CMS: covering all bases

***Simon Cedervall** and **Vladimir Piterbarg** develop a new vanilla model that directly links constant maturity swap (CMS) and payment convexity in general payouts to volatilities of swaptions of all relevant tenors, as well as prices of CMS spread options, while carefully controlling for potential sources of arbitrage. The model also accounts for the stochastic Libor-overnight indexed swap basis*

Constant maturity swap (CMS) convexity adjustments are driven by the covariance between the underlying swap rate, its associated annuity and the discount bond of the payment delay. This implicitly involves the volatility and correlations of rates of different tenors, and since there is a developed derivatives market in all these quantities (via caps, swaptions and CMS spread options, for instance), the size of the convexity adjustment can be linked to market-implied volatilities and correlations by a model that is sufficiently rich to capture this information. We develop such a 'fully implied' vanilla model in this article.

Most models popular for pricing single-rate CMS derivatives (see, for example, Brigo & Mercurio, 2001) make simplifying approximations and project all volatility risk on to swaptions on the underlying rate. This ignores volatility risks associated with tenors other than that of the primary rate, or attributes it to the wrong rate tenor. Some go further and make the link through other model parameters, such as mean reversion (see Hagan, 2003, and Andersen & Piterbarg, 2010, chapter 16), which allows the capture of risks, but does not project them easily on to the volatility grid. The model proposed in this article calculates all relevant volatility and correlation risks, and attributes them where they belong. Moreover, having developed a fully calibrated model, we are able to quantify the omissions made in standard approaches and highlight some cases in which volatility term structure and rate correlations have a large impact.

The motivation for our work comes from the fact that a versatile model that has a flexible projection of volatility risk and takes rate (de)correlations into account is of significant importance for the risk management of a large and varied book of CMS derivatives. It is also useful for risk-managing interest rate exotics, since long payment delays often appear in exotic products that cannot be valued with simple vanilla models such as the one presented here. The models used to price such products are often calibrated to the full term structure of volatility, while at the same time imperfectly calibrated to the volatility smile of the underlying rate. A model that takes all volatilities into account enables a more accurate delineation of volatility risks, and hedging portfolios to be spread across maturities.

The second part of our article explains how to adapt CMS pricing to the post-crisis world. During the financial crisis of 2007–08, a persistent basis developed between the curve used to forecast Libor rates and the discounting curve. While we initially derive our model in the 'classic' setting where Libor rates and discount factors are calculated from the same curve, we return to the multi-curve case later and show how to account for a forecast-discount basis that may be stochastic.

While most CMS research has focused – and rightfully so – on the impact of the volatility smile, we add no new contributions here on that point. In fact, our model can be used with any smile model suitable for pricing swaptions. That said, while we do not consider the impact of the underlying volatility smile in this article, we should point out that the final combination of our convexity adjustment model and a smile model can never be arbitrage-free if the smile is not. If it is given directly by a formula for volatilities rather than generated from a probability distribution, as is the case for instance with the SABR model, then special care is required in how the smile is extrapolated for low and high strikes. We refer the reader to specialist literature on the subject, with Andersen & Piterbarg (2010, chapter 16.9) and Benaim, Dodgson & Kainth (2009) being representative examples.

Notation

We consider rates that fix at time T, today being time zero. The schedule $T \le T_0 < T_1 < \dots$ is such that T_0 is the spot date or swap start date corresponding to the fixing date T, and $T_1, T_2, \dots$ are the fixed-leg payment dates of swaps starting at T_0. Accrual fractions τ_i correspond to the fixed payments at T_i.

The discount bond $P(t, T_i)$ is the value at time t of a bond paying one at time T_i, and the forward discount bond is $P(t, T_j, T_i) = P(t, T_i)/P(t, T_j)$. For brevity we write $P_i(t) = P(t, T_0, T_i)$.

Forward annuities are defined in terms of forward bonds:

$$A_i(t) = \sum_{j=1}^{i} \tau_j P_j(t) \tag{1}$$

and forward swap rates are given by:

$$S_i(t) = \frac{1 - P_i(t)}{A_i(t)} \tag{2}$$

For values at time zero, we generally drop the time parameter, $P_i = P_i(0)$, $A_i = A_i(0)$ and $S_i = S_i(0)$.

Using bonds $P(t, T_j)$ and annuities $P(t, T_0)A_i(t)$ as numeraires, we have probability measures P^{T_j} and P^{A_i}, where $E^{T_0}[P_i(T)] = P_i$ and $E^{A_i}[S_i(T)] = S_i$ for all $i \ge 1$. The distribution of the swap rate in the annuity measure is inferred from forward swaption prices $A_i E^{A_i}[(S_i(T) - k)^+]$, and we let σ_i be the normal volatility implied from that distribution at strike $k = S_i$ (that is, at-the-money). We also introduce (terminal) swap rate correlations $\rho_{i,j}$ between $S_i(T)$ and $S_j(T)$, which we assume are implied from the market in CMS spread options or, more pragmatically, marked as direct model inputs.

