EXPLORING THE OPERAD STRUCTURE APPLIED TO WELDED BRAIDS

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ABSTRACT. This paper defines and discusses the Operad structure, which generalises the composition of multi-linear functions. More generally, it admits a composition operation between objects which express some form of self-similarity.

We apply Operad theory to Welded braids, an extension of classical braids which incorporates an "impossible crossing," by allowing strands to cross without passing each other.

Further, we define composition on braids as the replacement of a strand of a braid with another braid. Classical Braids form an Operad under this composition. This essay explores whether welded braids form an operad structure under the generalisation of this composition to welded braids.

CHAPTER 1

Operads

The Operad structure generalises the idea of composition for operators which input n objects and output 1. Operators of this form are given the name n-ary operators and we compose n-ary operators using the composition operation $\circ_i$ which composes the second operator into the ith input to the first operator.

1.1. Examples of Operads

Example 1.1 (Operad of real multi-linear functions)**.** We will define

$$M(n) = \{ f \mid f : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ where } f \text{ is a real multi-linear function} \}$$

as the set of n-ary multi-linear functions in the real numbers. As an example we designate $A \in M(3)$ and $B \in M(2)$ and express the composition of B into the second argument of A as.

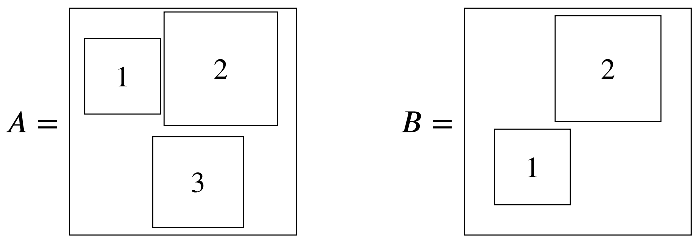
$$A \circ_2 B(x_1, x_2, x_3, x_4) = A(x_1, B(x_2, x_3), x_4)$$

With an intuitive notion of composition of functions. It's of note that $A \circ_2 B \in M(3 + 2 - 1)$ because the 2 inputs of B replaced one of the 3 inputs of A.

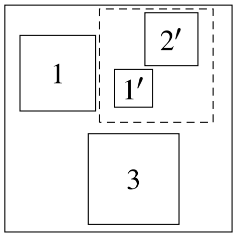
Example 1.2 (Little 2-cubes Operad)**.** Let's first define a solid square with side length $l \in \mathbb{R}$ and centre $\omega \in \mathbb{C}$ by

$$S_{\omega,l} = \{ z \in \mathbb{C} \mid | Re(z) - \omega| \leq l/2 \text{ and } |Im(z) - \omega| \leq l/2 \}$$

The n-squares Operad $\{S(n)\}_{n \geq 1}$ is described by [Spi13] as disjoint squares in $S_{0,1}$, where $S(n)$ represents the configuration of n disjoint squares in $S_{0,1}$ and composition of square configurations is defined by replacing squares with configurations. For example, we might describe $A \in S(3)$ and $B \in S(2)$ by the following diagrams.



If we do so then we can define composition $A \circ_i B$ as replacing square A_i with configuration B. For example the composition of B into 2 is the following, where the dotted line represents the outline of where 2 was.

$A \circ_2 B =$


Exercise 1.3. Determine the precise formula for the composition described above in terms of the location and side lengths of the original configurations of squares.

1.2. Definition of an Operad

There exist many definitions of an Operad which equivalently define the same structure, which generally includes the following data.

a) A sequence of collections of operators $\{\mathcal{O}(n)\}_{n\geq 0}$ of various arity.
b) Permutations maps for the inputs to each operator.
c) A composition operation $\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m-1)$ between these operators.
d) A unit, $e \in \mathcal{O}(1)$, a 1-ary function which acts as an identity.

Remark 1.4. We will consider a specialisation of the definition of the operad by [**Mer21**] which acts similarly to the definition by [**Spi13**].

1.2.1. Objects and Permutations. An Operad $\mathcal{O}$ over a category C is a functor from the permutation category $\mathbb{S}$ to the category C. $\mathbb{S}$ contains natural numbers as objects and morphisms as permutations, i.e. $\mathrm{Ob}(\mathbb{S}) := \mathbb{N}$ and $\mathrm{Hom}_{\mathbb{S}}(n, m) := \mathbb{S}_n$ if $n = m$, $\emptyset$ otherwise.

The morphisms of C are the image of the permutations in $\mathbb{S}$ and so C has a natural concept of permutation, i.e. for a given permutation in $\mathbb{S}$ which permutes $[n]$, this maps to a morphism in C which permutes the inputs of the operators by that permutation.

