liv11.tex Week 11: 29.4.2013

§4. Girsanov's Theorem

Consider first ([KS], §3.5) independent $N(0,1)$ random variables $Z_1, \cdots, Z_n$ on $(\Omega, \mathcal{F}, \mathcal{P})$. Given a vector $\mu = (\mu_1, \cdots, \mu_n)$, consider a new probability measure $\tilde{P}$ on $(\Omega, \mathcal{F})$ defined by

$$\tilde{P}(d\omega) = \exp\{\Sigma_1^n \mu_i Z_i(\omega) - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(d\omega).$$

This is a positive measure as $\exp\{.\} > 0$, and integrates to 1 as $\int \exp\{\mu_i Z_i\} dP = \exp\{-\frac{1}{2}\mu_i^2\}$, so is a probability measure. It is also equivalent to P (has the same null sets), again as the exponential term is positive. Also

$$\begin{aligned}
\tilde{P}(Z_i \in dz_i, \quad i = 1, \cdots, n) &= \exp\{\Sigma_1^n \mu_i Z_i - \frac{1}{2}\Sigma_1^n \mu_i^2\}.P(Z_i \in dz_i, \quad i = 1, \cdots, n) \\
&= (2\pi)^{-\frac{1}{2}n} \exp\{\Sigma \mu_i z_i - \frac{1}{2}\Sigma \mu_i^2 - \frac{1}{2}\Sigma z_i^2\} \Pi dz_i \\
&= (2\pi)^{-\frac{1}{2}n} \exp\{-\frac{1}{2}\Sigma (z_i - \mu_i)^2\} dz_1 \cdots dz_n.
\end{aligned}$$

This says that if the Z_i are independent $N(0,1)$ under P, they are independent $N(\mu_i, 1)$ under $\tilde{P}$. Thus the effect of the *change of measure* $P \to \tilde{P}$, from the original measure P to the *equivalent* measure $\tilde{P}$, is to *change the mean*, from $0 = (0, \cdots, 0)$ to $\mu = (\mu_1, \cdots, \mu_n)$.

This result extends to infinitely many dimensions - i.e., from random vectors to stochastic processes, indeed with random rather than deterministic means. We quote:

THEOREM (Girsanov's Theorem). Let $(\mu_t : 0 \le t \le T)$ be an adapted (e.g., left-continuous) process with $\int_0^T \mu_t^2 dt < \infty \quad a.s.$, and such that the process $(L_t : 0 \le t \le T)$ defined by

$$L_t = \exp\{-\int_0^t \mu_s dW_s - \frac{1}{2}\int_0^t \mu_s^2 ds\}$$

is a martingale. Then, under the probability P_L with density L_T relative to P, the process W^* defined by

$$W_t^* := W_t + \int_0^t \mu_s ds, \qquad (0 \le t \le T)$$

is a standard Brownian motion.

Here, L_t is the Radon-Nikodym derivative of P_L w.r.t. P on the σ-algebra $\mathcal{F}_t$.

In particular, for μ_t constant ($= \mu$), change of measure by introducing the Radon-Nikodym derivative $\exp\{-\mu W_t - \frac{1}{2}\mu^2\}$ corresponds to a change of drift from 0 to μ.

Girsanov's Theorem (or the Cameron-Martin-Girsanov Theorem) is formulated in varying degrees of generality, and proved, in [KS, §3.5], [RY, VIII].

Consider now the Black-Scholes model, with dynamics

$$dB_t = rB_t dt, \qquad dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Discounting the prices by e^{rt}, the discounted asset prices $\tilde{S}_t := e^{-rt} S_t$ have dynamics given, as before, by

$$\begin{aligned} d\tilde{S}_t &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= -r\tilde{S}_t dt + \mu \tilde{S}_t dt + \sigma \tilde{S}_t dW_t \\ &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dW_t. \end{aligned}$$

Now the drift term – the dt term – here prevents $\tilde{S}_t$ being a martingale; the noise – dW_t – term gives a stochastic integral, which is a martingale. Girsanov's theorem suggests the change of measure from P to the equivalent martingale measure (or risk-neutral measure) P^* that makes the discounted asset price a martingale. This
(i) gives directly the continuous-time version of the Fundamental Theorem of Asset Pricing: *to price assets, take expectations of discounted prices under the risk-neutral measure*;
(ii) allows a probabilistic treatment of the Black-Scholes model, avoiding the detour via PDEs of §2, §3.

