SSY281 Model Predictive Control
Assignment 3 – Optimization basics and QP problems

Instructions

The assignments comprise an important part of the examination in this course. Hence, it is important to comply with the following rules and instructions:

- The assignment is pursued and reported individually.
- The findings from each assignment are described in a short report, written by each student independently.
- The report should provide clear and concise answers to the questions, including your motivations, explanations, observations from simulations, etc. Conclusions should be supported by relevant results if applicable; e.g., the system is stable since the eigenvalues, $[0.5\ , 0.2 + 0.5j\ , 0.2 - 0.5j]$, are inside the unit circle. Figures included in the report should have legends, should be readable, should have proper scaling to illustrate the relevant information, and axes should be labeled. Try to verify your solutions if possible; e.g., plot the inputs and outputs and see whether they respect the constraints.
- Since the assignments are part of the examination in the course, plagiarism is of course not allowed. If we observe that this happens anyway, it will be reported.
- The report should be uploaded to Canvas *before the deadline*. A report uploaded a second or a day after the deadline are penalized equally. Name the report as A3.pdf.
- A MATLAB code should be uploaded which reproduces all numbers and figures in your report. Make sure that one can run your code and see your results without any error. Name the MATLAB script as A3.m.

Table 1: Points per question

Question:	1	2	3	4	Total
Points:	3	3	6	3	15

1. [3p] **Constrained optimization**

   Consider the optimization problem

$$\begin{aligned} &\min_x f(x) \\ \text{s.t.} \quad & g(x) \leq 0 \\ & h(x) = 0 \end{aligned} \tag{1}$$

   where

$$\begin{aligned} f &: \mathbb{R}^n \to \mathbb{R} \\ g &: \mathbb{R}^n \to \mathbb{R}^m \\ h &: \mathbb{R}^n \to \mathbb{R}^q \end{aligned}$$

   and let x_1 and x_2 be feasible points.

   (a) What is a convex/strictly convex function?

   (b) What is a convex set?

   (c) Under what conditions on functions $f,\ g,\ h$, (1) becomes a convex optimization problem?

2. [3p] **Convexity**

   Which of the following sets are convex? Motivate your answers.
   Note: Use definition of a convex set and prove the convexity mathematically (not verbally) or give an example that contradicts convexity of the set.

   (a) The set

$$S_1 = \{x \in \mathbb{R}^n | \alpha \leq a^\top x \leq \beta\}.$$

   (b) The set

$$S_2 = \{x | \ \|x - y\| \leq f(y) \text{ for all } y \in S\},$$

   where $S \subseteq \mathbb{R}^n$, $f(y) \geq 0$, and $\|.\|$ is an arbitrary norm.
   Hint: one may use the triangle inequality, i.e., $\|a + b\| \leq \|a\| + \|b\|$.

   (c) The set

$$S_3 = \{(x, y) | \ y \leq 2^x, \quad \text{for all } (x, y) \in \mathbb{R}^2\}.$$

3. [6p] **Norm problems as linear programs**

   Linear programs have the general form:

$$\begin{aligned} &\min_z \ c^\top z \\ \text{s.t.} \quad & Fz \leq g \end{aligned} \tag{2}$$

   and one can use available solvers to find the solution for the optimization problems that are presented in this form. For instance, one can solve

$$\min_x \|Ax - b\|_\infty \tag{3}$$

   using linear programs. Note that the infinity norm is defined as

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}, \quad \forall x \in \mathbb{R}^n. \tag{4}$$

   In order to solve (3) using linear programs, one can write

$$\begin{aligned} &\min_{x,\epsilon} \epsilon \\ \text{s.t.} \quad & -\epsilon \leq (Ax - b)_i \leq \epsilon, \quad \forall i \in \{1, ..., n\}, \end{aligned} \tag{5}$$

   where $(Ax - b)_i$ means element i of the vector $(Ax - b)$.

(a) Explain briefly why (5) and (3) yield the same results.

(b) Assuming $z^\top = [x^\top \ \ \epsilon]$, what are $c^\top$, F, g if one wants to represent (5) as in (2).

(c) Consider

$$A = \begin{bmatrix} 0.4889 & 0.2939 \\ 1.0347 & -0.7873 \\ 0.7269 & 0.8884 \\ -0.3034 & -1.1471 \end{bmatrix}, \quad b = \begin{bmatrix} -1.0689 \\ -0.8095 \\ -2.9443 \\ 1.4384 \end{bmatrix},$$

and solve (3) using linear programs.

(d) Write the dual of (2) and explain how you arrived at this form.

(e) Use the numerical values of A and b and solve the dual problem.

(f) Use the solution of the dual problem to find the solution of the primal one. Motivate your approach and compare it with the primal solution that you found before.

Note: You should use `linprog` function in Matlab to solve the primal and the dual problems.

4. [3p] **Quadratic programming**

Consider the following QP problem:

$$\begin{aligned} \min_{x,u} f(x,u) &= \frac{1}{2}(x_1^2 + x_2^2 + u_0^2 + u_1^2) \\ \text{s.t.} \quad & 2.5 \le x_1 \le 5 \\ & -0.5 \le x_2 \le 0.5 \\ & -2 \le u_0 \le 2 \\ & -2 \le u_1 \le 2 \end{aligned} \tag{6}$$

resulting from a finite-time constrained optimal control problem for the SISO process

$$x_{k+1} = 0.4x_k + u_k,$$

with initial state $x_0 = 1.5$.

(a) Solve the QP using MATLAB.

(b) Do the KKT conditions hold at the solution found in (a)? Which constraints are active?

(c) What would happen in the optimization problem if we remove the lower bound on x_1, and what if we remove the upper bound on x_1? Why?

Note: You should use the MATLAB command `quadprog` for this question.