π, e, and Other Irrational Numbers
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Reviewed work(s):
Source: *The American Mathematical Monthly*, Vol. 93, No. 9 (Nov., 1986), pp. 722-723
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2322291
Accessed: 09/03/2012 10:15

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This in essence is all that is required to show $A^2 = 0$. Since R is semiprime we therefore obtain $A = 0$, which means R must be associative. This completes the proof of the theorem.

How can one construct an example of a ring all of whose associators lie in its nucleus? One starts with the free nonassociative ring on k generators, $k \geq 1$. Then set all words of length five or more equal to zero. Since the hypothesis of all associators in the nucleus is a trio of identities, each involving words of length five or more, we have produced such a ring. Incidentally in this ring $(x, x, x) \neq 0$, for each generator x.

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π, e, AND OTHER IRRATIONAL NUMBERS

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In [1] Niven gave a clever, short proof that π is irrational. We would like to show how his proof can be generalized to prove substantially more:

(a) If $0 < |r| \leqslant \pi$ and if $\cos(r)$ and $\sin(r)$ are rational, then r is irrational.
(b) If r is positive and rational, $r \neq 1$, then $\ln(r)$ is irrational.

For example, (a) shows that π is irrational. If $a^2 + b^2 = c^2$ for rational numbers a, b, c, with $bc \neq 0$, then (a) shows that $\arccos(a/c)$ is irrational. Taking the contrapositive in (b) with $r = e$, we see that e is irrational.

Of course, by a famous theorem of Lindemann, all the numbers in (a) and (b) are not just irrational, but transcendental. The novelty of our argument is not so much the conclusion, but that our proof is elementary and can be effectively presented to students of calculus, for we require nothing beyond integration by parts and knowledge of the limit $\lim_{k \to \infty} M^k/k!$. Then again, at the beginning level, the fact that certain real numbers which occur naturally are irrational is interesting enough to present for its own sake.

We will apply the following theorem to prove (a) and (b). Its proof is a generalization of Niven's argument in [1].

THEOREM. *Let c be a positive real number and let $f(x)$ be a continuous function on $[0, c]$, positive on $(0, c)$. Suppose there are (antiderivatives) $f_1(x), f_2(x), \ldots$ with $f_1'(x) = f(x)$ and with $f_k'(x) = f_{k-1}(x)$ for all $k \geqslant 2$, and such that $f_k(0), f_k(c)$ are integers for all $k \geqslant 1$. Then c is irrational.*

Proof. Let P be the set of all polynomials $g(x)$ with real coefficients such that $g(0), g(c), g'(0), g'(c), \ldots, g^{(k)}(0), g^{(k)}(c), \ldots$ are all integers.

CLAIM 1. *If $g(x)$ is in P, then $\int_0^c f(x)g(x)\,dx$ is an integer.*

Proof. Successive integrations by parts give

$$\int_0^c f(x)g(x)\,dx = \left[f_1 \cdot g - f_2 \cdot g' + f_3 \cdot g'' - \cdots + (-1)^d f_{d+1} \cdot g^{(d)}\right]_0^c,$$

where d is the degree of $g(x)$. This proves the claim.

We will also need the following easy fact.

(1) If $g(x)$ and $h(x)$ are in P, then so is $g(x)h(x)$.

Now assume that c is rational, and write $c = m/n$, where m, n are positive integers. Then one verifies:

(2) $m - 2nx$ is in P.

Let $g_k(x) = x^k(m - nx)^k/k!$ for $k = 0, 1, 2, \ldots$.

CLAIM 2. *$g_k(x)$ is in P for all k.*

Proof. Induction on k: $g_0(x) = 1$ is an element of P. For $k \geqslant 1$,

$$g_k'(x) = g_{k-1}(x)(m - 2nx).$$

By induction, g_{k-1} is in P, by (2) $m - 2nx$ is in P, and thus by (1) g_k' is in P. Since also $g_k(0)$ and $g_k(c)$ are 0, we have that g_k is in P.

Observe that $g_k(x) > 0$ on $(0, c)$, a property shared by $f(x)$, so that $\int_0^c f(x)g_k(x)\,dx > 0$. By Claim 1, the integral is also an integer; therefore

(3) $\int_0^c f(x)g_k(x)\,dx \geqslant 1$ for all k.

Let M be the maximum for $x(m - nx)$ on $[0, c]$, and L that for $f(x)$, then

$$\int_0^c f(x)g_k(x)\,dx \leqslant \int_0^c L \cdot \frac{M^k}{k!}\,dx = c \cdot L \cdot \frac{M^k}{k!}.$$

But $\lim_{k \to \infty} M^k/k! = 0$, contradicting (3). We are forced to conclude that c is irrational.

To prove the statement (a) mentioned at the beginning, observe that if $\cos(r)$ and $\sin(r)$ are rational, so are $\cos(|r|)$ and $\sin(|r|)$, and thus we can find a positive integer n such that $n \cdot \sin(|r|)$ and $n \cdot \cos(|r|)$ are integers. Apply the theorem, with $c = |r|$ and $f(x) = n \cdot \sin(x)$, to conclude that $|r|$ is irrational, hence that r is irrational.

To prove (b), observe that $r > 1$ without loss of generality, so that $\ln(r) > 0$. Write $r = m/n$ for some positive integers m, n, and apply the theorem with $c = \ln(r)$ and $f(x) = n \cdot e^x$.

Reference

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ANSWER TO PHOTO ON PAGE 715

Kazimir Kuratowski.