THEOREM (Representation Theorem for Brownian Martingales). Let $(M_t : 0 \le t \le T)$ be a square-integrable martingale with respect to the Brownian filtration $(\mathcal{F}_t)$. Then there exists an adapted process $H = (H_t : 0 \le t \le T)$ with $E \int H_s^2 ds < \infty$ such that

$$M_t = M_0 + \int_0^t H_s dW_s, \qquad 0 \le t \le T.$$

That is, all Brownian martingales may be represented as stochastic integrals with respect to Brownian motion.

We refer to, e.g., [KS], [RY] for proof. The multidimensional version of the result also holds, and may be proved in the same way.

The economic relevance of the Representation Theorem is that it shows that the Black-Scholes model is *complete* – that is, that equivalent martingale measures are unique. Mathematically, the result is purely a consequence of properties of the Brownian filtration. The desirable mathematical properties of Brownian motion are thus seen to have hidden within them desirable economic and financial consequences of real practical value.

§5. American Options; Exotic Options

American Calls.

As in discrete time, these are equivalent to European calls - there is no advantage in exercise before expiry. See e.g. [SKKM], II, §8, esp. p. 94.

American Puts.

The results on Snell envelopes, least supermartingale majorants etc. extend to continuous time. See [SKKM] II, or for a survey, and references,
[M] MYNENI, R. (1992): The pricing of the American option. *Annals of Applied Probability* **2**, 1-23.

Pricing American calls is an *optimal stopping problem*: one wants to choose the exercise time so as to maximise the payoff. There is a whole subject on optimal stopping; see e.g. the book by Peskir & Shiryaev, [PS]. There are links with real (investment) options (below).

Exotic options.

The options considered so far (put/call, European/American) are so standard now as to be commonly called *vanilla options*. More complicated types of option are called *exotic options*. We turn to some of the commoner types below.

Asian options.

Here the payoff is a function of the *average price* of the underlying between contract time and expiry time. Asian options are widely used in practice - for instance, for oil and foreign currencies. The averaging complicates the mathematics, but, e.g., protects the holder against speculative attempts to manipulate the asset price near expiry. For details and references, see e.g.
[RS] ROGERS, L. C. G. & SHI, Z. (1995): The value of an Asian option. *J.*

Applied Probability **32**, 1077-1088.

Asian options are *pathwise* options, as their payoff depends on the whole sample path of the price process, rather than just the terminal value at expiry.

Lookback options.

It is natural to look back with the benefit of hindsight, and wish that we had acted optimally throughout – bought at the low, sold at the high, etc. How nice it would be to have a piece of paper that entitled us to the benefits that would have resulted if we had ... Such things exist, and are called *lookback options.* These are again exotics, and – like the Asian options above – are pathwise options. Their theory involves the powerful *reflection principle* (below) for Brownian motion (loosely: if we reflect a Brownian path in a mirror, we get another Brownian path).

Barrier options.

Another common type of pathwise exotic is that of *barrier options*, where whether or not an option ends in the money depends on whether or not some price level is crossed up to expiry. They may 'knock in' or 'knock out' (become in or out of the money), and the barrier may be crossed from above or below. So there are four types: 'up and in', 'up and out', 'down and in', 'down and out'. Again, their theory involves the reflection principle.

Sometimes there are *two* barriers, one above and one below. One can use the reflection principle at each barrier. As one might expect (from sitting in a barber's shop, where one has mirrors in front and behind, and sees an infinite sequence of reflections), this involves infinite summations.

