

Notes regarding lab format

We will use Matlab Livescript for this lab. Livescript allows switching between text and code cells.

You will find the entire lab manual in this file. Some exercises require you to write a text answer, others require you to write code. You should not define functions inside this file. Instead save functions to the functions folder and call them from the code cells in this notebook.

Your finished lab report should be a .zip-file containing the data folder, your functions folder and this livescript file. As usual, you should also provide a pdf of the result of running the live script (in the Live Editor, you can **export to pdf** under Save) where all result images should be visible.

Since we need to access the functions and data folder the first step is to add these two locations MATLAB's path.

```
addpath('./functions');
addpath('./data');
warning off;
clear all;
```

Lab 4 - Triangulation

Using RANSAC

This whole lab is basically concerned with the camera equation

$$\lambda u = PU$$

For the uncalibrated case, u is a 3-vector with the coordinates of a point in the image (and an added 1)

$$u = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

and U is a 4-vector containing the coordinates of the corresponding 3D point along with a 1 in its last element. A function

```
sigma = 0.0;
[Ps, us, U_true] = triangulation_test_case(sigma)
```

Ps = 1x2 cell

	1	2
1	3x4 double	3x4 double

us = 2x2

129.6049 181.4136
179.7829 -95.1819

U_true = 3x1

0.5502
0.6225
0.5870

is provided for creating a simple test case where you know the correct answer (**U_true**). Gaussian noise of standard deviation **sigma** is added to the image points. Use this example to evaluate your minimal solver.

Ex 4.1 Make a minimal solver for the triangulation problem, that is, a function

```
U = minimal_triangulation(Ps, us)
```

```
U = 3x1  
0.5502  
0.6225  
0.5870
```

```
% In the case when sigma=0, the minimal solver computes U as U_true. The  
% function thus works correctly.
```

that takes two camera matrices, **Ps**, and two image points, **us**, and triangulates a 3D point. The image points are a 2×2 array whereas the camera matrices is a cell list with one camera matrix in each cell.

Recall that λ is the *depth*. Points with negative depth would lie behind the camera, so negative depths indicate that something is wrong.

Ex 4.2 Make a function

```
positive = check_depths(Ps, U)
```

```
positive = 1x2  
1 1
```

that takes N camera matrices, **Ps**, and a 3D point, **U**, and checks the depth of **U** in each of the cameras. The output should be a an array of boolean values of length N that indicates which depths were positive. (Matlab will print boolean values as zeros and ones, so don't be confused by this.)

Ex 4.3 Make a function

```
errors = reprojection_errors(Ps, us, U)
```

```
errors = 2x1  
10^-12 x  
0.0284  
0.1705
```

that takes N camera matrices, **Ps**, N image points, **us**, and a 3D point, **U**, and computes a vector with the reprojection errors, that is, the lengths of the reprojection residuals. If a point has negative depth, set the reprojection error to Inf.

Ex 4.4 Make a function

```
threshold = 2;  
U
```

```
U = 3x1  
0.5502  
0.6225  
0.5870
```

```
[U, nbr_inliers] = ransac_triangulation(Ps, us, threshold)
```

```
U = 3x1  
0.5502  
0.6225  
0.5870  
nbr_inliers = 2
```

that implements triangulation using RANSAC. Use the number of outliers as loss function. A measurement is deemed as an outlier if the depth is negative or if the reprojection error is larger than threshold.

In `sequence.mat` you find a struct array `triangulation_examples` with triangulation examples. Each example (3D point) has a cell list of camera matrices `Ps` and a $2 \times N$ -array `us` with image points. It will take some time to triangulate all 32183 examples so start with the first 1000 or so. We used a RANSAC threshold of 5 pixels.

