Assignment #4

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"I pledge my honor that I have abided by the Stevens Honor System." N. Colonna

Problem 1: AR(2) model

$$r_t = \varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + a_t \quad , \quad a_t \sim N(0, \sigma^2)$$

a) $E[r_t | F_{t-1}] = E[\varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + a_t | F_{t-1}]$

$= E[\varphi_0 | F_{t-1}] + E[\varphi_1 r_{t-1} | F_{t-1}] + E[\varphi_2 r_{t-2} | F_{t-1}] + \underbrace{E[a_t | F_{t-1}]}_{=0}$

$= \varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + 0$

$$E[r_t | F_{t-1}] = \varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2}$$

b) $Var[r_t | F_{t-1}] = E[(r_t - E[r_t | F_{t-1}])^2 | F_{t-1}]$

$= E[(r_t - \varphi_0 - \varphi_1 r_{t-1} - \varphi_2 r_{t-2})^2 | F_{t-1}]$

$= E[a_t^2 | F_{t-1}]$

$= E[a_t^2]$

$$Var[r_t | F_{t-1}] = \sigma^2$$

Problem 2: AR(2) $\rightarrow$ $r_t = \varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + a_t \quad , \quad a_t \sim N(0, \sigma^2)$

a) $E[r_{t+2} | F_t]$

$\longrightarrow r_{t+2} = \varphi_0 + \varphi_1 r_{t+1} + \varphi_2 r_t + a_{t+2}$

$E[r_{t+2} | F_t] = E[\varphi_0 + \varphi_1 r_{t+1} + \varphi_2 r_t + a_{t+2} | F_t]$

$= E[\varphi_0 | F_t] + E[\varphi_1 r_{t+1} | F_t] + E[\varphi_2 r_t | F_t] + E[a_{t+2} | F_t]$

$= \varphi_0 + \varphi_1 \underbrace{E[r_{t+1} | F_t]}_{\text{same idea as } E[r_t | F_{t-1}]} + \varphi_2 r_t + 0$

$$E[r_{t+2} | F_t] = \varphi_0 + \varphi_1(\varphi_0 + \varphi_1 r_t + \varphi_2 r_{t-1}) + \varphi_2 r_t$$

b) $\phi_0 = 0.001$ $\quad r_t = 0.015$

$\phi_1 = 0.5$ $\quad r_{t-1} = 0.005$

$\phi_2 = 0.3$ $\quad \sigma^2 = 0.0001$

$$E[r_{t+2}|F_t] = \phi_0 + \phi_1(\phi_0 + \phi_1 r_t + \phi_2 r_{t-1}) + \phi_2 r_t$$

