SSY281 Model Predictive Control
Assignment 4 – MPT and persistent feasibility

Instructions

The assignments comprise an important part of the examination in this course. Hence, it is important to comply with the following rules and instructions:

- The assignment is pursued and reported individually.
- The findings from each assignment are described in a short report, written by each student independently.
- The report should provide clear and concise answers to the questions, including your motivations, explanations, observations from simulations, etc. Conclusions should be supported by relevant results if applicable; e.g., the system is stable since the eigenvalues, $[0.5\ , 0.2 + 0.5j\ , 0.2 - 0.5j]$, are inside the unit circle. Figures included in the report should have legends, should be readable, should have proper scaling to illustrate the relevant information, and axes should be labeled. Try to verify your solutions if possible; e.g., plot the inputs and outputs and see whether they respect the constraints.
- Since the assignments are part of the examination in the course, plagiarism is of course not allowed. If we observe that this happens anyway, it will be reported.
- The report should be uploaded to Canvas *before the deadline*. A report uploaded a second or a day after the deadline are penalized equally. Name the report as A4.pdf.
- A MATLAB code should be uploaded which reproduces all numbers and figures in your report. Make sure that one can run your code and see your results without any error. Name the MATLAB script as A4.m.

Table 1: Points per question

Question:	1	2	3	Total
Points:	3	8	4	15

1. **MPT and polyhedral sets**

   After installing the Multi-Parametric Toolbox 3 (MPT) in Matlab, run `mpt_demo_sets1` and `mpt_demo2` in MATLAB to study the two demos of the MPT package.

   (a) [1p] Use the commands in MPT to find the V- and the H-representation of the following polyhedron, plot them and explain the differences in the report.

   $$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

   Note: do not consider the tail in the V representation case, i.e., only consider the corner points to define the polyhedron.

   (b) [2p] Use the commands in MPT and the Minkowski sum and the Pontryagin difference operations to compute

   $$P,\ Q, P+Q,\ P-Q,\ (P-Q)+Q,\ (P+Q)-Q,\ (Q-P)+P,\ (Q+P)-P.$$

   Plot the above polyhedrons and comment on your observations.

   $$P = \{x : A_1 x \leq b_1,\ x \in \mathbb{R}^2\}$$

   $$Q = \{x : A_2 x \leq b_2,\ x \in \mathbb{R}^2\}$$

   $$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad b_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

   $$A_2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

   Note: Assume $S_1 = \{\},\ S_2 \neq \{\}$; While $S_1 \oplus S_2 = \{\}$, MPT incorrectly returns S_2!

2. **Forward and backward reachability**

   (a) [2p] Consider the system

   $$x^+ = Ax, \quad A = \begin{bmatrix} 0.8 & 0.4 \\ -0.4 & 0.8 \end{bmatrix}. \tag{1}$$

   Show that the set S, defined as follows, is positively invariant for this system:

   $$S := \{x :\ A_{in} x \leq b_{in},\ x \in \mathbb{R}^2\},$$

   $$A_{in} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad b_{in} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1.5 \\ 1.5 \\ 1.5 \\ 1.5 \end{bmatrix}.$$

   (b) [3p] Consider the system

   $$x^+ = Ax + Bu, \quad A = \begin{bmatrix} 0.8 & 0.4 \\ -0.4 & 0.8 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2}$$

with $-1 \leq u \leq 1$.

Calculate the one-step reachable set from S, defined in (a). Depict S and the reachable set you have calculated in the same plot.

Hint. You can use `alpha` in your plot option to adjust transparency of the polyhedrons in the figure.

Note. Do not use the `reachableSet` command to compute the set.

(c) [3p] Calculate one step "Pre" of S for system (2). Depict S and the "Pre" set you have calculated in the same plot.

Hint. Use `projection` command in MPT to compute the set.

3. [4p] **Persistent feasibility**

Consider

$$x^+ = Ax + Bu, \quad A = \begin{bmatrix} 0.9 & 0.4 \\ -0.4 & 0.9 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3}$$

and design an RH controller with $P_f = \mathbf{0}$, $Q = I_{2\times 2}$, and $R = 1$. Let $x(0) = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top$ be the initial condition and consider the following constraints:

$$|x_i(k)| \leq 3, \; |u(k)| \leq 0.1 \;\; \forall k \in \{0, 1, 2, \ldots\}, \; \forall i \in \{1, 2\}.$$

(a) Let $\mathbb{X}_f = \mathbf{0}_{2\times 1}$. Find the shortest prediction horizon N such that the RH controller is feasible.

(b) Set $N = 2$ and choose $\mathbb{X}_f$ as the maximal control invariant set for the system. Is the RH controller still feasible until convergence to the origin?

Hint. You can use `invariantSet` command in MPT.

(c) Plot the set of feasible initial states for the controllers designed at the previous two points. How do you explain the difference? What is the size of the optimization problems (amount of optimization variables and amount of equality and inequality constraints) underlying the two RH controllers?

Hint. One may use `reachableSet` command in MPT to calculate feasible initial states.