Theoretical Problems

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"I pledge my honor that I have abided by the Stevens Honor System." N. Colonna

1. P(reservation cancelled) = P(C) = 0.20

P(reservation used) = P(U) = 0.80

• Accepted 52 reservations, only 50 tables

a) $P(\text{not enough tables}) = \underbrace{P(0 \text{ cancelled}) + P(1 \text{ cancelled})}_{\text{if 2 or more cancel, there's room}}$

$$= \binom{52}{0}(0.2)^0(0.8)^{52} + \binom{52}{1}(0.2)^1(0.8)^{51}$$

$$= (0.8)^{52} + 52(0.2)(0.8)^{51}$$

$$= 0.00000913 + 0.00011874$$

$\boxed{P(\text{not enough tables}) = 0.000127877}$

b) $P(\text{more than 3 empty}) = 1 - P(3 \text{ or less empty}) = 1 - P(5 \text{ or less cancel})$

$$= 1 - [P(0 \text{ empty}) + P(1 \text{ empty}) + P(2 \text{ empty}) + P(3 \text{ empty})]$$

$$= 1 - \Big[\binom{52}{0}(0.2)^0(0.8)^{52} + \binom{52}{1}(0.2)^1(0.8)^{51} + \binom{52}{2}(0.2)^2(0.8)^{50} + \binom{52}{3}(0.2)^3(0.8)^{49} + \binom{52}{4}(0.2)^4(0.8)^{48} + \binom{52}{5}(0.2)^5(0.8)^{47}\Big]$$

$$= 1 - [1(.2)^0(.8)^{52} + 52(.2)^1(.8)^{51} + 1326(.2)^2(.8)^{50} + 22100(.2)^3(.8)^{49} + 270725(.2)^4(.8)^{48} + 2598960(.2)^5(.8)^{47}]$$

$$= 1 - [0.00000913 + 0.00011874 + 0.00075701 + 0.00315421 + 0.00965979 + 0.02318349]$$

$$= 1 - 0.03688237$$

$\boxed{P(\text{more than 3 empty}) = 0.96311763}$

2. $f: \mathbb{R} \to \mathbb{R}$ such that $f(x) = \dfrac{2}{\pi(1+x^2)} \mathbb{1}_{[0,\infty]}(x)$ $\quad \mathbb{1}_{[0,\infty]}(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases}$

Show that f is a PDF.

* We know that the integral of a pdf $= 1$, so lets integrate.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \frac{2}{\pi(1+x^2)} \mathbb{1}_{[0,\infty]}(x)\, dx = \int_0^{\infty} \frac{2}{\pi(1+x^2)} dx$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \frac{2}{\pi} \tan^{-1}(x) \Big]_0^{\infty}$$

$$= \frac{2}{\pi} \tan^{-1}(\infty) - \frac{2}{\pi}(0)$$

$$= \frac{2}{\pi}\left[\lim_{x \to \infty}(\tan^{-1}(\infty))\right]$$

$$= \frac{2}{\pi} \times \left(\frac{\pi}{2}\right)$$

$$= 1$$

$\int f(x) = 1$ $\therefore$ f is a probability density function

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3. $X \sim \exp(\lambda)$, $\lambda > 0$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Prove that for every $s > 0$ and $t > 0$,

$$P(X \geq s+t \,|\, X \geq s) = P(X \geq t)$$

Using the CDF of the exponential distribution:

$$P(X < s+t \,|\, X \geq s) = \frac{P(X < s+t \cap X \geq s)}{P(X \geq s)}$$

$$= \frac{P(s \leq X < s+t)}{P(X \geq s)}$$

$$= \frac{e^{-\lambda s} - e^{-\lambda(s+t)}}{e^{-\lambda s}}$$

$$= \frac{e^{-\lambda s}}{e^{-\lambda s}} - \frac{e^{-\lambda s-\lambda t}}{e^{-\lambda s}}$$

$$= 1 - \frac{e^{-\lambda s}e^{-\lambda t}}{e^{-\lambda s}}$$

$$P(X<s+t \mid X\geq s) = 1-e^{-\lambda t}$$

Therefore,

$$P(X\geq s+t \mid X\geq s) = P(X\geq t)$$

$$1-P(X<s+t \mid X\geq s) = P(X\geq t)$$

$$1-[1-e^{-\lambda t}] = e^{-\lambda t}$$

$$e^{-\lambda t} = e^{-\lambda t}$$

∴ I have proven that $P(X\geq s+t/X\geq s) = P(X\geq t)$

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4. AR(1) model: $r_t = 0.1 + 0.3r_{t-1} + a_t$, $\sigma_a = 0.025$

