ma414soln1.tex

MA414 SOLUTIONS 1. 23.1.2012

Q1. Being either 1 on the set or 0 off it, an indicator is determined by its parity (odd = 1, even = 0), or its value modulo 2.

$$I_{A\Delta B} = I_A + I_B - 2IA \cap B = I_A + I_B = I_{B\Delta A} \qquad \text{(mod 2)}.$$

For, on $A \setminus B$ and $B \setminus A$ both sides are 1; both sides are 0 on $A^c \cap B^c$ and 0 and 1+1-2 = 0 on $A \cap B$. Since addition (hence also addition mod 2) is associative, Δ is associative:

$$I_{A\Delta(B\Delta C)} = I_A + I_{B\Delta C} = I_A + I_B + I_C = I_{(A\Delta B)\Delta C} \quad \text{mod } 2,$$

so $A\Delta(B\Delta C) = (A\Delta B)\Delta C$. So we may write either side as $A\Delta B\Delta C$ omitting brackets, and similarly for $A_1\Delta \ldots \Delta A_n$.

Assume by induction that $A_1\Delta \ldots \Delta A_n = \{x : x \text{ is in an odd number of the sets}\}$. Then $A_1\Delta \ldots \Delta A_{n+1} = (A_1\Delta \ldots \Delta A_n)\Delta A_{n+1}$ is the set of points in an even number of the first n sets and the last, or an odd number of the first n and not the last, i.e. is the set of points in an odd number of the first $n+1$ sets, completing the induction.

Since $I_\emptyset = 0$, $A\Delta\emptyset = \emptyset\Delta A = A$, for all A. Combining: the set $\mathcal{P}(\Omega)$ of all subsets of Ω is an additive abelian group under Δ, with $\emptyset$ as 0 element.

Since $I_A.I_B = I_{A\cap B}$, $\mathcal{P}(\Omega)$ is an associative system with $\cap$ as multiplication. Since $A \cap \Omega = A = \Omega \cap A$, Ω serves as identity, 1. Since

$$I_{A\cap(B\Delta C)} = I_A.I_{B\cap C} = I_A(I_B + I_C) = I_{A\cap B} + I_{A\cap C} = I_{(A\cap B)\Delta(A\cap C)} \quad \text{(mod 2)},$$

$$A \cap (B\Delta C) = (A \cap B)\Delta(A \cap C),$$

showing that $\cap$ as multiplication is distributive over Δ as addition. Combining, $\mathcal{P}(\Omega)$ is a ring under these operations (called the *Boolean ring*), with $\emptyset$ as 0 and Ω as 1. Since $A \cap A = A$, and $\cap$ is multiplication, each $x^2 = x$. $A\Delta A = \emptyset$, each set A is its own additive inverse.

Note. A ring in Algebra is a set with two operations, called addition and multiplication, which is (a) an abelian group under addition, (b) an associative system (not necessarily commutative) under multiplication, and the distributive law holds. Prototypes: integers; polynomials; matrices.

Q2. If the sets are $\{A_n\}_{n=1}^{\infty}$ and $A_n = \{x_{n,k}\}_{k=1}^{\infty}$, display the point $x_{n,k}$ at the point (n, k) in the first quadrant. By 'diagonal sweep', enumerate this double sequence in a single sequence $(x_{1,1}; x_{2,1}, x_{1,2}; x_{3,1}, x_{2,2}, x_{1,3}; \ldots)$. This shows that $\cup A_n$ is countable [Cantor's proof that the rationals are countable].

Q3. If A_n are μ-null, $\mu(\cup_1^{\infty} A_n) \leq \sum_1^{\infty} \mu(A_n) = \sum_1^{\infty} 0 = 0$.

Q4. (i). Write $B_n := A_n \setminus A_{n+1}$. Then the B_n are disjoint, $A_n = \cup_1^n B_k$ and $\cup A_n = \cup B_n$, so

$$\mu(A_n) = \mu(\cup_1^n B_k) = \sum_1^n \mu(B_k) \uparrow \mu(\sum_1^{\infty} \mu(B_k).$$

But as μ is a measure,

$$\mu(\cup_1^{\infty} A_n) = \mu(\cup_1^{\infty} B_k) = \sum_1^{\infty} \mu(B_k).$$

So

$$\mu(A_n) \uparrow \mu(\cup A_n).$$

(ii) As $A_n \downarrow$ and $\mu(A_N) < \infty$: for $n \geq N$, $A_n \subset A_N$ and $(A_N \setminus A_n) \uparrow$. So by (i),

$$\begin{aligned} \mu(A_N \setminus A_n) &= \mu(A_N) - \mu(A_n) \uparrow \mu(\cup_{n \geq N} A_N \setminus A_n) \\ &= \mu(A_N \setminus \cap_{n \geq N} A_n) = \mu(A_N) - \mu(\cap_{n \geq N} A_n). \end{aligned}$$

So

$$\mu(A_n) \downarrow \mu(\cap A_n).$$

Q5. Let $B_n := \cap_{k \geq n} A_k$. Then $B_n \subset A_n$, so $\mu(B_n) \leq \mu(A_n)$, liminf $\mu(B_n) \leq$ liminf $\mu(A_n)$. But $B_n \uparrow$, so by Q4(i),

$$\text{liminf } \mu(B_n) = \lim \mu(B_n) = \mu(\cup B_n) = \mu(\text{liminf } A_n).$$

Combining,

$$\mu(\text{liminf } A_n) \leq \text{liminf } \mu(A_n), \quad //$$

giving (i). Part (ii) follows similarly from Q4(ii), or by taking complements of (i) w.r.t. $\cup_{k \geq N} A_k$. //

Q6. By Q5(i), (ii),

$$\mu(\lim A_n) = \mu(\text{liminf } A_n) \leq \text{liminf } \mu(A_n) \leq \text{limsup } \mu(A_n) \leq \mu(\text{limsup } A_n) = \mu(\lim A_n).$$

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