Project 3

Least Squares

Consider the equation system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \tag{3.1}$$

where $A \in \mathbb{C}^{m\times n}$ is full-rank and $m \geq n$. The objective is to find the solution vector $\boldsymbol{x}$.

> Does a solution for (3.1) always exist? Which condition should be fulfilled to obtain an exact solution?

Task 3.1

Given

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3.2}$$

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \tag{3.3}$$

> check whether we can obtain an exact solution for $\boldsymbol{x}$.

Task 3.2

In case no exact solution can be obtained, the least squares approach can give an approximate solution defined as

$$\boldsymbol{x}_{\mathrm{LS}} := \underset{\boldsymbol{x}}{\operatorname{argmin}} \left\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\right\|_2^2 .$$

Give the least squares solution $\boldsymbol{x}_{\mathrm{LS}}$ as a function of $\boldsymbol{A}$ and $\boldsymbol{b}$.

TASK 3.3

Calculate the least squares solution $\boldsymbol{x}_{\mathrm{LS}}$ for the given $\boldsymbol{A}$ and $\boldsymbol{b}$.

TASK 3.4

To avoid the high computational complexity of matrix inversion, the least squares solution can be computed using three different factorization methods, viz.,

- Cholesky factorization
- QR decomposition
- Singular-value decomposition (SVD).

3.1 Least Squares with Cholesky Factorization

The Cholesky factorization of the Gram matrix $\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$ is given as

$$\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^{\mathrm{H}},$$

with lower triangular $\boldsymbol{L} \in \mathbb{C}^{n \times n}$ and diagonal $\boldsymbol{D} \in \mathbb{C}^{n \times n}$.

How does the solution $\boldsymbol{x}_{\mathrm{LS}}$ look like if we apply the Cholesky factorization?

TASK 3.5

Finish the implementation in `ls_cf.m` for computing the least squares solution using the Cholesky factorization. Functions from previous projects can be used. You can test for correctness by running `ls_cf_test.m`.

PROGRAMMING TASK 3.6

3.2 Least Squares with QR Decomposition

How does the solution $\boldsymbol{x}_{\mathrm{LS}}$ look like if we apply the reduced QR decomposition to the matrix $\boldsymbol{A}$?

TASK 3.7

Finish the implementation in `ls_qr.m` for computing the least squares solution using the QR decomposition. Functions from previous projects can be used. You can test for correctness by running `ls_qr_test.m`.

PROGRAMMING TASK 3.8

3.3 Least Squares with SVD

The reduced SVD of $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \boldsymbol{U}_{\text{red}} \boldsymbol{\Sigma}_{\text{red}} \boldsymbol{V}^{\text{H}},$$

with sub-unitary $\boldsymbol{U}_{\text{red}} \in \mathbb{C}^{m \times n}$, diagonal $\boldsymbol{\Sigma}_{\text{red}} \in \mathbb{R}^{n \times n}$ and unitary $\boldsymbol{V} \in \mathbb{C}^{n \times n}$.

How does the solution $\boldsymbol{x}_{\text{LS}}$ look like if we apply the reduced SVD to the matrix $\boldsymbol{A}$?

TASK 3.9

Finish the implementation in `ls_svd.m` for computing the least squares solution using SVD. Functions from previous projects can be used. You can test for correctness by running `ls_svd_test.m`.

PROGRAMMING TASK 3.10

3.4 Comparison of Different Approaches

Given

$$\boldsymbol{A} = \begin{bmatrix} 10^{-5} & 10^{-15} \\ 10^{-5} & -10^{-15} \\ 1 & 0.5 \end{bmatrix} \tag{3.4}$$

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \tag{3.5}$$

calculate analytically the least squares solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Neglect the terms that belong to $\mathcal{O}\left(10^{-19}\right)$.

TASK 3.11

Compute the least squares solution using the three methods. How do the obtained solutions compare to the analytical solution?

Programming Task 3.12

What is the critical step in the approach with the largest error?

Task 3.13

Calculate analytically the condition number of $\boldsymbol{A}$. You can additionally compute it using the MATLAB function `cond`.

Task 3.14