The numerical examples we give use model inputs where, unless otherwise stated, we have $T = 10$ years, a flat initial swap term structure of $S_i = 5\%$, $\sigma_i = 1\%$ and $\rho_{i,j} = 1$ for all $i, j \ge 1$, while the swap schedules are idealised so that $T_i = T + i$ and $\tau_i = 1$. The distribution of $S_i(T)$ is assumed to be normal in the A_i annuity measure, $S_i(T) \sim \mathcal{N}(S_i, \sigma_i^2 T)$.

Vanilla models for CMS

For a payout $V(S_i(T))$, made at time T_k and depending on a single interest rate fixing $S_i(T)$ (a so-called single-rate derivative), we can calculate its forward value at time T_k as:

$$E^{T_k}\left[V\left(S_i(T)\right)\right] = E^{A_i}\left[\frac{P_k(T)}{P_k}\frac{A_i}{A_i(T)}V\left(S_i(T)\right)\right] \tag{3}$$

In principle, a term structure model is required to evaluate (3) as it involves bonds of multiple maturities. To use a vanilla model (that is, a model where only one swap rate is modelled directly), we can rewrite this as:

$$E^{T_k}\left[V\left(S_i(T)\right)\right] = E^{A_i}\left[M_{i,k}\left(S_i(T)\right)V\left(S_i(T)\right)\right] \tag{4}$$

where the annuity mapping function $M_{i,k}(x)$ is defined as:

$$M_{i,k}(x) \triangleq E^{A_i}\left[\frac{P_k(T)}{P_k}\frac{A_i}{A_i(T)}\middle| S_i(T) = x\right] \tag{5}$$

The crux of vanilla modelling for CMSs lies in specifying the annuity mapping functions in (5). Having defined them, valuation of single rate payouts with the formula (4) would then be performed using standard swaption replication (for details, see Andersen & Piterbarg, 2010, chapter 16.6.1).

While a vanilla model would ultimately specify the distribution of a single rate, $S_i(T)$, only, we still want to capture all the risks, and in particular all the volatility and correlation sensitivities, that a full term structure model would show. It should be clear that a term structure model such as the Libor market model would generate vegas (volatility sensitivities) of (3) to not just a single point in the swaption grid (corresponding to the swaption on $S_i(T)$) but to other swaptions with the same expiry as well. Similarly, it would show that (3) depends on correlations. Clearly, in a vanilla model such dependencies must come from the specification of the annuity mapping functions in (5).

Aside from capturing all these volatility and correlation dependencies, there are a number of no-arbitrage requirements that annuity mapping functions must satisfy:

$$E^{A_i}\left[M_{i,k}\left(S_i(T)\right)\right] = 1, \quad k \geq 0 \tag{6}$$

$$\sum_{a=1}^{i}\tau_a P_a M_{i,a}(x) = A_i, \quad \text{all } x \tag{7}$$

$$M_{i,0}(x) - P_i M_{i,i}(x) = A_i x, \quad \text{all } x \tag{8}$$

The first condition states that the value of a fixed payment in the model must be the same as observed from the initial yield curve. The second expresses the fact that any amount depending on the swap rate that is paid out as the corresponding annuity can be statically replicated using swaptions. Finally, the third condition comes from the relationship $1 - P_i(T) = S_i(T)A_i(T)$; an amount received at the start of the swap and eventually repaid at the end of it is equivalent to the same payment times the swap rate being received as an annuity. Note that the model-independent expressions for standard Libor, $M_{1,1}(x) = 1$, Libor in arrears, $M_{1,0}(x) = (1 + \tau_1 x)/(1 + \tau_1 S_1)$, and perpetual rates in arrears, $M_{\infty,0}(x) = x/S_\infty$, follow directly from (7) and (8).

It turns out that building a vanilla model that generates a vega profile consistent with the Libor market model, while avoiding arbitrage as in (6)–(8), is surprisingly difficult. We are unaware of any successful candidates except, perhaps, Auber (2005), which links CMS forward rates to a full set of volatilities and correlations, but does so in a way that cannot easily be extended to more general CMS-linked payouts. In the following sections, we will show how to construct arbitrage-free annuity mapping functions $M_{i,k}(x)$ using a full set of market information, which can then be used to value arbitrary CMS payouts with any length of payment delay.

Fully implied mapping

As we said earlier, our primary motivation for designing new mapping functions is the desire to capture all the volatility and correlation sensitivities of a single-rate derivative. With that in mind, we will express discount bonds in swap rate co-ordinates, utilising swap rates of all relevant tenors, while being mindful of the no-arbitrage constraints. By describing the curve consistently in terms of swap rates, constraints (7) and (8) will be automatically satisfied.

As we make no assumptions about the distribution of $S_i(T)$ at this point, we cannot hope to satisfy (6) in general. We derive a linear annuity mapping that is automatically normalised. Then we derive a non-linear mapping function and show how to adjust it so that (6) is satisfied, while still ensuring that (7) and (8) hold. Finally, we give examples of the sensitivities to volatilities and correlations that the model generates.

