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# TAUBERIAN AND MERCERIAN THEOREMS FOR SYSTEMS OF KERNELS

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## §1. Introduction

In [BI3], we introduced the notion of *ratio Mercerian theorem*, to improve the Mercerian theorem for Fourier and Hankel transforms first proved in [BI1]. In the first half of this paper, we extend (and correct) the ratio Mercerian theorem, and apply it to the proofs of Tauberian theorems for kernels of Korenblum type. The latter can be applied to analytic number theory; this was in fact the motivation for the present paper. The application to analytic number theory will be given in a separate paper [BI5]. In the second half of this paper, we prove a Mercerian counterpart to one of the Tauberian theorems - one in the boundary case. This is done via a further extension of our previous extension of the Drasin-Shea-Jordan theorem (see e.g. BGT, Ch. 5]). Throughout the paper, the idea of proving assertions for a *system of kernels*, rather than a single kernel, plays a key role.

For measurable functions $f, g : (0,\infty) \to \mathbf{R}$, the Mellin convolution $f * g$ is the function defined by

$$(f * g)(x) := \int_0^\infty f(x/t)g(t)dt/t,$$

for those $x > 0$ for which the integral converges absolutely. Given a measurable kernel $k : (0,\infty) \to \mathbf{R}$, let

$$\check{k}(z) := \int_0^\infty t^{-z}k(t)dt/t$$

be its Mellin transform for $k \in \mathbf{C}$ such that the integral converges absolutely.

A positive measurable function $f : [X,\infty) \to (0,\infty)$ is called *regularly varying with index $\rho \in \mathbf{R}$*, written $f \in \mathbf{R}_\rho$, if for all $\lambda > 0$

$$\lim_{x\to\infty} f(\lambda x)/f(x) = \lambda^\rho.$$

When the index $\rho$ is zero, we say that the function is *slowly varying*. A generic slowly varying function is usually written $\ell$.

The ratio Mercerian theorem of [BI3] asserts that, under adequate conditions on $k^1, k^2$ and $f$,

$$\frac{k^2 * f(x)}{k^1 * f(x)} \to c \neq 0 \qquad (x \to \infty) \tag{1.1}$$

implies $c = \check{k}^2(\rho)/\check{k}^1(\rho)$ and $f \in \mathbf{R}_\rho$ for $\rho := \limsup_{x\to\infty} \log f(x)/\log x$ the upper order of $f$. One of the key assumptions is that $\rho$ is the only zero of $\check{k}^1(\rho)\check{k}^2(z) - \check{k}^2(\rho)\check{k}^1(z)$ in some vertical strip $a \leq \Re z \leq b$ such that $\rho \in (a, b)$ (see the Remark in §2 below for an error in [BI3, Th. 3] on this point).

We can use this type of theorem to prove Tauberian theorems, as we now explain. Suppose that we want to prove the implication from

$$(k * f)(x) \sim x^\rho \ell(x)\check{k}(\rho) \qquad (x \to \infty) \tag{1.2}$$

with $\ell \in R_0$ to

$$f(x) \sim x^\rho \ell(x) \qquad (x \to \infty) \tag{1.3}$$

under adequate assumptions on $k, f$ and $\rho$. Let $\lambda > 0$. Then from (1.2) we have

$$\frac{(k * f)(\lambda x)}{(k * f)(x)} \to \lambda^\rho \qquad (x \to \infty),$$

or

$$\frac{(k_\lambda^2 * f)(x)}{(k_1 * f)(x)} \to \lambda^\rho \qquad (x \to \infty), \tag{1.4}$$

where

$$k^1(x) := k(x), \qquad k_\lambda^2(x) := k(\lambda x) \qquad (0 < x < \infty).$$

Thus we have the same setting as (1.1), and so the ratio Mercerian theorem provides a route towards $f \in R_\rho$, which is close to the desired conclusion (1.3).

However, there is a problem here: since now

$$\check{k}^1(\rho)\check{k}_\lambda^2(z) - \check{k}_\lambda^2(\rho)\check{k}^1(z) = (\lambda^z - \lambda^\rho)\check{k}(\rho)\check{k}(z), \tag{1.5}$$

the function on the left has infinitely many zeros $z = \rho + \frac{2n\pi}{\log\lambda} i$ $(n = \pm 1, \pm 2, \ldots)$, other than $z = \rho$, on the vertical line $\Re z = \rho$, hence in any strip $a \leq \Re z \leq b$ containing this line. Thus the key assumption above on the kernel seems to be unavoidably violated. Fortunately, one may bypass this problem, by considering not one $\lambda$ but more than one. In fact, two logarithmically incommensurable $\lambda$s suffice: choose $\lambda_1, \lambda_2 > 1$ so that $\log \lambda_2 / \log \lambda_1$ is irrational. Then we have

$$\{\rho + \frac{2\pi n}{\log \lambda_1} i : n \in \mathbb{Z}\} \cap \{\rho + \frac{2\pi m}{\log \lambda_2} i : m \in \mathbb{Z}\} = \{\rho\}.$$

This suggests that if we extend the ratio Mercerian theorem suitably to a system of kernels, then we would be able to follow the above line to prove Tauberian theorems of the form (1.2) $\Rightarrow$ (1.3).

We prove the extended ratio Mercerian theorem in §2 almost in parallel to the proof of [BI3, Th. 3], except for one point. At that point, we take a more direct route. As a result, we obtain another improvement on [BI, Th. 3], that is, the conclusion $E_1 * f \in R_\rho$ in Th. 2.1 below rather than $E_2 * k_1 * f \in R_\rho$ in [BI3, Th. 3].

We turn to our Tauberian results. In §3, we prove a Tauberian theorem, Th. 3.1, of the type (1.2) $\Rightarrow$ (1.3) for kernels of Korenblum type - that is, for kernels $k$ for which $\check{k}(z)$ has no zeros in a strip $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$, and satisfies the Nyman-Korenblum decay condition

$$\frac{\log |\check{k}(\sigma + it)|}{\exp\left(\frac{\pi |t|}{2\epsilon}\right)} \to 0 \qquad (t \to \pm\infty); \tag{1.6}$$

see e.g. Gurarii [G] for background and references ($\check{k}(\sigma + it) \to 0$ as $t \to \pm\infty$ in the strip, by the Riemann-Lebesgue lemma; (1.6) requires that this decay should not be too rapid). Here we choose $\sigma$ and $\epsilon$ so that $\rho \in (\sigma - \epsilon, \sigma + \epsilon) \setminus \{\sigma\}$. We note that what we actually do is to prove a little more than this: we prove, for a *system* of kernels $k_\lambda$ ($\lambda \in \Lambda$) the implication from

$$(k_\lambda * f)(x) \sim x^\rho \lambda(x) \check{k}_\lambda(\rho) \qquad (x \to \infty) \qquad \forall \lambda \in \Lambda \tag{1.7}$$

to (1.3). This setting is needed in §4, but we consider here the simplest case, when the set $\Lambda$ is a singleton. The point of the Tauberian theorem obtained is that it is optimal as regards the Tauberian conditions needed - as good as the classically studied case of the Laplace transform, for example, which serves as a test case here - and allows Tauberian conditions that can be easily checked. This point will be important in the applications to analytic number theory in [BI5].

