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MA414 STOCHASTIC ANALYSIS: SOLUTIONS TO MOCK EXAMINATION, 2011

Q1. (i) Q is non-negative as P is, and Q has total mass 1. If A is the disjoint union of $A_n \in \mathcal{A}$,

$$\begin{aligned} Q(A) &= E[XI(A)]/E[X] \\ &= E[X \sum_n I(A_n)]/E[X] \qquad (I(A) = \sum_n I(A_n)) \\ &= \sum_n E[XI(A_n)]/E[X] \qquad \text{(by monotone convergence applied to the partial sums)} \\ &= \sum_n Q(A_n), \end{aligned}$$

giving σ-additivity of Q. So Q is a probability measure. **[2]**
(ii) $Q(A) = \int_A X dP/E[X]$, so if $P(A) = 0$, then also $Q(A) = 0$: Q is absolutely continuous w.r.t. P. **[2]**
(iii) For each $A \in \mathcal{A}$,

$$\int_A dQ = Q(A) = E[XI(A)]/E[X] = \int_A (X/E[X])dP.$$

Since this holds for all A, $dQ = (X/E[X])dP$: $dQ/dP = (X/E[X])$. **[2]**
(iv) $E_Q[Z] = \int Z dQ = \int Z(dQ/dP)dP = \int (ZX/E[X])dP$
$= E[ZX]/E[X]$. **[2]**

Jensen's Inequality says that for ϕ convex and $X, \phi(X) \in L_1$,
$\phi(E[X]) \le E[\phi(X)]$. **[2]**

For $p > 1$, $q > 1$ also, so $\phi(x) := x^q$ is convex. So $(E_Q[Z])^p \le E_Q[Z^q]$:

$$(E[ZX^p]/E[X^p])^q \le E[Z^q X^p]/E[X^p].$$

Taking $Z := Y/X^{p-1}$ as suggested, and using $q(p-1) = p$:

$$(E[XY]/E[X^p])^q \le E[Y^q.X^p/X^{q(p-1)}]/E[X^p] = E[Y^q]/E[X^p].$$

Taking qth roots, $E[XY]/E[X^p] \le (E[Y^q])^{1/q}.E[X^p]^{-1/q} = \|Y\|_q.E[X^p]^{-1/q}$. That is,

$$E[XY] \le \|Y\|_q.E[X^p]^{1-1/q} = \|Y\|_q.E[X^p]^{1/p} = \|Y\|_q.\|X\|_p :$$

$$\|XY\|_1 \le \|X\|_p.\|Y\|_q,$$

giving Hölder's Inequality, as required. // **[15]**

Q2. *Fatou's Lemma*: if $X_n \geq 0$, $E[\liminf X_n] \leq \liminf E[X_n]$. **[3]**

Given $X, Y \geq 0$, $Y \in L_p$ for $p > 1$, and

$$xP(X \geq x) \leq E[YI(X \geq x)] \qquad \text{for all } x \geq 0. \qquad (*)$$

To prove: (i) $X \in L_p$; (ii) $\|X\|_p \leq \frac{p}{p-1}\|Y\|_p$.

Proof. Let $X_n := \min(X, n)$. Then $P(X_n \geq x) = I(n \geq x)P(X \geq x)$ and $I(X_n \geq x) = I(n \geq x)I(X \geq x)$. So if (X, Y) satisfy $(*)$ (given), so do (X_n, Y) (both sides are 0 if $x > n$ and as in $(*)$ if not), and X_n is (bounded, so) in L_q (we use Hölder's Inequality below). **[3]**
For $x \geq 0$,

$$x^p = p\int_0^x u^{p-1}du = p\int_0^\infty u^{p-1}I(x \geq u)du. \qquad [3]$$

Replace x by $X_n := \min(X, n)$ and take expectations:

$$\begin{aligned}
E[X_n^p] &= p\int_0^\infty u^{p-1}P(X_n \geq u)du \\
&\leq p\int_0^\infty u^{p-2}E[YI(X_n \geq u)]du \qquad \text{(by } (*)) \\
&= pE[Y\int_0^\infty u^{p-2}I(X_n \geq u)du] \\
&= \frac{p}{p-1}E[Y.X_n^{p-1}].
\end{aligned}$$

[7]

By Hölder's Inequality, with $q = p/(p-1)$ the conjugate index to p,

$$E(Y.X_n^{p-1}) \leq (E[Y^p])^{1/p}.(E[X_n^{(p-1)q}]^{1/q} = (E[Y^p])^{1/p}.(E[X_n^p]^{(p-1)/p} = \|Y\|_p.\|X_n\|_p^{p-1}.$$

[3]

Combining,

$$\|X_n\|_p^p \leq \frac{p}{p-1}\|Y\|_p.\|X_n\|_p^{p-1},$$

giving

$$\|X_n\|_p \leq \frac{p}{p-1}\|Y\|_p. \qquad // \qquad [3]$$

Now let $n \to \infty$: $X_n \to X$, so by Fatou's Lemma (applied to X_n^p), this extends from X_n to X, giving (i) and (ii). **[3]**