Bonds from swap rates. Changing the measure of the conditional expectation that defines the annuity mapping (5) gives the following useful expression:

$$\begin{aligned} M_{i,k}\left(S_i(T)\right) &= E^{A_i}\left[\frac{P_k(T)}{P_k}\frac{A_i}{A_i(T)}\middle| S_i(T)\right] \\ &= \frac{A_i}{P_k}\frac{E^{T_0}\left[P_k(T)\middle|S_i(T)\right]}{E^{T_0}\left[A_i(T)\middle|S_i(T)\right]} \\ &= \frac{A_i}{P_k}\frac{\Pi_{i,k}\left(S_i(T)\right)}{\sum_{j=1}^{i}\tau_j\Pi_{i,j}\left(S_i(T)\right)} \end{aligned} \tag{9}$$

where the bond mapping functions are defined by (note we use the T_0 forward measure and not the annuity one):

$$\Pi_{i,j}(x) \triangleq E^{T_0}\left[P_j(T)\middle|S_i(T) = x\right] \tag{10}$$

The no-arbitrage conditions can be expressed in terms of Π's; (8) becomes:

$$1 - \Pi_{i,i}(x) = x\sum_{j=1}^{i}\tau_j\Pi_{i,j}(x) \tag{11}$$

Equation (7) is trivially satisfied in this parameterisation, and (6) becomes:

$$E^{A_i}\left[\frac{\Pi_{i,k}\left(S_i(T)\right)}{\sum_{j=1}^{i}\tau_j\Pi_{i,j}\left(S_i(T)\right)}\right] = \frac{P_k}{A_i} \tag{12}$$

To derive an expression for (10) based solely on swap rates, we start by noting that:

$$A_i(T) = A_{i-1}(T) + \tau_i P_i(T) = A_{i-1}(T) + \tau_i\left(1 - S_i(T)A_i(T)\right)$$

so:

$$A_i(T) = \frac{1}{1+\tau_i S_i(T)}\left(\tau_i + A_{i-1}(T)\right)$$

and, unwrapping that recursion for $A_i(T)$, we get:

$$A_i(T) = \sum_{a=1}^{i}\tau_a\prod_{b=a}^{i}\frac{1}{1+\tau_b S_b(T)} \tag{13}$$

$$P_i(T) = 1 - S_i(T)\sum_{a=1}^{i}\tau_a\prod_{b=a}^{i}\frac{1}{1+\tau_b S_b(T)} \tag{14}$$

Note that, as promised earlier, $A_i(T)$ and $P_i(T)$ can be expressed in terms of all forward swap rates $S_b(T)$ for $b = 1, \ldots, i$.

Approximating:

$$\begin{aligned}\Pi_{i,j}(x) &= E^{T_0}\left[P_j(T)\middle|S_i(T)=x\right] \\ &\approx 1 - Y_{i,j}(x)\sum_{a=1}^{j}\tau_a\prod_{b=a}^{j}\frac{1}{1+\tau_b Y_{i,b}(x)}\end{aligned} \tag{15}$$

where:

$$Y_{i,b}(x) \triangleq E^{T_0}\left[S_b(T)\middle|S_i(T)=x\right] \tag{16}$$

(note that the approximation would be exact in a Markovian one-factor term structure model) we see that we can express the bond and annuity mapping functions in terms of $Y_{i,b}(x)$, the projections of swap rates on (other) swap rates. Such a representation is useful because $\Pi_{i,j}$'s defined via $Y_{i,b}$'s as in (15) will always satisfy (11). Moreover, this representation will now allow us to derive the mappings, by approximating $Y_{i,b}$'s in a number of different ways.

Linear annuity mapping. It is often technically convenient to have annuity mapping functions that are linear in the swap rate, not least because then the cumulative distribution function of $S_i(T)$ in the T_k forward measure can be directly inferred from a swaption and a digital swaption (see Andersen & Piterbarg, 2010, chapter 16.6.9). To get a linear mapping, as a first step we use a linear approximation for $Y_{i,b}$ in (16) inspired by a formula for a conditional expected value for Gaussian random variables:

$$Y_{i,b}(x) = S_b + \rho_{i,b}\frac{\sigma_b}{\sigma_i}(x - S_i) \tag{17}$$

Substituting this in (15), we linearise the mapping (9) derived from (15) around S_i to get a linear annuity mapping that satisfies all the constraints (6)–(8) while preserving a dependence on all the swaption volatilities and correlations. A short calculation gives:

$$\frac{d}{dx}\Pi_{i,k}(x)\bigg|_{x=S_i} = -\rho_{i,k}\frac{\sigma_k}{\sigma_i}A_k + S_k\sum_{a=1}^{k}\rho_{i,a}\frac{\sigma_a}{\sigma_i}\frac{\tau_a A_a}{\prod_{b=a}^{k}(1+\tau_b S_b)} \tag{18}$$

