SSY281 - Model predictive control
Assignment 2

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Question 1: Set-point tracking

(a) Calculate state and input set-points (x_s, u_s)
Since this system has the same amount of inputs as outputs, we should consider the set-point tracking for that case. We want the stady state x_s to fulfill the condition for setpoint tracking as repeated in equation (1).

$$\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix} \tag{1}$$

Inserting the numerical values for matrices A, B and C and the setpoint y_{sp} we can calculate the solution according to equation (2) by taking the inverse of the large matrix. The solution for state and input set-points x_s, u_s is seen in expression (3).

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y_{sp} \end{bmatrix} \tag{2}$$

$$x_s = \begin{bmatrix} 0.1745 \\ 0.0722 \\ -3.1416 \\ 0.0008 \end{bmatrix} \quad u_s = \begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix} \tag{3}$$

(b) Assume that only the second input is available...
Now we can only utilize one of the inputs which menas that we have less inputs than outputs. Thereby we need to find the solution which minimizes ($\mid Cx_s - y_{sp} \mid^2_{Q_s}$). This is done with Matlab with `quadprog` and the solution is seen in expression (4).

$$x_s = \begin{bmatrix} 0.0000 \\ 0.0000 \\ -3.1416 \\ 0.0000 \end{bmatrix} \quad u_s = -1.2096e-13 \tag{4}$$

Evidently with the control u_s we are not able to reach the set-point $y_s = \begin{bmatrix} \frac{\pi}{18} & -\pi \end{bmatrix}^\top$ but can only reach $y_s = \begin{bmatrix} 0 & -\pi \end{bmatrix}^\top$. Hence the system output cannot settle at the desired states when we can only rely on one control signal...

(c) Assume that only one state is measurable...
In this last scenario we have a system with more inputs than controlled outputs. This implies that there may be more than one solution and thus we want to find feasable stade-state targets that minimize ($\mid u_s - u_{sp} \mid^2_{R_s} + \mid Cx_s - y_{sp} \mid^2_{Q_s}$). This is also solved with Matlab with `quadprog` and the solution is seen in expression (5).

$$x_s = \begin{bmatrix} 0.1745 \\ 0.0722 \\ 0.0000 \\ 0.0008 \end{bmatrix} \quad u_s = \begin{bmatrix} -0.0153 \\ -0.1427 \end{bmatrix} \tag{5}$$

As shown, we are able to settle the selected controlled system output at $y_s = \frac{\pi}{18}$

Question 2: Controll of a ball and wheel system

(a) Construct an augmented model

To create an augmented system we will use equation (58) in the lecture notes. To determine the augmented systems detectability we have to fulfill two conditions, namely:

$$\operatorname{rank}\big(\mathcal{O}(A, C)\big) = n, \quad \text{and} \quad \operatorname{rank}\left(\begin{bmatrix} I - A & -B_d \\ C & C_d \end{bmatrix} \right) = n + n_d$$

Where n is the number of states and n_d is the number of disturbances. i.e The original system must be detectable and the augmented system must be full rank.

A function `construct_aug_model()` was created which returns the augmented system matrices A_a, B_a, C_a and checks wether the two conditions are fullfilled. As it turns out, the conditions are only satisfied for the distrubance models 1. and 3. For model 2. the augmented system is rank deficient and thus not detectable. The numerical values for the matrices A_a, B_a, C_a for the disturbance models can be seen in the Matlab script.

(b) Design a Kalman filter for the detectable systems

Since only disturbance models 1 and 3 are detectable, we will only design Kalman filters for those two. This task is simple enough. We want to solve the discrete algebraic riccati equations for our augmented systems and determine the optimal observer gain L^*. For the two augmented systems 1 and 3, we solve for the stationary estimation error covariance P and in turn the observer gain given by equations (6).

$$\begin{aligned} &P = APA^\top - APC^\top \left(CPC^\top + R\right)^{-1} CPA^\top + Q \\ &L = PC^\top \left(CPC^\top + R\right)^{-1} \end{aligned} \tag{6}$$

Solving with $A = A_a$, $C = C_a$ and having process disturbances covariance$Q$ and measurement noise covariance R as identity, we obtain the observer gains for the two disturbance models given in expression (7).

