An Alternative Proof for the NP – completeness of a matching problem in Graph Theory

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Let us define the "Friend Matching" problem. There are n people. Each pair of people u and v either know or don't know each other. A "Friend Matching of size k" is a set of pairs $\{u_1, v_1\}$, $\{u_2, v_2\}$, ... , $\{u_k, v_k\}$ such that: (a) u_i and v_i know each other, (b) u_i is the only person that v_i knows amongst $\{u_1, v_1, \ldots , u_k, v_k\}$, (c) v_i is the only person that u_i knows amongst $\{u_1, v_1, \ldots , u_k, v_k\}$. The objective is to decide if there is a Friend Matching of size k. We can model this problem using an undirected graph G, where the vertices correspond to people, and the edges correspond to pairs who know each other. Let us define the decision problem FRIENDMATCHING = { $\langle$G,k$\rangle$: G has a friend matching of size k }. Traditionally, this problem can be tackled with a reduction from Independent Set. In this paper, we briefly describe an unique and elegant approach to prove the NP-completness of this problem.

Given a subset of vertices S, we can check in polynomial time that S is a Friend Matching of size k. For a yes instance, we use a Friend Matching of size k. And for a no instance, clearly no such set exist : $FriendMatching \in NP$. To show that $FriendMatching \in NP - hard$, consider a reduction from Minimum Vertex Cover to Friend Matching. MINVC $\leq_{\text{p}}$ FM. Claim: There is a Minimum Vertex Cover of size k in Graph G if and only if there is a Friend Matching of size k. To determine whether the vertices V are a Minimum Vertex Cover of G we can partition graph G into two parts: A and B. If n is even and there are $\frac{n}{2}$ distinct edges (hence a perfect matching) then take

$$k = min(|A|, |B|)$$

as the size of Minimum Vertex Cover (i.e. the size of Friend Matching for G). Alternatively, if there are missing vertices then we can label the unmatched vertices in A with S and let U be the set of vertices in A reachable from S by *alternating paths* (i.e. path reachable from S to U where subsequent edges alternate in and out of matching. We define the situation when an edge does not exist between two vertices as “out of matching”). Let T be the neighbor vertices of U. Thus G has a Minimum Vertex Cover of size

$$k = (A - S - U) \cup T$$

and therefore a Friend Matching of size k.