The Reflection Principle.

There is a brief account of this in [BK] 6.3.3, in connection with barrier options. As mentioned there, the method goes back to Kelvin's *method of images* in electrostatics in 1848, although it is often known by the name of Désiré André (1887).

The idea of reflecting a Brownian path in a mirror can be formalised, by using the Markov property to re-start the process from the time when it hits the mirror (at level $b > 0$, say). But the time, τ_b say, that BM takes to reach level b is random. So one actually needs the *strong Markov property* – the Markov property applied at a stopping time. This can be justifeid: BM does have the strong Markov property (and so, more generally, do Lévy processes).

Another approach is to work discretely, and use simple random walk as a discrete alternative (or approximation) to BM. Here things are simpler and more elementary: one can reduce the problem to *combinatorics* (counting paths). See e.g. [GS] §3.10. Incidentally, this book (though not the 1st or 2nd editions) contains a brief account of Black-Scholes theory (§13.10).

6. Real options (Investment options)

The options considered in Ch. IV concern financial *derivatives* (so called because they derive from the underlying fundamentals such as stock). We turn now to options of another kind, concerned with business decision-making. Typically, we shall be concerned with the decision of whether or not to make a particular investment, and if so, when. Because these options concern the real economy (of manufacturing, etc.) rather than financial markets such as the stock market, such options are often called *real options.* But because they typically concern investment decisions, they are also often called *investment options.* There is a good introductory treatment in [D&P].

The key features are as follows. We are contemplating making some major investment – buying or building a factory, drilling an oil well, etc. While a speculator may consider buying a firm which he thinks is undervalued, or whose assets he thinks are under-utilized, breaking it up, and selling off the parts at a profit ('asset stripping'), we confine attention here to a more conventional situation – the management of a firm is considering some major investment to further their core economic activity. While if the decision goes wrong it may be possible to recoup some of the cost, much or most of it will usually be irrecoverable (a sunk cost – as with an oil well). So the investment is it irreversible – at least in part. Just as stock prices in Ch. IV are uncertain – so we model them as random, using some stochastic process – here too, the future profitability of the proposed investment is *uncertain.* Finally, we do not have to act now, or at any specified time. We may choose to delay investment,
(a) to gather more information, to help us assess the project, or
(b) to continue to generate interest on the capital we propose to invest.
So we must recognize, and feed into the decision process, the value of *waiting for further information.* When we commit ourselves and make the decision to invest, it is not just the sunk cost that we lose – we lose the valuable option to wait for new information.

This situation is very reminiscent of the American options of IV.9 with an infinite time-horizon. With such an American *call*, we have the right to buy at a specified price at a time of our choosing (or indeed, not to buy). There, we carried out a full analysis. We formulated an optimal stopping problem, and solved it as a free boundary problem, using the principle of smooth fit. We can apply the same method here (as is done in detail in [D&P], Ch. 5.

We suppose the cost of the investment is I. We suppose that the value of the project is given by a geometric Brownian motion, $X = (X_t) \sim GBM(\mu, \sigma)$ (the value of a project is uncertain for the same reasons that stock prices are uncertain; we model them both as stochastic processes; GBM is the default option here, just as in the Black-Scholes theory of Ch. IV). If τ is the investment time we choose, we want to maximize

$$V(X) := \max_{\tau} \ E[(X_\tau - I)e^{-r\tau}],$$

with r the riskless rate (discount rate) as before. Now if $\mu \leq 0$ the value of the project will fall, so we should invest immediately if $X_0 > I$ and not invest if not. If $\mu > r$, the growth of X will swamp the investment cost I and more than offset the discounting, so we should invest and there is no point in waiting. So we take $\mu \in (0, r]$. The analogues here of (i)-(v) in Ch. IV are