On the other hand, there is one weak point in Th. 3.1: we have to assume that some kernel $k_\lambda$ is non-negative, for technical reasons. It would be desirable to relax this assumption.

In §4, we prove another Tauberian theorem, Th. 4.1, involving Π-variation, using Theorem 3.1. For $\ell \in R_0$, the de Haan class $\Pi_\ell$ is the class of real measurable $g$, defined on some neighbourhood $[X, \infty)$ of infinity, satisfying

$$\lim_{x\to\infty}(g(\lambda x) - g(x))/\ell(x) = c\log\lambda \qquad \forall\lambda > 0$$

for some constant $c \in R$ called the $\ell$-index of $g$. For the associated de Haan theory, see e.g. [BGT, Ch. 3]; for the (simpler) Karamata theory of the classes $R_\rho$, see e.g. [BGT, Ch. 1,2].

Let $U(x) := \int_0^x u(t)dt$, where $u$ is monotone. Then de Haan's monotone density theorem ([BGT, Th. 3.6.8]) asserts that $U \in \Pi_\ell$ with $\ell$-index $c$ implies $u(x) \sim cx^{-1}\ell(x)$ as $x \to \infty$ (the converse also holds). This is the simplest of Tauberian theorems in which we pass from Π-variation to regular variation. De Haan's Tauberian theorem for Laplace transforms [BGT, Th. 3.9.1] may also be viewed as of this type. So too may the Tauberian theorems for Fourier series and integrals [I] and for Hankel transforms [BI2].

For kernels of Korenblum type, we prove that

$$x^{-\rho}(k * f)(x) \in \Pi_\ell \text{ with } \ell - \text{index } c \tag{1.8}$$

implies (1.3) under adequate conditions. Thus we can now give some sufficient conditions on $k$ for the Tauberian theorem of the above type to hold. The key assumption is that *the analytic continuation of* $\check{k}(z)$ *has a simple pole at* $z = \rho$ *with residue* $c$.

We thus see that (1.3) and (1.8) imply

$$\frac{(\lambda x)^{-\rho}(k * f)(\lambda x) - x^{-\rho}(k * f)(x)}{x^{-\rho}f(x)} \to c\log\lambda \qquad (x \to \infty) \qquad \forall\lambda > 0. \tag{1.9}$$

In §6, we prove the converse, Th. 6.1: under suitable conditions, (1.9) implies $f \in R_\rho$. This may be viewed as a Mercerian counterpart to the Tauberian theorem (1.8) ⇒ (1.3) (Theorem 4.1).

To prove the Mercerian theorem above, we need the following extension (Th. 5.2) of [BI4, Th. 1]: for a system of kernels $k_\lambda$ ($\lambda \in \Lambda$), under suitable conditions,

$$(k_\lambda * f)(x)/f(x) \to c_\lambda \neq 0 \qquad (x \to \infty) \qquad \forall\lambda \in \Lambda \tag{1.10}$$

implies $c_\lambda = \check{k}_\lambda(\rho)$ ($\lambda \in \Lambda$), and $f \in R_\rho$ for the upper order $\rho$ of $f$. Here the key assumption is that $z = \rho$ is the only common zero of $\check{k}_\lambda(z) - \check{k}_\lambda(\rho)$ ($\lambda \in \Lambda$) on the line $\Re z = \rho$. We note that [BI4, Th. 1] itself is an extension of the Drasin-Shea-Jordan theorem described in [BGT, Ch. 5]. We prove this version of the Drasin-Shea-Jordan theorem in §5, using Theorem 5.1, which is itself an extension of Titchmarsh [T, Th. 146]. We note that this extended Drasin-Shea-Jordan theorem may be regarded as the Mercerian counterpart of the Tauberian theorem (1.7) ⇒ (1.3) (Theorem 3.1). It will be found again in the proofs in §5-6 how useful the idea of using a system of kernels is.

*Remarks.* 1. The logarithmic incommensurability condition $(\log \lambda_1)/(\log \lambda_2)$ irrational is equivalent, by Kronecker's theorem ([HW, Ch. XXIII]) to assuming that the multiplicative group $\{\lambda_1^m \lambda_2^n : m, n \in \mathbb{Z}\}$ generated by $\lambda_1, \lambda_2$ is dense in $(0, \infty)$; cf. [BGT, §1.10, §3.2.1].
2. The occurrence of the de Haan class $\Pi$, and of kernels $k_j$ in Theorems 4.1 and 6.1 below, both reflect *differencing*. One reason why differencing is effective here is that it can serve to *eliminate poles* in the Mellin transform of a kernel. For example, in analytic number theory - particularly the Prime Number Theorem (PNT) and related results - we encounter the Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$, with simple pole at $s = 1$ of residue 1. Three different methods of using Tauberian theory to prove PNT all involve differencing: the Wiener-Ikehara theorem (Widder [W, VI.17]), Ingham's method ([H, §12.11] - which also involves a condition of logarithmic incomensurability), and Delange's method. For a detailed account of Delange's method, and background on the other two, see e.g. [BGT, §6.2].
3. All three Tauberian methods mentioned above use some form of the Wiener Tauberian theory. The key condition here on the kernel - Wiener's condition - is non-vanishing of the Mellin (or Fourier) transform on the relevant *line* in the complex plane. By contrast, the powerful extension of the Wiener theory due to Nyman and Korenblum ([K1, K2]; [G]) needs instead non-vanishing of the transform in an appropriate *strip*, together with the Nyman-Korenblum decay condition (1.6). We make essential use of the Korenblum theory here, in Th. 3.1 and 4.1, and in [BI1], [BI3].

We have some choice here as between the Wiener and Korenblum theories, involving a trade-off between conditions on the kernel $k$ and Tauberian conditions on the function $f$. Our motivating example involves a particular kernel (see Remark 4 below), for which the extra conditions for the Korenblum theory are easy to check. On the other hand, we are content in applications with Tauberian conditions such as $f$ non-decreasing. This is trivial to check, whereas a Tauberian condition of the form in [BGT, Th. 4.8.3] - the natural condition for Wiener theory - would be awkward to check in general. See [BGT, §4.8.5] for background here. So it is natural to prefer Korenblum theory in this context, where applicable.

