T_n^N$ above takes values $1+b, 1+a$, $X_n^N := \log(T_n^N/(1+r))$ takes values $\log((1+b)/(1+r))$, $\log((1+a)/(1+r)) = \pm\sigma/\sqrt{N}$ (so has second moment σ^2/N). Its mean is

$$\mu_N := \log\Big(\frac{1+b}{1+r}\Big)(1-p^*) + \log\Big(\frac{1+a}{1+r}\Big)p^* = \frac{\sigma}{\sqrt{N}}(1-p^*) - \frac{\sigma}{\sqrt{N}}p^* = (1-2p^*)\sigma/\sqrt{N}$$

(we shall see below that $1 - 2p^* = O(1/\sqrt{N})$, so the Lemma will apply). Now (recall $r = RT/N = O(1/N)$)

$$a = (1+r)e^{-\sigma/\sqrt{N}} - 1, \qquad b = (1+r)e^{\sigma/\sqrt{N}} - 1,$$

so $a, b, r \to 0$ as $N \to \infty$, and

$$1 - 2p^* = 1 - 2\frac{(b-r)}{(b-a)} = 1 - 2\frac{[(1+r)e^{\sigma/\sqrt{N}} - 1 - r]}{[(1+r)(e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}})]}$$

$$= 1 - 2\frac{[e^{\sigma/\sqrt{N}} - 1]}{[e^{\sigma/\sqrt{N}} - e^{-\sigma/\sqrt{N}}]}.$$

Now expand the two $[\cdots]$ terms above by Taylor's theorem: they give

$$\frac{\sigma}{\sqrt{N}}(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots), \qquad \frac{2\sigma}{\sqrt{N}}(1 + \frac{\sigma^2}{6N} + \cdots).$$

So, cancelling $\sigma/\sqrt{N}$,

$$1 - 2p^* = 1 - \frac{2(1 + \frac{1}{2}\frac{\sigma}{\sqrt{N}} + \cdots)}{2(1 + \frac{\sigma^2}{6N} + \cdots)} = -\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N) :$$

$$N\mu_N = N.\frac{\sigma}{\sqrt{N}}.(-\frac{1}{2}\frac{\sigma}{\sqrt{N}} + O(1/N)) \to \mu := -\frac{1}{2}\sigma^2 \qquad (N \to \infty).$$

We are thus in the situation of the Lemma, with $\mu = -\frac{1}{2}\sigma^2$. In (1), we have $Y_N \to Y$ in distribution and $(1 + \frac{RT}{N})^{-N} \to e^{-RT}$ as $N \to \infty$. This suggests that

$$C_0^{(N)} \to E[(S_0 e^Y - e^{-RT}K)_+],$$

where E is the expectation for the distribution of Y, which is $N(-\frac{1}{2}\sigma^2, \sigma^2)$. This can be justified, by standard properties of convergence in distribution (see e.g. [W]). So if $Z := (Y + \frac{1}{2}\sigma^2)/\sigma$, $Z \sim N(0,1)$, $Y = -\frac{1}{2}\sigma^2 + \sigma Z$, and

$$C_0^{(N)} \to \int_{-\infty}^{\infty} [S_0 \exp\{-\frac{1}{2}\sigma^2 + \sigma x\} - e^{-RT}K]_+ \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} dx \qquad (N \to \infty).$$

To evaluate the integral, note first that $[...] > 0$ where

$$S_0 \exp\{-\frac{1}{2}\sigma^2 + \sigma x\} > e^{-RT}K, \qquad -\frac{1}{2}\sigma^2 + \sigma x > \log(K/S_0) - RT :$$

$$x > [\log(K/S_0) + \frac{1}{2}\sigma^2 - RT]/\sigma = c, \quad \text{say.}$$

So writing $\Phi(x)$ for the standard normal distribution function,

$$C_0 = S_0 \int_c^{\infty} e^{-\frac{1}{2}\sigma^2}.\exp\{-\frac{1}{2}x^2 + \sigma x\} dx/\sqrt{2\pi} - Ke^{-RT}[1 - \Phi(c)].$$

The remaining integral is

$$\int_c^{\infty} \exp\{-\frac{1}{2}(x-\sigma)^2\} dx/\sqrt{2\pi} = \int_{c-\sigma}^{\infty} \exp\{-\frac{1}{2}u^2\} du/\sqrt{2\pi} = 1 - \Phi(c - \sigma).$$

So the option price is given as a function of the initial price S_0, strike price K, expiry T, interest rate R and variance σ^2 by

$$C_0 = S_0[1 - \Phi(c - \sigma)] - Ke^{-RT}[1 - \Phi(c)], \qquad c = [\log(K/S_0) + \frac{1}{2}\sigma^2 - RT]/\sigma.$$

To compare with our later work, it is convenient now to replace σ^2 by $\sigma^2 T$; thus σ^2 is now the variance per unit time. Its square root, σ, is called the *volatility* of the stock. Then $c - \sigma$, c above become $c_\pm$, where

$$c_\pm := [\log(K/S_0) - (R \pm \frac{1}{2}\sigma^2)T]/\sigma\sqrt{T}.$$

The result extends immediately to give the price of the option at time $t \in (0, T)$, by replacing T by $T - t$, S_0 by S_t.