$$\frac{d}{dx}\sum_{j=1}^{i}\tau_j\Pi_{i,j}(x)\bigg|_{x=S_i} = -\sum_{a=1}^{i}\rho_{i,a}\frac{\sigma_a}{\sigma_i}\frac{\tau_a A_a}{\prod_{b=a}^{i}(1+\tau_b S_b)} \tag{19}$$

so we obtain a linear annuity mapping function:

$$M_{i,k}(x) = 1 + \gamma_{i,k}(x - S_i) \tag{20}$$

where:

$$\begin{aligned}\gamma_{i,k} = &\sum_{a=1}^{i}\rho_{i,a}\frac{\sigma_a}{\sigma_i}\frac{\tau_a}{\prod_{b=a}^{i}(1+\tau_b S_b)}\frac{A_a}{A_i} \\ &-\rho_{i,k}\frac{\sigma_k}{\sigma_i}\frac{A_k}{P_k} + S_k\sum_{a=1}^{k}\rho_{i,a}\frac{\sigma_a}{\sigma_i}\frac{\tau_a}{\prod_{b=a}^{k}(1+\tau_b S_b)}\frac{A_a}{P_k}\end{aligned} \tag{21}$$

Non-linear annuity mapping. A linear mapping is technically convenient but leads to negative bond prices in some states of the world at time T, which is clearly undesirable. While for shorter expiries and tenors the effect is not significant and we would have no qualms using a linear mapping, the problem becomes more troublesome for longer expiries, and in particular for long payment delays. Fortunately, our approach allows us to derive a non-linear annuity mapping that is always positive in a tractable, numerically efficient way while still preserving the no-arbitrage conditions.

To obtain a positive annuity mapping, one can simply try to use (17) in (15). A quick experiment shows that a mapping thus obtained will violate (6) to a significant degree. It is not hard to see the main problem with (17) – it ignores the drifts of swap rates in the T_0 forward measure. While this problem does not manifest itself in the linear case due to (20) satisfying (6) automatically, we are not so lucky in the non-linear case.

We deal with this problem by replacing the projection (17) with one where the drifts of swap rates in the T_0 forward measure can be determined from suitable no-arbitrage conditions. While many choices of the functional form could be made, our choice is based on numerical tractability.

We link all swap rates to the same stochastic driver X via:

$$1 + \tau_j S_j(T) = (1 + \tau_j S_j)e^{\mu_j + \nu_j\sqrt{T}X} \tag{22}$$

where $X \sim \mathcal{N}(0, 1)$ in the T_0-forward measure. This gives:

$$A_j(T) = \sum_{a=1}^{j}\tau_a\exp\left(-\sum_{b=a}^{j}\left(\mu_b + \nu_b\sqrt{T}X\right)\right)\prod_{b=a}^{j}\frac{1}{1+\tau_b S_b} \tag{23}$$

and we see that, thanks to the functional form (22), $E^{T_0}[A_j(T)]$ can be calculated easily in this model. Given that we know that this expected value should be equal to A_j, we can use these no-arbitrage conditions to determine drifts μ_j, $1 \le j \le i$, from:

$$\begin{aligned}A_j &= E^{T_0}\left[A_j(T)\right] \\ &= \sum_{a=1}^{j}\tau_a\exp\left(-\sum_{b=a}^{j}\mu_b + \frac{T}{2}\left(\sum_{b=a}^{j}\nu_b\right)^2\right)\prod_{b=a}^{j}\frac{1}{1+\tau_b S_b}\end{aligned}$$

by solving for each μ_j, $1 \le j \le i$, in turn, having determined μ_k, $k \le j - 1$, on previous steps.

The parameters ν_j in (22) can be linked to normal volatilities and correlations by projecting the rates similarly to (17):

$$\nu_j = \frac{\tau_j}{1+\tau_j S_j}\rho_{i,j}\sigma_j \tag{24}$$

Note that we could write (22) with vector-valued X and ν_j, replace (24) with the relevant expression for $\nu_j^T\nu_i$, and then get (10) in closed form and from that the annuity mapping, but it is hard to justify such a careful handling of the correlations given the approximation involved in using (22) for the underlying rate to begin with. Instead, through (24) we effectively project the other rates on to $S_i(T)$ before calibrating the one-factor model (22), which then allows us to calculate the annuity mapping directly using (13) and (14). The mappings $\tilde{M}_{i,j}$ (the reason for the tilde will be clear shortly) thus defined satisfy (6) to a high degree of accuracy, which is not surprising given the careful handling of the swap rate drifts, but there is one final step that we can do to make sure (6) holds exactly. The functions $\tilde{M}_{i,j}$ satisfy (7) and (8), and it follows that if:

$$E^{A_i}\left[\tilde{M}_{i,k}(S_i(T))\right] = 1 + c_k$$

then we must have $\Sigma_{a=1}^{i}\tau_a P_a c_a = 0$ and $c_0 - P_i c_i = 0$. To ensure that (6) holds, we pick a function $\varphi(x)$, such that $E^{A_i}[\varphi(S_i(T))] = 1$, and finally define the full annuity mapping:

$$M_{i,k}(x) = \tilde{M}_{i,k}(x) - c_k\varphi(x) \tag{25}$$

This mapping function satisfies all the constraints (6)–(8).

