MAS-1
Study Review

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- Probability Review
- Stochastic Processes
- Life Contingencies
- Simulation
- Statistics
- Extended Linear Model
- Time Series

Lesson 1 : Probability Review

- **Bernouilli Shortcut :** If a random variable can only assume two values a and b with probability q and $1-q$, then is variance is $q(1-q)(b-a)^2$

Lesson 2 : Parametric Distributions

- **Transformations :**
  - Transformed : $\tau > 0$
  - Inverse : $\tau = -1$
  - Inverse-Transformed : $\tau < 0, \tau \neq 1$

Lesson 4 : Markov Chains

- **Chapman-Kolmogorov :**
$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$$
- **Gambler's ruin :**
$$p_j = \begin{cases} \frac{j}{N} & , r = 1 \\ \frac{r^j - 1}{r^N - 1} & , r \neq 1 \end{cases}$$
où $r = \dfrac{q}{p}$, p : winning prob.
- **Algorithmic efficency :** with N_j = number of steps from j^{th} solution to best solution.
$$\mathrm{E}[N_j] = \sum_{i=1}^{j-1} \frac{1}{i}$$
$$\mathrm{Var}(N_j) = \sum_{i=1}^{j-1} \left(\frac{1}{i}\right)\left(1 - \frac{1}{i}\right)$$
As $j \to \infty$, $\mathrm{E}[N_j] \to \ln j$, $\mathrm{Var}(N_j) \to \ln j$

Lesson 5 : Markov Chain Classification

- An **absorbing** state is one that cannot be exited.
- State j is **accessible**$(i \to j)$ from state i if $p_{ij}^n > 0$, $\forall n \geq 0$.
- Two states **communicate** if $i \leftrightarrow j$.
- A **class** of states is a maximal set of state that communicate with each other.
- A Markov chain is **irreductible** if it has only one class.
- A state (class) is **recurrent** if the probability of reentering the state is 1. $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$
- A state (class) si **transcient** if it is not recurrent. $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$
- A finite Markov Chain must have at least one recurrent class. If it is irreductible, then it is recurrent.

Lesson 6 : Markov Chains Limiting Probability

- A chain is **positive recurrent** is the expected number of transitions until the state occur is finite, **null recurrent** otherwise. Null recurrent mean that the long-term proportion of time in each state is 0.
- A chain is **periodic** when states occur every n periods for $n > 1$.
- A chain is **aperiodic** when the period is 1. In other world, $P_{ii}^{(1)} > 0, \forall i$
- A chain is **ergodic** when the chain is aperiodic and positive irreductible recurrent.
- **Stationary probability :**
$$\pi_j = \sum_{i=1}^{n} P_{ij}\pi_i \qquad \sum_{i=1}^{n} \pi_i = 1$$
- **Limiting probabilities :** if the chain is ergodic, then
$$\mathbf{P}^{(\infty)} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \\ \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

Lesson 7 : Time in Transient States

- Tips : Inverting a matrix
- $\mathbf{S} = (\mathbf{I} - \mathbf{P}_{\text{transcient}})^{-1}$, where s_{ij} is the time in state j given that the current state is i.
- $f_{ij} = \frac{s_{ij} - \delta_{i,j}}{s_{jj}} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$, where f_{ij} is the probability that state i ever transitions to state j.

Lesson 8 : Branching Processes

- A branching process is a special type of Markov chain representing the growth or extinction of a population.
- $\mathrm{E}[X_n] = \mathrm{E}[Z]^n$, where $\mathrm{E}[Z]$ is the expected number of people born in a generation.
- $\mathrm{Var}(X_n) = \mathrm{Var}(Z) \cdot \mathrm{E}[Z]^{n-1} \sum_{k=1}^{n} \mathrm{E}[Z]^{k-1}$
- If $X_0 \neq 1$ mean and variance of X_n need to be multiplicated by X_0.
- **Probability of extinction :**
$$\pi_0 = \sum_{j=1}^{\infty} p_j \pi_0^j$$
  - $\mu \leq 1 \Rightarrow \pi_0 \geq 1$,if $X_0 = 1$.
  - $\mu > 1 \Rightarrow \pi_0 < 1$,if $X_0 = 1$.

  For cubic equation, it guaranteed to factor $(\pi_0 - 1)$. Tips : Synthetic Division

Lesson 9 : Time Reversible

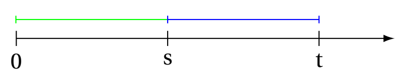
- If **Q** is the reverse-time Markov chain for ergodic **P**, then
$$\pi_i Q_{ij} = \pi_j P_{ji}$$
with $P_{ii} = Q_{ii}$ and if $p_{ij} = 0 \Leftrightarrow Q_{ji} = 0$
- If $\mathbf{Q} = \mathbf{P}$, then **P** is said to be **time-reversible**.