Q3. (i) *Markov's Inequality* says that for $X \geq 0$ in L_1, $\lambda > 0$,

$$P(X \geq \lambda) \leq E[X]/\lambda. \qquad [\mathbf{2}]$$

Proof.

$$E[X] = \int_\Omega X dP = \int XI(X \geq \lambda)dP + \int XI(X \leq \lambda)dP \geq \int XI(X \geq \lambda)dP$$

$$\geq \lambda \int I(X \geq \lambda)dP = \lambda E[I(X \geq \lambda)] = \lambda P(X \geq \lambda). \quad // \qquad [\mathbf{4}]$$

(ii) *L_1-convergence implies convergence in measure.* If $X_n, X \in L_1$, $X_n \to X$ in L_1, $E[|X_n - X|] \to 0$, so by Markov's Inequality

$$P(|X_n - X| \geq \epsilon) \leq E[|X_n - X|]/\epsilon \to 0,$$

so $X_n \to X$ in probability. **[5]**

(iii) *Conditional expectation is a contraction.* If $\mathcal{C}$ is a sub-σ-field and $X \in L_1$, $E|E[X|\mathcal{C}]| \leq E[|X|]$.
Proof. If $A := \{E[X|\mathcal{C}] \geq 0\}$, then $A \in \mathcal{C}$, and

$$E|E[X|\mathcal{C}]| = E[E[X|\mathcal{C}]I(A)] + E[E[X|\mathcal{C}]I(A^c)] = E[E[X]I(A)] + E[E[X]I(A^c)],$$

by definition of conditional expectation, as $A \in \mathcal{C}$. As

$$|E[XI(A)]| \leq E[|X|I(A)], \qquad |E[XI(A^c)]| \leq E[|X|I(A^c)]$$

and $I(A) + I(A^c) = 1$, this gives

$$E|E[X|\mathcal{C}]| \leq E[|X|]. \quad // \qquad [\mathbf{8}]$$

(iv) (X_n) is *uniformly integrable* (UI) if $\sup_n\{E[X_n I(|X_n| \geq x)\} \to 0$ ($n \to \infty$). **[2]**

(v) As X_n is UI and converges to X a.s., it converges to X in L_1 (lectures). So by (iii),

$$|E[X_n|\mathcal{C}] - E[X_n|\mathcal{C}]| \leq E[|X_n - X||\mathcal{C}] \leq E[|X_n - X|] \to 0,$$

so $E[X_n|\mathcal{C}] \to E[X|\mathcal{C}]$ in L_1, so by (i) in probability. // **[4]**

Q4. (i) A *stopping time* τ for a filtration $\{\mathcal{F}_t\}$ is a random variable such that for each t, $\min(t, \tau)$ (or $t \wedge \tau$) $\in \mathcal{F}_t$. **[2]**

A *local martingale* $X = (X_t)$ is a stochastic process such that for some *localising sequence* of stopping times $\tau_n \uparrow \infty$, $(X_{t\wedge\tau_n})$ is a martingale. **[2]**

If X is a local martingale and τ is a stopping time: let τ_n be a localising sequence. Then $X_{t\wedge\tau_n}$ is a mg for each n. By Doob's Stopping Time Theorem, $X_{t\wedge\tau_n\wedge\tau}$ is a mg, for each n. This says that $Y_{t\wedge\tau_n}$ is a mg for each n, which in turn says that Y is a local mg. **[4]**

(ii) Choose a localising sequence $\tau_n \uparrow \infty$. For $t \geq 0$ and $s \in [0, t]$, the mg property of $X_{t\wedge\tau_n}$ gives

$$E[X_{t\wedge\tau_n}|\mathcal{F}_s] = X_{s\wedge\tau_n}.$$

As $n \to \infty$, $X_{t\wedge\tau_n} \to X_t$, and $X_{s\wedge\tau_n} \to X_s$. As X is bounded, $|X_.| \leq M < \infty$ say, we can let $n \to \infty$ above and use dominated convergence to get

$$E[X_t|\mathcal{F}_s] = X_s.$$

This says that X is a mg. **[5]**

(iii) For τ_n a localising sequence, by the local mg property,

$$X_{s\wedge\tau_n} = E[X_{t\wedge\tau_n}|\mathcal{F}_s] \qquad (0 \leq s \leq t).$$

Let $n \to \infty$: by Fatou's Lemma,

$$X_s \geq E[X_t|\mathcal{F}_s] \qquad (0 \leq s \leq t). \qquad (*)$$

This says that X is a supermg. **[6]**

(iv) If now the time-set is $[0, T]$, then the above holds with $0 \leq s \leq t \leq T$. Taking expectations gives

$$E[X_0] \geq E[X_s] \geq E[X_t] \geq E[X_T].$$

We are given $E[X_0] = E[X_T]$, so the inequalities above are all equalities. If we had strict inequality in $(*)$, we would get $E[X_s] > E[X_t]$ on taking expectations. As we do not, we must have *equality* in $(*)$. This says that X is a mg. **[6]**