4. The particular context that motivated our work comes from analytic number theory [BI5]. It involves a particular kernel, the *Pólya kernel* $k(t) := I_{(0,\infty)}(x).[x]/x$ ([.] is the integer part), with $\check{k}(s) = \zeta(1+s)/(1+s)$. However, Korenblum theory cannot always be used here. This is because the best zero-free region known for $\zeta(s)$ is an open region containing $\sigma := \Re s \geq 1$, but not containing $\sigma > 1-\epsilon$ for any $\epsilon > 0$ ([Iv, Ch. 6, Th. 12.3]), as would be needed to use Korenblum theory in the boundary case. Accordingly, in Th. 5.2 and 6.1 here, and in §5 of the sequel [BI5], we restrict ourselves to Wiener rather than Korenblum Tauberian theory. In the motivating case, [BI5, Th. 5.3], we are able to check Tauberian conditions without difficulty (because $\rho = 0$ in the Tauberian condition of [BI5, Th. 4.2]).

§**2. The ratio Mercerian theorem**.

In this section, we prove a ratio Mercerian theorem for a system of kernels. We recall [BGT, §2.1.2] the Matuszewska indices of a positive function $f$. The *upper Matuszewska index* $\alpha(f)$ is the infimum of those $\alpha$ for which there exists a constant $C = C(\alpha)$ such that for each $\Lambda > 1$,

$$f(\lambda x)/f(x) \leq C\{1+o(1)\}\lambda^{\alpha} \qquad (x \to \infty) \text{ uniformly in } \lambda \in [1, \Lambda];$$

the *lower Matuszewska index* $\beta(f)$ is the supremum of those $\beta$ for which, for some constant $D = D(\beta) > 0$ and all $\Lambda > 1$,

$$f(\lambda x)/f(x) \geq D\{1+o(1)\}\lambda^{\beta} \qquad (x \to \infty) \text{ uniformly in } \lambda \in [1, \Lambda].$$

One says that $f$ has *bounded increase*, written $f \in BI$, if $\alpha(f) < \infty$, *bounded decrease*, written $f \in BD$, if $\beta(f) > -\infty$. The *upper order* $\rho(f)$ of a positive function is defined by

$$\rho(f) := \limsup_{x \to \infty} \frac{\log f(x)}{\log x}.$$

**Theorem 2.1.** *Let $\sigma \in \mathbf{R}$, $\epsilon > 0$, and $\rho \in (\sigma - \epsilon, \sigma + \epsilon)$. Let $k^1_\lambda : (0,\infty) \to [0,\infty)$ $(\lambda \in \Lambda)$ and $k^2_\lambda : (0,\infty) \to \mathbf{R}$ $(\lambda \in \Lambda)$ be systems of measurable kernels such that all $\check{k}^i_\lambda$ $(\lambda \in \Lambda, i = 1,2)$ converge absolutely in the strip $\sigma - \epsilon \leq \Re z \leq \sigma + \epsilon$. We set*

$$k^0_\lambda(t) := \check{k}^1_\lambda(\rho)k^2_\lambda(t) - \check{k}^2_\lambda(\rho)k^1_\lambda(t) \qquad (0 < t < \infty, \lambda \in \Lambda). \tag{2.1}$$

*We assume the following:*
*(k1) the only common zero of $\check{k}^0_\lambda(z) := \check{k}^1_\lambda(\rho)\check{k}^2_\lambda(z) - \check{k}^2_\lambda(\rho)\check{k}^1_\lambda(z)$ $(\lambda \in \Lambda)$ in the strip is*

*$z = \rho$,*
*(k2) for some $\lambda \in \Lambda$, the first and second derivatives of $\check{k}^0_\lambda(z)$ do not both vanish at $z = \rho$,*
*(k3) $k^0_\lambda$ satisfies the Nyman-Korenblum decay condition (1.6) for some $\lambda \in \Lambda$.*
*Let $f$ be non-negative and locally bounded on $[0, \infty)$, vanish in a neighbourhood of zero, have upper order $\rho$, and $f \in BD \cup BI$. If*

$$\frac{(k^2_\lambda * f)(x)}{(k^1_\lambda * f)(x)} \to c_\lambda \neq 0 \qquad (x \to \infty), \tag{2.2}$$

*then*

$$c_\lambda = \check{k}^2_\lambda(\rho)/\check{k}^1_\lambda(\rho) \qquad (\lambda \in \Lambda) \tag{2.3}$$

*and*

$$E_1 * f \in R_\rho, \tag{2.4}$$

*where $E_1(x) := I_{(1,\infty)}(x).x^{\sigma-\epsilon}$ for $x \in (0, \infty)$.*

*Proof.* The assertion $c_\lambda = \check{k}^2_\lambda(\rho)/\check{k}^1_\lambda(\rho)$ $(\lambda \in \Lambda)$ follows immediately from [BI3, Th. 2].

Take $\gamma \in (\rho, \sigma + \epsilon)$. Then there exists $d(\gamma) \in (0, \infty)$ such that $f(x) \leq d(\gamma)x^\gamma$ $(0 < x < \infty)$. From this, we have

$$|k^i_\lambda| * f(x) \leq d(\gamma)|\check{k}^i_\lambda(\gamma)|x^\gamma \qquad (0 < x < \infty, \quad \lambda \in \Lambda, \quad i = 1, 2).$$

Set

$$E_1(x) := I_{(1,\infty)}(x)x^{\sigma-\epsilon}, \qquad E_2(x) := I_{(0,1)}(x)x^{\sigma+\epsilon}.$$

Then the convergence strips of $\check{E}_1(z)$ and $\check{E}_2(z)$ are $\Re z > \sigma - \epsilon$ and $\Re z < \sigma + \epsilon$ respectively, and so $E_1 * k^i_\lambda * f$ $(\lambda \in \Lambda,\ i = 1, 2)$ are well-defined.

Now

$$(E_1 * k^i_\lambda * f)(x) = x^{\sigma-\epsilon} \int_0^x (k^i_\lambda * f)(t)dt/t^{1+\sigma-\epsilon}$$

and

$$\int_0^\infty (k^1_\lambda * f)(t)dt/t^{1+\sigma-\epsilon} = \check{k}^1_\lambda(\sigma - \epsilon).\check{f}(\sigma - \epsilon) = \infty$$

(see [BI1, Lemma 7] and the remarks after the proof of [BGT, Th. 5.2.3]). From this, we have

$$\frac{(k^2_\lambda * g)(x)}{(k^1_\lambda * g)(x)} \to \frac{\check{k}^2_\lambda(\rho)}{\check{k}^1_\lambda(\rho)} \qquad (x \to \infty) \qquad \forall \lambda \in \Lambda, \tag{2.5}$$

where

$$g(x) := (E_1 * E_2 * f)(x) \qquad (0 < x < \infty).$$

For $g$, we have the nice estimate

$$g(ux)/g(x) \leq \max(u^{\sigma-\epsilon}, u^{\sigma+\epsilon}) \qquad (0 < x < \infty, \quad 0 < u < \infty)$$

(see (4.12) of [BI3]).

