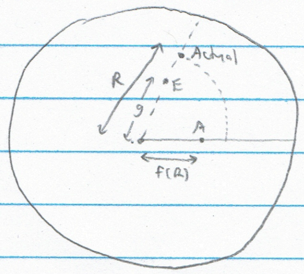
Jane Street Puzzle - April 2024 (Robot Capture-the-Flag); Solution by Nicholas Patel



Without loss of generality assume Aaron moves to position $(f, 0)$ in polar coordinates and Erin moves to position $(g, 0)$ in polar coordinates, where $f = f(R)$ and g is constant, and the true position of the flag is (R, θ).

$P(R \le r) = \frac{\pi r^2}{\pi \cdot 1^2} = r^2$ for $0 \le r \le 1$, so $f_R(r) = 2r$ for $0 \le r \le 1$.

$\theta \sim \text{Uniform}(0, \pi)$ so $P(\theta \le \alpha) = \frac{\alpha}{\pi}$ for $0 \le \alpha \le \pi$.

Let D_E and D_A be the distance from the flag to each robot.

Then, $D_E = |g - R|$ and $D_A = \sqrt{R^2 + f^2 - 2Rf\cos\theta}$ (using the Cosine rule)

$$\Rightarrow P(\text{Aaron wins}) = P(D_A < D_E) = P(D_A^2 < D_E^2)$$
$$= P(R^2 + f^2 - 2Rf\cos\theta < g^2 + R^2 - 2gR)$$
$$= P\left(\cos\theta > \frac{f^2 + g(2R - g)}{2Rf}\right)$$
$$= \int_0^1 P\left(\cos\theta > \frac{f^2 + g(2R-g)}{2Rf} \,\middle|\, R = r\right) \cdot 2r \, dr$$

Since both bots play optimally, assume Aaron knows the value of g, and he also knows the value of r.

If $g > 2r$, choosing $f = 0$ gives $P\left(\cos\theta > \frac{f^2 + g(2r-g)}{2rf}\right) = P(\cos\theta > -\infty) = 1$

If $g < 2r$, choosing $f = \sqrt{g(2r-g)}$ gives the minimum value of $\frac{f^2 + g(2r-g)}{2rf}$, and $P\left(\cos\theta > \frac{f^2 + g(2r-g)}{2rf}\right) = \cos^{-1}\left(\sqrt{\frac{g}{r}\left(2 - \frac{g}{r}\right)}\right) \cdot \frac{1}{\pi}$.

$$\Rightarrow P(\text{Aaron wins}) = \int_0^{\frac{1}{2}g} 1 \cdot 2r \, dr + \int_{\frac{1}{2}g}^1 \frac{2r}{\pi} \cos^{-1}\left(\sqrt{\frac{g}{r}\left(2 - \frac{g}{r}\right)}\right) dr$$

Substituting $r = gu$ in the second integral,

$$P(\text{Aaron wins}) = \frac{1}{4}g^2 + \frac{2g^2}{\pi}\int_{\frac{1}{2}}^{\frac{1}{g}} u \cos^{-1}\left(\sqrt{\frac{1}{u}\left(2 - \frac{1}{u}\right)}\right) du$$

Dividing by g^2 and differentiating with respect to g,

$$\frac{d}{dg}\left(\frac{P}{g^2}\right) = \frac{d}{dg}\left(\frac{1}{4}\right) + \frac{d}{dg}\left[\frac{2}{\pi}\int_{\frac{1}{2}}^{\frac{1}{g}} u \cos^{-1}\left(\sqrt{\frac{1}{u}\left(2 - \frac{1}{u}\right)}\right) du\right]$$

$$\frac{g^2 \frac{dP}{dg} - 2Pg}{g^4} = \frac{2}{\pi} \cdot -\frac{1}{g^2} \cdot \frac{1}{g} \cos^{-1}\left(\sqrt{g(2-g)}\right)$$

Erin chooses g that minimises $P \Rightarrow \frac{dP}{dg} = 0$ gives $P = \frac{1}{\pi}\cos^{-1}\left(\sqrt{g(2-g)}\right)$ at the optimal value of g

Thus, at the critical value $g = g'$,

$$\frac{1}{\pi}\cos^{-1}\left(\sqrt{g'(2-g')}\right) = \frac{1}{4}g'^2 + \frac{2g'^2}{\pi}\int_{\frac{1}{2}}^{\frac{1}{g'}} u \cos^{-1}\left(\sqrt{\frac{1}{u}\left(2 - \frac{1}{u}\right)}\right) du$$

$$\Rightarrow \frac{\pi}{8} - \frac{1}{2g'^2}\cos^{-1}\left(\sqrt{g'(2-g')}\right) + \int_{\frac{1}{2}}^{\frac{1}{g'}} u \cos^{-1}\left(\sqrt{\frac{1}{u}\left(2 - \frac{1}{u}\right)}\right) du = 0$$

Using that $0 < g' < 1$ we can solve this via any root-finding software, giving the solution $g' = 0.5013069942$, with corresponding probability

$$\boxed{P(\text{Aaron wins}) = 0.1661864865 \quad (10\text{dp})}$$