liv9.tex Week 9: 15.4.2013

§5. Stochastic Integrals (Itô Calculus)

Stochastic integration was introduced by K. ITÔ in 1944, hence its name Itô calculus. It gives a meaning to $\int_0^t XdY = \int_0^t X_s(\omega)dY_s(\omega)$, for suitable stochastic processes X and Y, the *integrand* and the *integrator*. We shall confine our attention here to the basic case with integrator Brownian motion: $Y = B$. Much greater generality is possible: for Y a continuous martingale, see [KS] or [RY]; for a systematic general treatment, see
MEYER, P.-A. (1976): Un cours sur les intégrales stochastiques. *Séminaire de Probabilités X: Lecture Notes on Math.* **511**, 245-400, Springer.

The first thing to note is that stochastic integrals with respect to Brownian motion, *if* they exist, must be *quite different* from the measure-theoretic integral of Ch. II.2. For, the Lebesgue-Stieltjes integrals described there have as integrators the difference of two monotone (increasing) functions, which are locally of *bounded variation*. But we know from §4 that Brownian motion is of *infinite (unbounded)* variation on every interval. So Lebesgue-Stieltjes and Itô integrals must be fundamentally different.

In view of the above, it is quite surprising that Itô integrals can be defined at all. But if we take for granted Itô's fundamental insight that they *can* be, it is obvious how to begin and clear enough how to proceed. We begin with the simplest possible integrands X, and extend successively in much the same way that we extended the measure-theoretic integral of Ch. II.

1. *Indicators.*
If $X_t(\omega) = I_{[a,b]}(t)$, there is exactly one plausible way to define $\int XdB$:

$$\int_0^t XdB, \quad \text{or} \quad \int_0^t X_s(\omega)dB_s(\omega), := \begin{cases} 0 & \text{if } t \leq a, \\ B_t - B_a & \text{if } a \leq t \leq b, \\ B_b - B_a & \text{if } t \geq b. \end{cases}$$

2. *Simple functions.* Extend by linearity: if X is a linear combination of indicators, $X = \Sigma c_i I_{[a_i,b_i]}$, we should define

$$\int_0^t XdB := \Sigma c_i \int_0^t I_{[a_i,b_i]}dB.$$

Already one wonders how to extend this from constants c_i to suitable random variables, and one seeks to simplify the obvious but clumsy three-line expressions above. It turns out that finite sums are not essential: one can have infinite sums, but now we take the c_i uniformly bounded.

We begin again, this time calling a *stochastic process* X *simple* if there is an infinite sequence

$$0 = t_0 < t_1 < \cdots < t_n < \cdots \to \infty$$

and uniformly bounded $\mathcal{F}_{t_n}$-measurable random variables ξ_n ($|\xi_n| \leq C$ for all n and ω, for some C) if $X_t(\omega)$ can be written in the form

$$X_t(\omega) = \xi_0(\omega) I_{\{0\}}(t) + \Sigma_{i=0}^{\infty} \xi_i(\omega) I_{(t_i, t_{i+1}]}(t) \qquad (0 \leq t < \infty, \omega \in \Omega).$$

The only definition of $\int_0^t X dB$ that agrees with the above for finite sums is, if n is the unique integer with $t_n \leq t < t_{n+1}$,

$$\begin{aligned} I_t(X) := \int_0^t X dB &= \Sigma_0^{n-1} \xi_i (B(t_{i+1}) - B(t_i)) + \xi_n (B(t) - B(t_n)) \\ &= \Sigma_0^{\infty} \xi_i (B(t \wedge t_{i+1}) - B(t \wedge t_i)) \qquad (0 \leq t < \infty). \end{aligned}$$

We pause here to note some properties of the stochastic integral defined so far:

A. $I_0(X) = 0 \qquad P - a.s.$

B. *Linearity.* $I_t(aX + bY) = aI_t(X) + bI_t(Y)$.
Proof. Linear combinations of simple functions are simple.

C. $E[I_t(X)|\mathcal{F}_s] = I_s(X) \qquad P - a.s. \qquad (0 \leq s < t < \infty):$
$I_t(X)$ is a *continuous martingale.*
Proof. There are two cases to consider.
(i) Both s and t belong to the same interval $[t_n, t_{n+1})$. Then

$$I_t(X) = I_s(X) + \xi_n (B(t) - B(s)).$$

But ξ_n is $\mathcal{F}_{t_n}$-measurable, so $\mathcal{F}_s$-measurable ($t_n \leq s$), so independent of $B(t) - B(s)$ (independent increments property of B). So

$$E[I_t(X)|\mathcal{F}_s] = I_s(X) + \xi_n E[B(t) - B(s)|\mathcal{F}_s] = I_s(X).$$