Assume $r_{1000} = 0.001$
- Calculate the 1-step & 2-step ahead forecast at $t=1000$
- Calculate the standard deviation of corresponding forecast errors
- Compute the numerical values for the 1, 2, and 3 lag autocorrelation

$X_t = r_t$

1 step ahead forecast: at time $t=1000$

$$\hat{X}_{1000}(1) = E[r_{1001}/F_{1000}] = E[0.1+0.3r_{1000}+a_{1001}/F_{1000}]$$

$$= 0.1 + 0.3E[r_{1000}/F_{1000}] + E[a_{1001}/F_{1000}]$$

independent

$$= 0.1 + 0.3(0.001) + \underbrace{E[a_{1001}]}_{=0}$$

$$\boxed{\hat{X}_{1000}(1) = 0.1003}$$

1 step forecast Standard Deviation

1 step forecast error $\rightarrow e_{1000}(1) = X_{1001} - \hat{X}_{1000}(1)$

$$e_{1000}(1) = X_{1001} - 0.1003$$

$$V(e_{1000}(1)) = V((X_{1001} - 0.1003)) = V(a_{n+1}) = \sigma_a^2 = 0.025^2 = 0.000625$$

$$\sigma_{e_{1000}(1)} = \sqrt{V(e_{1000}(1))} = \sqrt{\sigma_a^2}$$

$$\boxed{\sigma_{e_{1000}(1)} = \sigma_a = 0.025}$$

2 steps ahead forecast: at time $t = 1000$

$$\hat{X}_{1000}(2) = E[X_{1002} | F_{1000}] = E[0.1 + 0.3X_{1001} + a_{1002} | F_{1000}]$$

$$= 0.1 + 0.3E[X_{1001} | F_{1000}] + E[\underbrace{a_{1002} | F_{1000}}_{\text{independent}}]$$

$$= 0.1 + 0.3[\hat{X}_{1000}(1)] + \underbrace{E[a_{1002}]}_{=0}$$

$$= 0.1 + 0.3(0.1003)$$

$$\boxed{\hat{X}_{1000}(2) = 0.13009}$$

2 steps forecast Standard Deviation

2 steps forecast error $\rightarrow e_{1000}(2) = X_{1002} - \hat{X}_{1000}(2)$

$$e_{1000}(2) = X_{1002} - 0.13009$$

$$V(e_{1000}(2)) = V(X_{1002} - 0.13009)$$

$$= V(0.1 + 0.3X_{1001} - 0.13009 + a_{1002})$$

$$= V(0.1 + 0.3(X_{1001} - \hat{X}_{1000}(1)) + a_{1002})$$

$$= V(0.1) + 0.09V(X_{1001} - \hat{X}_{1000}(1)) + V(a_{1002})$$

$$= 0.09\,V(e_{1000}(1)) + \sigma_a^2$$

$$= 0.09\sigma_a^2 + \sigma_a^2$$
$$= 1.09\sigma_a^2$$
$$V(e_{1000}(2)) = 1.09(.025)^2 = 0.00068125$$

$$\sigma_{e_{1000}(2)} = \sqrt{V(e_{1000}(2))} = \sqrt{0.00068125}$$

$$\boxed{\sigma_{e_{1000}(2)} = 0.0261}$$

Lag Autocorrelations

$$\rho_\ell = \frac{\gamma_\ell}{\gamma_0}$$

$$\gamma_\ell = \operatorname{cov}(r_t, r_{t-\ell}) = E[(r_t - \mu)(r_{t-\ell} - \mu)]$$
$$\gamma_\ell = \varphi\gamma_{\ell-1} = 0.3\gamma_\ell$$
$$\gamma_0 = V(r_t) = \frac{\sigma_a^2}{1-\varphi^2} = \frac{.025^2}{1-.3^2} = 0.00068681$$
$$\gamma_1 = 0.3 \times \gamma_0 = 0.3(.00068681) = 0.00020604$$
$$\gamma_2 = 0.3 \times \gamma_1 = 0.3(.00020604) = 0.00006181$$
$$\gamma_3 = 0.3 \times \gamma_2 = 0.3(.00006181) = 0.00001854$$

Lag 1: $\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{0.00020604}{0.00068681} \Rightarrow \boxed{\rho_1 = 0.3}$

Lag 2: $\rho_2 = \frac{\gamma_2}{\gamma_0} = \frac{0.00006181}{0.00068681} \Rightarrow \boxed{\rho_2 = 0.09}$

Lag 3: $\rho_3 = \frac{\gamma_3}{\gamma_0} = \frac{0.00001854}{0.00068681} \Rightarrow \boxed{\rho_3 = 0.027}$