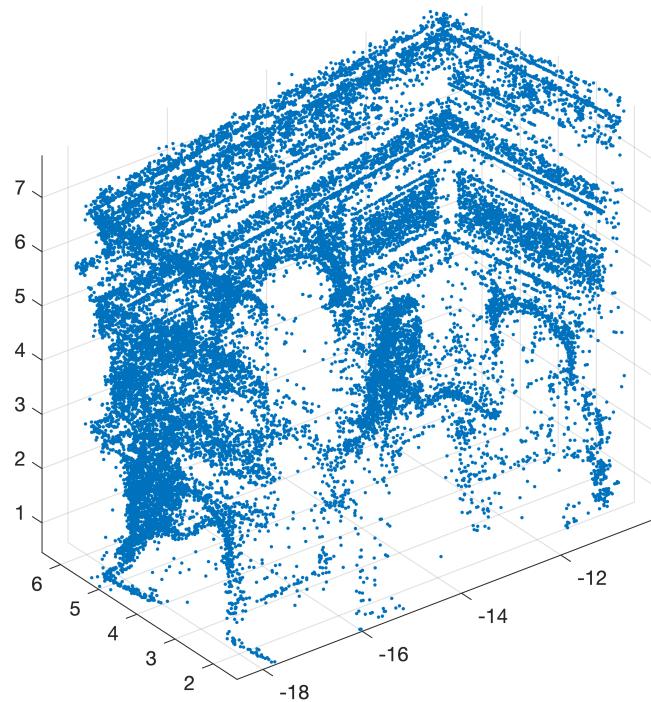
Ex 4.5 Make a script `triangulate_sequence` that runs `ransac_triangulation` for all (or at least 1000) of the examples from `sequence.mat`. Store all triangulated points with at least two inliers and plot them using `scatter3`. There will always be a few outliers among the estimated 3D points that make it harder to view the plot. You can use the provided `clean_for_plot.m` to clean it up a bit.

```
triangulate_sequence % Script that you need to create
```

```
Uc = clean_for_plot(Us);
```

```
Uc = 3x30266  
-17.5124 -18.1858 -15.7993 -14.0330 -16.7203 -16.4987 -14.9020 -17.9696 ...  
4.3631 4.9165 5.4013 5.5663 5.2964 5.2247 5.5230 5.2878  
6.4928 4.9135 3.2267 4.5489 7.0833 2.0639 4.7213 6.9711
```

```
scatter3(Uc(1,:),Uc(2,:),Uc(3,:),'.')  
axis equal
```



Can you recognize the building?

Write your answer with explanation as a comment here.

```
% I believe it is Arc de Triomphe in Paris.
```

Least squares triangulation

In this part of the lab you will solve the triangulation problem through least squares. Just as in the case of registration, the following pipeline is recommended:

- Use RANSAC to obtain a rough estimate of the parameters (U).
- Remove all measurements which are outliers with respect to these parameters.
- Estimate the least squares parameters using the remaining measurements.

Note that in this case a *measurement* is a pair consisting of an image point u_i and a camera matrix P_i . Don't forget that points with negative depths should be outliers.

Ex 4.6 Consider a camera matrix

$$P_i = \begin{bmatrix} \leftarrow a_i^T \rightarrow \\ \leftarrow b_i^T \rightarrow \\ \leftarrow c_i^T \rightarrow \end{bmatrix},$$

a 3D point U and an image point u_i . Write the formula for the reprojection error $r_i(U)$ below.

Your answer here:

$r_i(U) = \dots$ I honestly dont understand how this mathwriter works... here is the expression written on a piece of paper. This is used in the function `reprojection_errors()`.

$$r_i(U) = \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{a_i^T U}{c_i^T U} - x \\ \frac{b_i^T U}{c_i^T U} - y \end{pmatrix}$$

As you can see, the residuals are no longer linear, so computing a least squares solution will be significantly harder than in the previous lab. In fact, we cannot be sure to find the least squares solution. What we can do is to use local optimization to reduce the sum of squared residuals. We start at the solution produced by Ransac and use a few Gauss-Newton iterations.

Ex 4.7 Make a function

```
all_residuals = compute_residuals(Ps, us, U);
```

that takes a cell list **Ps** with N cameras, a $2 \times N$ array **us** of image points and a 3×1 array **U**, and computes a $2N \times 1$ array with all the reprojection residuals stacked into a single vector/array. A reprojection residual is a 2D vector corresponding to the difference between the original measurement and the projected point. The stacked vector is the \bar{r} from the lecture notes (page 99).

```
% usage example
points3D = triangulation_examples;

all_residuals = compute_residuals(points3D(1).Ps, points3D(1).xs, Us(:,1))

all_residuals = 32x1
-9.5400
 1.7561
-2.2038
 0.5214
-0.5831
-3.0967
 2.1423
-0.5882
```

0.7437

-3.4368

⋮

Ex 4.8 Find formulas for the partial derivatives in the Jacobian of \vec{r} . The Jacobian should be a $2N \times 3$ -matrix.
(Hint: you can start by finding the formula to the Jacobian of the i-th residual, which is a 2×3 matrix.)