(ii) $s < t$ belongs to a different interval from t: $s \in [t_m, t_{m+1})$ for some $m < n$. Then

$$\begin{aligned} E[I_t(x)|\mathcal{F}_s] &= E(E[I_t(X)|\mathcal{F}_{t_n}]|\mathcal{F}_s) \qquad \text{(iterated conditional expectations)} \\ &= E(I_{t_n}(X)|\mathcal{F}_s), \end{aligned}$$

since ξ_n $\mathcal{F}_{t_n}$-measurable and independent increments of B give

$$E[\xi_n(B(t) - B(t_n))|\mathcal{F}_{t_n}] = \xi_n E[B(t) - B(t_n)|\mathcal{F}_{t_n}] = \xi_n.0 = 0.$$

Continuing in this way, we can reduce successively to t_{m+1}:

$$E[I_t(X)|\mathcal{F}_s] = E[I_{t_m}(X)|\mathcal{F}_s].$$

But $I_{t_m}(X) = I_s(X) + \xi_m(B(s) - B(t_m))$; taking $E[.|\mathcal{F}_s]$ the second term gives zero as above, giving the result. //

Note. The stochastic integral for simple integrands is essentially a martingale transform, and the above is essentially the proof of Ch. III that martingale transforms are martingales.

We pause to note a property of martingales which we shall need below. Call $X_t - X_s$ the *increment* of X over $(s, t]$. Then for a *martingale* X, *the product of the increments over disjoint intervals has zero mean.* For, if $s < t \leq u < v$,

$$\begin{aligned} E[(X_v - X_u)(X_t - X_s)] &= E[E[(X_v - X_u)(X_t - X_s)|\mathcal{F}_u]] \\ &= E[(X_t - X_s)E[(X_v - X_u)|\mathcal{F}_u]], \end{aligned}$$

taking out what is known (as $s, t \leq u$). The inner expectation is zero by the martingale property, so the LHS is zero, as required.

D. $E[(I_t(X))^2] = E\int_0^t X_s^2 ds$.
Proof. The LHS above is $E[I_t(X).I_t(X)]$, i.e.

$$E[(\Sigma_{i=0}^{n-1}\xi_i(B(t_{i+1}) - B(t_i)) + \xi_n(B(t) - B(t_n)))^2].$$

Expanding out the square, the cross-terms have expectation zero by above, leaving

$$E[\Sigma_{i=0}^{n-1}\xi_i^2(B(t_{i+i} - B(t_i))^2 + \xi_n^2(B(t) - B(t_n))^2].$$

Since ξ_i is $\mathcal{F}_{t_i}$-measurable, each ξ_i^2-term is independent of the squared Brownian increment term following it, which has expectation $var(B(t_{i+1})-B(t_i)) = t_{i+1}-t_i$. So we obtain

$$\Sigma_{i=0}^{n-1} E[\xi_i^2](t_{i+1}-t_i) + E[\xi_n^2](t-t_n).$$

This is $\int_0^t E[X_u^2]du = E\int_0^t X_u^2 du$, as required.

E. $E[(I_t(X)-I_s(X))^2|\mathcal{F}_s] = E[\int_s^t X_u^2 du] \qquad P-a.s.$
Proof: as above.

F. *Quadratic variation.* The quadratic variation of $I_t(X) = \int_0^t X_u dB_u$ is $\int_0^t X_u^2 du$.

This is proved in the same way as the case $X \equiv 1$, that B has quadratic variation process t.

Integrands.

The properties above suggest that $\int_0^t XdB$ should be defined only for processes with

$$\int_0^t EX_u^2 du < \infty \qquad \text{for all} \quad t.$$

We shall restrict attention to such X in what follows. This gives us an L_2-theory of stochastic integration (compare the L_2-spaces introduced in Ch. II), for which Hilbert-space methods are available.

