Arithmetic Fed Funds

April 2016

1 Arithmetic-Average Coupon OIS

A Fed Funds arithmetic-average coupon OIS is a swap where the floating leg pays coupons corresponding to the average of daily overnight rates L^{ON} over three months periods against a specific rate. Suppose we have such a swap with maturity T, which pays the floating coupons in $\{T_p^1, T_p^2, ..., T_p^N = T\}$. Assuming L^{ON} as the ON rate and F^{ON} the forward ON rate, the NPV of each coupon could be written as:

$$
\begin{aligned}
C(t_0, T_p^i) &= E^Q\left[D(t_0, T_p^i)\sum_{k=h}^{T_p^i}\frac{\tau(t_k, t_{k+1})L^{ON}(t_k, t_{k+1})}{\tau(t_h, T_p)}\tau(t_h, T_p)\right] \\
&= E^Q\left[D(t_0, T_p^i)\sum_{k=h}^{T_p^i}\left(\frac{P(t_k, t_k)}{P(t_k, t_{k+1})} - 1\right)\right] \\
&= \sum_{k=h}^{T_p^i} E^Q\left[D(t_0, T_p^i)\left(\frac{P(t_k, t_k)}{P(t_k, t_{k+1})} - 1\right)\right] \\
&= \sum_{k=h}^{T_p^i} P(t_0, T_p^i)\left(E^{T_p^i}\left[\frac{P(t_k, t_k)}{P(t_k, t_{k+1})}\right] - 1\right) \\
&= \sum_{k=h}^{T_p^i} P(t_0, T_p^i)\left(ConvAdj(t_k)\frac{P(t_0, t_k)}{P(t_0, t_{k+1})} - 1\right) \\
&= \sum_{k=h}^{T_p^i} P(t_0, T_p^i)\left(ConvAdj(t_k)\left[1 + \tau(t_k, t_{k+1})F^{ON}(t_0, t_k, t_{k+1})\right] - 1\right)
\end{aligned}
$$

It will be exaplained in the following chapters how to evaluate the convexity adjustment choosing an Hull-White short rate model. Actually this kind of instruments are quoted as spread S_T against USD Libor 3M, so we will have

$$\sum_{i=1}^{N} \tau(T_p^{i-1}, T_p^i) F^{3M}(t_0, T_p^{i-1}, T_p^i) P(t_0, T_p^i) = \sum_{i=1}^{N} (C(t_0, T_p^i) + \tau(T_p^{i-1}, T_p^i) S_T P(t_0, T_p^i))$$

In order to use these instruments in the building of the OIS rate curve we need to bootstrap this curve together with the 3M Libor forwarding curve. Because it could be a bit complex, suppose that swaps USD Libor 3M against fixed rate are quoted in the market. Each market quote represents the fixed rate making the two legs of the swap equal. We can then substitute the USD Libor 3M with the fixed rate of the specific swap maturing in T

$$\sum_{i=1}^{N} \tau(T_p^{i-1}, T_p^i) R_T^{3M} P(t_0, T_p^i) = \sum_{i=1}^{N} (C(t_0, T_p^i) + \tau(T_p^{i-1}, T_p^i) S_T P(t_0, T_p^i))$$

By means of the synthetic instrument above we can bootstrap the OIS curve using the spread S_T without the USD Libor 3M forwarding term structure.

2 Hull-White Short Rate Model

Suppose to use an Hull-White model to describe the implicit risk neutral Q-dynamic of the short rate

$$dr_t = [\theta(t) - ar_t]\, dt + \sigma dW_t$$

Its explicit Q-dynamic could be written as

$$r_t = r_s e^{-a(t-s)} + \int_s^t e^{-a(t-u)} \theta(u) du + \sigma \int_s^t e^{-a(t-u)} dW_u$$

The stochastic discount factor at time t for the maturity T is simply define as $D(t,T) = e^{-\int_t^T r(u)du}$. We can derive the bond price representation at time t for the maturity T, $P(t,T) = e^{-\int_t^T f(t,u)du}$ (where $f(t,u) = F(t,u,u+du)$ is the istantaneus forward rate observed in t) as

$$P(t,T) = \exp[A(t,T) - B(t,T) r_t]$$

where the two terms $A(t,T)$ and $B(t,T)$ are defined as follows

$$\begin{aligned} B(t,T) &= \frac{1}{a}\left[1 - e^{-a(T-t)}\right] \\ A(t,T) &= \int_t^T \frac{\sigma^2}{2} B^2(u,T) du - \int_t^T \theta(u) B(u,T) du \end{aligned}$$