1.2.2. Composition. We define composition for two arities $n, m \in \mathbb{N}$ as a bi-functor

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(n) \longrightarrow \mathcal{O}(n+m-1)$$

Where associativity rules hold, i.e. for $n, m, k \in \mathbb{N}$, indices $i_1, i_2 \in [n]$ where $i_1 \neq i_2$, and operators $A \in \mathcal{O}(n)$, $B \in \mathcal{O}(m)$, and $C \in \mathcal{O}(k)$

$$(A \circ_{i_1} B) \circ_{i_2} C = (A \circ_{i_2} C) \circ_{i_1} A$$

And for indices $i \in [n]$ and $j \in [m]$

$$(A \circ_i B) \circ_j C = A \circ_i (B \circ_j C)$$

Where indices of the inputs to the operators aren't changed after composition, similarly to example 1.2.

1.2.3. Unit. The unit $e \in \mathcal{O}(1)$ is a 1-ary function which for a given arity $n \in \mathbb{N}$, for any $i \in [n]$ and operator $f \in \mathcal{O}(n)$ we know

$$f = f \circ_i e = e \circ_1 f$$

1.2.4. The components of an Operad. We now have a sequence of collections of operators $\{\mathcal{O}(n)\}_{n\geq 0}$ such that $\mathcal{O}(n)$ is the n-ary operators in C, a composition operation $\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \longrightarrow \mathcal{O}(n+m-1)$, the morphisms of C as images of permutations is our interactions with permutations, and our unit e has been defined appropriately to admit the natural notions of the identity.

Exercise 1.5. Show that our examples from section 1.1 are Operads.

CHAPTER 2

Classical Braids

We first consider the set of classical braids of n strands, uB_n, and specifically $B \in uB_n$ for $n \in \mathbb{N}_{\geq 0}$. We define a braid B as a continuous injective function from n intervals $I_i \subset \mathbb{R}$, $i \in [n]$ to a unit box $[0,1]^3$. And so $B : \bigsqcup_{i=1}^{n} I_i \to [0,1]^3$ is a braid where the image of I_i under B is the ith strand of the braid. The starting positions and ending positions of the strands are linearly spaced along the bottom and top of the cube, i.e. the image of all left and right bounds of the intervals are $\left\{ \left(\frac{i}{n+1}, \frac{1}{2}, 0 \right) \mid i \in [n] \right\}$ and $\left\{ \left(\frac{i}{n+1}, \frac{1}{2}, 1 \right) \mid i \in [n] \right\}$ respectively.

When B is restricted to a given interval I_i, $B_n|_{I_i} : I_i \to [0,1]^3$ is monotonically increasing in the z component. Intuitively, this means that any single strand cannot "decrease in height."

Finally, we consider the set of all n-strand braids uB_n, and we consider these braids modulo isotopy. Under isotopy we can deform any braid B such that the braids do not touch the side of the box $[0,1]^3$. We will assume this condition is satisfied from now on.

We finally define the set of all classical braids with any number of strands as uB, which is the union of all uB_n for $n \geq 0$.

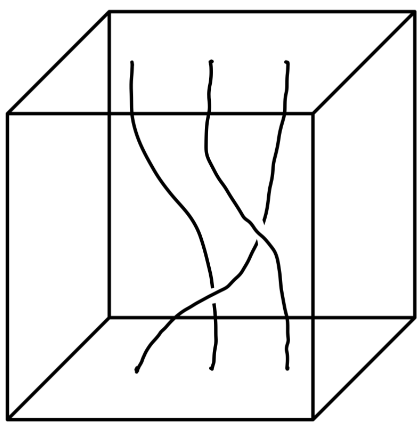


FIGURE 1. The Image of a Braid in uB_3

2.1. Visualisation of Classical Braids

We want to be able to generalise braids into the fourth dimension, as well as be able to study aspects of them such as the crossings. We do this by mapping the braid into different visualisations.

The movie visualisation is used to give intuition for 4 dimensional braids, braid diagrams give intuition for crossings of a braid, and the braid group gives intuition as to the structure that crossings make. As they are defined and shown to be equivalent, we will use these visualisations interchangeably after this section.

2.1.1. Braids as a movie. Let's consider a time parameter $t \in [0, 1]$ and define a frame of B at t, $F_t(B)$ as the projection of the intersection of $\{(x, y, z) \in [0, 1]^3 \mid z = t\}$ with $Im(B)$ into the xy-plane, i.e.

$$F_t(B) = \{(x, y) \in [0, 1]^2 \mid (x, y, t) \in Im(B)\}$$

In which case we can re-represent B as a movie where t is increasing from 0 to 1. Each one of these frames will be n points within $[0, 1]^2$, where each point represents a strand of the braid at that t.