3. *Approximation.*

Recall steps 1 (indicators) and 2 (simple integrands). By analogy with the integral of Ch. II, we seek a suitable class of integrands suitably approximable by simple integrands. It turns out that:
(i) The suitable class of integrands is the class of left-continuous adapted processes X with $\int_0^t EX_u^2 du < \infty$ for all $t > 0$ (or all $t \in [0,T]$ with finite time-horizon T, as here),
(ii) Each such X may be approximated by a sequence of simple integrands X_n so that the stochastic integral $I_t(X) = \int_0^t XdB$ may be defined as the limit of $I_t(X_n) = \int_0^t X_n dB$,
(iii) The stochastic integral $\int_0^t XdB$ so defined still has properties A-F above.

It is not possible to include detailed proofs of these assertions in a course of this type [recall that we did not construct the measure-theoretic integral of Ch. II in detail either - and this is harder!]. The key technical ingredient needed is the *Kunita-Watanabe inequalities.* For details, see e.g. [KS], §§3.1-2.

One can define stochastic integration in much greater generality.
1. *Integrands.* The natural class of integrands X to use here is the class of *predictable* processes. These include the left-continuous processes to which we confine ourselves above.
2. *Integrators.* One can construct a closely analogous theory for stochastic integrals with the Brownian integrator B above replaced by a *continuous martingale* integrator M. The properties above hold, with D replaced by

$$E[(\int_0^t X_u dM_u)^2] = E\int_0^t X_u^2 d\langle M\rangle_u.$$

See e.g. [KS], [RY] for details.

One can generalise further to *semimartingale* integrators: these are processes expressible as the sum of a martingale and a process of (locally) bounded variation. See e.g. [RW1] or Meyer (1976) for details.

§6. Stochastic Differential Equations (SDEs) and Itô's Lemma

Suppose that U, V are adapted processes,with U locally integrable (so $\int_0^t U_s ds$ is defined as an ordinary integral, as in Ch. II), and V is left-continuous with $\int_0^t EV_u^2 du < \infty$ for all t (so $\int_0^t V_s dB_s$ is defined as a stochastic integral, as in §5). Then

$$X_t := x_0 + \int_0^t U_s ds + \int_0^t V_s dB_s$$

defines a stochastic process X with $X_0 = x_0$. It is customary, and convenient, to express such an equation symbolically in differential form, in terms of the *stochastic differential equation*

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0. \tag{SDE}$$

Now suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is a function, continuously differentiable once in its first argument (which will denote time), and twice in its second argument (space): $f \in C^{1,2}$. The question arises of giving a meaning to the stochastic differential $df(t, X_t)$ of the process $f(t, X_t)$, and finding it.

Recall the Taylor expansion of a smooth function of several variables, $f(x_0, x_1, \cdots, x_d)$ say. We use suffices to denote partial derivatives: $f_i :=$

$\partial f/\partial x_i$, $\qquad f_{i,j} := \partial^2 f/\partial x_i \partial x_j$ (recall that if partials not only exist but are continuous, then the order of partial differentiation can be changed: $f_{i,j} = f_{j,i}$, etc.). Then for $x = (x_0, x_1, \cdots, x_d)$ near u,

$$f(x) = f(u) + \Sigma_{i=0}^{d}(x_i - u_i)f_i(u) + \frac{1}{2}\Sigma_{i,j=0}^{d}(x_i - u_i)(x_j - u_j)f_{i,j}(u) + \cdots$$

In our case (writing t_0 in place of 0 for the starting time):

$$f(t, X_t) = f(t_0, X(t_0)) + (t-t_0)f_1(t_0, X(t_0)) + (X(t)-X(t_0))f_2 + \frac{1}{2}(t-t_0)^2 f_{11} +$$

$$(t - t_0)(X(t) - X(t_0))f_{12} + \frac{1}{2}(X(t) - X(t_0))^2 f_{22} + \cdots,$$

which may be written symbolically as

$$df(t, X(t)) = f_1 dt + f_2 dX + \frac{1}{2}f_{11}(dt)^2 + f_{12}dtdX + \frac{1}{2}f_{22}(dX)^2 + \cdots.$$