The choice of the normalising function φ is to some extent arbi-

A. CMS forward adjustments $E^{T_k}[S_i(T)] - S_i$ in basis points, for $i = 10$ and $k = 0, 1, \ldots, 10$

$i = 10; k =$	0	1	2	3	4	5	6	7	8	9	10
Swap–yield	48.4	38.8	29.3	19.9	10.5	1.05	–8.39	–17.9	–27.4	–37.1	–46.9
Linear	48.6	39.0	29.5	20.0	10.5	0.94	–8.58	–18.1	–27.6	–37.2	–46.7
Non-linear	48.3	38.9	29.5	20.1	10.6	1.12	–8.42	–18.0	–27.6	–37.3	–47.1
Normalised	48.3	38.9	29.5	20.1	10.6	1.12	–8.42	–18.0	–27.7	–37.4	–47.1

Note: using a standard swap–yield model, the linear mapping function and the non–linear mapping function with and without normalisation. As expected with the flat term structure of the inputs, there is little difference between the models

B. Convexity adjustments and vega buckets for $S_{10}(T)$ paid at T_k

k	F_k	C_k	$V_{k,1}$	$V_{k,2}$	$V_{k,3}$	$V_{k,4}$	$V_{k,5}$	$V_{k,6}$	$V_{k,7}$	$V_{k,8}$	$V_{k,9}$	$V_{k,10}$	$\Sigma_j V_{k,j}$
0	5.48%	48	0.7	1.5	2.3	3.2	4.1	5.0	6.0	7.1	8.2	**58**	96
1	5.39%	37	**–8.2**	1.5	2.3	3.1	3.9	4.8	5.8	6.8	7.9	**46**	74
2	5.30%	27	1.1	**–16**	2.2	3.0	3.8	4.6	5.5	6.5	7.5	**35**	54
3	5.20%	17	1.0	2.1	**–24**	2.8	3.6	4.4	5.3	6.2	7.2	**26**	35
4	5.11%	8.7	1.0	2.0	3.1	**–31**	3.5	4.3	5.1	5.9	6.9	**17**	18
5	5.01%	0.9	1.0	1.9	3.0	4.1	**–38**	4.1	4.9	5.7	6.6	**8.4**	2.0
6	4.92%	–6.3	0.9	1.9	2.9	3.9	5.0	**–44**	4.7	5.4	6.3	**0.9**	–12
7	4.82%	–13	0.9	1.8	2.7	3.7	4.8	5.9	**–50**	5.2	6.0	**–6.0**	–25
8	4.72%	–19	0.8	1.7	2.6	3.5	4.5	5.6	6.7	**–56**	5.7	**–12**	–37
9	4.63%	–24	0.8	1.6	2.5	3.4	4.3	5.3	6.4	7.5	**–62**	**–18**	–48
10	4.53%	–29	0.7	1.5	2.3	3.2	4.1	5.0	6.0	7.1	8.2	**–97**	–58
$\Sigma_{k=1}^{10} x_k$		–0.0	–0.0	0.0	0.0	0.0	–0.0	0.0	0.0	0.0	0.0	0.0	0.0
$x_0 - x_{10}$		77	–0.0	–0.0	–0.0	–0.0	0.0	–0.0	–0.0	–0.0	–0.0	154	154

Note: the risk buckets corresponding to the tenor of the pay delay and the underlying rate are in a bold font. Here $F_k = E^{T_k}[S_{10}(T)]$, $C_k = P_k(F_k - S_{10})$ (in basis points), $V_{k,j} = dC_k/d\sigma_j$ (basis points per percentage point), and x_k is either C_k or $V_{k,j}$ depending on the column. Values correct to one decimal place

trary. The simplest choice is to pick the constant one, but this will in general make the mapping function go negative for long payment delays and high rates. We find that a more practical choice is to take $\varphi(x) \propto e^{-(x/S_i)^2}$.

Sensitivity to volatility and correlation. To ensure that none of our assumptions have been too drastic, in particular the normalisation in (25), we perform a quick test using the example inputs from the Notations section. In this we include a simple swap-yield model (see Hagan, 2003, or Andersen & Piterbarg, 2010, chapter 16.3.4), for which:

$$M_{i,k}(x) = \frac{(1+x)^{-k}}{(1+S_i)^{-k}} \frac{\sum_{j=1}^{i}(1+S_i)^{-j}}{\sum_{j=1}^{i}(1+x)^{-j}}$$

In this idealised scenario, with a flat curve and flat volatility term structure, we expect all the models to be in rough agreement. In table A, which shows the convexity adjustments $E^{T_k}[S_i(T)] - S_i$ for $i = 10$ and $k = 1, \ldots, 10$, we see that this is indeed the case.