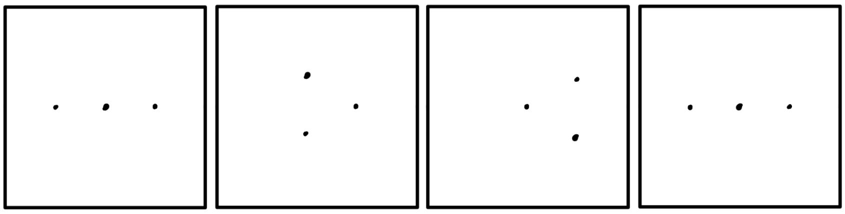


FIGURE 2. Approximate frames of the movie of fig. 1 at times $t = 0, 1/3, 2/3, 1$ (Left to Right)

Let M be the set of all possible movies of braids and therefore we will denote the map for the movie visualisation of the braid B as $M : uB \to F$ where:

$$M(B) = \{F_t(B) \mid t \in [0, 1]\}$$

Remark 2.1. For a given movie of a braid $M(B)$, we can determine the image of the braid $Im(B)$ by stacking braids on top of each other, and therefore determine the braid B modulo isotopy.

2.1.2. Braid Diagrams and Crossings. Let's consider instead a braid B and consider it's projection onto the xz-plane $\text{proj}_{xz}(B)$ such that there are no points where multiple braids are crossing in the same instance or braids are intersecting in parallel, i.e. not more than 3 braids crossing simultaneously and no tangential points.

$$\text{proj}_{xz}(B) = \{(x, z) \in [0, 1]^2 \mid \exists t \in [0, 1] : (x, t, z) \in B\}$$

We want to encode the crossings onto the braid diagram and we therefore denote the set of all crossings, with position of intersections in $[0, 1]$ and crossing type as $C(B)$.

Classical braids only admit two types of crossings, an overcrossing and an undercrossing. These are defined only in terms of an orientation. We will arbitrarily define the orientation in the direction of projection, in this case the direction of the negative y-axis.

Let Br be the set of all possible braid diagrams and let's denote the map for the braid diagram of the braid B as $Br : uB \to Br$ where:

$$Br(B) = (\text{proj}_{xy}(B), C(B))$$

Remark 2.2. Let's consider the braid diagrams as a graphical tool where we we denote overcrossings and undercrossings as

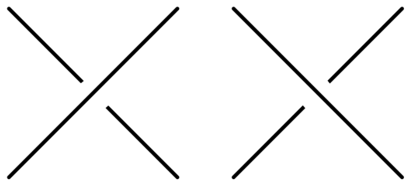


FIGURE 3. Overcrossing (left) and Undercrossing (right)

Therefore the braid diagram of fig. 1 would be

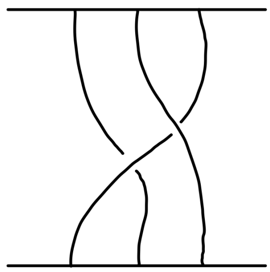


FIGURE 4. Braid Diagram of fig. 1

Noting that braid diagrams are modulo isotopy due to the braids themselves being considered under isotopy.

Remark 2.3. For a given braid diagram $Br(B)$ we can determine the image of the braid B modulo isotopy by performing the inverse projection, this is possible because at the crossing points we know their relative order in the y direction, which is all the information we need modulo isotopy.

2.1.3. The Braid Group. Let's consider the braids $A, B \in uB_n$ and define a multiplication operation by considering stacking B on A. We can consider the x, y, and z components as A_x, A_y, and A_z, similarly for B, we therefore define.

$$AB := \begin{cases} (A_x(2t), A_y(2t), \frac{1}{2}A_z(2t)) & \text{for } t \in [0, \frac{1}{2}] \\ (B_x(2t-1), B_y(2t-1), \frac{1}{2}B_z(2t-1) + \frac{1}{2}) & \text{for } t \in (\frac{1}{2}, 1] \end{cases}$$

Under stacking, braids form a group, where inverses is a flip along the xz plane or equivalently where $t \mapsto 1 - t$, and the identity 1 is the braid without crossings.

Let's consider a braid $B \in uB_n$ and specifically it's braid diagram $Br(B)$. We will then consider the sequence of crossings of the braid B where the order of the sequence is the order in which the crossings occur from the bottom to the top of the braid diagram, and the information of the index of the braids being crossed is recorded.

Let's denote an overcrossing of strand i over strand $i+1$ as σ_i and an undercrossing of strand i under strand $i+1$ as σ_i^{-1}. We can therefore denote any braid in uB_n

as a sequence of crossings and convert the sequence of crossings equivalently as a concatenation of words σ_i and σ_i^{-1}.

Under our multiplication of the braid group we can define the multiplication of the words σ_i and σ_i^{-1} as concatenation, with the inverse of an overcrossing σ_i as an undercrossing σ_i^{-1} and the 1 as the identity.

Example 2.4. Consider the braid diagram fig. 4, this is described by $\sigma_1\sigma_2^{-1}$.

Example 2.5. Consider the following stacking operation in uB_3, where $A = \sigma_1\sigma_2$ and $B = \sigma_1\sigma_2^{-1}$. We compute $AB = \sigma_1\sigma_2\sigma_1\sigma_2^{-1}$ by stacking the braid diagrams on top of each other.

