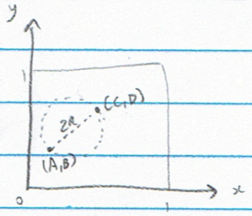
Jane Street Puzzle – February 2024 (Some Off Square) ; Solution by Nicholas Patel

This approach is clunky but works; a more elegant approach could integrate over possible radii / orientations / centre points.



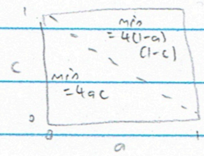
$A, B, C, D \sim \text{Uniform}(0,1)$, and $R = \frac{1}{2}\sqrt{(A-C)^2 + (B-D)^2}$

We want to find $\mathbb{P}\left(\frac{A+C}{2} + R > 1 \text{ or } \frac{A+C}{2} - R < 0 \text{ or } \frac{B+D}{2} + R > 1 \text{ or } \frac{B+D}{2} - R < 0\right)$

$= 1 - \mathbb{P}\left(R < \frac{A+C}{2} < 1 - R \text{ and } R < \frac{B+D}{2} < 1 - R\right)$

Conditioning on $A = a$ and $C = c$, $\mathbb{P}\left(R < \frac{A+C}{2} < 1-R \text{ and } R < \frac{B+D}{2} < 1-R \,\middle|\, A = a, C = c\right) = \mathbb{P}\left((B-D)^2 < \min\{4ac, 4(1-a)(1-c), 1-(a-c)^2\} \text{ and } (a-c)^2 < \min\{4BD, 4(1-B)(1-D), 1-(B-D)^2\}\right)$

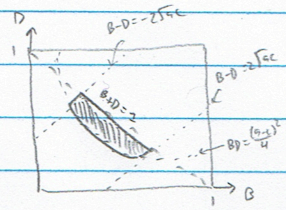
Without loss of generality assume $A + C < 1$. (since $(A,C) \leftrightarrow (1-A, 1-C)$ is equivalent).



Then, $\min\{4ac, 4(1-a)(1-c), 1-(a-c)^2\} = 4ac$.

Similarly, assume $B + D < 1$, so $\min\{4BD, 4(1-B)(1-D), 1-(B-D)^2\} = 4BD$.

$\Rightarrow$ probability simplifies to $\mathbb{P}\left((B-D)^2 < 4ac \text{ and } (a-c)^2 < 4BD\right)$.



This is the proportion of Δ area bound by the lines $B+D=1$, $B-D=2\sqrt{ac}$, $B-D=-2\sqrt{ac}$ and the ray of the hyperbola $D = \frac{(a-c)^2}{4} \cdot \frac{1}{B}$

Integrating gives that this area is $\sqrt{ac}\cdot(2-a-c) - \frac{(a-c)^2}{4} \cdot \ln\left[\left(\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}\right)^2\right]$

And so $P\left(R < \frac{A+C}{2} < 1-R \text{ and } R < \frac{B+D}{2} < 1-R\right) = 4\int_0^1 \int_0^{1-c} \sqrt{ac}\cdot(2-a-c) - \frac{(a-c)^2}{4} \cdot \ln\left(\left(\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}\right)^2\right) \, da\, dc$

$= 4 \cdot \left[\frac{5\pi}{96} - \frac{\pi}{96}\right]$ (after some grunt work)

$= \frac{\pi}{6}$

So, the probability the circle cuts outside the square is $1 - \frac{\pi}{6} = \boxed{\frac{6-\pi}{6}}$