Now, the point of all this was not to do lots of calculations and end up with the same result as from a simple swap-yield model, so the interesting question is: what happens when the term structure of volatility changes? Since both of the annuity mapping functions (20) and (25) depend on the volatilities of all swap rates $S_b(T)$ for $b = 1, \ldots, \max(i, k)$, and their correlations with $S_i(T)$ through (17) or (24), a single-rate derivative will exhibit vega in the whole 'row' (corresponding to expiry T) of the swaption grid, as well as sensitivities to CMS spread option prices.

To see what such a vega profile looks like, we set the non-linear model up with model inputs as explained in the Notations section, and consider the convexity-adjusted forwards $F_k = E^{T_k}[S_{10}(10)]$, the T_0 rebased convexity adjustments $C_k = P_k(F_k - S_{10})$ and the vegas $V_{k,j} = dC_k/d\sigma_j$ for $k = 0, \ldots, 10, j = 1, \ldots, 10$. The results are shown in table B, with C_k given in basis points and the vega risk scaled in terms of a percentage point move in σ_j. Note that the table is not a swaption grid – the rows are indexed by payment delay, and the time to expiry is constant.

A significant proportion of the total vega in table B sits 'off-column', that is, on tenors shorter than 10 years. The last two rows illustrate constraints (7) and (8) – when received as an annuity, the convexity adjustments cancel out, and when received at T_0 then paid at T_{10} it can be replicated with swaptions on $S_{10}(10)$ so has vega only in the last column.

It is worth emphasising what we said in the introduction: few, if any, of the other available CMS models will ascribe vega to all the grid entries as in table B, and even fewer will show zeros in the last two rows.

Next, with the same model inputs as before, in table C we consider the convexity adjustments for $S_1(10)$ and $S_{10}(10)$ with payment delay $k = 0, 1, \ldots, 10$, and compare the perfectly correlated case with a market-like correlation structure. The correlations are generated using a simple interpolation scheme, and are set at a level where $\rho_{2,10} = 80\%$.

We see that the impact is relatively small, except for the case of a short rate tenor together with a long payment delay. Intuitively, if the payment delay and rate tenor are different in length they may very well be decorrelated, but only if the payment delay is long will the associated discount bond be volatile enough to have a large impact on the convexity adjustment.

Accounting for Libor-OIS basis

We have assumed so far that the forward swap rate $S_i(t)$ can be calculated using the simple formula (2), which is only correct if

C. CMS forward adjustments $E^{T_k}[S_i(T)] - S_i$ in basis points, for $i = 1, 10$ and $k = 0, 1, \ldots, 10$

$i = 10; k =$	0	1	2	3	4	5	6	7	8	9	10
$\rho_{i,k}$		75%	80%	84%	88%	92%	95%	97%	99%	100%	100%
One-factor	48	39	30	20	11	1.1	–8.4	–18	–28	–37	–47
Decorrelated	46	39	31	22	13	2.4	–8.2	–19	–30	–40	–51
Impact	–2.1	0.2	1.6	2.2	2.1	1.3	0.2	–1.1	–2.3	–3.1	–3.4
$i = 1; k =$	0	1	2	3	4	5	6	7	8	9	10
$\rho_{i,k}$		100%	100%	99%	97%	95%	92%	89%	85%	80%	75%
One-factor	9.5	–0.0	–9.5	–19	–29	–38	–48	–58	–68	–77	–87
Decorrelated	9.5	–0.0	–9.5	–19	–28	–36	–43	–49	–54	–58	–60
Impact	0.0	0.0	0.1	0.3	1.1	2.5	4.8	8.3	13	20	27

Note: in the one-factor case with all correlations equal to one, in the decorrelated case with correlations $\rho_{i,k}$ as indicated

D. Impact of stochastic Libor-OIS basis on forward value and vega risk for payer swaptions and a CMS caplet on the 10-year rate

	FV = 974		Vega = 9.74	
β_{10}	Swaption	CMS	Swaption	CMS
100%	969	1,167	9.60	13.62
80%	938	1,127	9.03	12.83
60%	907	1,089	8.47	12.05
40%	877	1,050	7.92	11.27
20%	846	1,012	7.37	10.50
0%	816	974	6.83	9.74

Note: the parameter β_{10} is the regression coefficient of the OIS swap rate on the market swap rate, and the values at the top of the table are for the physically settled swaption

the same curve is used for calculating forward Libor rates and discounting future cashflows. However, the market has moved to using a curve based on overnight indexed swap (OIS) rates for discounting, as this reflects the interest paid or received on posted collateral, and separate curves for calculating forward Libor rates of different terms. We refer to Piterbarg (2010) for background and Pallavicini & Tarenghi (2010) for a study of how this is reflected in market prices for CMS swaps.