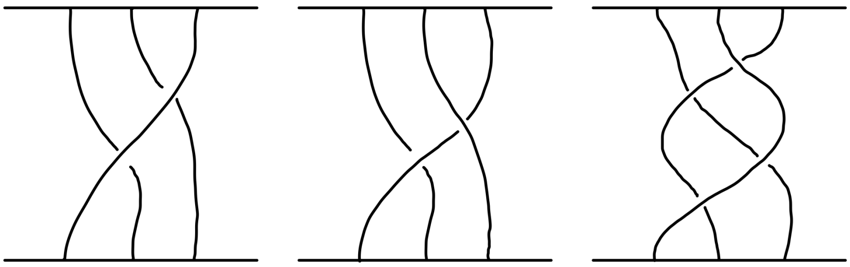


FIGURE 5. A (Left), B (Centre), AB (Right)

Definition 2.6 (The Artin Relations for Braids)**.** *The relations defined on the braid group are the Artin Relations*

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = 1 \\ \sigma_i\sigma_j &= \sigma_j\sigma_i \qquad \textit{where } |i-j| > 1 \end{aligned}$$

Exercise 2.7. Explain why uB_2 is isomorphic to $\mathbb{Z}$.

Exercise 2.8. Prove $\sigma_1^{-1}\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2^{-1}$ in uB_3, verify this using a braid diagram.

2.2. Operadic Composition for Classical Braids

We want to define $A \circ_i B$ for braids $A \in uB_n$ and $B \in uB_m$ and strand index $i \in [n]$. We're going to do this by considering the braid diagram of A and consider a small region around the ith strand, and then replacing this region with the braid diagram B.

There are alternate ways of defining this operation. Specifically, we could consider a region around the ith strand of A and this with B at each height, however, the definition we will give generalises to welded braids, and infact is the more intuitive definition of composition of braids.

Consider the braid diagrams $Br(A)$ and $Br(B)$. We're going to define the notation $Br_i(A)$ to be the braid diagram of the ith strand of A.

Consider $Br_i(A)$ translated in the positive x direction by r, call this $Br_i^r(A)$. Let $\varepsilon > 0$ be such that both curves $Br_i^{\pm\varepsilon}(A)$ are equivalent under isotopy to $Br_i(A)$. We know this exists as if it were not to exist, then with a small perturbation of $Br_i(A)$ in x will cause a extra crossing to occur or for a crossing not to occur, which can occur only if the curve $Br_i(A)$ was tangent to another braid, which is not possible for any braid diagram.

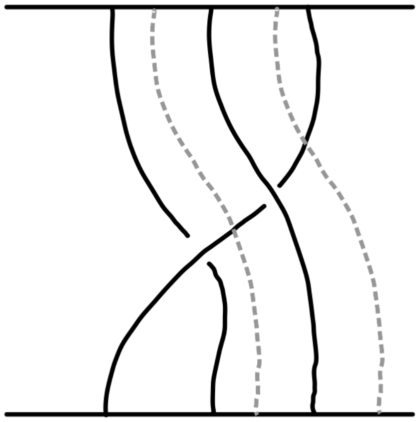


FIGURE 6. $Br_3^{\pm\varepsilon}(A)$ (grey dashed) for $A = \sigma_1\sigma_2^{-1} \in uB_3$

Let's now consider the region enclosed by the curves $Br_i^{\pm\varepsilon}(A)$, call this $R_i^{\varepsilon}(A)$. We then consider for all heights $t \in [0, 1]$, replacing the slice of the region $R_i^{\varepsilon}(A)$ at that height with the slice of the horizontal height of $Br(B)$.

We now want to consider the crossings after this replacement. We can distinguish two types of crossings on the inputted braid B, the crossings formed by the the strands of B with themselves, and the crossings formed by the strands of B with A The crossings of the former aren't effected, however the crossings of the latter are defined by the order in which the the crossings of $Br_i(A)$ occurred from top to bottom. We're guaranteed this is possible, because we know that since $Br_i^{\pm\varepsilon}(A)$ are both equivalent to $Br_i(A)$, then anything in between them must be.

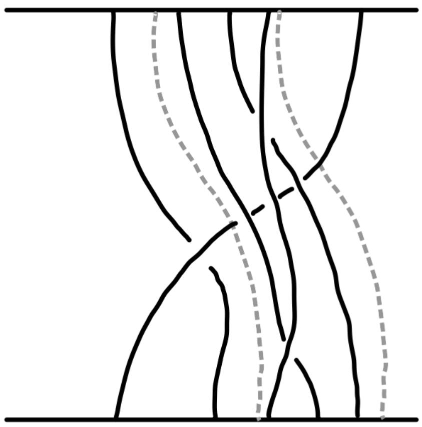


FIGURE 7. $A \circ_3 B$, where A and B are defined in example 2.5.

