Operad Structure of 4D Braids

Nicholas Bertollo

Dr. Zsuzsanna Dancso

University of Sydney

October 22, 2024

Itinerary

1. What are Structures?
2. What are Operads?
3. 3D Braids!
4. The 3D Canonical Braid Operad
5. 4D Braids!
6. Do 4D Braids form a Canonical Operad Structure?

Structures

Consider the integers under addition:

$$\cdots - 5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \cdots$$

And question what properties they have?

We can name many properties, however here are some.

Consider the integers a, b, and c.

1. $a + b$ is an integer (Closure)
2. $(a + b) + c = a + (b + c)$ (Associative)
3. $a + 0 = a$ (Identity)
4. $a + (-a) = 0$ (Inverses)

$\boxed{\text{Objects with these properties are called Groups.}}$

A Group is a type of structure.

Consider the integers a, b, and c.

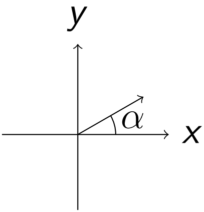
1. $a + b$ is an integer (Closure)
2. $(a + b) + c = a + (b + c)$ (Associative)
3. $a + 0 = a$ (Identity)
4. $a + (-a) = 0$ (Inverses)

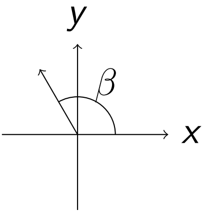
Objects with these properties are called Groups.

A Group is a type of structure.

But... why do we care?

Consider Rotations in 2D under the operation of composition, e.g.

$\delta_\alpha =$ 

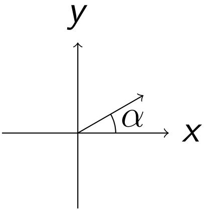
$\delta_\beta =$ 

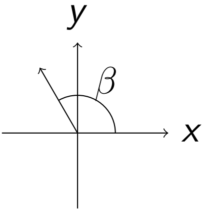
Composing gets us:

$\delta_\alpha \circ \delta_\beta =$

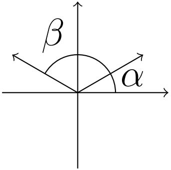
$= \delta_{\alpha+\beta}$

Consider Rotations in 2D under the operation of composition, e.g.

$\delta_\alpha =$ 

$\delta_\beta =$ 

Composing gets us:

$\delta_\alpha \circ \delta_\beta =$  $= \delta_{\alpha+\beta}$

Let's think about the structure of rotations.
Consider the rotations δ_α, δ_β, and δ_γ.

1. $\delta_\alpha \circ \delta_\beta$ is an rotation (Closure)
2. $(\delta_\alpha \circ \delta_\beta) \circ \delta_\gamma = \delta_\alpha \circ (\delta_\beta \circ \delta_\gamma)$ (Associative)
3. $\delta_\alpha \circ \delta_0 = \delta_\alpha$ (Identity)
4. $\delta_\alpha \circ (\delta_{-\alpha}) = \delta_0$ (Inverses)

The exact same as before!

Rotations in 2D form a Group!

Let's think about the structure of rotations.
Consider the rotations δ_α, δ_β, and δ_γ.

1. $\delta_\alpha \circ \delta_\beta$ is an rotation (Closure)
2. $(\delta_\alpha \circ \delta_\beta) \circ \delta_\gamma = \delta_\alpha \circ (\delta_\beta \circ \delta_\gamma)$ (Associative)
3. $\delta_\alpha \circ \delta_0 = \delta_\alpha$ (Identity)
4. $\delta_\alpha \circ (\delta_{-\alpha}) = \delta_0$ (Inverses)

The exact same as before!

$\boxed{\text{Rotations in 2D form a Group!}}$

Point

We do not have to study situations, we can instead study the structure of the situation.

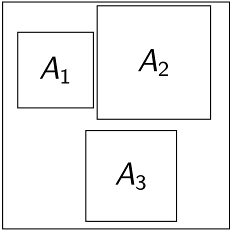
Definition

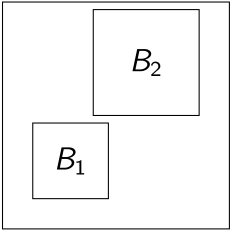
A structure is a set of objects with some rules defined on them.

An Operad is a type of structure.

Operads

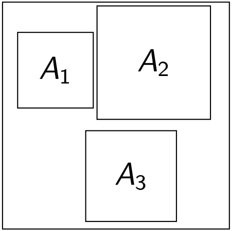
Operads are best learnt by example.
Consider some named squares located inside a square.