Your answer here:

Jacobian of the i-th residual:

$$J_i = \begin{bmatrix} \frac{\partial r_{i,x}}{\partial u_1} & \frac{\partial r_{i,x}}{\partial u_2} & \frac{\partial r_{i,x}}{\partial u_3} \\ \frac{\partial r_{i,y}}{\partial u_1} & \frac{\partial r_{i,y}}{\partial u_2} & \frac{\partial r_{i,y}}{\partial u_3} \end{bmatrix}$$

and since we have N residuals we get jacobian with dimensions $2N \times 3$...

$$J = \begin{bmatrix} \frac{\partial r_{i,x}}{\partial u_1} & \frac{\partial r_{i,x}}{\partial u_2} & \frac{\partial r_{i,x}}{\partial u_3} \\ \frac{\partial r_{i,x}}{\partial u_1} & \frac{\partial r_{i,x}}{\partial u_2} & \cdot \\ \frac{\partial r_{i+1,x}}{\partial u_1} & \cdot & \cdot \\ \vdots & \vdots & \vdots \\ \frac{\partial r_{N,x}}{\partial u_1} & \frac{\partial r_{N,x}}{\partial u_2} & \frac{\partial r_{N,x}}{\partial u_3} \\ \frac{\partial r_{N,y}}{\partial u_1} & \frac{\partial r_{N,y}}{\partial u_2} & \frac{\partial r_{N,y}}{\partial u_3} \end{bmatrix}_{2N \times 3}$$

given the expression for r , we can define the jacobian for the i-th residual as:

$$J_i = \frac{\partial r_i}{\partial U} = \begin{bmatrix} \frac{a_i^T(c_i^T U) - (a_i^T U)c_i^T}{(c_i^T U)^2} \\ \frac{b_i^T(c_i^T U) - (b_i^T U)c_i^T}{(c_i^T U)^2} \end{bmatrix}$$

The above expression has the dimensions 2x3. Doing this for each N we get the total jacobian with size 2Nx3

$$J = \left[\begin{array}{c} \end{array} \right]$$

Ex 4.9 Make a function

```
jacobian = compute_jacobian(Ps, U)
```

that computes the Jacobian given a 3 x 1- vector **U** and a cell array of camera matrices **Ps**.

```
% usage example
warning off
jacobian = compute_jacobian(points3D(1).Ps, Us(:,1))
```

```
jacobian = 32x3
-17.1402  74.5577  148.2198
-163.2305 -16.6054   -7.6179
-24.2161  75.6598  150.6913
-164.8040 -25.6219   -9.9308
-28.5342  71.4114  147.9102
-160.7433 -32.7292  -11.1013
-29.3419  69.7052  147.4278
-158.4709 -38.8637   -8.9062
-29.9871  70.3175  148.3120
-158.4231 -43.2657   -7.3672
:
:
```

Ex 4.10 Use these functions to make a function

```
U = refine_triangulation(Ps, us, Uhat)
```

that uses an approximate 3D point **Uhat** as a starting point for Gauss-Newton's method. Use five GaussNewton iterations. Print the sum of squared residuals after each Gauss-Newton step to verify that it decreases.

```
% usage example
Uref = refine_triangulation(points3D(1).Ps, points3D(1).xs, Us(:,1))
```