Finally we can deform the current braid-like object to be linearly spaced along the top and the bottom of the diagram.

Our definition is flawed however. If the two braids are both under isotopy, the order in which the crossings occur could be different under the composition of different representatives. Let's decide to choose a representative A where crossings occur linearly spaced along the height of the braid.

Example 2.9. For the braid $\sigma_1\sigma_2 \in uB_3$, there will be two crossings at the heights $t = \frac{1}{3}$ and $t = \frac{2}{3}$ under this specification.

The issue with our composition operation occurs when two crossings occur simultaneously at the same height. If this occurs, then we arbitrarily decide on the order in which the crossings occur. This is justified, as any decision would be equivalent modulo isotopy.

Example 2.10. Consider $A = \sigma_1 \in uB_2$ and $B = \sigma_2^{-1} \in uB_3$, $A \circ_2 B$ is computed diagrammatically as

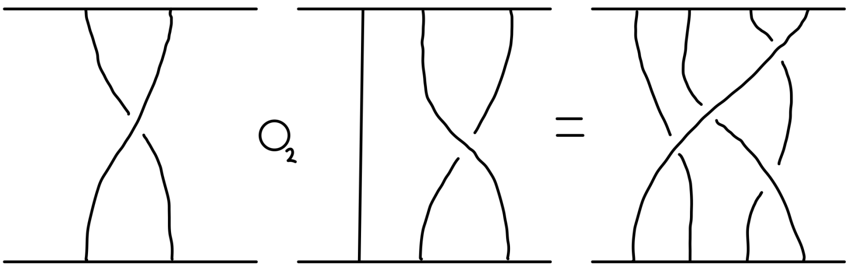


FIGURE 8. Composition of B (middle) into the 2nd strand of A (left) which computes $A \circ_2 B$ (right).

Therefore getting the braid $A \circ_2 B = \sigma_3^{-1}\sigma_1\sigma_2\sigma_3 \in uB_4$.

Exercise 2.11. Determine the braid diagram for $A = \sigma_1\sigma_2 \in uB_3$, $B = \sigma_2^{-1}$, and $A \circ_2 B$.

Exercise 2.12. Let $A = \sigma_2\sigma_3\sigma_1^{-1}\sigma_3 \in uB_4$ and $B = \sigma_1^3 \in uB_3$. Using braid diagrams, compute: $A \circ_2 B$, $B \circ_1 A$, $A \circ_2 1$ where $1 \in uB_2$, and $1 \circ_2 B$ where $1 \in uB_5$.

Exercise 2.13. Show that classical braids form a canonical operad structure under this composition we described.

CHAPTER 3

Welded Braids

Welded Braids are defined in terms of an extension of the classical braid group which described in section 3.1.2, however we will explore the flying ring visualisation of welded braids described by [**BND16**] to gain intuition.

Since welded braids can be described in numerous ways, it is also of note that welded braids can additionally be considered as a generalisation of braids into the 4th dimension, where strands are tubes monotonically increasing in the w direction. This is possibly the most natural way of understanding welded braids due to it's explicit generalisation of classical braids, however, we lack the dimension to understand it completely.

We aim to define the welded braid diagram so that we can generalise our composition operation of classical braids effectively. I will denote the set of welded braids with n strands as wB_n and denote the set of welded braids as wB.

3.1. Visualisation of Welded Braids

We will consider the flying rings visualisation, which is a generalisation of the movie visualisation of classical braids. From there we will show how this is related to the welded braid group. And then we will consider the welded braid diagram, which we will create artificially from the welded braid group to be able to define the composition operation effectively.

3.1.1. Flying Rings Visualisation. The flying ring visualisation is a generalisation of the movie visualisation of classical braids, where for a braid in wB_n, each frame is a box $[0, 1]^3$ with n rings horizontal to the xy plane.

Consider a welded braid $B \in wB_n$. Each frame consists within it n rings placed parallel to the xy plane in $[0, 1]^3$ such that at $t = 0$ and $t = 1$ each rings is linearly space along the bottom of the box on the line $\{(x, y, z) \in [0, 1]^3 \mid x = 1/2 \text{ and } z = 0\}$ This is similar to the classical braid definition, where the points begin and end linearly spaced in $[0, 1]^2$.

We want to consider crossings between strands i and $i+1$ of a welded braids. An overcrossing between these strands in the flying rings visualisation looks like ring $i + 1$ going through ring i. Similarly, an undercrossing between these strands would consist of a ring i going through the ring $i+1$. Finally, we consider a crossing called a virtual crossing, this crossing swaps rings i and $i + 1$.

Virtual crossings are what makes welded braids fundamentally different, if we consider the braid diagram of classical braids, we could not admit a crossing without the lines intersecting, basically by definition, however welded braids allow a crossing which isn't an overcrossing or an undercrossing.