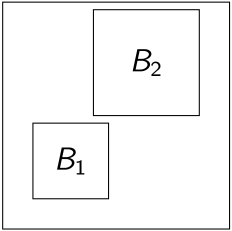
$A =$ 

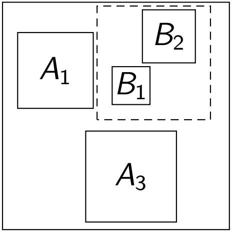
$B =$ 

What happens when you compose B within A_2?

We denote composition into the second spot as $\circ_2$, let's perform $A \circ_2 B$.

$A \circ_2 B =$ 

$\circ_2$ 

$A \circ_2 B =$ 

The dotted line is not apart of $A \circ_2 B$, it is there purely for explanatory reasons.

Point

An Operad generalises composition for multi-dimensional objects.

Definition

We call a function n-ary if it has n inputs and 1 output.

Let's define:

$$\text{A 3-ary function, } A(x_1, x_2, x_3)$$
$$\text{And a 2-ary function, } B(y_1, y_2)$$

$A \circ_2 B$ is a 4-ary function such that:

$$A \circ_2 B(x_1, y_1, y_2, x_3) = A(x_1, \underbrace{B(y_1, y_2)}_{\text{Replaces } x_2}, x_3)$$

This forms an Operad[1].

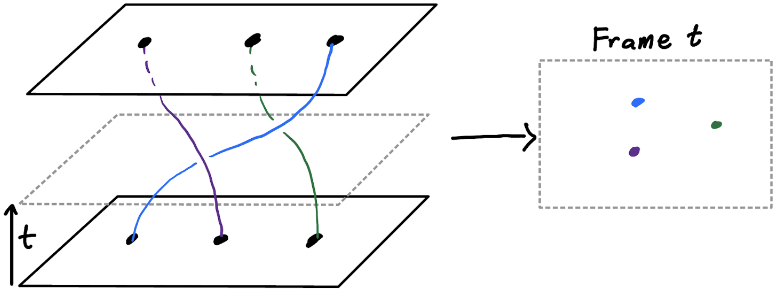
Question

Can you see the parallel between this and the squares?

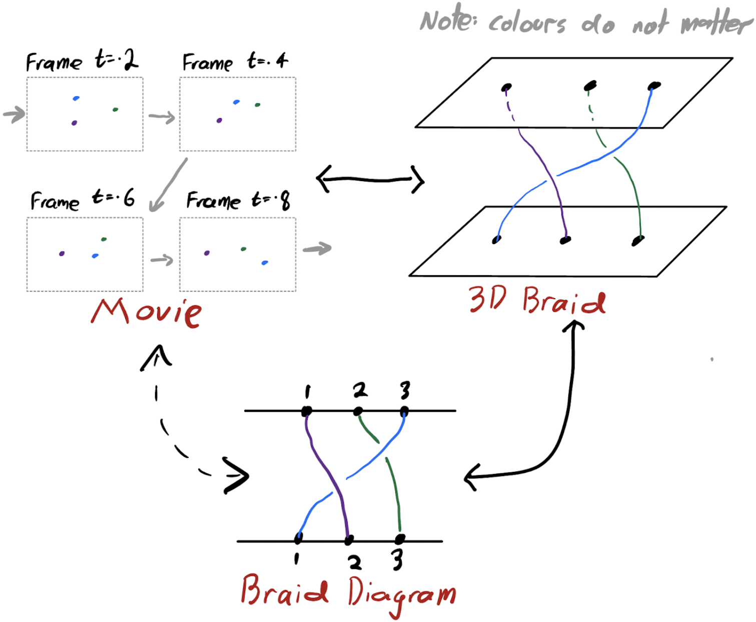
[1] Conditions/Rigour Applies.

3D Braids

Think of 3D braids as a movie, where time is our third dimension.

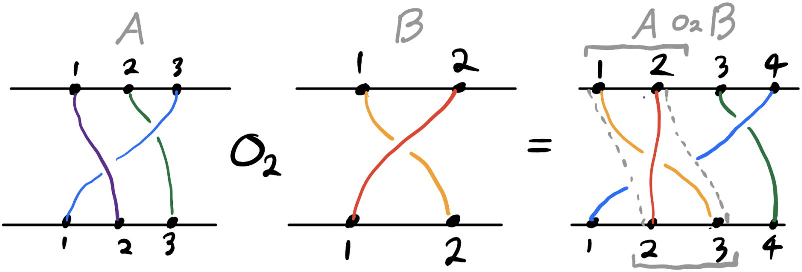


At time t, we have a snapshot of what the braid looks like.