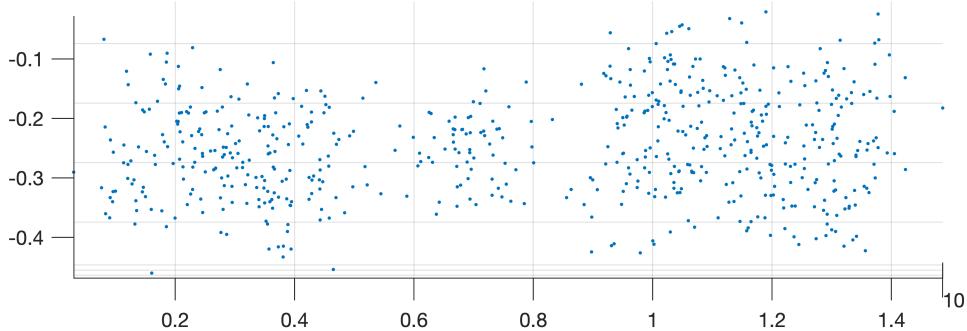
```
Residual for iter=1: 3950.15
Residual for iter=2: 2117.26
Residual for iter=3: 2114.53
Residual for iter=4: 2114.53
Residual for iter=5: 2114.53
Uref = 3x1
-17.4131
 3.8202
 6.7842
```

```
% compare to Uref to Us
Us(:,1)
```

```
ans = 3x1
-17.5124
 4.3631
 6.4928
```

Ex 4.11 Try your refine_triangulation on the data in gauss_newton.mat. First we will plot the points given in **Uhat**, then refine each point using your function and plot the results using scatter3. You should see an improvement.

```
% don't forget to plot your results
load gauss_newton
scatter3(Uhat(1,:),Uhat(2,:),Uhat(3,:),'.' )
axis equal
view([0 5])
```



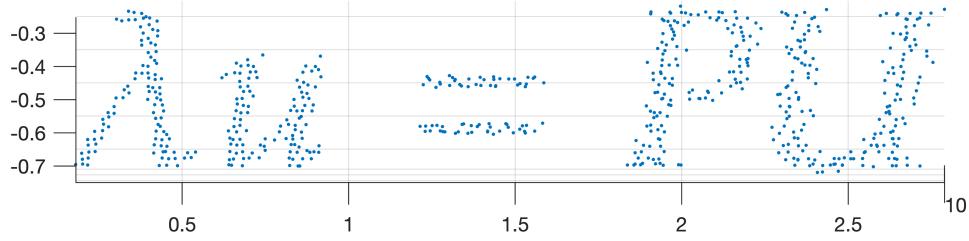
```
% Gather camera matrices in cell
Ps = {P,P_tilde};

% Number of triangulated points
N = size(Uhat,2);

% Initialize Urefined vector
Uref = zeros(3,N);

% Iterate through the points
for i = 1:N
    % refine estimation of U
    Uref(:,i) = refine_triangulation(Ps, [u(:,i), u_tilde(:,i)], Uhat(:,1));
end

% Plot
scatter3(Uref(1,:),Uref(2,:),Uref(3,:),'.')
axis equal
view([0 5])
```



Ex* 4.12 Compute the camera positions for the data in `gauss_newton.mat` and plot them together with the estimated 3D points. Try to understand why the noise in the estimated points looks as it does.

```
% YOUR CODE HERE; don't forget to plot your results
```

```
% Find the pose of the cameras
```

```
% Number of cameras
```

```
N_cam = length(Ps);
```

```
% Initialize
```

```
C = zeros(4, N_cam);
```

```
t = zeros(3, N_cam);
```

```
R = zeros(3, N_cam);
```

```
% Iterate through the cameras
```

```
for i=1:N_cam
```

```
    % Camera position is the nullspace of P
```

```
C(:,i) = null(Ps{i});
```

```
    % Translation t of camera
```

```
t(:,i) = C(1:3,i) / C(4,i);
```

```

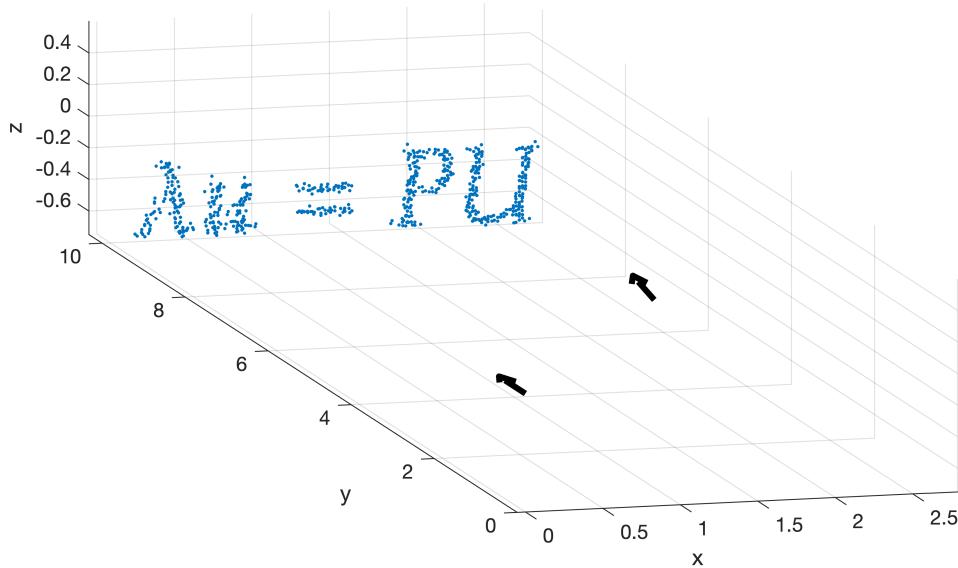
% Rotation of camera (direction)
R(:,i) = Ps{i}(3,1:3);
end

% Plot triangulated points U
scatter3(Uref(1,:),Uref(2,:),Uref(3,:),'.')
hold on

% Plot pose of Cameras
quiver3(t(1,:),t(2,:),t(3,:),R(1,:), R(2,:), R(3,:),['k','--'], 'LineWidth',3, 'MaxHeads',1)

% Set axis and view
axis equal
hold off
view([-15 10])
xlabel('x')
ylabel('y')
zlabel('z')