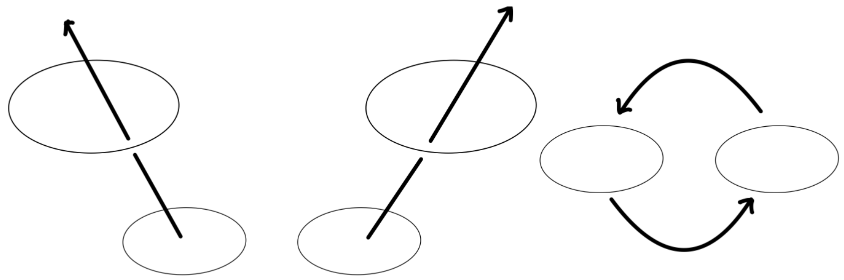


FIGURE 1. Overcrossing (Left), Undercrossing (Centre), Virtual Crossing (Right)

Remark 3.1. Let's consider our 4 dimensional braid, where each strand is a tube. Let's consider our fourth dimension as time, this implies that each frame of our movie visualisation is in $[0, 1]^3$ and our time component t is in $[0, 1]$. Therefore for a given welded braid $B \in wB_n$ we define.

$$F_t(B) = \{(x, y, z) \in [0, 1]^3 \mid (x, y, z, t) \in im(B)\}$$

Similarly to our classical braid visualisation, where we're intersecting the space where $w = t$ with the image of B.

The intuition around this is that we're performing an intersection of a tube that is increasing in the w component with a space that's constant in the w component, and so each frame must be a ring embedded in $[0, 1]^3$.

3.1.2. The Welded Braid Group. Let's consider the flying ring visualisation and consider the stacking of welded braids by performing one movie after another. I.e. we let $A, B \in wB_n$ and define.

$$AB := \begin{cases} F_{2t}(A) & \text{for } t \in [0, \frac{1}{2}] \\ F_{2t-1}(B) & \text{for } t \in (\frac{1}{2}, 1] \end{cases}$$

This forms a group, from which we can consider the Welded Braid Group described by [**BND16**].

Let's consider the sequence of overcrossings, undercrossings, and virtual crossings in order of when they occurred in time. We will denote an overcrossing between strands i and $i + 1$ as σ_i and similarly an undercrossing between strands i and $i + 1$ as σ_i^{-1}. We will similarly denote a virtual crossing between strands i and $i + 1$ as s_i. We can describe a welded braid $B \in wB_n$ as a concatenation of the words $\sigma_i, \sigma_i^{-1}, s_i$ for $i \in [n - 1]$ where the order of words aligns with the sequence of crossings.

This forms the group wB_n for any $n \in \mathbb{N}$ and thus is a natural extension of uB_n. Infact if we restrict welded braids such that virtual crossings do not occur, then the group formed by the stacking (performing a sequence of crossings and then another sequence of crossings) of welded braids is isomorphic to the our classical braid group with n strands uB_n. Therefore all our relations in uB_n are imposed on wB_n.

We now impose relations pertaining to our virtual crossings.

$$s_i^2 = 1 \tag{3.2}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \tag{3.3}$$

$$s_i s_j = s_j s_i \quad \text{where } |i-j| > 1 \tag{3.4}$$

And mixed relations.

$$s_i \sigma_{i+1}^{\pm 1} s_i = s_{i+1} \sigma_i^{\pm 1} s_{i+1} \tag{3.5}$$

$$s_i \sigma_j = \sigma_j s_i \quad \text{where } |i-j| > 1 \tag{3.6}$$

There is one more relation which we haven't defined, it is called the *Overcrossings commute* OC relation, and it breaks the symmetry between overcrossings and undercrossings. This is specifically interesting because it makes overcrossings and undercrossings fundamentally different.

$$\sigma_i \sigma_{i+1} s_i = s_{i+1} \sigma_i \sigma_{i+1} \tag{3.7}$$

Exercise 3.8. Describe what eq. (3.2) and eq. (3.7) represent in their flying rings visualisation.

Exercise 3.9. We call a classical braid PURE if the skeleton of the braid (braid without crossing information and thus is a permutation) is the identity permutation. For example, the skeleton of $\sigma_1\sigma_2 \in uB_3$ is $(12)(23) = (132) \in \mathbb{S}_3$ which isn't the identity and so $\sigma_1\sigma_2$ isn't a pure braid.

Describe pure welded braids in terms of the flying ring visualisation of welded braids.

Exercise 3.10. Using the flying rings visualisation, explain why the UNDERCROSSINGS COMMUTE UC relation $s_i\sigma_{i+1}, \sigma_i = \sigma_{i+1}\sigma_i s_{i+1}$ is false.