$$\begin{aligned} L_1 &= \begin{bmatrix} 1.1835 & 9.7171 & 0.4284 & 0.4089 & -0.3508 \\ -0.4358 & -3.5104 & 0.2323 & -0.1008 & 0.4763 \end{bmatrix} \\ L_3 &= \begin{bmatrix} 1.3594 & 11.5489 & 0.0140 & 0.3197 & -0.4815 & -0.0102 \\ 0.0128 & 0.3198 & 0.2478 & 2.0989 & -0.0090 & 0.5109 \end{bmatrix} \end{aligned} \tag{7}$$

(c) find matrix M_{ss} for each detectable system

By using the expression found in lecture notes section 5.2 (equation number missing), of the case with distrubance modeling for more inputs than controlled outputs. We get the correlation given by expression (8).

$$\begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} B_d\hat{d} \\ z_{sp} - HC_d\hat{d} \end{bmatrix} \tag{8}$$

We can rewrite (8) with $z_{sp} = 0$ and extracting $\hat{d}$ from the right hand side matrix and obtain the final expression in (9) which shows what the matrix M_{ss} is.

$$\begin{bmatrix} x_s \\ u_s \end{bmatrix} = \underbrace{\begin{bmatrix} I-A & -B \\ HC & 0 \end{bmatrix}^{-1} \begin{bmatrix} B_d \\ -HC_d \end{bmatrix}}_{M_{ss}} \hat{d}(k) \tag{9}$$

For models 1 and 3 we simply insert the correspoing matrices in (9) and we then achieve the M_{ss} matrices given in expression (10).

$$M_{ss_1} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \qquad M_{ss_3} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{10}$$

(d) Design a RHC for each detectable system

Disturbance model 1: By simulating the system we get the result shown in Figure 1. The disturbance is applied at the red cross in the plots (when $k = 50$). Evidently the controller is <u>not</u> able to remove the off-set for the third state governing the wheel angle θ_2. After the disturbance is applied, the wheel angle drifts away from its setpoint, never to return.

This can be explained by a poor choice of B_d and C_d. Since these matrices are unknown to the designer, one must assume the values of these. For this specific case, it was assumed that $B_d = 0_{4x1}$ and $C_d = 1_{1x2}$ i.e that the disturbance does not act as a process disturbance, but only as a measurement noise. This assumption is wrong since the actual system, described by eq (1a-1b) in assignment, has both process disturbances $B_p p(k)$ and measurement noise $C_p p(k)$. This controller is thus not able to track the off-set in the third state...

Nevertheless, since the controller succeeds to remove the off-set in the first state, ball angle θ_1, the controller can still accomplish to balance the ball on top of the wheel...