$$= .001 + .5 \times (.001 + .5 \times .015 + .3 \times .005) + .3 \times .015$$

$$= .001 + .5 \times (.01) + .0045$$

$$E[r_{t+2}|F_t] = 0.0105$$

Assignment 4

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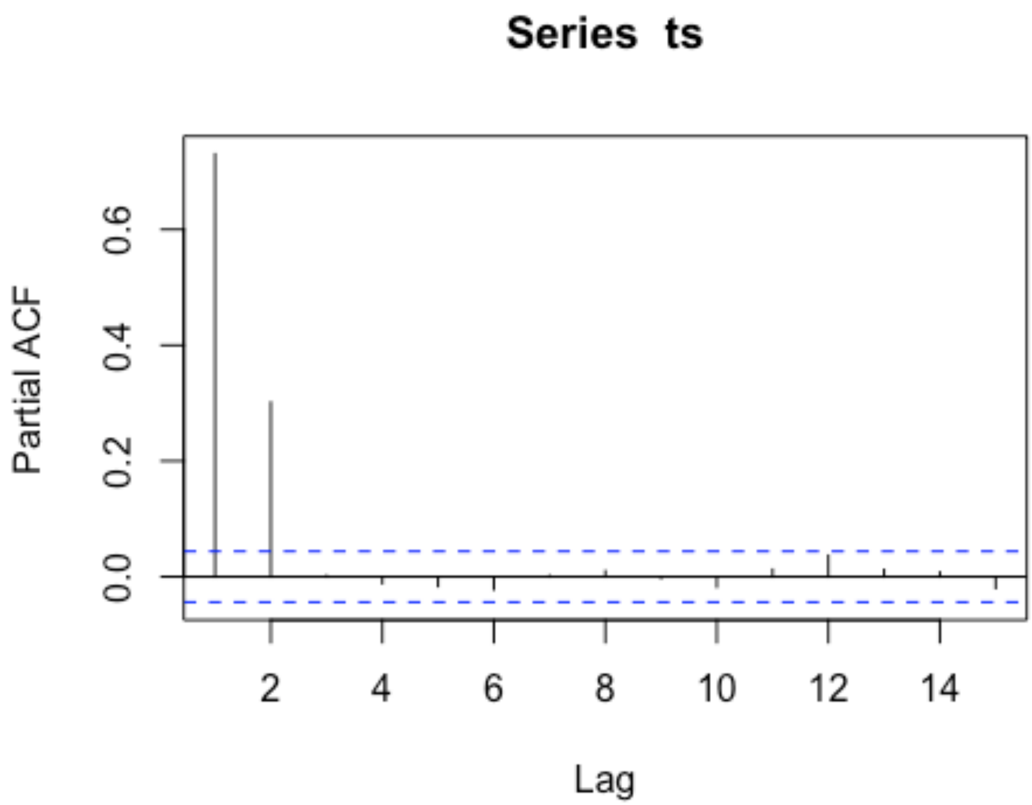
Problem 2

```
#2c
#phi0=0.001, phi1=0.5, phi2=0.3, rt=0.015, rt-1=0.005 var=0.0001

sum <- 0
for(i in 1:1000){
  rs <- c(0.005, 0.015)  #(rt-1, rt)
  a <- rnorm(1, 0, sqrt(0.0001))
  for(j in 1:2){  #generates next 2 values, making rs<-(rt-1, rt, rt+1, rt+2)
    rs <- c(rs, 0.001+0.5*rs[j+1]+0.3*rs[j]+a)
  }
  sum <- sum + rs[4]  #adds only the term for rt+2 to the sum variable
}


avgRp2 <- sum / 1000
avgRp2

## [1] 0.01051579
```

```
#2d
ts <- c(0.005, 0.015)
for(i in 1:2000){     #iterates to generate timeseries, starting at rt+1 and ending at rt+2000
  at <- rnorm(1, 0, sqrt(0.0001))
  ts <- c(ts, 0.001+0.5*ts[i+1]+0.3*ts[i]+at)
}


summary(ts)

##      Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
## -0.045140 -0.005756  0.005772  0.005333  0.016300  0.055830
```

```r
#2e
pacf(ts, lag.max=15)
```




```r
# Given the results of the PACF, the recommended AR order is 2




#2f
ar2model <- ar(ts, method="mle")
ar2model$ar

## [1] 0.5098432 0.3014436

ar2model$var.pred

## [1] 0.0001063846

# These parameter do a pretty good job approximating the real parameter
values. The real parameters are phi1=0.5, phi2=0.3, and var=0.0001. The
estimated parameters were phi1=0.50984, phi2=0.30144, and var=0.0001064,
which are very close to the actual values.
```

Bonus 1 : AR(2) $\rightarrow$ $r_t = \varphi_0 + \varphi_1 r_{t-1} + \varphi r_{t-2} + a_t$, $a_t \sim N(0, \sigma^2)$

a) $E[r_t | F_{t-2}] = E[\varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + a_t | F_{t-2}]$

$= E[\varphi_0 | F_{t-2}] + E[\varphi_1 r_{t-1} | F_{t-2}] + E[\varphi_2 r_{t-2} | F_{t-2}] + \underbrace{E[a_t | F_{t-2}]}_{=0}$

$= \varphi_0 + E[\varphi_1 r_{t-1} | F_{t-2}] + \varphi_2 r_{t-2} + 0$

$E[r_t | F_{t-2}] = \varphi_0 + \varphi_1 [\varphi_0 + \varphi_1 r_{t-3} + \varphi_2 r_{t-3}] + \varphi_2 r_{t-2}$

b) $Var[r_t | F_{t-2}] = Var[\varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} + a_t | F_{t-2}]$

