pfsl23.tex

Lecture 23. 28.11.2013 (half-hour: Problems)

So take $\sigma = 0$ below.

FV (Finite Variation, on finite time-intervals, a.s.).

$$\int \min(1, |x|)\mu(dx) < \infty.$$

IA (Infinite Activity). Here there are infinitely many jumps in finite time-intervals, a.s.: μ has infinite mass, equivalently $\int_{-1}^{1} \mu(dx) = \infty$: $\mu(\mathbf{R}) = \infty$.

FA (Finite Activity). Here there are only finitely many jumps in finite time, a.s., and we are in the compound Poisson case: $\mu(\mathbf{R}) < \infty$.

Stable processes. Note that all stable processes have infinite activity, as $\int dx/|x|^{1+\alpha}$ diverges at the origin since $\alpha > 0$. As $\int dx/|x|^{\alpha}$ diverges at 0 if $\alpha \geq 1$ but converges if $0 < \alpha < 1$, stable processes have IV for $1 \leq \sigma \leq 2$, FV for $0 < \alpha < 1$ (as for the stable subordinator, where the paths are not just FV but monotone).

VII. MARKOV CHAINS

1. Notation and Examples.

Recall that a *Markov process* in discrete time is a stochastic process $X = (X_n)$ with

$$P(X_n \in A | X_m, B) = P(X_n \in A | X_m)$$

for time $m < n$, where B denotes an event depending on values of X for time $< m$ (think of m here as the present, $n > m$ as in the future, and B as in the past). In words: the conditional probability of the future given the present and the past is that same as that of the future given the present only. That is, where you are is all that counts, not how you got there.

The values taken by the process $X = (X_n)$ may be discrete or continuous. The discrete case is easier, so we begin with it. The X-values form a finite or countable set, $\{x_n\}$. It is usually possible to disregard the precise values x_n and replace them by *labels*, n. Usually the label set will be the natural numbers $\mathbf{N}$, $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, $\mathbf{N}_n := \{1, \ldots, n\}$, $\mathbf{N}_n^0 := \mathbf{N}_n \cup \{0\}$, or $\mathbf{Z}$, depending on context. In general, write E_k for state k.

Example: Simple random walk on $\mathbf{Z}$: the label set is $\mathbf{Z}$, and so is the value set.

It is conventional to refer to a Markov process with both time and state discrete as a *Markov chain.* To describe such a Markov chain, we need the *transition probabilities* $P(X_{n+1} = j|X_n = i)$. We confine ourselves here, for simplicity, to the most important special case, when these transition probabilities are *stationary* – do not depend on n:

$$p_{ij} := P(X_{n+1} = i|X_n = j) = P(i \to j),$$

in an obvious notation. We assemble these transition probabilities (p_{ij}) into a *transition (probability) matrix*

$$P := (p_{ij})$$

(the matrix whose (i, j) element is p_{ij}). Similarly, we define the *n-step* transition probabilities

$$p_{ij}^{(n)} := P(X_{m+n} = j|X_m = i) = P(i \to j \text{ in } n \text{ steps})$$

(by stationarity, this does not depend on m), and form the *n-step transition (probability) matrix* $P(n) := (p_{ij}^{(n)})$:

$$P^{(n)} := (p_{ij}^{(n)}).$$

Note. 1. Here i and j run through the possible states of the chain. Usually, these will be labelled $\{1, 2, \ldots, N\}$ in the finite case, $\{1, 2, \ldots\}$ in the (countably) infinite case. It pays to keep the notation flexible, to cover both cases.
2. Much of what we will cover applies to both finite and infinite chains. Finite chains have certain special properties (VII.4). We are also much more familiar with finite matrices than with infinite ones. Bear in mind that in the infinite case, matrices and sums over states are both infinite.
3. A matrix is called *stochastic* if its entries are non-negative and sum to 1. The transition probability matrix $P = (p_{ij})$ of a Markov chain is stochastic, as

$$p_{ij} = P(i \to j) \geq 0,$$
$$\sum\nolimits_j p_{ij} = \sum\nolimits_j P(i \to j) = 1$$

– as the chain has to go somewhere. Infinite matrices are difficult in general, but stochastic matrices are much simpler, and are often no harder to handle than finite matrices.

Theorem (Chapman-Kolmogorov equations).

$$P^{(n)} = P^n :$$

the n-step transition probability matrix is the nth matrix power of the (1-step) transition probability matrix.