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# Qubit, An Intuition #2 — Inner Product, Outer Product, and Tensor Product in Bra-ket Notation

*The beauty of Quantum Mechanics for Quantum Computation, featuring IBM Quantum*

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TL;DR;
2 qubits inner product, outer product, and tensor product in bra-ket
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notation, with examples.

Please refer to the previous article (published in July 12 2021), "*Qubit, An Intuition #1 — First Baby Steps in Exploring the Quantum World*" for a discussion on a single qubit as a computing unit for quantum computation.

> *For an introductory helicopter view of the overall six articles in the series, please visit this link "Embarking on a Journey to Quantum Computing — Without Physics Degree."*

In the first article, "*Qubit, An Intuition #1 — First Baby Steps in Exploring the Quantum World*," we discussed the intuition of a quantum bit (qubit) as a computing unit in a Quantum computer, in comparison to a binary digit (bit) in a Classical computer. We also discussed Bra-ket notation, Bloch sphere, along with the hybrid classical-quantum approach.

In this follow-on article, we will discuss the inner product, outer product, and tensor product. These are some of the basic mathematical foundations to understand quantum computation.

## Quantum Bit

Indeed, a single quantum bit (qubit) is exciting with its nature of being in superposition. Hence $|\Psi> = \alpha|0> + \beta|1>$. The quantum state *Psi* ($|\Psi>$) is in the linear combination of $|0>$ and $|1>$ with the probability of being in state $|0>$ is $|\alpha|^2$, and the probability of being in state$|1>$ is $|\beta|^2$.

$\alpha$ and $\beta$ are the probability amplitudes while $\alpha^*$ and $\beta^*$ are the complex conjugates of $\alpha$ and $\beta$, respectively. It fulfills the normalization constraint, such that $|\alpha|^2+|\beta|^2=1$ or $\alpha^*\alpha+\beta^*\beta=1$, where $\alpha, \beta \in \mathbb{C}^2$.

## Inner Product, Outer Product, and Tensor Product

However, we need more qubits to do meaningful quantum computation. Let's say that we have the following two qubits, *Psi*$|\Psi>$ and *Phi* $|\Phi>$. Each has its respected quantum state.

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$|\Phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

Note that the relationship for bra and ket for a quantum state. The bra is a complex conjugate transpose of ket and vice versa.

$$\langle\Psi| = \left(|\Psi\rangle\right)^{\dagger} = \left(|\Psi\rangle\right)^{*T}$$
$$\langle\Phi| = \left(|\Phi\rangle\right)^{\dagger} = \left(|\Phi\rangle\right)^{*T}$$

We can have the following valid operations for|Ψ> and |Φ>. We call the operation between bra <Ψ| and ket |Φ>as the **inner product**, ket |Ψ>and bra <Φ|as the **outer product**, and ket |Ψ> and ket |Φ> as the **tensor product**.

$$\langle\Psi||\Phi\rangle = \langle\Psi\Phi\rangle \quad \text{"Inner Product"}$$
$$|\Psi\rangle\langle\Phi| = |\Psi\Phi| \quad \text{"Outer Product"}$$
$$|\Psi\rangle|\Phi\rangle = |\Psi\Phi\rangle \quad \text{"Tensor Product"}$$

Bra-ket notation for two-qubit operations: inner product, outer product, and tensor product.

However, the following operation is invalid:

$$\langle\Psi|\langle\Phi| \qquad \textit{Invalid Operation}$$

Let's give a value for α, β, γ, and δ (note that α, β, γ, δ ∈ $\mathbb{C}^2$). By assigning α=-4/5i, β=3/5, γ=1/2, and δ=sqrt(3)/2 for example, we can have some values to work on for ket |Ψ>, bra <Ψ|, ket |Φ>, and bra <Φ|.