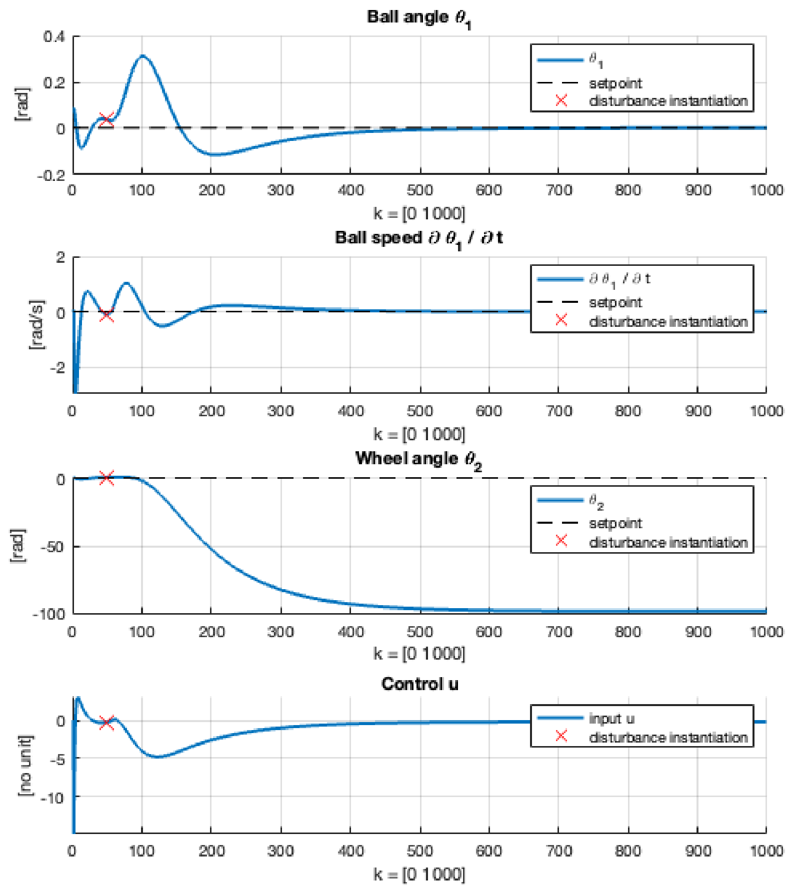

Figure 1: System 1 simulated from $k = 0 \rightarrow 1000$. Constant disturbance applied at $k = 50$

Disturbance model 3: By simulating the system we get the result shown in Figure 2. The disturbance is yet again applied at the red cross in the plots (when $k = 50$). Compared to the previous model, we are now able to remove the off-set in the third state θ_2 and the wheel angle succesfully reaches its setpoint at 0.

In this scenario, the designer has done better assumptions on B_d and C_d. For this disturbance modeling, it is assumed that there is a noise that acts as process disturbance. It is assumed that $B_d = [0_{4x1} \quad B_p]$ which is a more accurate assumption of how the disturbances affect the actual system. This controller is thus able to track the off-set in all states resulting in that all states reach their setpoint 0.

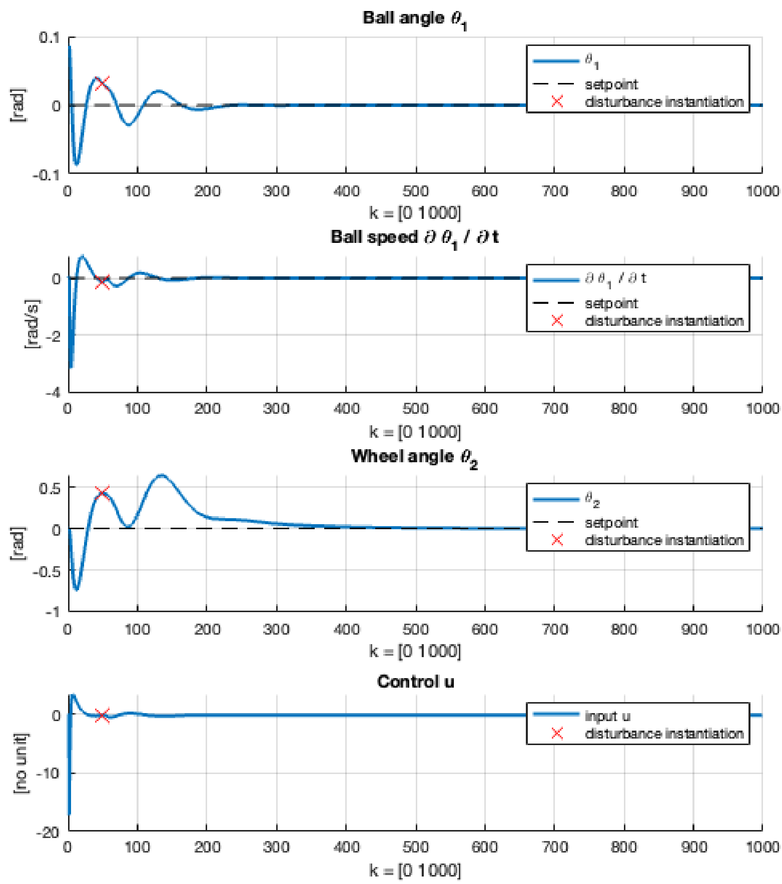

Figure 2: System 3 simulated from $k = 0 \rightarrow 1000$. Constant disturbance applied at $k = 50$

/Nicholas Granlund