SSY281 Model Predictive Control
Assignment 5 – Explicit MPC and minimum time control

The purpose of this assignment is to understand how finite time control can be constructed from a family of explicit MPC algorithms.

Instructions

The assignments comprise an important part of the examination in this course. Hence, it is important to comply with the following rules and instructions:

- The assignment is pursued and reported individually.
- The findings from each assignment are described in a short report, written by each student independently.
- The report should provide clear and concise answers to the questions, including your motivations, explanations, observations from simulations, etc. Conclusions should be supported by relevant results if applicable; e.g., the system is stable since the eigenvalues, $[0.5\ , 0.2 + 0.5j\ , 0.2 - 0.5j]$, are inside the unit circle. Figures included in the report should have legends, should be readable, should have proper scaling to illustrate the relevant information, and axes should be labeled. Try to verify your solutions if possible; e.g., plot the inputs and outputs and see whether they respect the constraints.
- Since the assignments are part of the examination in the course, plagiarism is of course not allowed. If we observe that this happens anyway, it will be reported.
- The report should be uploaded to Canvas *before the deadline*. A report uploaded a second or a day after the deadline are penalized equally. Name the report as A5.pdf.
- A MATLAB code should be uploaded which reproduces all numbers and figures in your report. Make sure that one can run your code and see your results without any error. Name the MATLAB script as A5.m.

Table 1: Points per question

Question:	1	2	Total
Points:	6	9	15

1. **Linear MPC design**

   Consider the following system

$$x(t+1) = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{1}$$

   The input and state constraints are

$$\mathcal{U} : -1 \leq u(k) \leq 1; \tag{2a}$$

$$\mathcal{X} : \begin{bmatrix} -15 \\ -15 \end{bmatrix} \leq x(k) \leq \begin{bmatrix} 15 \\ 15 \end{bmatrix} \tag{2b}$$

   (a) [2p] Consider $Q = I_2$, $R = 100$, and $N = 4$. Assume $\mathcal{X}_f = \mathbf{0}$, and find a terminal weight P_f for the constrained receding horizon controller to guarantee asymptotic stability for all $x_0 \in \mathcal{X}_N$. Motivate the choice of P_f in the report and plot $\mathcal{X}_N$.

   (b) [2p] Set $x(0) = [7 \ \ -4]^T$ and design an MPC controller for $N = 10,\ 15,\ 20$. Provide three figures in the report where each one is simulated for N time steps and contains the state predictions at time zero and the actual states when MPC is implemented. Analyze and explain the mismatch between predicted vs closed-loop trajectories (specifically at time N) as you increase N in the MPC design.

   (c) [1p] Assume $N = 20$, $\mathcal{X}_f = \{x : \ ||x||_\infty \leq 0.01\}$, and P_f as the Riccati solution. Find the explicit-MPC solution and plot the state-space partitions.

   (d) [1p] Assume $N = 1$. Can you choose a new $\mathcal{X}_f$ so that persistent feasibility is guaranteed for all x_0 belonging to C_∞? Motivate your answer in the report.

2. **Finite time control of a DC motor**

   Consider the model of a DC-servo, depicted below and with parameters listed in the table.

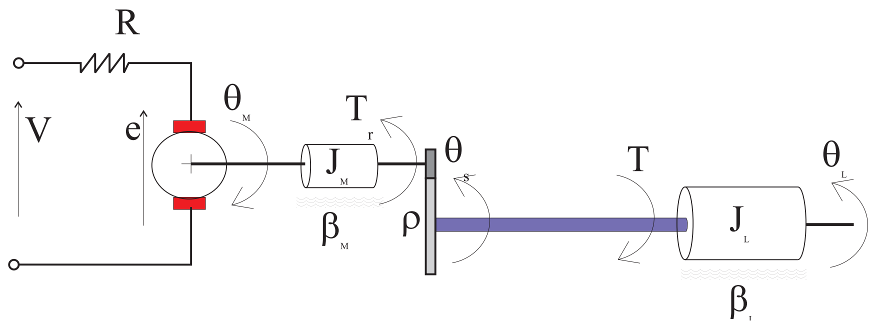

Figure 1: **DC-servomechanism.**

By defining the state $x = [\theta_L \ \dot{\theta}_L \ \theta_M \ \dot{\theta}_M]^\top$, the model can be described in state-space form

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_\theta}{J_L} & -\frac{\beta_L}{J_L} & \frac{k_\theta}{\rho J_L} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_\theta}{\rho J_M} & 0 & -\frac{k_\theta}{\rho^2 J_M} & -\frac{\beta_M + k_T^2/R}{J_M} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_T}{R J_M} \end{bmatrix} V \tag{3a}$$

$$T = \begin{bmatrix} k_\theta & 0 & -\frac{k_\theta}{\rho} & 0 \end{bmatrix} x. \tag{3b}$$

System output is the torsional torque T and input is the DC voltage V. The DC voltage is constrained within the following range:

$$|V| \leq 200\,\text{V}.$$

Table 2: Model parameters

Symbol	Value (MKS)	Meaning
L_S	1.0	shaft length
d_S	0.02	shaft diameter
J_S	negligible	shaft inertia
J_M	0.5	motor inertia
β_M	0.1	motor viscous friction coefficient
R	20	resistance of armature
k_T	10	motor constant
ρ	20	gear ratio
k_θ	1280.2	torsional rigidity
J_L	$50J_M$	nominal load inertia
β_L	25	load viscous friction coefficient

(a) [1p] Using sampling interval $h = 0.1\,\mathrm{s}$, find the matrices A, B, C in the following discrete-time model.

$$\begin{aligned} x(k+1) &= Ax(k) + BV(k), \\ T(k) &= Cx(k). \end{aligned} \tag{4}$$

(b) [2p] Starting from the initial state $x(0) = \begin{bmatrix} 0 & 2.5 & 0 & 75 \end{bmatrix}^\top$, design a minimum-time controller that brings the system to the target set $\dot{\theta}_L = \dot{\theta}_M = 0$, i.e. it brings the system to standstill. Provide the minimum time and plot the predicted system states and output.

Note: You have to design your own minimum time controller, without using a ready solution from the MPT toolbox or elsewhere. You do not need to design an explicit minimum-time controller.

(c) [2p] With the same initial state as in b), design a minimum-time controller to bring the rotating shafts to standstill, such that the torsional torque is constrained within the range

$$|T| \leq 150\,\mathrm{Nm}.$$

Provide the minimum time and plot the predicted system states and output.

(d) [4p] With the same initial state as in b) and constraint on torsional torque as in c), design a minimum time controller to bring the rotating shafts close to standstill, i.e.,

$$X_{\text{target}} = \{x \in \mathbb{R}^4 : \ |x_1| \leq 10, \ |x_2| \leq 0.01, \ |x_3| \leq 10, \ |x_4| \leq 0.01\}.$$

To this end, first find the largest control invariant set inside X_{target} such that $|V| \leq 200$ and $|T| \leq 150$ also hold. Provide the minimum time and plot the predicted system states and output considering the invariant set as your terminal set. Create a closed-loop controller and plot the states and output for at least 2 s. Note that the prediction horizon in the closed loop shrinks as time proceeds and it becomes one once the states are inside the invariant set, i.e., the system remains in its invariant set afterwards.