3.1.3. Welded Braid Diagrams. Our classical braid diagram took advantage of a projection with extra information, for welded braids this doesn't extend nicely and thus we will define our welded braid diagrams artificially by denoting overcrossings and undercrossings the same as classical braids, however denoting virtual crossings as

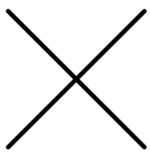


FIGURE 2. Virtual Crossing

We will denote the number of crossings of a braid B as k_B. We will consider the base of the diagram as being height 0 and the top of the diagram as being height 1. Consider $B \in wB_n$ and k_B values of height linearly spaced along $[0,1]$, i.e. $h_B = \left\{ \frac{i}{k_B+1} \in [0,1] \mid i \in [k_b] \right\}$. We will consider a crossing to occur instantaneously at these height values in the order in which they occur as a sequence of crossings. I.e. if σ_i is the kth crossing to occur for a braid B, then we place the diagram of σ_i between the i and $i+1$th braids.

Therefore for a given braid diagram we can determine the information about the sequence of crossings of the braid by considering the order from bottom to top of which crossings occur. We then consider the braid diagram under isotopy, subject to the welded braid relations described in section 3.1.2.

We therefore have an equivalence between the flying rings visualisation, the group, and the braid diagram of welded braids.

Example 3.11. Let $A = \sigma_1 s_2 \sigma_2 \in wB_3$ and $B = s_1 s_2 \sigma_2^2 \in wB_3$. Their braid diagram are described by

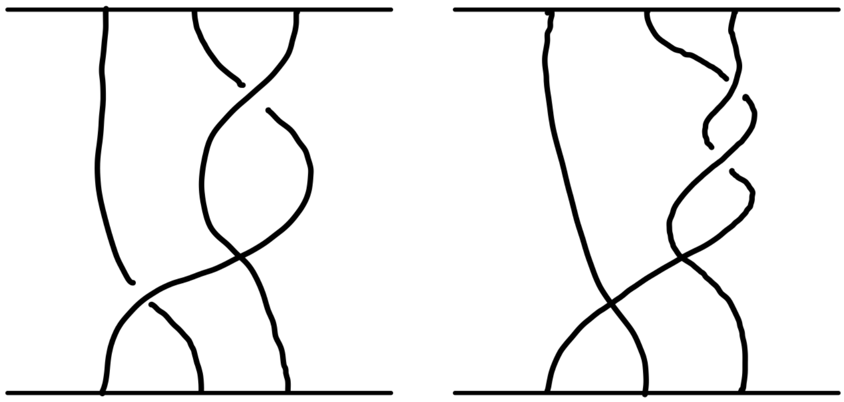


FIGURE 3. A (left) and B (right)

Exercise 3.12. Determine the skeleton and braid diagram for the braids described by $\sigma_1 s_1 s_2 \in wB_3$ and $\sigma_2 s_2 \sigma_1^{-1} \sigma_1 s_1 \in wB_3$.

Exercise 3.13. Show that OC is equivalent to $\sigma_i^{-1} s_{i+1} \sigma_i = \sigma_i s_i \sigma_{i+1}^{-1}$ algebraically, then by considering the braid diagram and flying rings visualisation.

CHAPTER 4

Operad of Welded Braids

We want to prove that welded braids form an operad, i.e. we want to see if we can define a composition operation $\circ_i$ on the set of braids wB which is a natural extension of the composition of classical braids. Thankfully in this case the same definition works as before, where we define an interval around the ith strand at each height and replace into our braid for each height.

4.1. The Composition of Welded Braids

Consider welded braids $A \in wB_n$ and $B \in wB_m$ and $i \in [n]$. We define $A \circ_i B$ by considering the classical braid definition of the $\circ_i$ operator and performing it analogously on welded braids.

Theorem 4.1. *For welded braids $A \in wB_n$ and $B \in wB_m$ and index $i \in [n]$. $A \circ_i B$ isn't well defined.*

Proof. We prove by counter example. Let $A = \sigma_1 \in wB_2$ and $B = s_1 \in wB_2$, and consider $A \circ_1 B$.

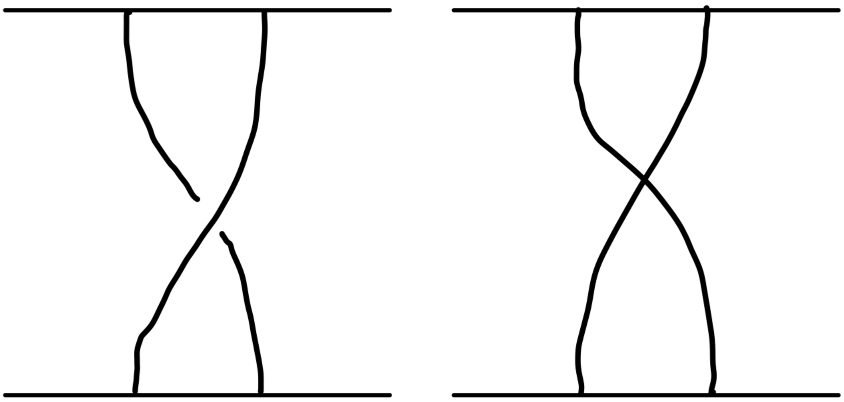


FIGURE 1. A (left) and B (right)

Because there is a single crossing for each braid this implies that there exists a single point at height $t = \frac{1}{2}$ where the crossings occur.