Choose any sequence $x_n \uparrow \infty$, and consider

$$h_n(u) := g(x_n u)/g(x_n) \qquad (0 < u < \infty).$$

Then by Helly selection we can find a sequence $n' \to \infty$ such that $h_{n'}(u)$ converges pointwise on $(0, \infty)$, to $h$, say. We see that $u^{-(\sigma-\epsilon)}h(u)$ is increasing, $h(1) = 1$, and

$$h(u) \leq \max(u^{\sigma-\epsilon}, u^{\sigma+\epsilon}) \qquad (0 < u < \infty).$$

Now, for $x > 0$ and $\lambda \in \Lambda$, we have

$$\begin{aligned} \frac{(k^2_\lambda * g)(x.xn')}{(k^1_\lambda * g)(x.x_{n'})} &= \frac{\int_0^\infty k^2_\lambda(t)h_{n'}(x/t)dt/t}{\int_0^\infty k^1_\lambda(t)h_{n'}(x/t)dt/t} \\ &\to \frac{(k^2_\lambda * h)(x)}{(k^1_\lambda * h)(x)} \qquad (n' \to \infty) \end{aligned}$$

(this is the point that is different from the proof of [BI3, Th. 3]). This and (2.5) give

$$(k^0_\lambda * h)(x) = 0 \qquad (0 < x < \infty) \qquad \forall \lambda \in \Lambda.$$

Since Korenblum's theorem [K1, K2] (see also [BI1, §5] and [BI3, §3]) can be applied to such a system of integral equations in the same way as to a single equation, we obtain, as in the proof of [BI3, Th. 3],

$$h(x) = a_1 x^\rho + a_2 x^\rho \log x \qquad a.e. \quad x \in (0, \infty)$$

for some $a_1, a_2 \in \mathbf{C}$. From this, we can easily deduce $h(x) = x^\rho$ for all $x \in (0, \infty)$. This says that the partial limit $u^\rho$ of $g(x_n)/g(x_n)$ does not depend on the particular sequence $(x_n)$ chosen. Thus

$$g(xu)/g(x) \to u^\rho \qquad (x \to \infty) \qquad \forall u > 0,$$

and so $g \in R_\rho$.

Finally, by a monotone density argument, we obtain $E_1 * f \in R_\rho$.

*Remark.* In [BI3, Th. 3], we assumed only that $\check{k}_0(z)$ has a unique zero $z = \rho$ on the vertical line $\Re z$, and wrongly stated that the strip-form condition such as (k1) follows from this. Instead, we should have *assumed* the strip form of the condition. Consequently, the correct (and slightly improved) statement of [BI3, Th. 3] is the case of Theorem 2.1 above with $\Lambda$ a singleton. Fortunately, Theorems 4 and 5 of [BI3], which were proved using [BI3, Th. 3], hold as they stand because the proofs of [BI3, Th. 4, 5] also show that the relevant $\check{k}_0(z)$ satisfies the strip form of the condition.

**§3. Tauberian theorem: regular variation**

In this section, we prove a Tauberian theorem of the type (1.7) ⇒ (1.3). Recall [BGT, §1.7.6] that a function $g : [X, \infty) \to \mathbf{R}$ is called *slowly decreasing* if

$$\lim_{\lambda\downarrow 1} \liminf_{x\to\infty} \inf_{t\in[1,\lambda]} (g(tx) - g(x)) \geq 0 \qquad (\text{hence } = 0).$$

**Lemma.** *Let $-\infty < a < \rho < b < \infty$, and $\ell \in R_0$. Let $k : (0, \infty) \to [0, \infty)$ be a non-negative measurable kernel, not a.e. zero, such that the Mellin transform $\check{k}(z)$ converges absolutely in $a \leq \Re z \leq b$. Let $f$ be non-negative, measurable and locally bounded on $[0, \infty)$, and vanish in a neighbourhood of zero. If one of the following Tauberian conditions holds:*

*(T1) $f$ is eventually positive and $\log f$ is slowly decreasing,*

*(T2) $f(x)/\{x^\rho \ell(x)\}$ is slowly decreasing,*

*(T3)*

$$\lim_{\lambda\downarrow 1} \liminf_{x\to\infty} \inf_{y\in[x,\lambda x]} \frac{y^{-\sigma} f(y) - x^{-\sigma} f(x)}{x^{\rho-\sigma}\ell(x)} \geq 0 \qquad \textit{for some } \sigma \in \mathbf{R},$$

*then (1.2) implies that the upper order $\rho(f)$ of $f$ is equal to $\rho$.*

*Proof.* We prove the lemma only under (T1); the cases (T2) and (T3) can be proved similarly (see the proof of [BGT, Th. 1.7.5]).

Suppose $\rho(f) < \rho$. Choose $\gamma \in (a, b) \cap (\rho(f), \rho)$. Then we have

$$f(x) \leq d(\gamma)x^\gamma \qquad (0 < x < \infty) \tag{3.1}$$

for some $d(\gamma) \in (0, \infty)$. This gives

$$(k * f)(x) \leq d(\gamma).\check{k}(\gamma).x^\gamma \qquad (0 < x < \infty), \tag{3.2}$$

which contradicts (1.2). Thus $\rho(f) \geq \rho$.

By (T1), we may choose $\delta \in (0, 1)$ and find $\lambda > 1$, $X > 0$ such that

$$f(y) \geq \delta f(x) \qquad (y \geq x \geq X, \quad y/x \leq \lambda).$$

For this $\lambda$, we can find $t_0 \in (0,\infty)$ such that $k$ is not a.e. zero on $[t_0, \lambda t_0]$. For, if we choose $s_1 \in (0,\infty)$ such that $k$ is not a.e. zero on $[s_1,\infty)$, then $k$ is not a.e. on at least one $[\lambda^m s_1, \lambda^{m+1} s_1)$ for $m = 1, 2, \ldots$ (as these have union $[s_1,\infty)$). Now for $x > \lambda t_0 X$,

$$(k * f)(x) \geq \int_{t_0}^{\lambda t_0} k(t) f(x/t) dt/t$$
$$\geq \delta f(x/\lambda t_0) \int_{t_0}^{\lambda t_0} k(t) dt/t.$$

This and (1.2) give $\rho(f) \leq \rho$. Combining, $\rho(f) = \rho$, as required.

**Theorem 3.1.** *Let $\sigma \in \mathbf{R}$, $\epsilon > 0$ and $\rho \in (\sigma - \epsilon, \sigma + \epsilon) \setminus \{\sigma\}$. Let $k_\lambda$ ($\lambda \in \Lambda$) be a system of real measurable kernels on $(0,\infty)$ such that all $\check{k}_\lambda(z)$ ($\lambda \in \Lambda$) converge absolutely in the strip $\sigma - \epsilon \leq \Re z \leq \sigma + \epsilon$. We assume that for some $\lambda \in \Lambda$, $k_\lambda$ is non-negative. We also assume that $\check{k}_\lambda(z)$ ($\lambda \in \Lambda$) have no common zeros in $\sigma - \epsilon \leq \Re z \leq \sigma + \epsilon$, that $\check{k}_\lambda(\rho) \neq 0$ for all $\lambda \in \Lambda$, and that for some $\lambda_0 \in \Lambda$, $k_{\lambda_0}$ satisfies the Nyman-Korenblum decay condition (1.6). Let $f$ be non-negative, measurable and locally bounded on $[0,\infty)$, and vanish in a neighbourhood of zero. Then (1.3) implies (1.7), and conversely, if $f$ satisfies one of the Tauberian conditions (T1)-(T3), (1.7) implies (1.3).*

*Proof.* Since $x^{-(\sigma-\epsilon)} f(x)$ is bounded on every interval $(0,a]$, the Abelian implication from (1.3) to (1.7) follows immediately from [BGT, Th. 4.1.6].