We re-write the formula in more customary notation. First, write r in place of R for the interest rate. Next, using the symmetry of the normal distribution, $1 - \Phi(c_\pm) = \Phi(-c_\pm) = \Phi(d_\pm)$, say, where

$$d_\pm := -c_\pm = [\log(S/K) + (r \pm \frac{1}{2}\sigma^2)(T-t)]/\sigma\sqrt{T-t} :$$

the price of the European call option is

$$c_t = S_t\Phi(d_+) - e^{-r(T-t)}K\Phi(d_-).$$

This is the famous *continuous Black-Scholes formula*. We shall return to it in Chapter VI, where we re-derive it by continuous-time methods (Brownian motion and Itô calculus).

Note. 1. The same argument (or put-call parity) gives the value of the European put option as $p_t = Ke^{-r(T-t)}\Phi(-d_-) - S_t\Phi(-d_+)$.

2. The proof above starts from a binomial distribution and ends with a normal distribution. The binomial distribution is that of a sum of independent Bernoulli random variables. That sums (equivalently, averages) of independent random variables with finite means and variances gives a normal limit is the content of the Central Limit Theorem or CLT (the *Law of Errors*, as physicists would say). The particular form of the CLT used here - normal approximation to the binomial - is the *de Moivre-Laplace limit theorem.*

The picture for this is familiar. The Binomial distribution $B(n, p)$ has a histogram with $n + 1$ bars, whose heights peak at the mode and decrease to either side. For large n, one can draw a smooth curve through the histogram. The curve looks like a normal density curve (with the appropriate location and scale, i.e. mean and variance). The result proved above, and the classical de Moivre-Laplace limit theorem, say that this is exactly right.

3. The Cox-Ross-Rubinstein binomial model above goes over in the passage to the limit to the geometric Brownian motion model of VI.1. We will later

re-derive the continuous Black-Scholes formula in Ch. VI, using continuous-time methods (Itô calculus), rather than using the method above of deriving the discrete Black-Scholes formula and going to the limit on the *formula*, rather than the *model*.
4. For similar derivations of the discrete Black-Scholes formula and the passage to the limit to the continuous Black-Scholes formula, see e.g. [CR], §5.6.
5. One of the most striking features of the Black-Scholes formula is that it does **not** involve the mean rate of return μ of the stock - only the riskless interest-rate r and the volatility of the stock σ. Mathematically, this reflects the fact that the change of measure involved in the passage to the risk-neutral measure involves a change of drift. This has the effect of eliminating the μ term; see Ch. VI.
6. The Black-Scholes formula involves the parameter σ (where σ^2 is the variance of the stock per unit time), called the *volatility* of the stock. In financial terms, this represents how sensitive the stock-price is to new information - how 'volatile' the market's assessment of the stock is. This volatility parameter is very important, *but* we do not know it; instead, we have to *estimate* the volatility for ourselves. There are two approaches:
(a) *historic volatility*: here we use Time Series methods to estimate σ from past price data. Clearly the more variability we observe in runs of past prices, the more volatile the stock price is, and given enough data we can estimate σ in this way.
(b) *implied volatility*: match observed option prices to theoretical option prices. For, the price we see options traded at tells us what the *market* thinks the volatility is (estimating volatility this way works because the dependence is monotone; see later).

If the Black-Scholes model were perfect, these two estimates would agree (to within sampling error). But discrepancies can be observed, which shows the imperfections of our model.

Volatility estimation is a major topic, both theoretically and in practice. We return to this in IV.7.3-4 below and VI.7.5-8. But looking ahead:
(i) trading is itself one of the major causes of volatility;
(ii) options like volatility [i.e., option prices go up with volatility].
Recalling Ch. I, this shows that volatility is a 'bad thing' from the point of view of the real economy (uncertainty about, e.g., future material costs is nothing but a nuisance to manufacturers), but a 'good thing' for financial markets (trading increases volatility, which increases option prices, which generates more trade ...) – at the cost of increased instability.