$$\frac{1}{2}\sigma^2 x^2 V''(x) + \mu x V'(x) - rV, \tag{i}$$

$$V(0) = 0, \tag{ii}$$

$$V(x^*) = x^* - I, \tag{iii}$$

$$V'(x^*) = 1 \qquad \text{(smooth pasting)}. \tag{iv}$$

(for (ii), the GBM does not hit 0, but if it approaches 0, so will the value of the project, so (ii) follows from this by continuity). For (iii), this is the value-matching condition: on investment, the firm receives the net pay-off $X^* - I$. As before, we use a trial solution $V(x) = Cx^p$. Substituting in (i), this is a solution if p satisfies the quadratic

$$Q(p) := \frac{1}{2}\sigma^2 p(p-1) + \mu p - r = 0.$$

The product of the roots is negative, and $Q(0) = -r < 0$, $Q(1) = \mu - r < 0$. So one root $p_1 > 1$ and the other $p_2 < 0$. The general solution is $V(x) = C_1 x^{p_1} + C_2 x^{p_2}$, but from $V(0) = 0$ we get $C_2 = 0$, so $V(x) = C_1 x^{p_1}$, or $V(x) = Cx^{p_1}$. If x^* is the critical value at which it is optimal to invest,

$$V(x^*) = x^* - I,$$

and 'smooth pasting' gives

$$V'(x^*) = 1.$$

From these two equations, we can find C and x^*. The second is

$$V'(x^*) = Cp_1(x^*)^{p_1-1} = 1, \qquad C = (x^*)^{1-p_1}/p_1.$$

Then the first gives

$$C(x^*)^{p_1} = x^* - I, \qquad x^*/p_1 = x^* - I, \qquad x^* = \frac{p_1}{(p_1-1)}I.$$

The main feature here is the factor

$$q := p_1/(p_1-1) > 1$$

by which the value must exceed the investment cost I before investment should be made (q is used because this is related to "Tobin's q" in Economics). One can check that q increases with σ (the riskier the project, the more reluctant we are to invest), and also q increases with r (as then investing our capital risklessly becomes more attractive). Then the critical threshold above which it is optimal to invest is

$$x^* = qI.$$

Also

$$C = (qI)^{1-p_1}/p_1, \qquad V(x) = (qI)^{1-p_1}x^{p_1}/p_1.$$

Note. 1. For an overview of real options, see e.g.
N. DUNBAR: The power of real options. *RISK* **13** (6) (2000), 20-22.
2. The results above show that the traditional *net present value* (NPV – accountancy-based) approach to valuing real options is misleading – see [DP].

§7. Extensions

1. Discontinuities in stock price.

The Black-Scholes model relies on stock-price movements being continuous. If stock prices jump – for instance, in response to abrupt events such as outbreaks of war/devaluations/natural disasters such as major earthquakes/the oil-price crisis of 1973, etc. – the Black-Scholes analysis fails. In particular, the market will no longer be complete, and it will no longer be possible to hedge against a contingent claim by replicating it. Because of incompleteness, there will be many equivalent martingale measures, so many prices. One should seek the optimal measure, which minimises risk or maximises payoff (minimal equivalent martingale measures – this involves the Föllmer-Schweizer decomposition).

One model for price discontinuities is a *Poisson* model, in which 'shocks' occur, but prices move in a Black-Scholes way between shocks.

Recent work by Barndorff-Nielsen and Shephard, by Eberlein, by Bingham & Kiesel and others, has focussed on *Lévy* models (stationary independent increments - generalising Brownian motion to include jumps). This is sensible because
(i) when prices are examined in sufficient detail, they are in fact seen to be discontinuous, the jumps resulting from the individual transactions by which the assets are traded,
(ii) the extra flexibility provided by the larger class of Lévy models gives more scope for model fitting to observed data. The subclass of *hyperbolic* models seems particularly well-suited here.

There is a whole field of such *Lévy finance*. For background and details, see e.g. [BK] §5.5.

