ltccprob4.tex
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PROBLEMS 4

Q1 *Symmetric stable processes.*

A symmetric stable process $X = (X_t)$ with index $\alpha \in (0, 2]$ is defined by its characteristic function,

$$E \exp\{isX_t\} = \exp\{-ts^\alpha\}.$$

Show that for $c > 0$ the process X_c, where $X_c(t) := c^{-1}X(c^\alpha t)$, is again a symmetric stable process with index α, and so has the same distribution as X. Deduce that, as for Brownian motion, the sample path of X is a fractal, and so is the zero set Z of X.

Q2 *Time inversion.*

If B is Brownian motion and

$$X_t := tB(1/t),$$

show that X has mean 0 and covariance

$$cov(X_s, X_t) = st cov(B(1/s), B(1/t)) = st \min(1/s, 1/t) = \min(t, s) = \min(s, t),$$

the same covariance as Brownian motion. Show also that X has continuous paths, and is a Gaussian process. Deduce that X *is* Brownian motion. (We say that X is obtained from B by *time inversion*).

Q3 *Zero set of Brownian motion.*

Deduce from Q2 and the fact that Brownian motion has zeros at times increasing to infinity that Brownian motion has zeros at times decreasing to zero. That is, if B is Brownian motion started at 0 at time 0, then B is zero at (random) times $t_n \downarrow 0$, with probability 1.

Note. 1. This means that it is impossible, even in principle, to *draw a Brownian path*!. The best we can do is to draw it approximately.

2. Brownian paths have other properties that may seem bizarre or counter-intuitive at first glance. For instance, it can be shown that, a.s., a Brownian path is (not only continuous but also) *nowhere differentiable.* Such behaviour is in fact typical, or *generic*: in a sense that can be made precise, *most* continuous functions are nowhere differentiable.

3. It is hard to construct explicit examples of such things. This opens up an important use of Brownian motion *in Analysis*: a Brownian path may have a property a.s. for which it is hard to find specific examples, so giving a *non-constructive existence proof.* NHB