mpc2soln7.tex

SOLUTIONS 7. 28.11.2011

Q1.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$= (3-\lambda)[-\lambda(3-\lambda)-4]-2[2(3-\lambda)-4.2]+4[2.2+4\lambda] = (3-\lambda)(\lambda^2-3\lambda-4)-2(-2\lambda-2)+4(4+4\lambda)$
$= (3-\lambda)(\lambda^2-3\lambda-4)+20\lambda+20 = -\lambda^3+\lambda^2(3+3)+\lambda(4-9+20)+(-12+20)$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8.$$

By inspection (from a few trial values), $\lambda = -1$ is a solution (RHS is 1+6-15+8 $= 0$). So RHS is

$$-(\lambda^3 - 6\lambda^2 - 15\lambda - 8) = -(\lambda + 1)(\lambda^2 + c\lambda - 8)$$

for some c. The coefficient of λ^2 gives $-6 = c + 1$, $c = -7$. The quadratic is

$$\lambda^2 - 7\lambda - 8 = (\lambda + 1)(\lambda - 8),$$

so the other roots are $\lambda = 8$ and -1:
the eigenvalues are 8 (simple) and -1 (double).
For $\lambda = 8$, we have to solve

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8x_1 \\ 8x_2 \\ 8x_3 \end{pmatrix}.$$

The e-vectors are determined only to within a scalar multiple. So we can take, say, $x_3 = 1$ (this will fail only if $x_3 = 0$, in which case we choose x_1 or x_2). We then only need two equations. The first two give

$$\begin{aligned} 5x_1 - 2x_2 &= 4, \\ 2x_1 - 8x_2 &= -2. \end{aligned}$$

Hence ["2(1) - 5(2)" to eliminate x_1] $(-4+40)x_2 = 18$, $36x_2 = 18$, $x_2 = 1/2$. Then back-substitution gives $x_1 = 1$ [check]. So the e-vector is $(1, 1/2, 1)$, or (doubling to clear fractions):

e-value $\lambda = 8$ has e-vector $(2, 1, 2)$.
For $\lambda = -1$, we get similarly

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix}.$$

All three of these equations reduce to

$$2x_1 + x_2 + 2x_3 = 0$$

[check]. So we can take x_1, x_3 arbitrarily, and solve for x_2. Two independent solutions are: $x_1 = 1$, $x_3 = 0$, and then $x_2 = -2$: e-vector $(1, -2, 0)$;
$x_1 = 0$, $x_3 = 1$, and then $x_2 = 0$: e-vector $(0, -2, 1)$.

Q2. Write a_n, b_n for the Fourier cosine and sine coefficients of the given f. Then

$$a_0 = \frac{1}{\pi}\int_0^\pi x dx = \frac{1}{\pi}[\frac{1}{2}x^2]_0^\pi = \frac{\pi}{2},$$

$$a_n = \frac{1}{\pi}\int_0^\pi x \cos \ nx dx = \frac{1}{n\pi}\int_0^\pi x d \sin nx = \frac{[x \sin nx]_0^\pi}{n\pi} - \frac{1}{n\pi}\int_0^\pi \sin nx dx$$

$$= \frac{1}{n^2\pi}[\cos nx]_0^\pi = \frac{\cos n\pi - 1}{n^2\pi} = \frac{(-1)^n - 1}{n^2\pi},$$

$$b_n = \frac{1}{\pi}\int_0^\pi x \sin \ nx dx = -\frac{1}{n\pi}\int_0^\pi x d \cos nx = -\frac{[x \cos nx]_0^\pi}{n\pi} + \frac{1}{n\pi}\int_0^\pi \cos nx dx$$

$$= -\frac{\cos n\pi}{n} + \frac{1}{n^2\pi}[\sin nx]_0^\pi = \frac{\cos n\pi}{n} = \frac{(-1)^n - 1}{n}.$$

So

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^\infty \frac{[(-)^n - 1]\cos nx}{\pi n^2} \cos nx - \sum_{n=1}^\infty \frac{(-)^n}{n} \sin nx.$$

Note. $(-)^n = 1$ (n even), -1 (n odd), so $(-)^n - 1 = 0$ (n even), -2 (n odd). So also

$$f(x) = \frac{\pi}{4} - 2\sum_{m=1}^\infty \frac{\cos(2m-1)x}{\pi(2m-1)^2} \cos nx - \sum_{n=1}^\infty \frac{(-)^n}{n} \sin nx.$$

NHB