This means that there is a basis $b_i(t)$ between the OIS swap rate $S_i^o(t)$, defined by (2), and the market swap rate:

$$S_i^m(t) = \frac{\sum_{k=1}^{n} \tau'_k P(t, T_0, T'_k) L_k(t)}{\sum_{k=1}^{i} \tau_k P_k(t)} = S_i^o(t) + b_i(t)$$

Here T'_k, τ'_k are the payment times and accrual fractions for the Libor leg, with $T'_0 = T_0$ and $T'_n = T_i$, and $L_k(t)$ is the forward Libor rate spanning T'_{k-1} and T'_k. All of $S_i^m(t)$, $S_i^o(t)$ and the basis $b_i(t)$ are martingales in the A_i annuity measure. Note that $b_i(t)$ is implied from curves fitted to market instruments, and in general different from the Libor-OIS basis quoted in the market. In particular, if the convention is to swap a different Libor tenor for OIS than for fixed, then b_i will be a combination of both (quoted) Libor-OIS basis and Libor tenor basis.

We generalise the previous argument by writing:

$$E^{T_k}\left[V\left(S_i^m(T)\right)\right] = E^{A_i}\left[M_{i,k}^m\left(S_i^m(T)\right) V\left(S_i^m(T)\right)\right]$$

where the modified mapping function $M_{i,k}^m$ is now the projection on $S_i^m(t)$:

$$M_{i,k}^m(x) = E^{A_i}\left[G_{i,k}(T) \middle| S_i^m(T) = x\right] \quad (26)$$

with:

$$G_{i,k}(T) = \frac{A_i}{A_i(T)} \frac{P_k(T)}{P_k}$$

Let $M_{i,k}^o(x) = E^{A_i}[G_{i,k}(T) | S_i^o(T) = x]$ be the single (OIS) curve annuity mapping function. If $b_i(t)$ is assumed to be constant then $M_{i,k}^m(x) = M_{i,k}^o(x - b_i)$, and we may calculate $M_{i,k}^o$ using the OIS curve together with implied volatilities and correlations of S^m rates. In practice, this is likely to be adequate for day-to-day risk management, since CMS risk usually sits far out on the curve (long expiries and/or long tenors) where the basis is relatively stable.

To consider the impact of a stochastic basis, we first make the approximation:

$$\begin{aligned} &E^{A_i}\left[G_{i,k}(T) \middle| S_i^m(T)\right] \\ &= E^{A_i}\left[E^{A_i}\left[G_{i,k}(T) \middle| S_i^o(T), S_i^m(T)\right] \middle| S_i^m(T)\right] \\ &\approx E^{A_i}\left[E^{A_i}\left[G_{i,k}(T) \middle| S_i^o(T)\right] \middle| S_i^m(T)\right] \\ &= E^{A_i}\left[M_{i,k}^o\left(S_i^o(T)\right) \middle| S_i^m(T)\right] \end{aligned} \quad (27)$$

which is equivalent to assuming that given $S_i^o(T)$, knowledge of $S_i^m(T)$ gives no additional information about the OIS curve. Then, since $M_{i,k}^o(x)$ is close to linear, we move the expectation inside:

$$\begin{aligned} M_{i,k}^m(x) &= E^{A_i}\left[M_{i,k}^o\left(S_i^o(T)\right) \middle| S_i^m(T) = x\right] \\ &\approx M_{i,k}^o\left(E^{A_i}\left[S_i^o(T) \middle| S_i^m(T) = x\right]\right) \end{aligned} \quad (28)$$

To calculate $M_{i,k}^o$, we need volatilities σ_a^o of $S_a^o(T)$ and correlations $\rho_{a,b}^{o,o}$ between $S_a^o(T)$ and $S_b^o(T)$, $a, b \geq 1$. We also need the projection of $S_i^o(T)$ on $S_i^m(T)$ in (28), for which we assume that:

$$E^{A_i}\left[S_i^o(T) \middle| S_i^m(T) = x\right] = S_i^o + \beta_i\left(x - S_i^m\right) \quad (29)$$

Now, if we were to continue in the spirit of the previous section, we would set $\beta_i = \rho_{i,i}^{o,m}\sigma_i^o/\sigma_i^m$, where $\rho_{i,i}^{o,m}$ is the terminal correlation between $S_i^o(T)$ and $S_i^m(T)$ and σ_i^m is the volatility of $S_i^m(T)$. However, none of the volatilities or correlations involving OIS rates are readily observable in the market, and a discussion of how to link OIS, Libor and basis volatility term structures is beyond the scope of this article (see, for example, Andersen & Piterbarg, 2010, chapter 15.5, or Mercurio, 2010). Instead, we will consider a more pragmatic approach.

