SSY281 - Model predictive control
Assignment 6

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Question 1: Lyapunov functions

(a) Find Q. What can you say about stability of the system given S and Q?

Given the numerical values for A, B and S we can calculate Q as:

$$Q = -(A^\top SA - S) = \begin{bmatrix} -0.6964 & -0.8415 & -0.0002 & -0.0396 \\ -0.8415 & -0.0083 & 0.0000 & 0.0091 \\ -0.0002 & 0.0000 & 0.0000 & -0.0098 \\ -0.0396 & 0.0091 & -0.0098 & 0.0726 \end{bmatrix}$$

Since the matrix S is symmetric and has positive eigenvalues (positive definite) we know that the conditions for S is fullfilled. However we notice that Q is not positive definite. This implies that the Lyapunov equation does not hold for the given matrices A and S. An intuitive explanation for what this implies is that, the potential energy of the system doesn't necessarily decrease for a given set of state trajectories. Therefor Q does not satisfy the conditions and it is not possible to draw any conclusions about the stability by the Lyapunov function alone. We need further analysis, so we compute the matrix A eigenvalues. The eigenvalues of A are:

$$eig(A) = \{1.0000, 1.0949, 0.9126, 0.9636\}$$

Which implies an unstable system since we have a pole outside the unit disc.

(b) considering $u(k) = -Kx(k)$. Find Q. What can you say about stability of the system given S and Q?

Now we have closed-loop feedback in the system, hence the system behaviour is given by $A_{cl} = A - BK$. Given the numerical values for A, B, K and S we can calculate Q as:

$$Q = -(A_{cl}^\top SA_{cl} - S) = \begin{bmatrix} -4.3342e+03 & -4.6845e+02 & 4.8917e+01 & 1.2221e+02 \\ -4.6845e+02 & -5.0419e+01 & 5.2801e+00 & 1.3404e+01 \\ 4.8917e+01 & 5.2801e+00 & -5.5212e-01 & -1.3826e+00 \\ 1.2221e+02 & 1.3404e+01 & -1.3826e+00 & -2.4944e+00 \end{bmatrix}$$

Here. once again the matrix S is symmetric and has positive eigenvalues. However, even with the closed-loop feedback we still don't get the matrix Q to be positive definite. Furthermore, in this case the eigenvalues of the closed-loop are:

$$eig(A - BK) = \{0.1894 + 0.0000i, 0.9156 + 0.0101i, 0.9156 - 0.0101i, 0.9901 + 0.0000i\}$$

Which are indeed inside the unit disc!

(c) considering $u(k) = -Kx(k)$ and $Q = I_4$. Find S. Is the closed-loop stable?

By finding the solution S for the Lyapunov equation with the help of Matlab we obtain with the given numerical values:

$$S = \begin{bmatrix} 3.9766e+04 & 4.3268e+03 & -9.3681e+02 & -1.0303e+03 \\ 4.3268e+03 & 4.7677e+02 & -1.0313e+02 & -1.1334e+02 \\ -9.3681e+02 & -1.0313e+02 & 1.2354e+02 & 2.5102e+01 \\ -1.0303e+03 & -1.1334e+02 & 2.5102e+01 & 2.8036e+01 \end{bmatrix}$$

This matrix S is both symmetric and has positive eigenvalues. The matrix Q was choosen as I_4 which is positive definite, and finally the eigenvalues in the closed-loop system $A - BK$ is inside the unit disc as stated by the previous question. Hence we can conclude that the system is finally stable!

Question 2: Stability with receding horizon control

(a) what is the effect of Q on the stability when $N = 1$

In the case when $N = 1$, Q has no effect on the stability. This is because the horizon only considers P_f when calculating the feedback gain K. $P(N) = P_f$

(b) Find the shortest N such that the RH controller stabilizes the system

By utilizing dynamic programming / batch approach and the functions we created in assignment 1, we can determine the shortest horizon N such that the feedback is stable. We get that the shortest horizon $N = 38$ with the feedback gain K given by.

$$K = \begin{bmatrix} 45.6628 & 5.1672 & -0.0068 & -0.3898 \end{bmatrix}$$

(c) Verify whether $V_N(x)$ is a Lyapunov function for the closed-loop system with the feedback gain K computed in the (b). Interpret the results.

To verify if $V_N(x)$ is the Lyapunov function, the function has to fulfill the conditions defined by Definition 11.4 in the lecture notes...

(d) Find a P_f for which the system is stable for any $N \geq 1$

Question 3: Receding horizon controller example

(a) Show that the controller cannot stabilize the system regardless of Q and R

Question 4: Stability for constrained systems

The plots for the different cases can be seen below in Figure 1. For clarification, the set X_f is the intersection between the terminal set constraint and the control-invariant set. We aim to regulate the system towards the control-invariant part of the terminal constraint.

Evidently, when we choose $P_f = I_2$, the state trajectory cannot be maintained inside X_f and Matlab also gives us a prompt that at $k = 7$, there is no control action in $u \in U$ that can stabilize the system, it is thus infeasible.

For the other case when we choose P_f as the Riccati solution, the controller is able to stabilize the system. This follows from Theorem 11.13 from the lecture notes, that if the terminal cost V_f is chosen as the value function of the corresponding unconstrained problem (P_f as riccati solution) then the origin is asymptotcally stable. This also explains that the state-trajectory doesn't just reach X_f, but they eventually reach the origin.

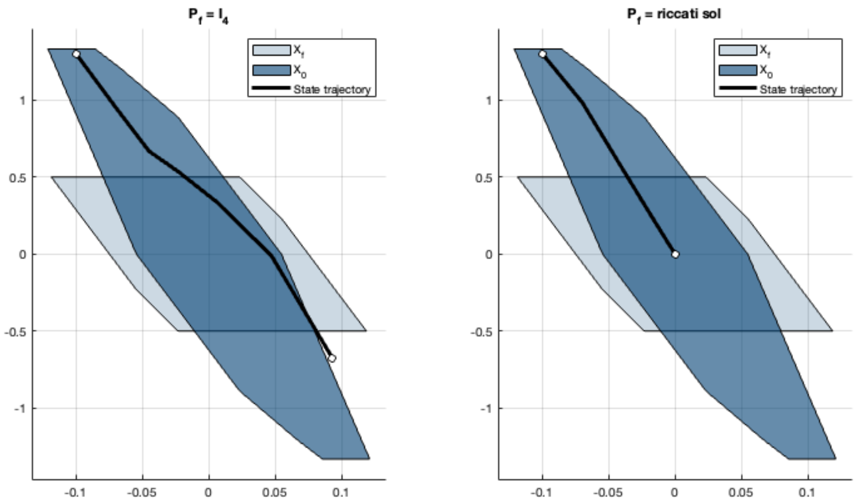

Figure 1: X_0, X_f and the closed-loop state trajectories for the two cases.

/ Nicholas Granlund