```



% As we can see. The first camera is located at $(0,0,0)$ and is pointing at $(0,1,0)$, mening that it is pointing straight along the y-axis.

% The second camera is located at $(0.83, 0.00, 0.55)$ and is directed % towards $(0.04, 0.99, 0.06)$. This means that the second camera is located % slightly above to the right of the first camera. The second camera is

```
% also pointed towards the expression lambda=PU but is slightly tilted  
% upwards to the right.
```

For the next exercise try the following way to match features,

```
matchFeatures(d1,d2,'Method','Approximate','MaxRatio',0.9,'MatchThreshold',100);
```

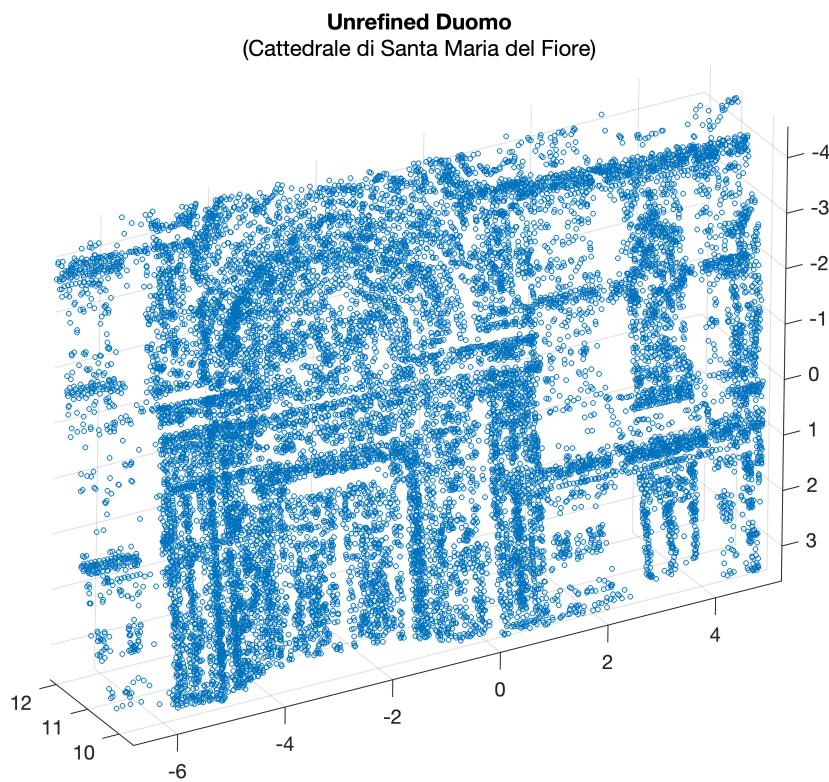
The approximate option makes it go significantly faster. A Lowe ratio of 0.9 is very high, but it makes the model more *dense*.

