pfsl20(15).tex

Lecture 20. 30.11.2015

This gives us an easy way, in the persistent case, to tell the two sub-cases of null and positive apart. If j is null,

$$u_{n,j} \to 0,$$

while if j is positive,

$$u_{n,j} \to 1/\mu_j > 0$$

(in the aperiodic case, with a similar statement in the periodic case). This will be useful below. It also explains the terms *null* and *positive*.

We introduce one more term (the motivation is from Physics, specifically Statistical Mechanics, to which we return later). A state is called *ergodic* if it is aperiodic and positive recurrent (= persistent).

When a chain is irreducible (so each state can be reached from every other state, eventually), we quote that all states have the same character: all aperiodic/periodic with the same period, all transient, all recurrent, all null, all positive, or all ergodic. Results of this type are called *solidarity theorems*; we shall assume them. We then call an irreducible chain aperiodic etc. if all its states are.

3. Limit distributions and invariant (= stationary) distributions

Recall that the transition probability matrix P of a Markov chain has row-sums 1. This means that if we post-multiply P by the column-vector $\mathbf{1}$ all of whose elements are 1,

$$P\mathbf{1} = \mathbf{1}.$$

This says that 1 is an *eigenvalue*, with right *eigenvector* $\mathbf{1}$.

It turns out that this eigenvalue is special, and that the long-term behaviour of the chain is dominated by the eigenstructure of P. The key result is the following classical theorem, which (perhaps surprisingly) is a result of Linear Algebra. It is due to Oskar PERRON (1880-1975) in 1907 and Georg FROBENIUS (1849-1917) in 1908 and 1912.

Theorem (Perron-Frobenius Theorem). Let P be the transition probability matrix of a finite irreducible Markov chain with period d.
(i) $\lambda_1 = 1$ is always an eigenvalue of P; if $d > 1$, so too are the other dth roots of unity, $\lambda_2 = \omega, \ldots, \lambda_d = \omega^{d-1}$, where $\omega := \exp\{2\pi i/d\}$.
(ii) All other eigenvalues λ_j have modulus $|\lambda_j| < 1$.

The eigenvalue (e-value) 1 is called the *Perron-Frobenius (PF)* e-value.

For proof of the PF theorem, see e.g. [HJ], 8.4, [Sen].

Theorem. In an ergodic chain (not necessarily finite):
(i) there exists

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j,$$

independent of i.
(ii) $\pi_j > 0$, and $\sum \pi_j = 1$.
(iii) $\pi_j = \sum_i \pi_i p_{ij}$ for each j, or writing π for the row-vector of the π_j,

$$\pi = \pi P.$$

Thus π is the *left* eigenvector for the Perron-Frobenius eigenvalue 1.

Conversely, if (ii), (iii) hold for an irreducible periodic chain, (i) holds, with $\pi_k = 1/\mu_k$, μ_k the mean recurrence time of state k, and the chain is ergodic.

Proof. $P_{ij}(s) = F_{ij}(s)P_{jj}(s)$. By the Erdös-Feller-Pollard theorem,

$$p_{jj}^{(n)} \to \pi_j = 1/\mu_j \qquad (n \to \infty),$$

and $\pi_j > 0$ as $\mu_j < \infty$ (the states are positive, as the chain is ergodic). So if

$$f_{ij} = F_{ij}(1) = P_i(\text{reach } j),$$

$$p_{ij}^{(n)} \sim f_{ij} p_{jj}^{(n)} \to f_{ij}\pi_j.$$

But here $f_{ij} = 1$ as the chain is irreducible (each state is accessible from every other), so

$$p_{ij}^{(n)} \to \pi_j \qquad (n \to \infty)$$

for each i, proving (i). Now

$$1 = \sum_{j=1}^{\infty} p_{ij}^{(n)} \geq \sum_{j=1}^{N} p_{ij}^{(n)}$$

for each N. Let $n \to \infty$:

$$1 \geq \sum_{j=1}^{N} \pi_j,$$

for each N. Let $N \to \infty$:

$$s := \sum_{j=1}^{\infty} \pi_j \leq 1.$$

Now

$$p_{ij}^{(n+1)} = \sum_k p_{ik}^{(n)} p_{kj} \geq \sum_{k=1}^{N} p_{ik}^{(n)} p_{kj},$$

for each N. Let $n \to \infty$:

$$\pi_j \geq \sum_{k=1}^{N} \pi_k p_{kj}.$$

Let $N \to \infty$:

$$\pi_j \geq \sum_k \pi_k p_{kj}. \tag{$*$}$$

Sum over j:

$$s = \sum_j \pi_j \geq \sum_j \sum_k \pi_k p_{kj} = \sum_k \pi_k \sum_j p_{kj} = \sum_k \pi_k = s.$$

So the inequality we got by summing $(*)$ is an *equality* (the extreme left and right are both s). So $(*)$ must itself be an equality (as inequality would contradict this). This proves (iii).

That $\sum_j \pi_j = 1$ follows formally from $\sum_j p_{ij}^{n)} = 1$ and $p_{ij}^{(n)} \to \pi_j$ $(n \to \infty)$ on interchanging $n \to \infty$ and $\sum_j$. This follows by dominated convergence – or see e.g. [GS] 6.4, pp 207-217. //

The distribution $\pi = (\pi_j)$ is called the *limit distribution* of the chain. It is also an *invariant distribution*, or *stationary distribution*, in the sense that if π is the *initial* or *starting distribution*, the distribution after one step is πP, which is also π as $\pi = \pi P$, and similarly after n steps. So:

Cor. If an ergodic chain is started in its invariant or limit distribution π, it stays in distribution π for all time.

Examples.

1. *Gambler's ruin.* There is no limit distribution. The chain is not irreducible. The extreme states 0, a are absorbing; the others are transient. There are two different invariant distributions: 'start in 0 and 'start in a'.

2. *Ehrenfest urn.* Again, there is no limit distribution: the chain is periodic with period 2. but apart from this, the chain comes as close to having a

limit distribution as possible: it has an *invariant* distribution, the *binomial* distribution

$$\pi = (\pi_j), \qquad \pi_j = 2^{-d}\binom{d}{j}.$$

Recurrence time.

The mean recurrence time of state j is $\mu_j = 1/\pi_j$. So here

$$\mu_0 = 1/\pi_0 = 1/2^{-d} = 2^d.$$

Now d is of the order of Avogradro's number (6.02×10^{23}), so 2^d is astronomically vast. So π_0 is astronomically vast – effectively infinite. This means that in practice, we will not see the chain return to its starting position if started at 0 – even though it does so (infinitely often, almost surely).

Rate of convergence.

The distribution at time n is governed by P^n by the Chapman-Kolmogorov theorem. In the periodic case, the d e-values that are dth roots of unity do not have nth powers that $\to 0$, but in the aperiodic case every e-value other than the PF e-value 1 does. From the Perron-Frobenius theorem, the *rate of convergence* is determined by the *spectral gap* $1 - |\lambda_2|$, where as usual we order the e-values in decreasing modulus:

$$\lambda_1 = 1 > |\lambda_2| \geq \ldots \geq \ldots$$

(recall there is only one e-value of modulus 1, the PF e-value 1).

Reversibility.

The chain is *reversible* if its probabilistic structure is invariant under time-reversal (i.e., the chain looks the same if run backwards in time). We quote (Kolmogorov's theorem, 1936) that this is the same as *detailed balance* (DB: Ludwig BOLTZMANN (1844-1906) in 1872):

$$\pi_i p_{ij} = \pi_j p_{ji} \qquad \text{for all } i,\ j. \tag{DB}$$

One can check (DB) here. So the Ehrenfest chain is reversible.

The interpretation of this in Statistical Mechanics is that μ_0 is the mean recurrence time of state 0, when all the $2d$ gas molecules are in one half of the container. Although this state is certain to recur, its mean recurrence time is so vast as to be effectively infinite – which explains why we do not see such states recurring in practice! This reconciles the theoretical reversibility of the model with the irreversible behaviour we observe when gases diffuse, etc. This was the Ehrenfests' motivation for their model, in 1907.