Jane Street Puzzle – August 2024 (Tree-edge Triage) ; Solution by Nicholas Patel

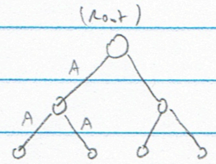
Let $p = \mathbb{P}(\text{an edge is labelled } A)$ and let $\theta = \theta(p) = \mathbb{P}(\text{Aaron wins where each edge } A \text{ with probability } p)$

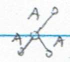
We wish to find the minimum p such that $\theta(p) > 0$.

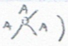
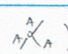
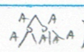
Consider the first turn of each player:

Aaron's turn →

Beren's turn →



For Aaron to win, there must be at least one  at the start/root.

$\mathbb{P}(1$  $) = 2p^3 - 2p^6$, $\mathbb{P}(2$  $) = \mathbb{P}($  $) = p^6$

Since both players know all the arcs, Beren will always choose the losing path (for Aaron) where possible. In other words, Aaron only wins if all possible actions for Beren lead to him winning.

Thus, recursing since the tree is infinitely big,

$$\theta = (2p^3 - 2p^6)\,\theta^2 + (p^6)\cdot\left[1 - (1-\theta^2)^2\right]$$

$$\Rightarrow \quad \theta = 2p^3\theta^2 - 2p^6\theta^2 + p^6\left[2\theta^2 - \theta^4\right]$$

$$\Rightarrow \quad \theta = 2p^3\theta^2 - p^6\theta^4$$

$\theta = 0$ is always a root, but we need to find the minimum p for which another positive root exists.

Graphically, this occurs at $\frac{dp}{d\theta} = 0$

$$\Rightarrow \quad 1 = 4\theta p^3 + 6\theta^2 p^2 \cancel{\frac{dp}{d\theta}}^{\,0} - 4\theta^3 p^6 - 6\theta^4 p^5 \cancel{\frac{dp}{d\theta}}^{\,0}$$

$$\Rightarrow \quad 1 = 4\theta p^3 - 4\theta^3 p^6$$

Originally, we had $\theta = 2p^3\theta^2 - p^6\theta^4 \Rightarrow 1 = 2p^3\theta - p^6\theta^3 \quad (\theta \neq 0)$

Solving simultaneously, $\theta^3 p^6 = \frac{1}{2}$ and $\theta p^3 = \frac{3}{4}$

Thus, $p^3 = \dfrac{(\theta p^3)^3}{\theta^3 p^6} = \dfrac{\left(\frac{3}{4}\right)^3}{\frac{1}{2}} = \dfrac{27}{32}$

Yielding $\boxed{p = \sqrt[3]{\frac{27}{32}}} \approx 0.945$ (3sf)