We prove the Tauberian implication from (1.7) to (1.3). By the lemma above, we have $\rho(f) = \rho$. We may choose $\gamma \in (\rho, \sigma + \epsilon)$ and find $d(\gamma) \in (0,\infty)$ such that (3.1) - and so (3.2) - holds with $k$ replaced by $|k_\lambda|$. So $x^{-(\sigma-\epsilon)}(k_\lambda * f)(x)$ is bounded on every interval $(0,a]$, and so, by [BGT, Th. 4.1.6] and Fubini's theorem,

$$(k_\lambda * g)(x) \sim x^\rho \ell(x) \check{k}_\lambda(\rho) \check{E}_1(\rho) \qquad (x \to \infty), \tag{3.3}$$

where $E_1(x) := I_{(1,\infty)}(x) x^{\sigma-\epsilon}$ as in §2 and $g := (E_1 * f)(x)$ (not as in §2). Since $x^{-(\sigma-\epsilon)} g(x) = \int_0^x f(t) dt/t^{1+\sigma-\epsilon}$ is increasing and $g$ vanishes in a neighbourhood of zero, we find from (3.3) and the lemma above that $\rho(g) = \rho$. We also find that $g \in BD$ (this is the reason why we consider $g$ rather than $f$).

It follows from (3.3) that for all $\lambda \in \Lambda$ and $\mu \in (1,\infty)$,

$$\frac{(k_\lambda * g)(\mu x)}{(k_\lambda * g)(x)} \to \mu^\rho \qquad (x \to \infty).$$

That is, writing

$$k^2_{\lambda,\mu}(x) := k_\lambda(\mu x), \qquad k^1_{\lambda,\mu}(x) := k_\lambda(x),$$

we have

$$\frac{(k^2_{\lambda,\mu} * g)(x)}{(k^1_{\lambda,\mu} * g)(x)} \to \mu^\rho \qquad (x \to \infty).$$

In order to apply Theorem 2.1, we form the differenced kernel

$$k^0_{\lambda,\mu}(t) := \check{k}^1_{\lambda,\mu}(\rho)k^2_{\lambda,\mu}(t) - \check{k}^2_{\lambda,\mu}(\rho)k^1_{\lambda,\mu}(t),$$

with Mellin transform

$$\check{k}^0_{\lambda,\mu}(z) = (\mu^z - \mu^\rho)\check{k}_\lambda(z)\check{k}_\lambda(\rho) \qquad (\sigma - \epsilon \le \Re z \le \sigma + \epsilon).$$

For each pair $(\lambda, \mu) \in \Lambda \times (1, \infty)$, the set of zeros of $\check{k}^0_{\lambda,\mu}(z)$ in the strip $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$ is, from above,

$$\{\rho + \frac{2\pi n}{\log \mu} i : n \in \mathbb{Z}\} \cup \{\text{the zeros of } \check{k}_\lambda(z) \text{ in } \sigma - \epsilon \le \Re z \le \sigma + \epsilon\}.$$

We show that there are no common zeros other than $\rho$ in the strip $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$. For, for every $z_0 \ne \rho$ in the strip, there exists at least one $\lambda \in \Lambda$ such that $\check{k}_\lambda(z_0) \ne 0$ by assumption. On the other hand, we may choose $\mu > 1$ so that $z_0$ falls outside $\{\rho + \frac{2\pi n}{\log \mu} : n \in \mathbb{Z}\}$. Then for this pair $(\lambda, \mu)$, we have $\check{k}^0_{\lambda,\mu}(z_0) \ne 0$, as claimed.

Since

$$0 < |\mu^\sigma - \mu^\rho| \le |\mu^{\sigma+it} - \mu^\rho| \le \mu^\sigma + \mu^\rho \qquad (t \in \mathbf{R}),$$

the factor $(\mu^z - \mu^\rho)$ is suitably controlled, and the Nyman-Korenblum decay condition transfers from $k^0_{\lambda_0}$ to $k^0_{\lambda_0,\mu}$.

Now

$$\frac{d}{dz}\check{k}^0_{\lambda,\mu}(z)|_{z=\rho} = \log \mu . \mu^\rho (\check{k}_\lambda(\rho))^2,$$

and so $\check{k}^0_{\lambda,\mu}(z)$ has non-vanishing derivative at $z = \rho$. We can now apply Theorem 2.1, to conclude $E_1 * g \in R_\rho$. From this and the monotone density theorem, we obtain $g \in R_\rho$.

This says that $g(x) \sim x^\rho L(x)$ for some $L \in R_0$. This and [BGT, Th. 4.1.6] give

$$(k_\lambda * g)(x) \sim x^\rho L(x)\check{k}_\lambda(\rho) \qquad (x \to \infty).$$

Comparing this with (3.3), we find

$$L(x) \sim \ell(x)\check{E}_1(\rho) \qquad (x \to \infty),$$

so

$$g(x) \sim x^\rho \ell(x)\check{E}_1(\rho) \qquad (x \to \infty).$$

Finally, this gives (1.3) by a further use of the monotone density theorem, completing the proof.

*Example.* Write [.] for the integer part, and consider the kernel

$$k(t) := I_{(1,\infty)}(t)[t]/t \qquad (0 < t < \infty).$$

Then $\check{k}(z)$ converges absolutely in $\Re z > 0$, and partial summation gives

$$\check{k}(z) = \frac{\zeta(z+1)}{(z+1)},$$

where $\zeta(s)$ is the Riemann zeta function. One may check that $k$ satisfies the conditions of Theorem 3.1 with $\Lambda$ a singleton. This kernel underlies the application to analytic number theory in [BI5]. It may be traced to work of Pólya in 1917 (see e.g. [BGT, §6.4.4]).

## §4. Tauberian theory: Π-variation

In this section, we prove a Tauberian theorem of the Π-variation type (1.8) ⇒ (1.0). The proof will use Theorem 3.1.