Q5. (i)

$$\begin{aligned}\psi(t) = E[e^{itY}] &= E[\exp\{it(X_1 + \ldots + X_N)\}] \\ &= \sum_n E[\exp\{it(X_1 + \ldots + X_N)\}|N = n].P(N = n) \\ &= \sum_n e^{-\lambda}\lambda^n/n!.E[\exp\{it(X_1 + \ldots + X_n)\}] \\ &= \sum_n e^{-\lambda}\lambda^n/n!.(E[\exp\{itX_1\}])^n \\ &= \sum_n e^{-\lambda}\lambda^n/n!.\phi(t)^n \\ &= \exp\{-\lambda(1 - \phi(t))\}.\end{aligned}$$

[7]

Differentiate:

$$\psi'(t) = \psi(t).\lambda\phi'(t),$$

$$\psi''(t) = \psi'(t).\lambda\phi'(t) + \psi(t).\lambda\phi''(t).$$

As $\phi(t) = E[e^{itX}]$, $\phi'(t) = E[iXe^{itX}]$, $\phi''(t) = E[-X^2e^{itX}]$. So ($\phi(0) = 1$ and) $\phi'(0) = i\mu$, $\phi''(0) = -E[X^2]$,

$$\psi'(0) = \lambda\phi'(0) = \lambda.i\mu,$$

and as also $\psi'(0) = iEY$, this gives $EY = \lambda\mu$. **[5]**

Similarly,

$$\psi''(0) = i\lambda\mu.i\lambda\mu + \lambda\phi''(0) = -\lambda^2\mu^2 - \lambda E[X^2],$$

and also ($\psi(0) = 1$, $\psi'(0) = i\lambda\mu$ and) $\psi''(0) = -E[Y^2]$. So

$$var\ Y = E[Y^2] - [EY]^2 = \lambda^2\mu^2 + \lambda E[X^2] - \lambda^2\mu^2 = \lambda E[X^2]. \qquad [5]$$

(ii) Given N, $Y = X_1 + \ldots + X_N$ has mean $NEX = N\mu$ and variance $N\ var\ X = N\sigma^2$. As N is Poisson with parameter λ, N has mean λ and variance λ. So by the Conditional Mean Formula,

$$EY = E[E(Y|N)] = E[N\mu] = \lambda\mu. \qquad [4]$$

By the Conditional Variance Formula,

$$var\ Y = E[var(Y|N)] + var\ E[Y|N] = E[Nvar\ X] + var[N\ EX]$$

$$= EN.var\ X + var\ N.(EX)^2 = \lambda[E(X^2) - (EX)^2] + \lambda.(EX)^2 = \lambda E[X^2]. \quad [4]$$

Q6. (i). Write $f(B,t) := (B^2 - t)^2$. By Itô's formula,

$$df = f_B dB + f_t dt + \frac{1}{2}[f_{BB}(dB)^2 + 2f_{Bt}dBdt + f_{tt}(dt)^2].$$

In the [...] on RHS, $(dB)^2 = dt$, $dBdt = 0$, $(dt)^2 = 0$. Also

$$f_B = 2.2B(B^2 - t), \quad f_t = -2(B^2 - t), \quad f_{BB} = 4(B^2 - t) + 4B.2B = 12B^2 - 4t.$$

So

$$df = 4B(B^2 - t)dB - 2(B^2 - t)dt + (6B^2 - 2t)dt = 4B(B^2 - t)dB + 4B^2 dt.$$

As $M = f - 4\int_0^t B_s^2 ds$, the stochastic differential of M is

$$dM = df - 4B_t^2 dt = 4B(B^2 - t)dB. \qquad \textbf{[8]}$$

(ii) So integrating, M is the Itô integral

$$M_t = 4\int_0^t B_s(B_s^2 - s)dB_s. \qquad \textbf{[5]}$$

The Itô integral on the RHS is a continuous local martingale starting from 0. Now $B_t =_d t^{1/2}.Z$ where Z is $N(0,1)$. As Z has all moments finite, each $E[B_t^n]$ is a polynomial in t. So the integrand $h = h(B_t, t)$ on RHS satisfies the integrability condition $\int_0^t E[h_s^2]ds < \infty$ for all t. So the RHS is a (true) continuous mg starting from 0. **[5]**

(iii) With $([M_t])$ the quadratic variation of M,

$$d[M]_t = (dM)_t^2; \qquad dM_t = 4B_t(B_t^2 - t)dB_t.$$

So

$$d[M]_t = 16B_t^2(B_t^2 - t)^2(dB_t)^2 = 16B_t^2(B_t^2 - t)^2 dt:$$

$$[M]_t = 16\int_0^t B_s^2(B_s^2 - s)^2 ds. \qquad \textbf{[7]}$$

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