In this, we
(i) substitute $dX_t = U_t dt + V_t dB_t$ from above,
(ii) substitute $(dB_t)^2 = dt$, i.e. $|dB_t| = \sqrt{dt}$, from §4:

$$df = f_1 dt + f_2(Udt + VdB) + \frac{1}{2}f_{11}(dt)^2 + f_{12}dt(Udt + VdB) + \frac{1}{2}f_{11}(Udt + VdB)^2 + \cdots$$

Now using $(dB)^2 = dt$,

$$\begin{aligned}(Udt + VdB)^2 &= V^2 dt + 2UV dtdB + U^2(dt)^2 \\ &= V^2 dt + \text{higher-order terms} :\end{aligned}$$

$$df = (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + Vf_2 dB + \text{higher-order terms}.$$

Summarising, we obtain *Itô's Lemma*, the analogue for the Itô or stochastic calculus of the chain rule for ordinary (Newton-Leibniz) calculus:

THEOREM (Itô's Lemma). If X_t has stochastic differential

$$dX_t = U_t dt + V_t dB_t, \qquad X_0 = x_0,$$

and $f \in C^{1,2}$, then $f = f(t, X_t)$ has stochastic differential

$$df = (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + Vf_2 dB_t.$$

That is, writing f_0 for $f(0, x_0)$, the initial value of f,

$$f(t, X_t)) = f_0 + \int_0^t (f_1 + Uf_2 + \frac{1}{2}V^2 f_{22})dt + \int_0^t V f_2 dB.$$

This important result may be summarised as follows: use Taylor's theorem formally, together with the rule

$$(dt)^2 = 0, \qquad dtdB = 0, \qquad (dB)^2 = dt.$$

Itô's Lemma extends to higher dimensions, as does the rule above:

$$df = (f_0 + \Sigma_{i=1}^d U_i f_i + \frac{1}{2}\Sigma_1^d V_i^2 f_{ii})dt + \Sigma_1^d V_i f_i dB_i$$

(where U_i, V_i, B_i denote the ith coordinates of vectors U, V, B, f_i, f_{ii} denote partials as above); here the formal rule is

$$(dt)^2 = 0, \qquad dtdB_i = 0, \qquad (dB_i)^2 = dt, \qquad dB_i dB_j = 0 \qquad (i \neq j).$$

COROLLARY. $Ef(t, X_t) = f_0 + \int_0^t E[f_1 + Uf_2 + \frac{1}{2}V^2 f_{22}]dt.$

Proof. $\int_0^t V f_2 dB$ is a stochastic integral, so a martingale, so its expectation is constant ($= 0$, as it starts at 0). //

Note. Powerful as it is in the setting above, Itô's Lemma really comes into its own in the more general setting of semimartingales. It says there that if X is a semimartingale and f is a smooth function as above, then $f(t, X(t))$ is also a semimartingale. The ordinary differential dt gives rise to the bounded-variation part, the stochastic differential gives rise to the martingale part. This closure property under very general non-linear operations is very powerful and important.

Example: The Ornstein-Uhlenbeck Process.

The most important example of a SDE for us is that for geometric Brownian motion (VI.1 below). We close here with another example.

Consider now a model of the velocity V_t of a particle at time t ($V_0 = v_0$), moving through a fluid or gas, which exerts
(i) a frictional drag, assumed propertional to the velocity,
(ii) a noise term resulting from the random bombardment of the particle by

the molecules of the surrounding fluid or gas. The basic model for processes of this type is given by the SDE

$$dV = -\beta V dt + c dB, \tag{OU}$$

whose solution is called the *Ornstein-Uhlenbeck* (velocity) process with *relaxation time* $1/\beta$ and *diffusion coefficient* $D := \frac{1}{2}c^2/\beta^2$. It is a stationary Gaussian Markov process (not stationary-increments Gaussian Markov like Brownian motion), whose limiting (ergodic) distribution is $N(0, \beta D)$ and whose limiting correlation function is $e^{-\beta|\cdot|}$.

If we integrate the OU velocity process to get the OU *displacement process*, we lose the Markov property (though the process is still Gaussian). Being non-Markov, the resulting process is much more difficult to analyse.