**Theorem 4.1.** *Let $\ell \in R_0$; let $\sigma \in \mathbf{R}$, $\epsilon > 0$ and $\rho \in (\sigma-\epsilon, \sigma+\epsilon)\backslash\{\sigma\}$. Let $k : (0,\infty) \to \mathbf{R}$ be a measurable kernel such that the Mellin transform $\check{k}(z)$ converges absolutely in the strip $\rho < \Re z \le \sigma + \epsilon$, satisfies the Korenblum decay condition, and*
*(k1): there exist $\lambda_1$, $\lambda_2 \in (0,\infty) \setminus \{1\}$ such that $\log \lambda_2 / \log \lambda_1$ is irrational and $(\lambda_j x)^{-\rho} k(\lambda_j x) - x^{-\rho} k(x) \ge 0$ for $0 < x < \infty$ and $j = 1, 2$;*
*(k2): $\check{k}(z)$ has an analytic continuation in $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$ with a unique singularity at $z = \rho$, and this singularity is a simple pole with residue $c \in \mathbf{R} \setminus \{0\}$;*
*(k3): $\check{k}(z)$ has no zeros in $\sigma - \epsilon \le \Re z \le \sigma + \epsilon$.*
*Let $f$ be non-negative, measurable and locally bounded on $[0,\infty)$, and vanish in a neighbourhood of zero. Then under one of the Tauberian conditions (T1) - (T3) of §3, (1.8) implies (1.3).*

*Proof.* We set

$$k_j(x) := (\lambda_j x)^{-\rho} k(\lambda_j x) - x^{-\rho} k(x) \qquad (0 < x < \infty) \qquad (j = 1, 2).$$

Then

$$\check{k}_j(z) = (\lambda_j^z - 1)\check{k}(z+\rho) \qquad (0 < \Re z \le \sigma - \rho + \epsilon). \tag{4.1}$$

By (k2), $\check{k}_j(z)$ $(j = 1, 2)$ can be continued to a holomorphic function in $D := \{z \in \mathbf{C} : \sigma - \epsilon - \rho \leq \Re z \leq \sigma + \epsilon - \rho\}$ and satisfies

$$\check{k}_j(0) = \lim_{x \downarrow 0} \frac{(\lambda_j^x - 1)}{x} . x\check{k}(x + \rho) = c \log \lambda_j. \tag{4.2}$$

Also, $k_j$ is non-negative, by (k1). So by the Vivanti-Pringsheim theorem (Doetsch [D, Ch. 4, §5]), the region of holomorphy will be limited by a singularity on the real axis. Thus in particular, $\check{k}_j(z) = \int_0^\infty t^{-z} k_j(t)dt/t$ $(j = 1, 2)$ converges absolutely in $D$.

By (4.1) and (k3), the zeros of $\check{k}(z)$ in $D$ are $(2\pi n/\log \lambda_j)i$ $(n \in \mathbb{Z} \setminus \{0\})$. Since $\log \lambda_2 / \log \lambda_1$ is irrational, we find that $\check{k}_1(z)$ and $\check{k}_2(z)$ have no common zeros in $D$.

Just as in §3, the Nyman-Korenblum decay condition transfers from the kernel $k$ to the kernels $k_j$ in the strip $\sigma - \rho - \epsilon \leq \Re z \leq \sigma - \rho + \epsilon$.

Set $\tilde{f}(x) := x^{-\rho} f(x)$ for $0 < x < \infty$. Then

$$(\lambda_j x)^{-\rho}(k * f)(\lambda_j x) - x^{-\rho}(k * f)(x) = (k_j * \tilde{f})(x),$$

and so, by (1.8) and (4.2), we have

$$(k_j * \tilde{f})(x) \sim \ell(x)\check{k}_j(0) \qquad (x \to \infty) \qquad (j = 1, 2).$$

Now one of the Tauberian conditions (T1) - (T3) for $f$ implies the same condition for $\tilde{f}$ with 0 and $\rho$ in place of $\sigma - \rho$ and $\sigma$. Now, by Theorem 3.1, we obtain (1.3) from (1.8), as required.

*Example: Stieltjes transform.* Let $a > 0$ and $\ell \in R_0$; let $f$ be a real function in $L^1_{loc}[0, \infty)$. Assume, say, the Tauberian condition (T1). We show that

$$f(x) \sim \ell(x)x^{-1} \qquad (x \to \infty) \tag{4.3}$$

if and only if

$$x^a \int_0^\infty \frac{f(y)}{(x + y)^a} dy \in \Pi_\ell \text{ with } \ell\text{-index } 1. \tag{4.4}$$

Choose $X > 0$ so that $f$ is positive and locally bounded on $[0, \infty)$. We set

$$g(x) := I_{[X,\infty)}(x)x^{1-a}f(x) \qquad (0 < x < \infty).$$

By the mean-value theorem, we have, for $\lambda > 1$, $x > 0$ and $y > 0$,

$$\left|\frac{(\lambda x)^a}{(\lambda x + y)^a} - \frac{x^a}{(x + y)^a}\right| < a(1 - \lambda^{-1})y/x,$$

hence

$$(\lambda x)^a \int_0^X \frac{f(y)}{(\lambda x + y)^a} dy - x^a \int_0^X \frac{f(y)}{(x+y)^a} dy = O(1/x) \qquad (x \to \infty).$$

So (4.4) is equivalent to

$$x^a (k * g)(x) \in \Pi_\ell \text{ with } \ell\text{-index } 1, \tag{4.5}$$

where

$$k(x) := 1/(1+x)^a \qquad (0 < x < \infty).$$

The Mellin transform $\check{k}(z)$ converges absolutely in $-a < \Re z < 0$, and equals $\Gamma(-z)\Gamma(z+a)/\Gamma(a)$ there. Since $x^a k(x)$ is increasing, $k$ satisfies (k1) with $-a$ as $\rho$, for any $\lambda_1, \lambda_2$ with $\log \lambda_1 / \log \lambda_2$ irrational. The analytic continuation of $\check{k}$ to $-a-1 < \Re z < 0$ has a pole $-a$ with residue 1 as the unique singularity and no zeros there. If we choose $\sigma \in (-a-1, 0)$ and $\epsilon > 0$ so that $-a-1 < \sigma - \epsilon < -a < \sigma + \epsilon < 0$, then, by the asymptotic behaviour of the Gamma function (Stirling's formula in the complex plane: see e.g. Rademacher [R, p. 38], or Copson [C, §9.55]), we find that $k$ satisfies (1.6). So by Theorem 4.1, (4.5) (or (4.4), by above) implies

$$g(x) \sim x^{-a} \ell(x) \qquad (x \to \infty),$$

hence (4.3). On the other hand, the Abelian implication from (4.3) to (4.4) follows easily from [BGT, Th. 4.1.6], and so (4.3) and (4.4) are equivalent.

The result above provides a $\Pi$-variation form of Tauberian theorem for the Stieltjes transform, complementing the classical work of Karamata and others, for which see [BGT, Th. 1.7.4]. It also complements de Haan's Tauberian theorem for the Laplace transform [BGT, Th. 3.9.1, Th. 3.9.3].

## §5. The Drasin-Shea-Jordan theorem for systems

In this section, we prove an extension of [BI4, Th. 1] to a system of kernels. We shall need to use this in the next section.