3 Convexity Adjustment

From the previous section the risk neutral Q-dynamic of the bond price $P_t(T) = P(t,T)$ is described as

$$
\begin{aligned}
dP_t(T) &= \left(\frac{\partial}{\partial t}A(t,T) - \frac{\partial}{\partial t}B(t,T)r_t\right)P_t(T)dt - B(t,T)P_t(T)dr_t + \frac{\sigma^2}{2}B(t,T)^2P_t(T)dt \\
dP_t(T) &= \left\{\frac{\partial}{\partial t}A(t,T) - \frac{\partial}{\partial t}B(t,T)r_t - B(t,T)[\theta(t) - ar_t] + \right. \\
&\quad \left. + \frac{\sigma^2}{2}B^2(t,T)\right\}P_t(T)dt - B(t,T)P_t(T)\sigma dW_t \\
dP_t(T) &= \left\{-\frac{\sigma^2}{2}B^2(t,T) + \theta(t)B(t,T) + [1 - aB(t,T)]r_t - B(t,T)[\theta(t) - ar_t] + \right. \\
&\quad \left. + \frac{\sigma^2}{2}B^2(t,T)\right\}P_t(T)dt - B(t,T)P_t(T)\sigma dW_t \\
dP_t(T) &= r_tP_t(T)dt - B(t,T)P_t(T)\sigma dW_t
\end{aligned}
$$

In order to evaluate the convexity adjustment shown in the first section, we define the process

$$
\frac{P_t(t_k)}{P_t(t_{k+1})} = e^{A(t,t_k) - A(t,t_{k+1}) - [B(t,t_k) - B(t,t_{k+1})]r_t}
$$

From Ito's Lemma we have the following Q-dynamic for the process above

$$
\begin{aligned}
d\left(\frac{P_t(t_k)}{P_t(t_{k+1})}\right) &= \frac{1}{P_t(t_{k+1})}dP_t(t_k) - \frac{P_t(t_k)}{P_t^2(t_{k+1})}dP_t(t_{k+1}) + \frac{1}{2}\left(2\frac{P_t(t_k)}{P_t^3(t_{k+1})}\right)dP_t^2(t_{k+1}) + \\
&\quad - \frac{1}{P_t^2(t_{k+1})}dP_t(t_k)dP_t(t_{k+1}) \\
d\left(\frac{P_t(t_k)}{P_t(t_{k+1})}\right) &= \Big[r_tdt - B(t,t_k)\sigma dW_t\Big]\frac{P_t(t_k)}{P_t(t_{k+1})} - \Big[r_tdt - B(t,t_{k+1})\sigma dW_t\Big]\frac{P_t(t_k)}{P_t(t_{k+1})} + \\
&\quad + B^2(t,t_{k+1})\sigma^2\frac{P_t(t_k)}{P_t(t_{k+1})}dt - B(t,t_k)B(t,t_{k+1})\sigma^2\frac{P_t(t_k)}{P_t(t_{k+1})}dt \\
d\left(\frac{P_t(t_k)}{P_t(t_{k+1})}\right) &= \sigma^2B(t,t_{k+1})\Big[B(t,t_{k+1}) - B(t,t_k)\Big]\frac{P_t(t_k)}{P_t(t_{k+1})}dt + \\
&\quad + \sigma\Big[B(t,t_{k+1}) - B(t,t_k)\Big]\frac{P_t(t_k)}{P_t(t_{k+1})}dW_t
\end{aligned}
$$

Moving to the T_p-forward measure (where T_p is the coupon payment date), we have $dW_t = dW_t^{T_p} - \sigma B(t,T_p)dt$ and so the T_p-dynamic is

$$
\begin{aligned}
d\left(\frac{P_t(t_k)}{P_t(t_{k+1})}\right) &= \sigma^2\Big[B(t,t_{k+1}) - B(t,T_p)\Big]\Big[B(t,t_{k+1}) - B(t,t_k)\Big]\frac{P_t(t_k)}{P_t(t_{k+1})}dt \\
&\quad + \sigma\Big[B(t,t_{k+1}) - B(t,t_k)\Big]\frac{P_t(t_k)}{P_t(t_{k+1})}dW_t^{T_p}
\end{aligned}
$$

It is then possible to evaluate the following

$$
\begin{aligned}
E^{T_p}\left[\frac{P_t(t_k)}{P_t(t_{k+1})}\right] &= \frac{P_0(t_k)}{P_0(t_{k+1})}\exp\left\{\int_0^t \sigma^2\Big[B(s,t_{k+1})-B(s,T_p)\Big]\Big[B(s,t_k)-B(s,t_{k+1})\Big]ds\right\}\\
&= \frac{P_0(t_k)}{P_0(t_{k+1})}\exp\left\{\frac{\sigma^2}{a^2}\int_0^t \Big[e^{-a(t_{k+1}-s)}-e^{-a(T_p-s)}\Big]\Big[e^{-a(t_k-s)}-e^{-a(t_{k+1}-s)}\Big]ds\right\}\\
&= \frac{P_0(t_k)}{P_0(t_{k+1})}\exp\left\{\frac{\sigma^2}{a^2}\int_0^t e^{2as}\Big[e^{-at_{k+1}}-e^{-aT_p}\Big]\Big[e^{-at_k}-e^{-at_{k+1}}\Big]ds\right\}\\
&= \frac{P_0(t_k)}{P_0(t_{k+1})}\exp\left\{\frac{\sigma^2}{2a^3}\Big(e^{2at}-1\Big)\Big[e^{-at_{k+1}}-e^{-aT_p}\Big]\Big[e^{-at_k}-e^{-at_{k+1}}\Big]\right\}
\end{aligned}
$$

