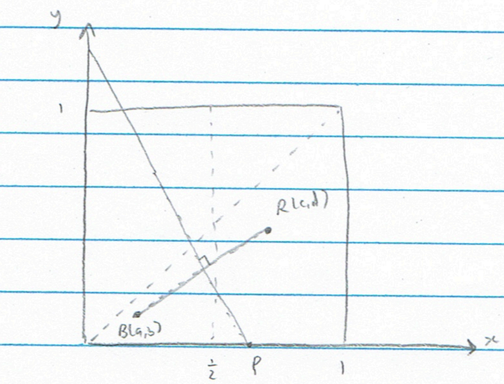
Jane Street Puzzle – November 2024 (Beside the Point); Solution by Nicholas Patel



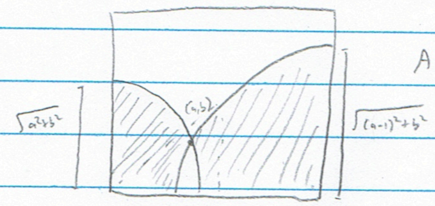
Without loss of generality assume the blue point $B(a,b)$ is uniformly in the triangle defined by $0 \le x \le \frac{1}{2}$ and $0 \le y \le x$.
Then, the side of the square closest to B is always the bottom.
Let the red point be $R(c,d)$ where $c, d \overset{iid}{\sim} \text{Uniform}(0,1)$.
There exists a point on the bottom side equidistant from B and R iff the perpendicular bisector of BR intersects the bottom side.

Perpendicular bisector: passes through $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$ with gradient $\frac{-1}{\left(\frac{d-b}{c-a}\right)} = \frac{a-c}{d-b} \Rightarrow y - \frac{b+d}{2} = \frac{a-c}{d-b}\left(x - \frac{a+c}{2}\right)$

We require P at $0 \le x \le 1 \Rightarrow 0 \le x = \frac{a^2+b^2-c^2-d^2}{2(a-c)} \le 1$.

This leads to the constraint $(c-1)^2+d^2 \le (a-1)^2+b^2$ if $a<c$ or $c^2+d^2 \le a^2+b^2$ if $a>c$.

The area of this region looks like, for a given a and b, the union of parts of 2 circles:



$$A = b + (a^2+b^2)\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{b}{a}\right)\right] + \left((a-1)^2+b^2\right)\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{b}{1-a}\right)\right]$$

Thus, the desired probability is

$$p = 8\int_0^{\frac{1}{2}}\int_0^a A \, db \, da \approx \boxed{0.4914075788} \quad (10dp)$$

If we want in exact form,

$$I_1 = \int_0^{\frac{1}{2}}\int_0^a b \, db \, da = \frac{1}{48}$$

$$I_2 = \int_0^{\frac{1}{2}}\int_0^a (a^2+b^2)\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{b}{a}\right)\right) db \, da = \int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{2}\sec\theta} r^2\left(\frac{\pi}{4}-\theta\right) \cdot r \, dr \, d\theta$$

$$I_3 = \int_0^{\frac{1}{2}}\int_0^a \left[(1-a)^2+b^2\right]\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{b}{1-a}\right)\right] db \, da = \int_0^{\frac{\pi}{4}}\int_{\frac{1}{2}\sec\theta}^{\frac{1}{\sin\theta+\cos\theta}} r^2\left(\frac{\pi}{4}-\theta\right) \cdot r \, dr \, d\theta$$

$$\Rightarrow I_2 + I_3 = \int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{\sin\theta+\cos\theta}} r^2\left(\frac{\pi}{4}-\theta\right) \cdot r \, dr \, d\theta = \frac{2\pi - 2\ln 2 - 1}{96} \quad \text{(after some grunt work)}$$

$$\Rightarrow p = 8(I_1 + I_2 + I_3) = \boxed{\frac{2\pi - 2\ln 2 + 1}{12}}$$