We start with an extension to systems of Titchmarsh's theorem [T2, Th. 146]. For a system of measurable kernels $k_\lambda : \mathbf{R} \to \mathbf{R}$ ($\lambda \in \Lambda$), we consider the system of integral equations

$$f(x) = \int_{-\infty}^{\infty} k_\lambda (x-y) f(y) dy \qquad (-\infty < y < \infty) \qquad \forall \lambda \in \Lambda. \tag{5.1}$$

We write $\hat{k}_\lambda$ for the Fourier transform of $k_\lambda$:

$$\hat{k}_\lambda (z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itz} k(t) dt$$

for those $z \in \mathbf{C}$ for which the integral converges absolutely. To assist comparison with [T2], we define the Fourier transform as there rather than in [BI1,3,4].

**Theorem 5.1.** *Let $0 < c < c'$, and let $e^{c'|x|}k_\lambda(x)$ belong to $L^1(\mathbf{R})$ for all $\lambda \in \Lambda$ and $e^{-c|x|}f(x)$ to $L^2(\mathbf{R})$. Then all solutions $f$ to (5.1) are of the form*

$$f(x) = \sum_{k=1}^{r} \sum_{p=1}^{q_r} c_{k,p} x^{p-1} e^{-iw_k x}, \tag{5.2}$$

*where $w_k$ $(k = 1, \ldots, r)$ are the common zeros of $1 - \sqrt{2\pi}\hat{k}_\lambda(w)$ $(\lambda \in \Lambda)$ in the strip $|\Im z| \le c$, the $c_{k,p}$ are complex constants, and $q_r$ is the largest $q$ such that all the $1 - \sqrt{2\pi}\hat{k}_\lambda(.)$ vanish to order $q$ at $w_k$.*

*Remark.* For each $\lambda \in \Lambda$, $\hat{k}_\lambda(x + iy)$ converges to 0 uniformly in $y \in [-c, c]$ as $x \to \pm\infty$, by the Riemann-Lebesgue lemma and Vitali's theorem [T1, §5.21]. Hence the number of zeros of $1 - \sqrt{2\pi}\hat{k}_\lambda(w)$ in the strip $|\Im w| \le c$ is finite.

*Proof.* For $\lambda \in \Lambda$, $\hat{k}_\lambda(w)$ is defined in $|\Im w| \le c'$. We set, as in [T2],

$$F_+(w) := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{iwt} f(t)dt \qquad (\Im w > c),$$

$$F_-(w) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 e^{iwt} f(t)dt \qquad (\Im w < -c).$$

Then $\hat{k}_\lambda(w)$ $(\lambda \in \Lambda)$ are analytic in $-c' < \Im w < c'$, $F_+(w)$ is analytic in $\Im w > c$, and $F_-(w)$ is analytic in $\Im w < -c$. Choose $a \in (c, c')$, $b \in (-c', -c)$. Then as in the proof of [T2, Th. 146], for each $\lambda$ $F_+(w)\{1 - \sqrt{2\pi}\hat{k}_\lambda(w)\}$ and $F_-(w)\{1 - \sqrt{2\pi}\hat{k}_\lambda(w)\}$ can be continued throughout the strip $b < \Im w < a$, and $F_+(w) = -F_-(w)$ holds in this strip. Here $F_+$ is regular in the strip except possibly for poles $w_k$, and the order of the pole $w_k$ is at most $q_k$. The rest of the proof is the same as that of [T2, Th. 146].

We are now ready to prove the extension of the Drasin-Shea-Jordan theorem to systems of kernels. Theorem 1 of [BI4] is the special case of Theorem 5.2 below with $\Lambda$ a singleton.

**Theorem 5.2.** *Let $-\infty < a < \rho < b < \infty$. Let $k_\lambda : (0, \infty) \to \mathbf{R}$ $(\lambda \in \Lambda)$ be a system of real measurable kernels such that all $\check{k}_\lambda(z)$ converge absolutely in the strip $a \le \Re z \le b$. Assume also that $z = \rho$ is the only common zero of $\check{k}_\lambda(z) - \check{k}_\lambda(\rho)$ $(\lambda \in \Lambda)$ on the line $\Re z = \rho$, and that for some $\lambda$,*

$$|\check{k}_\lambda'(\rho)| + |\check{k}_\lambda''(\rho)| > 0.$$

*Let $f$ be non-negative, measurable and locally bounded on $[0, \infty)$, vanish in a neighbourhood of zero, have finite upper order $\rho \in (a, b)$, and $f \in BD \cup BI$. Then (1.10) implies $c_\lambda = \check{k}_\lambda(\rho)$ $(\lambda \in \Lambda)$ and $f \in R_\rho$.*

*Proof.* As in Steps 1 and 2 of the proof of Jordan's theorem in [BGT, Th. 5.3.1], we obtain $c_\lambda = \check{k}_\lambda(\rho)$ $(\lambda \in \Lambda)$.

Choose one $\lambda' \in \Lambda$ arbitrarily. For any $\epsilon > 0$ such that $[\rho - 2\epsilon, \rho + 2\epsilon] \subset (a, b)$, $\check{k}_{\lambda'}(z)$ converges to zero uniformly in $\Re z \in [\rho - 2\epsilon, \rho + 2\epsilon]$ as $\Im z \to \pm\infty$. Hence, for $M > 0$ large enough, $\check{k}_{\lambda'}(z)$ does not take the value $\check{k}_{\lambda'}(\rho)$ in $\rho - 2\epsilon \le \Re z \le \rho + 2\epsilon$, $|\Im z| \ge M$.

On the other hand, since $D_\epsilon := \{z : \rho - 2\epsilon \le \Re z \le \rho + 2\epsilon, \quad |\Im z| \le M\}$ is compact, $\check{k}_{\lambda'}(z)$ takes the value $\check{k}_{\lambda'}(\rho)$ only at finitely many points in $D_\epsilon$. So for $\epsilon > 0$ small enough, $\check{k}_{\lambda'}(z)$ takes the value $\check{k}_{\lambda'}(\rho)$ only on the line $\Re z = \rho$ in the strip $\rho - 2\epsilon \le \Re z \le \rho + 2\epsilon$. For this $\epsilon$, the only common zero of all the $k_\lambda(z) - k_\lambda(\rho)$ in the strip $\rho - 2\epsilon \le \Re z \le \rho + 2\epsilon$ is $z = \rho$.

The rest of the proof is almost the same as that of [BI4, Th. 1]; we just use Theorem 5.1 instead of [T2, Th. 146]. See also the proof of Theorem 2.1. We omit the details.

## §6. Mercerian theorem for Π-variation

To close, we prove a Mercerian theorem of the Π-variation type (1.9) $\Rightarrow f \in R_\rho$.