$= Var[\varphi_0 + \varphi_1 r_{t-1} + \varphi_2 r_{t-2} | F_{t-2}] + Var[a_t | F_{t-2}]$

$= \underbrace{Var[\varphi_0 + \varphi_2 r_{t-2} | F_{t-2}]}_{=0} + \varphi_1 Var[\varphi_1 r_{t-1} | F_{t-2}] + Var[a_t | F_{t-2}]$

$= \varphi_1 \sigma^2 + \sigma^2$

$Var[r_t | F_{t-2}] = \sigma^2 (\varphi_1 + 1)$

Bonus 2: AR(1) $\rightarrow$ $r_t = \varphi_0 + \varphi_1 r_{t-1} + a_t$, $a_t \sim N(0, \sigma^2)$, r_t = stationary

a) $E[r_t]$ (unconditional expected value)

$E[r_t] = E[\varphi_0 + \varphi_1 r_{t-1} + a_t]$

$= E[\varphi_0] + E[\varphi_1 r_{t-1}] + E[a_t]$

$= \varphi_0 + \varphi_1 \underbrace{E[r_{t-1}]}_{\text{same as } E[r_t]} + 0$

$E[r_t] = \varphi_0 + \varphi_1 E[r_t]$

$E[r_t] - \varphi_1 E[r_t] = \varphi_0$

$E[r_t](1 - \varphi_1) = \varphi_0$

$E[r_t] = \dfrac{\varphi_0}{(1 - \varphi_1)}$

b) $Var[r_t]$ (unconditional variance)

$$Var[r_t] = Var[\varphi_0 + \varphi_1 r_{t-1} + a_t]$$
$$= Var[\varphi_0] + Var[\varphi_1 r_{t-1}] + Var[a_t]$$
$$= 0 + \varphi_1^2 \underbrace{Var[r_{t-1}]}_{\text{same as } Var[r_t]} + \sigma^2$$

$$Var[r_t] = \varphi_1^2 Var[r_t] + \sigma^2$$
$$Var[r_t] - \varphi_1^2 Var[r_t] = \sigma^2$$
$$Var[r_t](1-\varphi_1^2) = \sigma^2$$

$$Var[r_t] = \frac{\sigma^2}{(1-\varphi_1^2)}$$

Bonus 3: Prove that

$$\gamma_1 = cov(r_t, r_{t-1}) = E[(r_t-\mu)(r_{t-1}-\mu)]$$

and for a general l, $\gamma_l = E[(r_t-\mu)(r_{t-l}-\mu)]$

$r_t = \mu + \sum_{i=0}^{\infty} \varphi_i a_{t-i}$ where $\varphi_0 = 1$

ex/ $r_t = \mu + a_t + \varphi_1 a_{t-1} + \varphi_2 a_{t-2} - - -$

$r_{t-1} = \mu + a_{t-1} + \varphi_1 a_{t-2} + \varphi_2 a_{t-3} - - - -$

$$\gamma_1 = V(r_{t-1}) = E[(r_{t-1} - E[r_{t-1}])^2] = E[(r_{t-1}-\mu)^2]$$
$$= E[(\sum_{i=1}^{\infty} \varphi_{i-1} a_{t-i})^2]$$
$$= E[(\sum_{i=0}^{\infty} \varphi_i a_{t-i})(\sum_{j=1}^{\infty} \varphi_{j-1} a_{t-j})]$$
$$\gamma_1 = E[(r_t-\mu)(r_{t-1}-\mu)] = cov(r_t, r_{t-1})$$

In general,

$$\gamma_l = V(r_{t-l}) = E[(r_{t-l} - E[r_{t-l}])^2] = E[(r_{t-l}-\mu)^2]$$
$$= E[(\sum_{i=l}^{\infty} \varphi_{i-l} a_{t-i})^2]$$
$$= E[(\sum_{i=0}^{\infty} \varphi_i a_{t-i})(\sum_{j=l}^{\infty} \varphi_{j-l} a_{t-j})]$$
$$\gamma_l = E[(r_t-\mu)(r_{t-l}-\mu)]$$