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MA414 STOCHASTIC ANALYSIS: EXAMINATION, 2011

Q1. (i) State [**1**] and prove [**4**] the first Borel-Cantelli lemma, for events A_n.
(ii) State [**1**] and prove [**7**] the second Borel-Cantelli lemma, where the A_n are independent.
(ii) Prove the second Borel-Cantelli lemma when the A_n are pairwise independent [**12**].

Q2. By dyadic expansion, or otherwise, show how to pass from either of the following two random experiments to the other:
(a) sampling a random number uniformly from $[0, 1]$;
(b) tossing a fair coin infinitely often independently. [**9**]

Show how to simulate the following from a single realization of a random variable U uniformly distributed on $[0, 1]$:
(i) infinitely many independent uniform random variables; [**4**]
(ii) infinitely many independent standard normal random variables; [**4**]
(iii) Brownian motion; [**4**]
(iv) infinitely many independent Brownian motions. [**4**]

Q3. Define infinite divisibility, and state without proof the Lévy-Khintchine formula. [**2, 3**]

The *Cauchy density* is defined by $f(x) := 1/(\pi(1 + x^2))$.
(i) Show that it has characteristic function $\phi(t) = e^{-|t|}$. [**6**]
(ii) Deduce that it is infinitely divisible. [**3**]
(iii) If $X = (X_t)$ is the corresponding Lévy process (the *Cauchy process*), show that the Lévy measure μ of X has density $1/(\pi|x|^2)$ (you may assume that $\int_0^\infty ((\sin\ x)/x)dx = \pi/2$). [**8**]
(iv) If $X_1, X_2, \ldots$ are independent with the Cauchy distribution, show that $(X_1 + \ldots + X_n)/n$ and X_1 have the same distribution. Why does this not contradict the Strong Law of Large Numbers? [**3**]

Q4. (i) For $B = (B_t) = (B(t))$ standard Brownian motion, define

$$X_t := tB(1/t) \qquad (t \neq 0).$$

Show that $X = (X_t)$ is again standard Brownian motion. [**8**]

(ii) Hence or otherwise, show that

$$B_t/t \to 0 \qquad a.s. \qquad (t \to \infty). \qquad \textbf{[8]}$$

(iii) Define the *Brownian bridge* B_0 by $B_0(t) := B(t) - tB_1$. Find the expansion of B_0 in terms of the Schauder functions $\Delta_n(t)$. **[9]**

Q5. Define a *convex function*. State without proof Jensen's inequality, and its conditional form. **[1, 2, 2]**
(i) If $M = (M_t)$ is a martingale, and ϕ is a convex function such that each $\phi(M_t)$ is integrable, show that $\phi(M)$ is a submartingale. **[5]**
(ii) If M is a submartingale, and ϕ is non-decreasing on the range of M and convex with each $\phi(M_t)$ integrable, show that again $\phi(M)$ is a submartingale. **[5]**
(iii) Deduce that for $B = (B_t)$ Brownian motion, $B^2 = (B_t^2)$ is a submartingale. **[5]**
(iv) Find the increasing process in its Doob-Meyer decomposition. Deduce that Brownian motion has quadratic variation t. **[5]**

Q6. (i) Define the space $H^2 := H^2(0,T)$, and state without proof the Itô isometry for Itô integrals with integrands in H^2. **[2]**
(ii) Prove the *conditional Itô isometry*: for $0 \le s \le t \le T$, $f \in H^2$,

$$E[(\int_s^t f^2(\omega,u)dB_u)^2|\mathcal{F}_s] = E[\int_s^t f^2(\omega,t)dt|\mathcal{F}_s]. \qquad \textbf{[6]}$$

(iii) Show that for $f \in H^2$

$$M_t := (\int_s^t f(\omega,u)dB_u)^2 - \int_0^t f^2(\omega,u)du$$

is a martingale. **[13]**
(iv) Deduce that $B^2(t,\omega) - t$ is a martingale. **[4]**

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