The OU process is the prototype of processes exhibiting *mean reversion*, or a *central push*: the frictional drag acts as a restoring force tending to push the process back towards its mean position. It is important in many areas, including
(i) statistical mechanics, where it originated,
(ii) mathematical finance, where it appears in the *Vasicek model* for the term-structure of interest-rates,
(iii) *stochastic volatility* models, where the volatility σ itself is now a stochastic process σ_t, subject to an SDE of OU type.

Chapter VI. MATHEMATICAL FINANCE IN CONTINUOUS TIME

§1. Geometric Brownian Motion (GBM)

As before, we write B for standard Brownian motion. We write $B_{\mu,\sigma}$ for Brownian motion with *drift* μ and *diffusion coefficient* σ: the path-continuous Gaussian process with independent increments such that

$$B_{\mu,\sigma}(s+t) - B_{\mu,\sigma}(s) \text{ is } N(\mu t, \sigma^2 t).$$

This may be realised as

$$B_{\mu,\sigma}(t) = \mu t + \sigma B(t).$$

Consider the process

$$X_t = f(t, B_t) := x_0.\exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t\}.$$

Here, since

$$f(t,x) = x_0.\exp\{(\mu - \frac{1}{2}s^2)t + \sigma x\},$$

$$f_1 = (\mu - \frac{1}{2}\sigma^2)f, \qquad f_2 = \sigma f, \qquad f_{22} = \sigma^2 f.$$

By Itô's Lemma (Ch. V: $dX_t = U_t dt + V_t dB_t$ and f smooth implies $df = (f_1 + U f_2 + \frac{1}{2}V^2 f_{22})dt + V f_2 dB_t$) we have (taking $U = 0, \quad V = 1, \quad X = B$),

$$dX_t = df = [(\mu - \frac{1}{2}\sigma^2)f + \frac{1}{2}\sigma^2 f]dt + \sigma f dB_t :$$

$$dX_t = \mu f dt + \sigma f dB_t = \mu X_t dt + \sigma X_t dB_t :$$

X satisfies the SDE

$$dX_t = X_t(\mu dt + \sigma dB_t), \qquad (GBM)$$

and is called *geometric Brownian motion* (GBM). We turn to its economic meaning, and the role of the two parameters μ and σ, below.

We recall the model of Brownian motion from Ch. V. It was developed (by Brown, Einstein, Wiener, ...) in *statistical mechanics*, to model the irregular, random motion of a particle suspended in fluid under the impact of collisions with the molecules of the fluid.

The situation in *economics* and *finance* is analogous: the price of an asset depends on many factors (a share in a manufacturing company depends on, say, its own labour costs, and raw material prices for the articles it manufactures. Together, these involve, e.g., foreign exchange rates, labour costs – domestic and foreign, transport costs, etc. – all of which respond to the unfolding of events – economic data/political events/the weather/technological change/labour, commercial and environmental legislation/ ... in time. There is also the effect of individual transactions in the buying and selling of a traded asset on the asset price. The analogy between the buffeting effect of molecules on a particle in the statistical mechanics context on the one hand, and that of this continuous flood of new price-sensitive information on the other, is highly suggestive. The first person to use Brownian motion to model price movements in economics was Bachelier in his celebrated thesis of 1900.

Bachelier's seminal work was not definitive (indeed, not correct), either mathematically (it was pre-Wiener) or economically. In particular, Brownian motion itself is inadequate for modelling prices, as
(i) it attains negative levels, and
(ii) one should think in terms of *return*, rather than prices themselves.
However, one can allow for both of these by using *geometric*, rather than ordinary,
Brownian motion as one's basic model. This has been advocated in economics from 1965 on by Samuelson[1] – and was Itô's starting-point for his development of Itô or stochastic calculus in 1944 - -and has now become standard:
SAMUELSON, P. A. (1965): Rational theory of warrant pricing. *Industrial Management Review* **6**, 13-39,
SAMUELSON, P. A. (1973): Mathematics of speculative prices. *SIAM Review* **15**, 1-42.
Returning now to (GBM), the SDE above for geometric Brownian motion driven by Brownian noise, we can see how to interpret it. We have a risky asset (stock), whose price at time t is X_t; $dX_t = X(t+dt) - X(t)$ is the change in X_t over a small time-interval of length dt beginning at time t; dX_t/X_t is the gain per unit of value in the stock, i.e. the *return*. This is a sum of two components:

[1]Paul A. Samuelson (1915-2009), American economist; Nobel Prize in Economics, 1970