Note: colours do not matter
Frame t=.2
Frame t=.4
Frame t=.6
Frame t=.8
Movie
3D Braid
1
2
3
1
2
3
Braid Diagram

The Canonical 3D Braid Operad

But we want to compose braids!
Let's define composition as replacement, for example:



You can see we're replacing the second strand of A with the braid B.

It is known that 3D braids form an Operad Canonically. This means that there is only a single reasonable choice for how composition might be defined for braids in 3D. Does this work for 4D braids however?

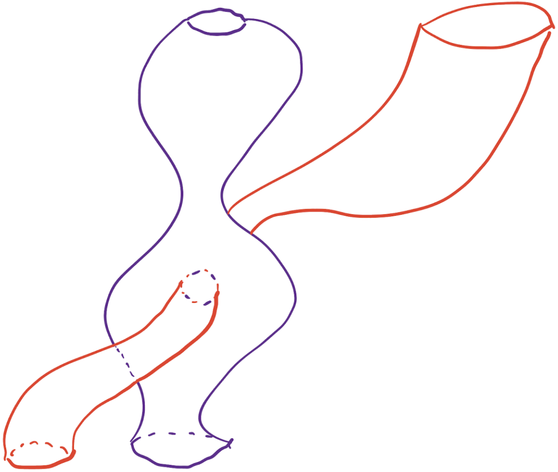
4D braids

You cannot braid 1D things in 4D!
As lines can "pass into the 4th dimension" and go around each other.
All 4D braids of 1D strands are uninteresting.

4D braids

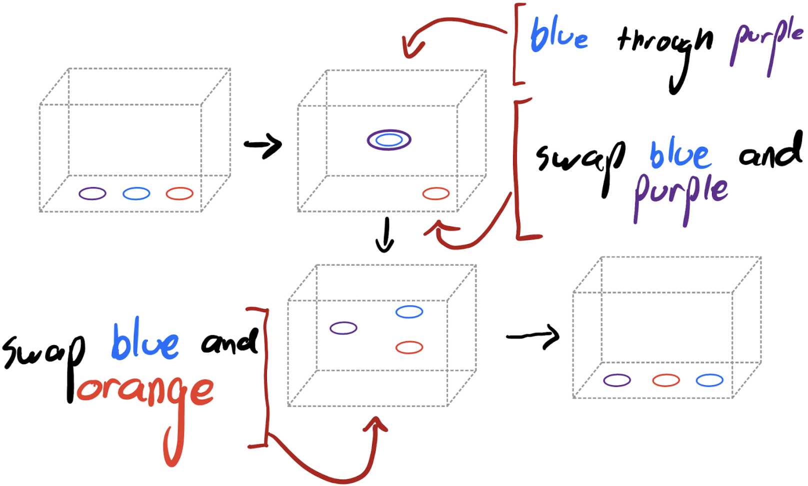
You cannot braid 1D things in 4D!
As lines can "pass into the 4th dimension" and go around each other.
All 4D braids of 1D strands are uninteresting.

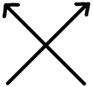
Therefore, we are infact braiding tubes in 4D.

