ullsoln6b.tex

SOLUTIONS 6b. 1.11.2017

Q1. *Renewal function of the exponential law.*

Recall the exponential law $E(\lambda)$: density and distribution function

$$f(x) = \lambda e^{-\lambda x}, \qquad F(x) = 1 - e^{-\lambda x} \quad (x > 0).$$

The Laplace transform of f (Laplace-Stieltjes transform, or LST, of F) is

$$\begin{aligned}\hat{F}(s) &= \int_{[0,\infty)} e^{-\lambda s} dF(x) \\ &= \int_{[0,\infty)} \lambda e^{-\lambda s}.e^{-\lambda x} dx \\ &= \lambda/(\lambda + s).\end{aligned}$$

The LST of the nth convolution F^{*n} of F is the nth power of this. Summing over n: for the renewal function

$$U(x) = \sum_{n=0}^{\infty} F^{*n}(x),$$

its LST is

$$\begin{aligned}\hat{U}(s) &= \sum_{n=0}^{\infty} (\lambda/(\lambda + s))^n = \frac{1}{(1 - \lambda/(\lambda + s))} \\ &= \frac{1}{s/(\lambda + s)} = (\lambda + s)/s :\end{aligned}$$

$$\hat{U}(s) = 1 + \lambda/s.$$

Now with δ_0 the Dirac measure at 0 (probability measure with all its mass 1 at the origin 0), its LST is 1. The LST of Lebesgue measure on $(0, \infty)$ (the measure with mass x on $(0, x)$) is, putting $u := sx$,

$$\int_0^{\infty} e^{-sx} dx = \int_)^{\infty} e^{-u} du/s = 1/s.$$

Combining, for $F = E(\lambda)$,

$$U(x) = \delta_0(x) + \lambda x \qquad (x \geq 0).$$

Interpretation. The first term 1 here just says that, by definition, there is always a renewal at time 0 (we always start with a new item). After that, because the hazard rate for $E(\lambda)$ is the constant λ, the expected number of renewals in $(0, x)$ is λx.

Q2. *Renewal Theorem.*

Recall that the mean μ of $E(\lambda)$ is, putting $u := \lambda x$ as above,

$$\mu = \int_0^\infty x.f(x)dx = \int_0^\infty x.\lambda e^{-\lambda x}dx = \int_0^\infty ue^{-u}du/\lambda = 1/\lambda$$

(the integral is $\Gamma(2) = 1! = 1$, or check by integration by parts). So the Renewal Theorem holds:

$$E(t) = E[N(t)] = 1 + t\lambda = 1 + t/\mu \sim t/\mu \quad (t \to \infty).$$

Blackwell's renewal theorem holds here with equality, as

$$U(x + h) - U(x) = h\lambda = h/\mu.$$

The Key Renewal Theorem holds, as

$$Z(t) = (z{*}U)(t) = \int_0^t z(u).u(t-u)du = \int_0^t z(u).\lambda \to \int_0^\infty z(u)du/\mu \quad (t \to \infty).$$

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