(i) a deterministic component μdt, equivalent to investing the money risklessly in the bank at interest-rate μ (> 0 in applications), called the *underlying return rate* for the stock,
(ii) a random, or noise, component σdB_t, with *volatility* parameter $\sigma > 0$ and driving Brownian motion B, which models the market uncertainty, i.e. the effect of noise.
Justification. For a recent treatment of this and other diffusion models via microeconomic arguments, see
[FS] FÖLLMER, H. & SCHWEIZER, M. (1993): A microeconomic approach to diffusion models for stock prices. *Mathematical Finance* **3**, 1-23.
Note. Observe the decomposition of what we are modelling into two components: a systematic component and a random component (driving noise). We have met such decompositions elsewhere – e.g. regression, and the Doob decomposition.

§2. The Black-Scholes Model

For the purposes of this section only, it is convenient to be able to use the 'W for Wiener' notation for Brownian motion/Wiener process, thus liberating B for the alternative use 'B for bank [account]'. Thus our driving noise terms will now involve dW_t, our deterministic [bank-account] terms dB_t.

We now consider an investor constructing a trading strategy in continuous time, with the choice of two types of investment:
(i) riskless investment in a bank account paying interest at rate $r > 0$ (the *short rate* of interest): $B_t = B_0 e^{rt} \quad (t \geq 0)$ [we neglect the complications involved in possible failure of the bank - though *banks do fail* - witness Barings 1995, or AIB 2002!];
(ii) risky investment in stock, one unit of which has price modelled as above by $GMB(\mu, \sigma)$. Here the volatility $\sigma > 0$; the restriction $0 < r < \mu$ on the short rate r for the bank and underlying rate μ for the stock are economically natural (but not mathematically necessary); the stock dynamics are thus given by

$$dS_t = S_t(\mu dt + \sigma dW_t).$$

Notation. Later, we shall need to consider several types of risky stock - d stocks, say. It is convenient, and customary, to use a *superscript* i to label stock type, $i = 1, \cdots, d$; thus $S^1, \cdots, S^d$ are the risky stock prices. We can then use a superscript 0 to label the bank account, S^0. So with one risky asset as above (Week 9), the dynamics are

$$\begin{aligned} dS_t^0 &= rS_t^0 dt, \\ dS_t^1 &= \mu S_t^1 dt + \sigma S_t^1 dW_t. \end{aligned}$$

We shall focus on pricing at time 0 of options with expiry time T; thus the index-set for time t throughout may be taken as $[0, T]$ rather than $[0, \infty)$.

We proceed as in the discrete-time model of IV.1. A *trading strategy* H is a vector stochastic process

$$H = (H_t : 0 \le t \le T) = ((H_t^0, H_t^1, \cdots, H_t^d)) : 0 \le t \le T)$$

which is *previsible*: each H_t^i is a previsible process (so, in particular, $(\mathcal{F}_{t-})$-adapted) [we may simplify with little loss of generality by replacing previsibility here by *left-continuity* of H_t in t]. The vector $H_t = (H_t^0, H_t^1, \cdots, H_t^d)$ is the *portfolio* at time t. If $S_t = (S_t^0, S_t^1, \cdots, S_t^d)$ is the vector of *prices* at time t, the *value* of the portfolio at t is the scalar product

$$V_t(H) := H_t.S_t = \Sigma_{i=0}^d H_t^i S_t^i.$$

The *discounted value* is

$$\tilde{V}_t(H) = \beta_t(H_t.S_t) = H_t.\tilde{S}_t,$$

where $\beta_t := 1/S_t^0 = e^{-rt}$ (fixing the scale by taking the initial bank account as 1, $S_0^0 = 1$), so

$$\tilde{S}_t = (1, \beta_t S_t^1, \cdots, \beta_t S_t^d)$$

is the vector of discounted prices.

Recall that

(i) in IV.1 H is a self-financing strategy if $\Delta V_n(H) = H_n.\Delta S_n$, i.e. $V_n(H)$ is the martingale transform of S by H,

(ii) stochastic integrals are the continuous analogues of martingale transforms.

We thus define the strategy H to be *self-financing*, $H \in SF$, if

$$dV_t = H_t.dS_t = \Sigma_0^d H_t^i dS_t^i.$$