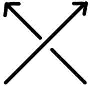


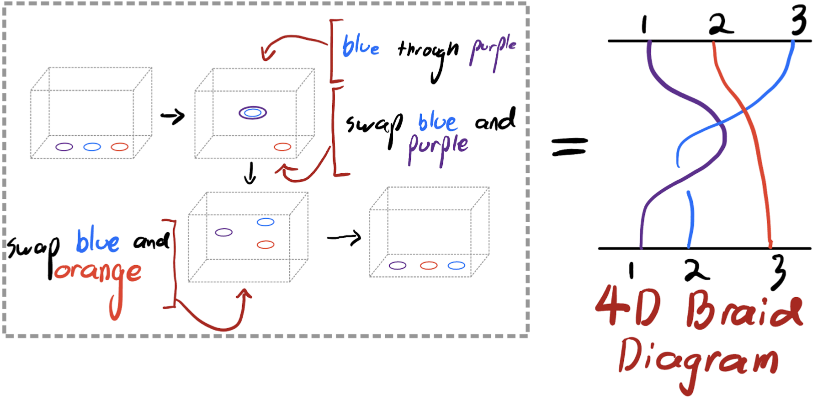
3D shadow of braid in 4D

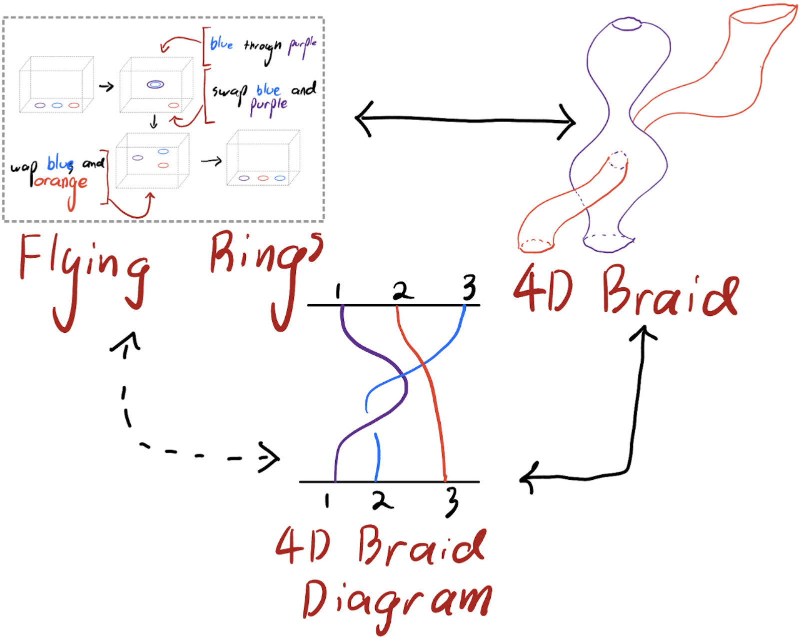
If we make our 4th dimension time, then we have a movie of rings in 3D.



“Swap ith and $(i+1)$th ring” $=$ 

“Cross $i+1$th ring through ith ring” $=$ 

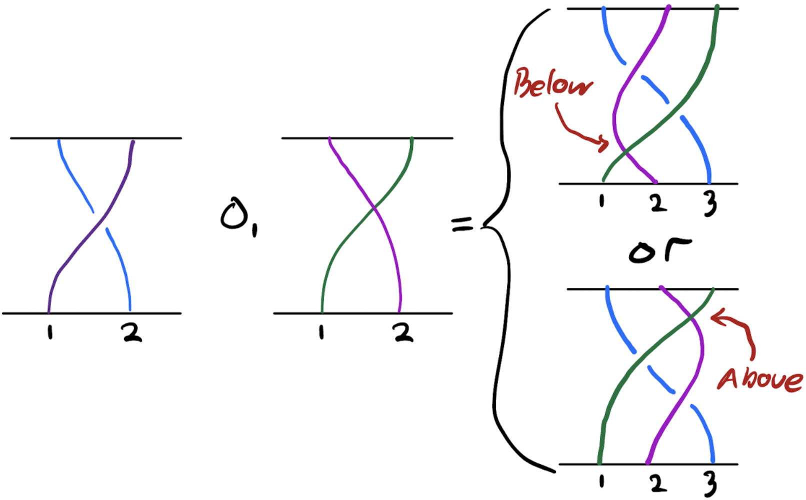



blue through purple
swap blue and purple
wap blue and orange
Flying Rings
4D Braid
1 2 3
1 2 3
4D Braid Diagram

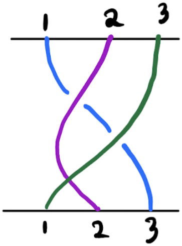
Question

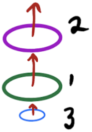
If we compose 4D Braids like we do with 3D Braids, does this form a Canonical Operad Structure?

Let's test via example:



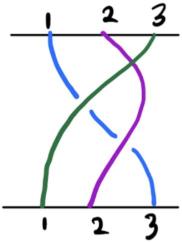
For this operation to be well defined, we need the two 4D braid diagrams to describe the same braid.

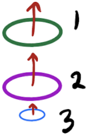




Blue through Green then Pink

These braids are <u>not</u> equal!





Blue through Pink then Green

With this contradiction it doesn't form an Operad...

However, if we make a convention that we always

"Do the crossings of the input braid first"

Then 4D Braids form an non-canonical Operad.

With this contradiction it doesn't form an Operad...
However, if we make a convention that we always

"Do the crossings of the input braid first"

Then 4D Braids form an non-canonical Operad.

Answer

Yes, 4D braids form an Operad.
No, it is not Canonical.

Relevance

3D braids forming an Operad is old news, however 4D braids form an Operad is a new, previously unproved, result. This being said, this result is possibly unsurprising due to it's geometric intuition.

Theorem

4D Braids form a non-canonical operad under replacement of strands.