Ex 4.13 There are images duomo.jpg and duomo_tilde.jpg in the lab folder and also a data file duomo.mat with two camera matrices. Extract SIFT features from the two images using detectSIFTFeatures and extractFeatures, then match them using matchFeatures. Also store the color of each SIFT point. It doesn't matter from which of the images you select the color information.

Use your triangulation code (don't forget refine_triangulation) to triangulate the points and plot them with the right color using scatter3. Again, you can use clean_for_plot.m to remove outliers among the estimated 3D points. Beware that triangulating all the points will take a little time so work with a subset until you are sure that the code works.

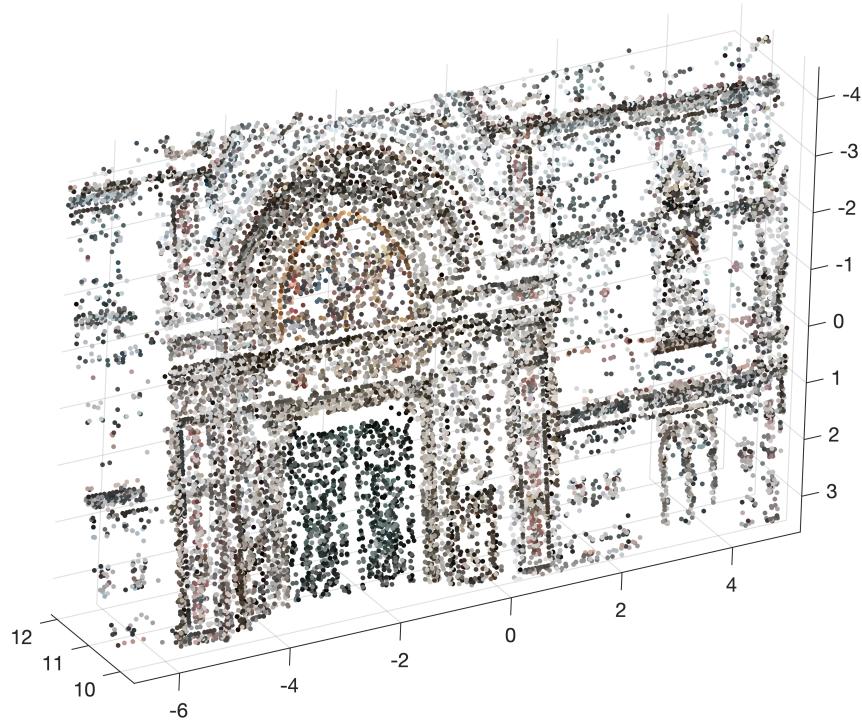
```
%Your code here  
load duomo  
  
% Load the images  
image1 = rgb2gray(imread("duomo.jpg"));  
image2 = rgb2gray(imread("duomo_tilde.jpg"));  
colors = imread("duomo.jpg");  
  
% Detect SIFT features  
points1 = detectSIFTFeatures(image1);  
[d1, validPointsA] = extractFeatures(image1,points1);  
  
points2 = detectSIFTFeatures(image2);  
[d2, validPointsB] = extractFeatures(image2,points2);  
  
% Identify matching features  
matches = matchFeatures(d1,d2,'Method','Approximate','MaxRatio',0.9,'MatchThreshold',100);  
  
% Extract position u and u_tilde of SIFT features  
u = validPointsA(matches(:,1)).Location';  
u_tilde = validPointsB(matches(:,2)).Location';  
loc = round(u);  
  
% Save RGB colors of matched SIFT features  
colorsSIFT = zeros(length(loc),3);  
for i=1:length(loc)  
    colorsSIFT(i,:) = colors(loc(2,i),loc(1,i),:);  
    colorsSIFT(i,:) = colorsSIFT(i,:)/255;  
end
```

```
% Run script (triangulate)  
triangulate_sequence_duomo;
```



```
% Run script (refine with colors)  
refine_duomo;
```

