Jane Street Puzzle - August 2023 (Single - Cross 2) : Solution by Nicholas Patel

The line segment is placed uniformly in space, so let one of it's endpoints be defined as (X, Y, Z) where

$$X, Y, Z \overset{iid}{\sim} \text{Uniform}(0, 1)$$

The line segment is chosen uniformly in orientation, thus the second endpoint is equally likely to be anywhere on the surface of the sphere centred at (X, Y, Z) with radius D.
Using Archimedes' Hat-Box Theorem, define the second endpoint (X', Y', Z') as

$$X' = X + D\sqrt{1-U^2}\cos\Theta$$
$$Y' = Y + D\sqrt{1-U^2}\sin\Theta$$
$$Z' = Z + DU,$$

where $U \sim \text{Uniform}(-1, 1)$ and $\Theta \sim \text{Uniform}(0, 2\pi)$.

Then, $P(\text{Exactly 1 cross}) = P(0 \le Y', Z' \le 1 \text{ and } -1 < X' < 0 \text{ or } 1 < X' < 2) +$
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$P(0 \le X', Z' \le 1 \text{ and } -1 < Y' < 0 \text{ or } 1 < Y' < 2)$ (Since (X, Y, Z) is in the 'first' unit cube, $0 \le X, Y, Z \le 1$)

Without loss of generality, exploiting symmetry, we can just consider non-negative $\Delta x, \Delta y, \Delta z$. That is $U \sim \text{Uniform}(0, 1)$ and $\Theta \sim \text{Uniform}(0, \frac{\pi}{2})$. It follows that $X', Y', Z' > 0$ so we need only consider when the rod crosses a plane in one of the 3 positive directions. And, by symmetry, we set

$$P(\text{Exactly 1 cross}) = 3P(0 \le X', Y' \le 1 \text{ and } 1 < Z' < 2)$$

Thus, $P(\text{Exactly 1 cross} \mid \Theta = \phi, U = u) = 3 \cdot P(0 \le X' \le 1 \text{ and } 0 \le Y' \le 1 \text{ and } 1 < Z' < 2 \mid \Theta = \phi)$
$= 3 \cdot P(X' \le 1 \text{ and } Y' \le 1 \text{ and } 1 < Z' < 2 \mid \Theta = \phi)$
$= 3 \cdot P(1 - X \ge D\sqrt{1-u^2}\cos\phi \text{ and } 1 - Y \ge D\sqrt{1-u^2}\sin\phi \text{ and } 1 < Z + Du < 2 \mid \Theta = \phi)$

Note: we condition on the random orientation, and use the distribution of the random position

Since $1-X, 1-Y, 1-Z \overset{iid}{\sim} \text{Uniform}(0, 1)$, by independence it follows that, assuming $D \le 1$,

$$P(\text{Exactly 1 cross} \mid \Theta = \phi, U = u) = 3 \cdot (1 - D\sqrt{1-u^2}\cos\phi) \cdot (1 - D\sqrt{1-u^2}\sin\phi) \cdot Du$$

Thus, $$P(\text{Exactly 1 cross}) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^1 3(1 - D\sqrt{1-u^2}\cos\theta)(1 - D\sqrt{1-u^2}\sin\theta) \cdot Du \, du \, d\theta = \frac{D(3D^2 - 16D + 6\pi)}{4\pi}$$ (after some grunt work)

Which is maximised when $D = \frac{16}{9} - \frac{\sqrt{256 - 54\pi}}{9} \approx 0.745$, with corresponding probability $p \approx 0.509$.
Note a simulation plot of p against D for $0 \le D \le \sqrt{3}$ shows a clear decrease for $D \ge 1$, justifying the assumption $D \le 1$.