2. Varying or random interest rates.

We have assumed that the interest rate r is a positive constant. It is more realistic, though more complicated, to let $r = r_t$ vary with time. More generally still, r may be random, i.e. $r = (r_t) = (r_t(\omega))$ may be a stochastic process.

A number of possible models for such interest-rate processes r have been proposed and studied. For background and details, see
[HJM] HEATH, D., JARROW, R. & MORTON, A. (1992): Bond pricing and the term structure of interest rates. A new methodology for contingent claims evaluation. *Econometrica* **60**, 77-106,
[M] MILTERSEN, K. R. (1994): An arbitrage theory of the term structure of interest rates. *Ann. Appl. Probab.* **4**, 953-967.

3. Transaction costs.

Real markets suffer from friction: there are actual costs in trading and making transactions, which complicate the theory. For further detail, see e.g.
[DN] DAVIS, M. H. A. & NORMAN, A. R. (1990): Portfolio selection with

transaction costs. *Math. Oper. Research* **15**, 676-713,
[SS] SHREVE, S. E. & SONER, H. M. (1994): Optimal investment and consumption with transaction costs. *Ann. Appl. Probab.* **4**, 609-692.

4. Higher interest rates for borrowing than lending.

Real financial markets have higher interest rates for borrowing than for lending (which is how banking works), and this introduces another kind of friction into the market. For further detail, see e.g.
[CK] CVITANOVIC, J. & KARATZAS, I. (1993): Hedging contingent claims with constrained portfolios, *Ann. Appl. Prob.* **3**, 652-681, §9.

5. Stochastic volatility.

The Black-Scholes theory above - in discrete or continuous time – has involved the volatility - the parameter that describes the sensitivity of the stock price to new information, to the market's assessment of new information. Volatility is so important that it has been subjected to intensive scrutiny, in the light of much real market data. Alas, such detailed scrutiny reveals that volatility is not really constant at all - the Black-Scholes theory over-simplifies reality. (This is hardly surprising: real financial markets are more complicated than the contents of this course, as they involve *investor psychology*, rather than straight mathematics!) One way out is to admit that volatility is *random* (stochastic), and then try to model the stochastic process generating it. Volatility exhibits *clustering*, linked to *mean reversion*, so *Ornstein-Uhlenbeck* models are useful here. Such *stochastic volatility models* are topical today.

6. Stochastic Volatility (SV); ARCH and GARCH

There are a number of *stylised facts* in mathematical finance. E.g.:
(i). Financial data show *skewness*. This is a result of the asymmetry between profit and loss (large losses are lethal!)
(ii). Financial data have much *fatter tails* than the normal (Gaussian). We have discussed this in I.5.
(iii) Financial data show *volatility clustering*. This is a result of the economic and financial environment, which is extremely complex, and which moves between good times/booms/upswings and bad times/slumps/downswings. Typically, the market 'gets stuck', staying in its current state for longer than is objectively justified, and then over-correcting. As investors are highly sensitive to losses (see (i) above), downturns cause widespread nervousness,

which is reflected in higher volatility. The upshot is that good times are associated with periods of growth but low volatility; downturns spark extended periods of high volatility (as well as stagnation, or shrinkage, of the economy).

ARCH and *GARCH*. We turn to models that can incorporate such features.

The model equations are (with Z_t ind. $N(0,1)$)

$$X_t = \sigma_t Z_t, \qquad \sigma_t^2 = \alpha_0 + \sum_1^p \alpha_i X_{i-1}^2, \qquad (ARCH(p))$$

while in $GARCH(p,q)$ the σ_t^2 term becomes

$$\sigma_t^2 = \alpha_0 + \sum_1^p \alpha_i X_{i-1}^2 + \sum_1^q \beta_j X_{t-j}^2. \qquad (ARCH(p))$$

The names stand for (generalised) autoregressive conditionally heteroscedastic (= variable variance). These are widely used in Econometrics, to model *volatility clustering* – the common tendency for periods of high volatility, or variability, to cluster together in time.