Refined Duomo + colors
(Cattedrale di Santa Maria del Fiore)



```
% Find the pose of the cameras

% Number of cameras
N_cam = length(Ps);

% Initialize
C = zeros(4, N_cam);
t = zeros(3, N_cam);
R = zeros(3, N_cam);

% Iterate through the cameras
for i=1:N_cam

    % Camera position is the nullspace of P
    C(:,i) = null(Ps{i});

    % Translation t of camera
    t(:,i) = C(1:3,i) / C(4,i);

    % Rotation of camera (direction)
    R(:,i) = Ps{i}(3,1:3);
end
```

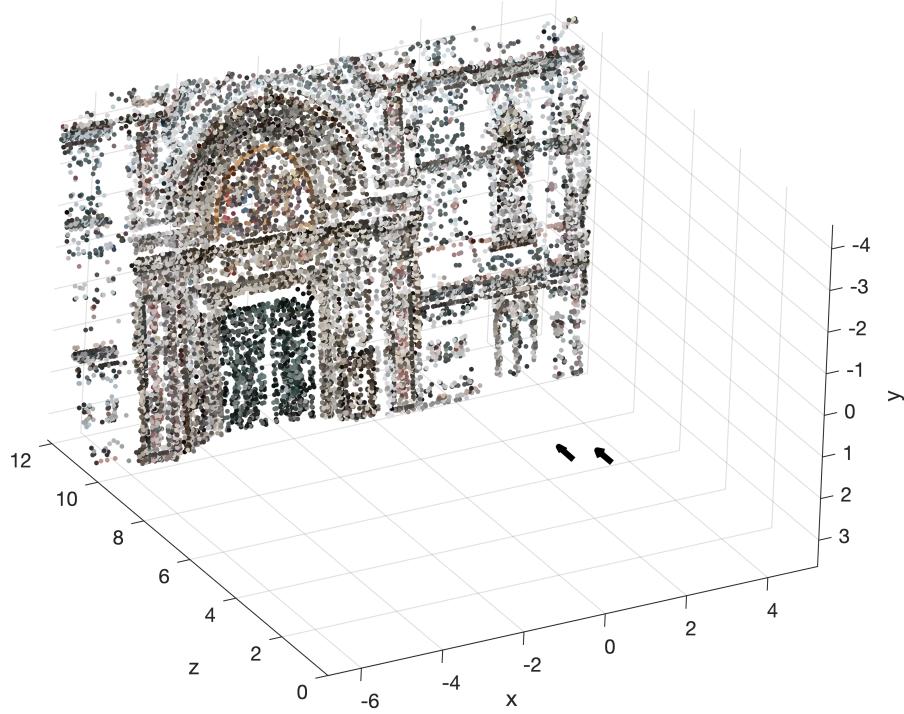
```

% Plot triangulated points U
s = 6;
scatter3(Uref(1,:),Uref(2,:),Uref(3,:),s,colors_inliers,'filled')
hold on

% Plot pose of Cameras
quiver3(t(1,:),t(2,:),t(3,:),R(1,:), R(2,:), R(3,:),['k','--'], 'LineWidth',3, 'MaxHeads',1)

% Set axis and view
axis equal
hold off
view([-45,-50])
camroll(50)
xlabel('x')
ylabel('y')
zlabel('z')

```



Essential Matrix

The two exercises below are optional and therefore marked with *.

Ex* 4.14 Load the data in P.mat. There you will find two calibrated camera matrices, P1 and P2. Make a function

```
E = essentialMatrix(P1,P2)
```

that computes the essential matrix E from P_1 and P_2 .

```
load P.mat
```

Ex* 4.15 Similar to the reprojection error in the triangulation problem, the essential matrix can also be used for outlier detection using the residual

$$r^2 = \frac{|\tilde{x}^T E x|}{\|S E x\|^2 + \|S E^T \tilde{x}\|^2}, \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where x and \tilde{x} are the image projections of a 3D point X using some arbitrary camera matrices P and \tilde{P} , respectively. Note that r does not depend on X . Assuming that you are using a similar threshold for outlier detection, for a set of pixel correspondences, you would get $I1$ inliers if you use the residual from the triangulation problem and $I2$ inliers if you used the residual from (4.4). Which one is greater, $I1$ or $I2$? Why?