**Theorem 6.1.** *Let $\sigma \in \mathbf{R}$, $\epsilon > 0$, and $\rho \in (\sigma - \epsilon, \sigma + \epsilon)$. Let $k : (0, \infty) \to \mathbf{R}$ be a measurable kernel such that $\check{k}(z)$ converges absolutely in the strip $\rho < \Re z \le \sigma + \epsilon$. We assume that $k$ satisfies (k2) of Theorem 4.1 and*
*(k4): there exist $\lambda_1, \lambda_2 \in (0, \infty) \setminus \{1\}$ such that $\log \lambda_1 / \log \lambda_2$ is irrational and $\check{k}_j(z)$ $(j = 1, 2)$ converge absolutely in the strip $\sigma - \epsilon + \rho \le \Re z \le \sigma + \epsilon + \rho$, where*

$$k_j(t) := (\lambda_j x)^{-\rho} k(\lambda_j x) - x^{-\rho} k(x) \qquad (0 < x < \infty) \qquad (j = 1, 2).$$

*Let $f$ be non-negative, measurable and locally bounded on $[0, \infty)$, vanish in a neighbourhood of zero, have finite upper order $q \in (\sigma - \epsilon, \sigma + \epsilon)$, and $f \in BD \cup BI$. If $(k * f)(x)$ converges for $0 < x < \infty$ and satisfies*

$$\frac{(\lambda_j x)^{-\rho}(k * f)(\lambda_j x) - x^{-\rho}(k * f)(x)}{x^{-\rho} f(x)} \to c \log \lambda_j \qquad (x \to \infty) \qquad (j = 1, 2), \tag{6.1}$$

*then $q = \rho$ and $f \in R_\rho$.*

*Proof.* As in the proof of Theorem 4.1, we have

$$\check{k}_j(z) = (\lambda_j^z - 1)\check{k}(z + \rho)$$

for $\sigma - \epsilon - \rho \le \Re z \le \sigma + \epsilon - \rho$ and $z \ne 0$, while

$$\check{k}_j(0) = c \log \lambda_j.$$

Set $\tilde{f}(x) := x^{-\rho} f(x)$ for $0 < x < \infty$. Then (6.1) may be written

$$\frac{(k_j * \tilde{f})(x)}{\tilde{f}(x)} \to c \log \lambda_j \qquad (x \to \infty) \qquad (j = 1, 2). \tag{6.2}$$

Since $\tilde{f}$ has upper order $q - \rho$ and belongs to $BD \cup BI$ as $f$ does, the argument of Steps 1 and 2 of [BGT, Th. 5.3.1] gives

$$\check{k}_j(q - \rho) = c \log \lambda_j \qquad (j = 1, 2).$$

We next show $q = \rho$. We have

$$\frac{(\lambda_2^{q-\rho} - 1)\check{k}(q)}{\log \lambda_2} = c = \frac{(\lambda_1^{q-\rho} - 1)\check{k}(q)}{\log \lambda_1}.$$

Since $c \neq 0$, $\check{k}(q) \neq 0$, and so

$$\frac{\lambda_2^{q-\rho} - 1}{\log \lambda_2} = \frac{\lambda_1^{q-\rho} - 1}{\log \lambda_1}. \tag{6.3}$$

We set

$$g(x) := \frac{\lambda_2^x - 1}{\log \lambda_2} - \frac{\lambda_1^x - 1}{\log \lambda_1} \qquad (0 < x < \infty).$$

Then $g'(x) = \lambda_2^x - \lambda_1^x$, so $g$ is strictly monotone since $\lambda_1 \neq \lambda_2$. This forces $q = \rho$, as otherwise (6.3) gives a repeated value for $g$.

Next, take the Taylor expansion of $z\check{k}(z + \rho)$:

$$z\check{k}(z + \rho) = c + dz + \ldots.$$

Then

$$\begin{aligned}
\check{k}_j(z) &= \frac{(\lambda_j^z - 1)}{z} . z\check{k}(z + \rho) \\
&= \{\log \lambda_j + \frac{1}{2}(\log \lambda_j)^2 z + \ldots\}(c + dz + \ldots) \\
&= c \log \lambda_j + \log \lambda_j . (d + \frac{1}{2} c \log \lambda_j) z + \ldots.
\end{aligned}$$

Since $c \neq 0$, $d + \frac{1}{2} c \log \lambda_j$ $(j = 1, 2)$ cannot be zero simultaneously. So at least one $\check{k}_j$ has non-zero derivative at the origin.

Finally we show that the common zeros of $\check{k}_j(z) - \check{k}_j(0)$ on the line $\Re z = 0$ are only $z = 0$. Suppose that $w$ with $\Re w = 0$, $w \neq 0$ were a common zero. Since $\check{k}_j(0) = c \log \lambda_j$, we have

$$\frac{(\lambda_2^w - 1)\check{k}(w + \rho)}{\log \lambda_2} = c = \frac{(\lambda_1^w - 1)\check{k}(w + \rho)}{\log \lambda_1}.$$

Since $c \neq 0$, $\check{k}(w+\rho) \neq 0$ and so

$$\frac{\lambda_2^w - 1}{\log \lambda_2} = \frac{\lambda_1^w - 1}{\log \lambda_1}. \tag{6.4}$$

Now for real $y$, the trajectory of $y \mapsto (\lambda_j^{iy} - 1)/\log \lambda_j \in \mathbf{C}$ is a circle with centre $-1/\log \lambda_j$ which passes through zero. So the trajectories for $\lambda_1$ and $\lambda_2$ meet only at zero, and so (6.4) implies, for $j = 1, 2$, $\lambda_j^w = 1$, i.e. $e^{w \log \lambda_j} = 1$, or

$$w \log \lambda_j = 2\pi i n_j \qquad (n_j \in \mathbb{Z}, \quad j = 1, 2).$$

This says that

$$\frac{\log \lambda_2}{\log \lambda_1} = \frac{n_2}{n_1},$$

which contradicts the assumption of logarithmic incommensurability in (k4). So $z = 0$ is the only common zero on $\Re z = 0$.

Now we can apply Theorem 5.2, and obtain $\tilde{f} \in R_0$, or $f \in R_\rho$, as required.

*Remark.* (k1) under (k2) implies (k4). See the proof of Theorem 4.1.

*Example: The Laplace transform.* For $k(t) := t^{-1}e^{-1/t}$ $(t > 0)$, $\check{k}(z) = \Gamma(1+z)$, absolutely convergent in $\Re z > -1$. So $\check{k}(z)$ has an analytic continuation in $\Re z > -2$ with unique singularity at $z = -1$, a simple pole with residue 1. Since $tk(t)$ is increasing, for $\rho = -1$ $k$ satisfies (4.1), hence (6.1), for any $\lambda_1$, $\lambda_2 > 1$ such that $\log \lambda_1 / \log \lambda_2$ is irrational. Thus, for suitable $\sigma$ and $\epsilon$, Theorem 6.1 can be applied to this kernel. This Mercerian result for the Laplace transform (new, to our knowledge) complements both the de Haan Tauberian theorem for Laplace transforms [BGT, Th. 3.9.1, 3.9.3] and the Mercerian theorems for Laplace transforms already known - for Karamata theory [BGT, Th. 5.2.4] and de Haan theory (Embrechts' theorem: [BGT, Th. 5.4.1]).

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