For the first qubit |Ψ> with α=-4i/5 and β=3/5, we find the ket |Ψ> and bra <Ψ|. First, we do a qubit validity test by verifying that $|\alpha|^2+|\beta|^2=1$.

$$\alpha = -\frac{4}{5}i\,,\ \beta = \frac{3}{5}$$

$$|\alpha|^2 + |\beta|^2 = 1?$$
$$|\alpha^*\alpha| + |\beta^*\beta| = 1?$$
$$\left(\left(\frac{4}{5}i\right) * \left(-\frac{4}{5}i\right)\right) + \left(\frac{3}{5} * \frac{3}{5}\right) = 1?$$
$$\frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1 \quad \text{"}\textit{a valid qubit}\text{"}$$

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$$
$$-\frac{4}{5}i\,|0\rangle + \frac{3}{5}\,|1\rangle$$
$$\langle\Psi| = \alpha^*\langle 0| + \beta^*\langle 1|$$
$$\frac{4}{5}i\,\langle 0| + \frac{3}{5}\,\langle 1|$$

Then, for the second qubit |Φ> with γ=1/2, and δ=sqrt(3)/2, we find the ket |Φ> and bra <Φ|. First, we do a qubit validity test by verifying $|\gamma|^2+|\delta|^2=1$.

$$\gamma = \frac{1}{2},\ \delta = \frac{\sqrt{3}}{2}$$

$$|\gamma|^2 + |\delta|^2 = 1?$$
$$|\gamma^*\gamma| + |\delta^*\delta| = 1?$$
$$\left(\frac{1}{2} * \frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2} * \frac{\sqrt{3}}{2}\right) = 1?$$

$$\frac{1}{4} + \frac{3}{4} = \frac{4}{4} = 1 \quad \text{"}\textit{a valid qubit}\text{"}$$

$$\begin{aligned} |\Phi\rangle &= \gamma|0\rangle + \delta|1\rangle \\ &\quad \frac{1}{2}\,|0\rangle + \frac{\sqrt{3}}{2}\,|1\rangle \\ \langle\Phi| &= \gamma\langle 0| + \delta\langle 1| \\ &\quad \frac{1}{2}\,\langle 0| + \frac{\sqrt{3}}{2}\,\langle 1| \end{aligned}$$

We can then illustrate the two-qubit operations for the inner product, outer product, and tensor product.

**Inner Product — <ΨΦ>**

A product of two quantum states bra *Psi* <Ψ| and ket *Phi* |Φ> is called an **inner product,** producing a **value**. An inner product is also called an *overlap,* the overlap between quantum states.

$$\langle \Psi || \Phi \rangle = \langle \Psi \Phi \rangle$$

$$\begin{pmatrix} \frac{4}{5}i & \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\frac{4}{10}i + \frac{3\sqrt{3}}{10}$$

**Outer Product — |ΨΦ|**

A product of two quantum states, ket *Psi*|Ψ> and bra *Phi* <Φ|, is called an **outer product**, producing a **matrix**. An outer product is also called a *projection*.

$$|\Psi\rangle\langle\Phi| = |\Psi\Phi|$$

$$\begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{4}{10} & -\frac{4\sqrt{3}}{10} \\ \frac{3}{10} & \frac{3\sqrt{3}}{10} \end{pmatrix}$$

**Tensor Product — |ΨΦ>**

A product of two quantum states, ket *Psi*|Ψ> and ket *Phi* |Φ> is called a **tensor product**, producing a **column vector** with length $2^n$ (where n is the number of qubits).

$$|\Psi\rangle|\Phi\rangle = |\Psi\rangle \otimes |\Phi\rangle = |\Psi\Phi\rangle$$

$$|\Psi\Phi\rangle = \left(\alpha|0\rangle + \beta|1\rangle\right) \otimes \left(\gamma|0\rangle + \delta|1\rangle\right)$$
$$\alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle$$

$$\begin{pmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{pmatrix} \otimes \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$
$$\begin{pmatrix} -\frac{4}{5} & \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \\ \frac{3}{5} & \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{4}{10} \\ -\frac{4\sqrt{3}}{10} \\ \frac{3}{10} \\ \frac{3\sqrt{3}}{10} \end{pmatrix}$$

$$|\Psi\Phi\rangle = -\frac{4}{10}|00\rangle - \frac{4\sqrt{3}}{10}|01\rangle + \frac{3}{10}|10\rangle + \frac{3\sqrt{3}}{10}|11\rangle$$

If we square all individual vector elements and sum them up, the total must be 1.

$$|-\frac{4}{10}|^2 + |-\frac{4\sqrt{3}}{10}|^2 + |\frac{3}{10}|^2 + |\frac{3\sqrt{3}}{10}|^2 = 1?$$
$$\frac{16}{100} + \frac{48}{100} + \frac{9}{100} + \frac{27}{100} = \frac{100}{100} = 1$$