Therefore we will arbitrarily consider which crossing occurs first by considering the cases where the crossings originating from A occur first, or the crossing originating from B occurs first. This is shown via the diagrams

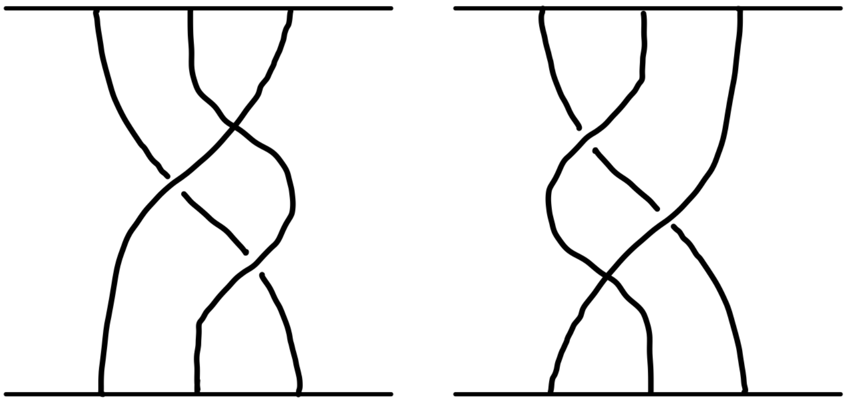


FIGURE 2. Two outputs of $A \circ_1 B$ depending on if the crossings in A occur first (left), or the crossings in B occur first (right).

Which can be represented by

$$A \circ_1 B = \begin{cases} \sigma_2 \sigma_1 s_2 & \text{If A occurs first} \\ s_1 \sigma_2 \sigma_1 & \text{If B occurs first} \end{cases}$$

However this is the *UC* relation, which was shown to explicitly not to be equivalent in exercise 3.10. And thus the $\circ_i$ operation we defined is not well defined and therefore doesn't form an operad. □

4.2. The Non-Canonical Operad of Welded Braids

We shall now consider how we might fix the problems presented in section 4.1.

We can consider a handful of decisions to make the composition operation well defined, however we're going to consider the arbitrary decision where if there is height in which two crossings occur simultaneously, then we arbitrarily decide to perform the crossing of the input braid first. E.g. for counter-example in section 4.1 this would result in $A \circ_1 B = s_1 \sigma_2 \sigma_1$, implying that this composition operation is well defined.

Theorem 4.2. *Welded Braids form a non-canonical Operad under this composition operation.*

Proof. Let's consider the category of welded braids $\mathcal{W}$, where welded braids are our objects $\mathrm{Ob}(\mathcal{W}) := wB$ and the permutations of the names of the strands are our morphisms.

We consider the functor $\mathcal{W} : \mathbb{S} \longrightarrow \mathcal{W}$, where $\mathcal{W}(n) = wB_n$ for $n \in \mathbb{N}$. We then consider a composition operation $\circ_i$ defined as mentioned above, and we show that the associativity rules hold.

For $n, m, k \in \mathbb{N}$, indices $i_1, i_2 \in [n]$ where $i_1 \neq i_2$, and braids $A \in \mathcal{W}(n)$, $B \in \mathcal{W}(m)$, and $C \in \mathcal{W}(k)$

$$(A \circ_{i_1} B) \circ_{i_2} C = (A \circ_{i_2} C) \circ_{i_1} A$$

This is because inputting into two different strands in a different order don't effect each other, as the composition is a property of the single strand only. And for indices $i \in [n]$ and $j \in [m]$

$$(A \circ_i B) \circ_j C = A \circ_i (B \circ_j C)$$

This is because composition is an operation on a single braid, and can be performed in place inside another braid.

Finally we consider the unit morphism e as the single braid $1 \in \mathcal{W}(1)$. Evidently, replacing a strand in a braid with a strand will result in the same braid. And replacing the singular strand in 1 with a braid will result in the braid. Thus for a braid $B \in WB(n)$ and $i \in [n]$

$$B = B \circ_i e = e \circ_i B$$

And therefore $\{\mathcal{W}(n)\}_{n \geq 0}$ forms a non-canonical operad. $\square$

Exercise 4.3. Consider $A = s_1 \sigma_2 s_2 \sigma_1^{-1} \in \mathcal{W}(3)$ and $B = \sigma_1 s_2 \sigma_3^{-1} \in \mathcal{W}(4)$. In the operad $\mathcal{W}$, compute $A \circ_1 B$, $A \circ_3 B$, and $B \circ_2 A$.

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