We can therefore define the convexity adjustment $ConvAdj$ due to the delayed payment of the coupon as:

$$
ConvAdj(t) = \exp\left\{\frac{\sigma^2}{2a^3}\Big(e^{2at}-1\Big)\Big[e^{-at_{k+1}}-e^{-aT_p}\Big]\Big[e^{-at_k}-e^{-at_{k+1}}\Big]\right\}
$$

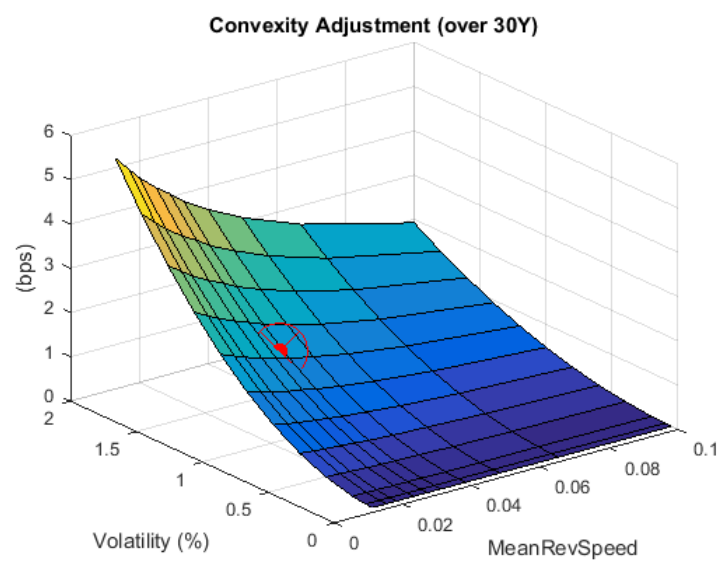

Figure 1: Convexity adjustment impact (bps) over 30Y arithemtic-average coupon OIS fair rate for different values of (a, σ)

. Red point is the value for calibrated parameters (see next chapter).

4 Calibration

Now that the convexity adjustment formula is defined, it is necessary to choose the parameters of the HW model (speed of mean reversion a and annual volatility σ). As we can see in figure 1, the impact of the parameters on the convexity adjustment could be not negligible. A good approach could be to calibrate parameters in order to bootstrap a curve coherent with the standard compounded OIS quotes. Suppose to bootstrap an OIS curve using arithmetic-average coupon OIS without convexity (quotes from 1Y to 10Y, 12Y, 15Y, 20Y, 25Y, 30Y). Then use this curve to price fair rates of the correspondig standard compounded OIS. Comparing the results with the compunded OIS market quotes, we will observe a strip of repricing errors increasing in the matutities (blue points in figure 2).

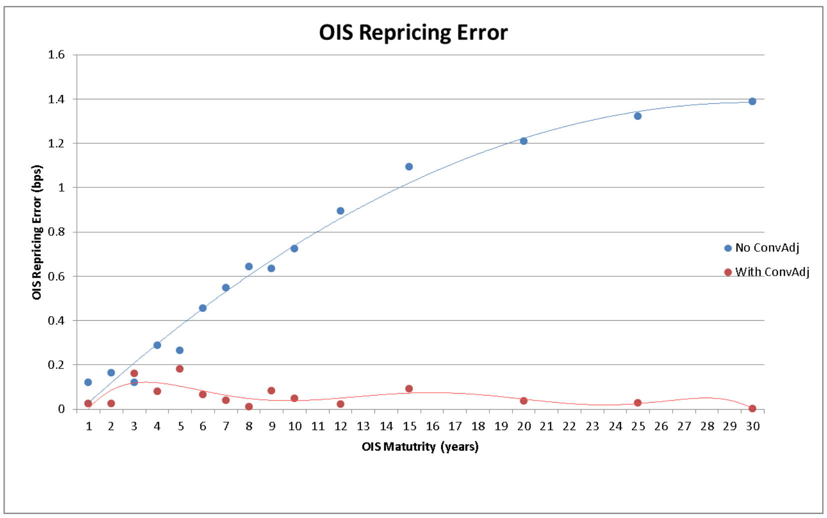

Figure 2: Compounded OIS fair rates repricing errors using a curve bootstrapped through arithmetic-average coupon OIS with convexity adjustment (red) or not (blue)

The first idea is then to find the parameters such that the curve bootstrapped through arithmetic-average coupon OIS (using the convexity adjustment) reprices compounded OIS fair rates as good as possible. Unfortunately this approach is computationally heavy. Another way could be instead to bootstrap an OIS curve through compounded OIS, then keeping the curve fixed, evaluate the arithmetic-average coupon OIS fair rates (using the convexity adjustment). We look for the parameters that minimize the repricing errors between the evaluated fair rates and the market quotes. Proceeding in this way, we can find out a couple of optimal parameters. Now it is possible to bootstrap again the OIS curve through arithmetic-average coupon OIS with convexity adjustment. In figure 2 we can observe how the compounded OIS repricing errors are minimized (red points).