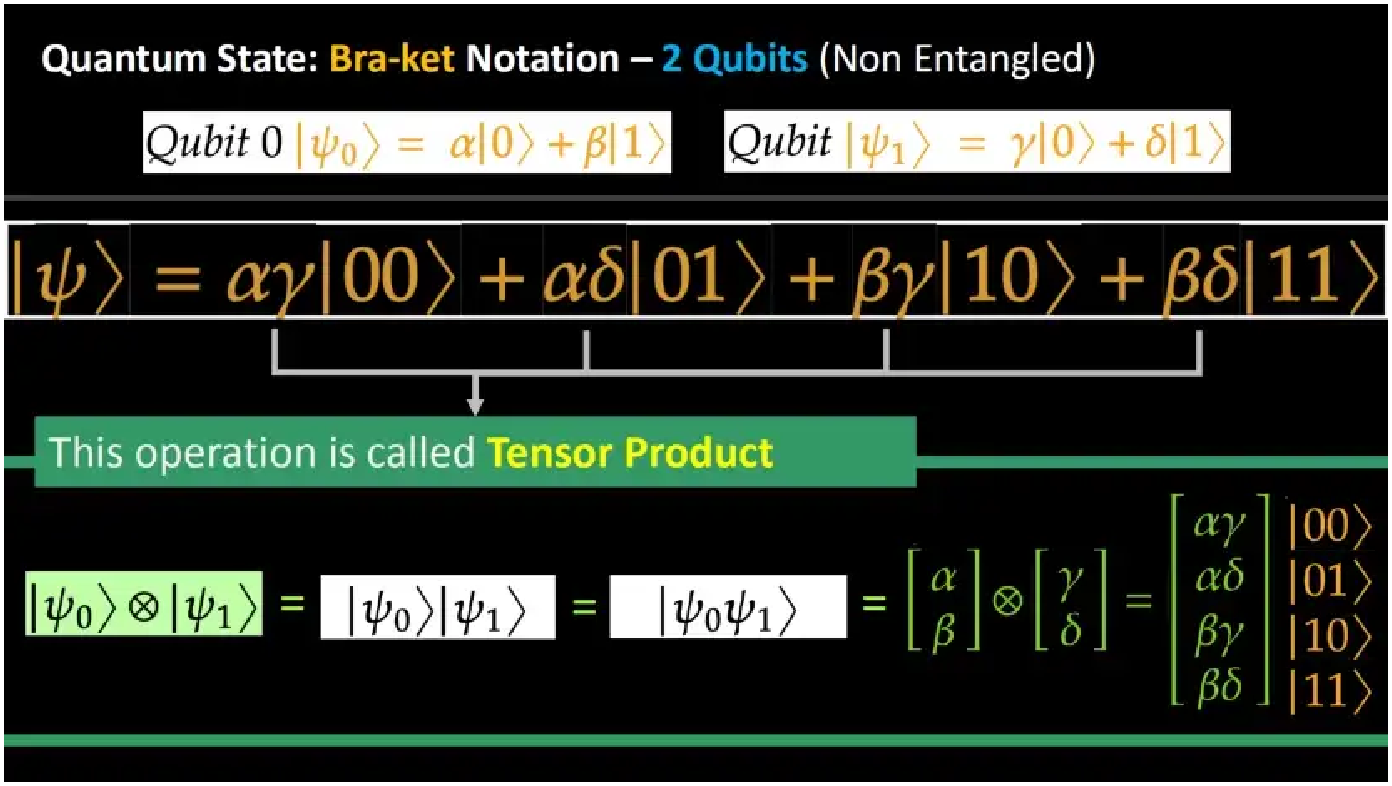
It means that the following normalization constraint applies:

$$|\alpha\gamma|^2 + |\alpha\delta|^2 + |\beta\gamma|^2 + |\beta\delta|^2 = 1$$

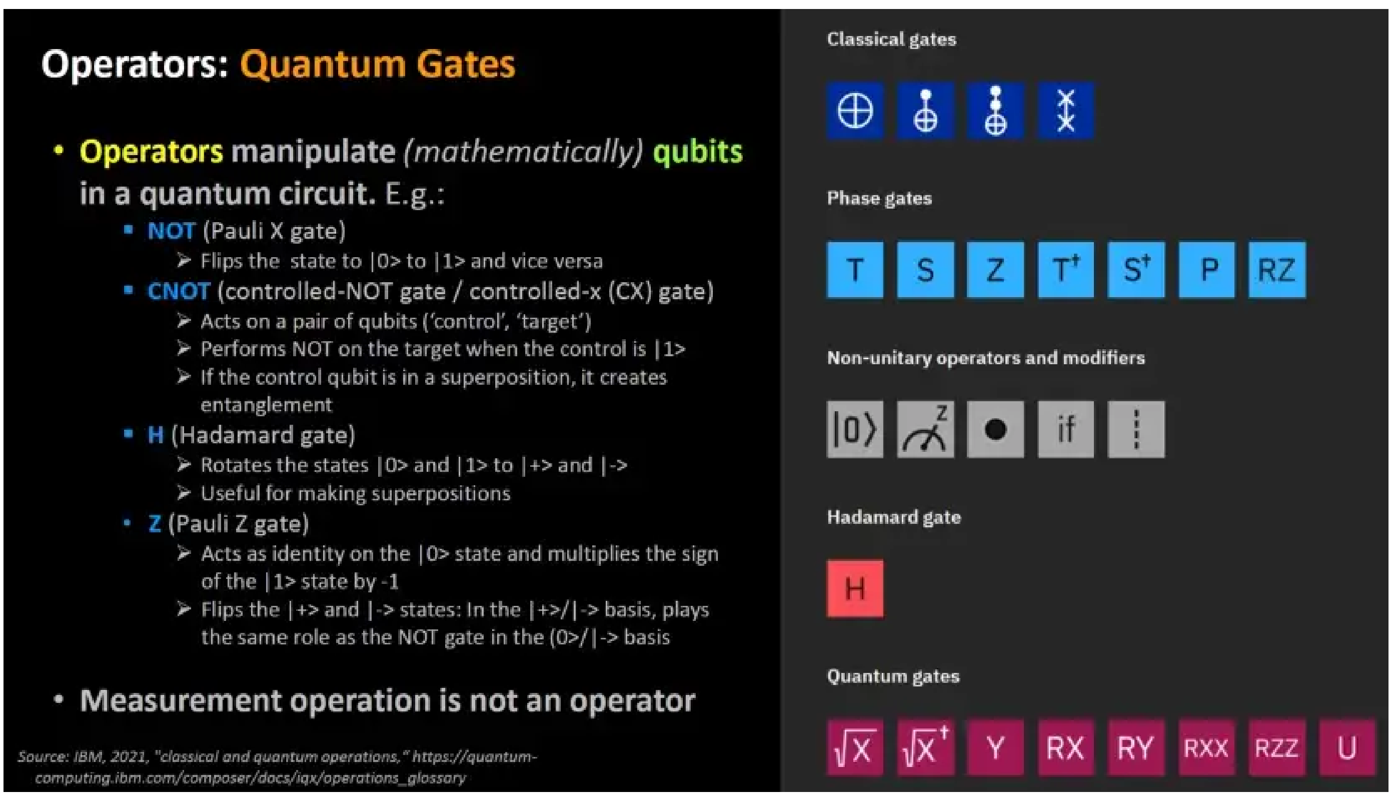
$|\alpha\gamma|^2$, $|\alpha\delta|^2$, $|\beta\gamma|^2$, and $|\beta\delta|^2$ represent the probability of measuring 00, 01, 10, and 11, respectively — which is 16%, 48%, 9%, and 27%.

$$\begin{aligned}
Probability\ of\ measuring\ 00 &= |-\frac{4}{10}|^2 &= \frac{16}{100} &= 16\% \\
Probability\ of\ measuring\ 01 &= |-\frac{4\sqrt{3}}{10}|^2 &= \frac{48}{100} &= 48\% \\
Probability\ of\ measuring\ 10 &= |\frac{3}{10}|^2 &= \frac{9}{100} &= 9\% \\
Probability\ of\ measuring\ 11 &= |\frac{3\sqrt{3}}{10}|^2 &= \frac{27}{100} &= 27\%
\end{aligned}$$

The following illustration describes the tensor product further:




We use the tensor product in a quantum circuit for combining multiple qubits (two or more qubits) with quantum operators to transform qubits from one state to another, therefore performing quantum computation.




An example of quantum gates: NOT (flip gate), CNOT (Controlled-NOT), H (Hadamard), and Z (phase flip gate). Quantum gates are used to perform quantum computation in a quantum circuit.

## Moving Forward

We can do a full or partial measurement of qubits. We will discuss this further in the next article, "*Qubit, An Intuition #3 — Quantum Measurement, Full and Partial Qubits*."

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Tensor Product