We consider β_i simply as the regression coefficient of $S_i^o(T)$ on $S_i^m(T)$, and make it the single parameter that captures the effect of

Libor-OIS decorrelation. To determine OIS volatilities and correlations without involving any further parameters, we take the view that if the volatility of $b_i(t)$ is low, then $\beta_i \approx 1$ and $\sigma^o_j \approx \sigma^m_j$, while if it is high, then $\beta_i \ll 1$ and $\sigma^o_j \ll \sigma^m_j$ (that is, we exclude the possibility that $\sigma^o_j \gg \sigma^m_j$). To get a model consistent with this, we set $\sigma^o_j = \beta_i \sigma^m_j$, $\rho^{o,o}_{i,j} = \rho^{m,m}_{i,j}$, and finally get:

$$M^m_{i,k}(x) = M^o_{i,k}\left(S^o_i + \beta_i\left(x - S^m_i\right)\right) \tag{30}$$

Note that the sensitivity of $M^o_{i,k}$ to OIS volatilities primarily comes from the relative term structure σ^o_j/σ^o_i, $j \geq 1$, (for the linear mapping (20), this is the only volatility dependence) so, in the absence of a model for the joint term structure of the two curves, the assumption regarding the scaling of OIS volatilities is not going to have a large impact. Rather, it is based on the practical consideration of having $M^m_{i,k}(x) \equiv 1$ when $\beta_i = 0$.

We have now reduced the impact of a stochastic Libor-OIS basis to a single parameter β_i, the effect of which is to scale the size of the convexity adjustment (for the linear mapping (20), it acts exactly as a multiplicative scaling). Furthermore, β_i is straightforward to estimate, either from historical data or trader intuition. While there is no liquid market that allows hedging the risk to β_i, it is still a useful risk measure since it can be marked on an expiry-tenor grid, where the sensitivity to each point in the grid gives a measure of Libor-OIS decorrelation risk associated with a particular forward swap rate.

Interestingly, in a stressed market situation where market swap rates S^m are driven primarily by the basis, that is, β_i is small, convexity adjustments will be small (and will, in fact, disappear completely if the forecasting curve is independent of the discounting curve). One practical application of the model is to force $\beta_i = 0$ in order to separate volatility risk due to payment convexity from that due to payout convexity. The sensitivity to β_i is also worth bearing in mind when trying to explain the sometimes observed misalignment between market CMS and swaption quotes – in addition to the commonly considered smile extrapolation issues mentioned in the introduction, Libor-OIS decorrelation gives another degree of freedom to CMS prices that cannot be implied directly from the market.

It is also instructive to note what potential decorrelation implies for the so-called cash-settled swaptions that use the standard (in euro) yield convention for calculating the settlement amount. Assuming the fixed leg frequency is f, a unit-notional payer swaption struck at K pays:

$$\left(S_i(T) - K\right)^+ \sum_{a=1}^{i \times f} \frac{1/f}{\left(1 + S_i(T)/f\right)^a}$$

at T_0. The motivation behind this convention is to pay an amount that approximates the value of the underlying swap using only the fixing $S_i(T)$, but we could equally well characterise it as a CMS payout with a payout convexity designed to offset the payment convexity. If Libor and OIS swap rates are decorrelated, CMS convexity adjustments are smaller than they would otherwise be, and cash-settled swaptions become short convexity to a similar extent – something worth bearing in mind when using cash-settled swaptions to hedge the volatility risk of Bermudan swaptions and other callable products that are by nature physically settled.

As an example, in table D we use the non-linear annuity mapping to price a cash-settled payer swaption and a CMS caplet on $S_{10}(10)$ for different values of β_{10}. Both options have strike $S_{10}(10)$, and the CMS option has notional A_{10} to be comparable with the swaption. Other inputs are as for previous examples, and $b_j(0) = 0$, $j \geq 1$.

Conclusion

Traditional single-rate models impose, rather than imply, volatility and correlation term structures, and therefore allocate some volatility risks to the wrong hedging instruments. We have developed a flexible and arbitrage-free vanilla model for pricing CMS and similar single-rate payouts that is fully implied from swap rate volatilities and correlations of all tenors, takes the Libor-OIS basis into account, and also provides a practical risk measure for handling a stochastic basis.

While it can be argued that this is more than what is needed for risk-managing standard CMS swaps and options, the various features (that is, sensitivities) of the model can be turned on and off easily. A simplified model can be used for day-to-day risk management, while less significant exposures can be checked on a less-frequent basis.

We used the model to highlight a number of cases where the risk of a single-rate derivative goes beyond the smile of the underlying itself:

- Combinations of payment delays of varying lengths, where the volatilities of shorter tenors must be included to get an accurate projection of volatility risk.
- Payment delays significantly longer than the underlying term, for which rate correlations have a large impact.
- Combinations of physically settled and cash-settled derivatives, in particular cash-settled swaptions using the simple yield convention, which have different sensitivities to the volatility of the Libor-OIS basis.

As an example of a product for which the model shows its strengths, consider a digital cap far out-of-the-money with the non-standard feature that all caplets pay at the end of the trade, for instance as a bonus coupon at the redemption of a structured note. The long payment delays will require a non-linear mapping function in order not to generate negative probabilities, the digital payout requires it to be properly normalised, the varying lengths of payment delay require the full projection of volatility risks in order to be efficiently hedged, the impact of swap rate correlations may be significant, and last but not least, any volatility of the Libor-OIS basis has a direct impact on volatility hedge ratios. Rather than resorting to a calibrated term structure model, assuming one has a model that can resolve all the above together with the swaption smile, the methods we have derived here allow for fast and robust pricing using simple replication with swaptions. ■

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