7. Volatility Modelling

In the standard Black-Scholes theory we have developed, volatility σ is *constant*. Thus a graph of volatility against strike K (or stock price S) should be flat. But typically it isn't, and displays curvature. Such volatility curves often turn upwards at both ends ('volatility smile'); there may well be asymmetry ('volatility smirk').

As above, it may be useful to model volatility stochastically, and use an SV model. However, the driving noise in this model will have a volatility of its own ('vol of vol'), etc.

Practitioners often use *computer graphics* to represent *volatility surfaces* – the three-dimensional equivalents of graphs, where e.g. σ is graphed against K and S.

The subject is too big to pursue further here; there is a good account (mixing theory with practice) in

[G] Jim GATHERAL: *The volatility surface: A practitioner's guide*. Wiley 2006.

8. Volatility Index (VIX).

Just as there are indices of stocks (FTSE, S&P, DAX etc.), there is also a volatility index (VIX). Just as options on the Footsie etc. can be traded, so too can options on VIX.

It may amuse you to know that VIX has already entered popular fiction. I recommend the novel
[H] Robert HARRIS: *The fear index*. Hutchinson, 2011.

9. Portfolios and Multivariate Time Series.

By Markowitzian diversification, we should carry a portfolio of risky assets. Its evolution over time involves two areas of Statistics, Time Series and Multivariate Analysis. Suitable stochastic models for such multivariate financial time series are still being studied. Note for now that the more Statistics you can learn here, the better.

Postscript.

1. One recent book on Financial Mathematics describes the subject as being composed of three strands:
arbitrage – the core economic concept, which we have used throughout,
martingales - the key probabilistic concept, which we have used from Ch. III on,
numerics. Finance houses in the City use *models*, which they need to *calibrate to data* – a task involving both statistical and numerical skills, and in particular an ability to *programme*. Numerical skills and programming ability are at a premium here.
2. You will probably already have experience with at least one general mathematics package (e.g., Mathematica and/or Maple) (if not: get it, a.s.a.p.!).

You may already have some knowledge of Numerical Analysis. This is the theory behind computation. Each branch of mathematics you have studied – e.g., Linear Algebra, PDEs – has its numerical counterpart; so too do SDEs.

You may have encountered *simulation*, also known as *Monte Carlo*, and/or a branch of Probability and Statistics called *Markov Chain Monte Carlo (MCMC)*. These are computer-intensive methods to investigate the efficiency of proposed procedures, where these are too complicated for one to do the relevant mathematics and find an explicit solution.

The leaders of R & D teams in the City need to be expert at both stochastic modelling (e.g., to propose new products), and simulation (to evaluate how these perform). Most of the ones I know use Matlab for this.

At a lower level, quantitative analysts (quants) working under such leaders will certainly need expertise in a computer language; C++ is the industry

standard. If you are thinking of a career in Mathematical Finance, you are strongly advised to learn C++, as soon as possible, and for academic credit.
3. The contents of this course have been concerned with *equity markets* – with *stocks*, and financial derivatives of them – options on stocks, etc. The relevant mathematics – complicated though some of it is, at least first time around – is *finite-dimensional*. Lurking in the background is the corresponding theory of *bond markets* ('money markets': bonds, gilts etc., where *interest rates* dominate), and the relevant options – *interest-rate derivatives*, together with questions of *foreign exchange* between different currencies ('forex'). This is where a lot of the interest in the financial sector lies. Alas, the resulting mathematics (which is highly topical, and so in great demand in the City!) is *infinite-dimensional*, and so much harder than the equity-market theory we have done. However, the underlying principles are basically the same. One has to learn to walk before one learns to run, and equity markets serve as a preparation for money markets.

The aim of this lecture course is simple. It is to familiarize the student with the basics of Black-Scholes theory, as the core of modern finance, and with the mathematics necessary to understand this. The motivation driving the ever-increasing study of this material is the financial services industry and the City. I hope that any of you who seek City careers will find this introduction to the subject useful in later life. NHB, 2013