Lesson 10 : Exponential Distribution

- **Lack of memory :**
$$\Pr(X > k + x | X > k) = \Pr(X > x)$$
- **Minimum :** if $X_i \sim \mathrm{Exp}(\lambda_i)$, then
$$\min(X_1, X_2, ..., X_n) \sim \mathrm{Exp}\left(\sum_{i=1}^{n} \lambda_i\right)$$
- The sum of 2 Exponentials randoms variables is the sum of the maximum and the minimum, since one must be the min and the other the max.
$$X_1 + X_2 = \min(X_1, X_2) + \max(X_1, X_2)$$

Lesson 11 : Poisson Process

- $X(t) \sim \mathrm{Poisson}[m(t)]$,where $m(t)$ is **mean value function** representing the mean of the number events before time t.
- Poisson process can't deacrease over time. $N(t) \geq N(s)$
- $N(0) = 0$
- Increament are **independent** :



$$\Pr[N(t) - N(s) = n | N(s) = k] = \Pr[N(t) - N(s) = n]$$
- **Non-homogeneous Poisson process :**
$$m(t) = \int_0^t \lambda(u)\, \mathrm{d}u$$
where $\lambda(t)$ is the **intensity function**
- **Homogeneous Poisson process :** The Poisson process is said to be homogeneous when the intensity function is a constant.
$$m(t) = \int_0^t \lambda\, \mathrm{d}u = \lambda t$$
We then say that the process have **stationary increments**.
$$\Pr[N(s)] = \Pr[N(t) - N(s)]$$

Lesson 12 : Poisson Process Time To Next Events

- T_n is the time between the n[e] event and the (n-1)[e] event.
- $S_n = \sum_{i=1}^{n} T_i$, is the time for the n[e] event.
- $F_{T_1}(t) = 1 - e^{-\int_0^t \lambda(u)\, \mathrm{d}u}$
- For homogeneous process :
$$T_n \sim \mathrm{Exp}(\lambda)$$
$$S_n \sim \mathrm{Gamma}(n, \lambda)$$

Lesson 13 : Poisson Process Counting Special Type

- If event of type 1 occur with probability $\alpha_1(t)$, then the event follow a Poisson process with intensity $\lambda(t) \cdot \alpha_1(t)$.
$$m(t) = \int_0^t \lambda(u)\alpha_1(u)\,\mathrm{d}u$$

Lesson 14 : Poisson Process Other Characteristics

- Only for homogeneous Poisson processes.
- The probability of k event from process 1 is given by :
$$k \sim \text{Binomial}\left(k+l-1, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$
Then the probability that k event from process 1 occur before l from process 2 is :
$$\sum_{i=k}^{k+l-1} \binom{k+l-1}{i} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^i \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k+l-1-i}$$
- Given that exactly $N(t) = k$ Poisson events occured before time t, the joint distribution of event time is the joint distribution of k independent uniform random variables on $(0, t)$.
$$F_{S_1,\ldots,S_n|n(t)}(s_1,\ldots s_n|k) = \frac{k!}{t^k}$$
- For k independent uniform random variable on $(0, t)$, the expected value of the j^{th} order statistics is : $E[T^{(j)}] = \frac{jt}{(k+1)}$.
- Tips : Statistic Order

Lesson 15 : Poisson Process Sums and Mixtures

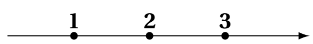
- A **Sums** of independent Poisson random variables is a Poisson random with intensify function $\lambda(t) = \sum \lambda_i(t)$. Warning : Substraction don't give a Poisson random variable.
- A **Mixture** of Poisson processes is not a Poisson processes.
  - **Discrete** mixture :
  $$F_{X(t)}(t) = \sum_i w_i F_{X_i(t)}(t)$$
  where $w_i > 0, \sum w_i = 1$
  - **Continuous** mixture :
  $$F_{X(t)}(t) = \int F_{\{X_u(t)\}}(t) f(u)\,\mathrm{d}u$$
  - If $N(t)|\lambda$ is a Poisson random variable and $\lambda \sim \text{Gamma}(\alpha, \theta)$, then $N(t) \sim \text{NegBin}(r = \alpha, \beta = \theta t)$.

Lesson 16 : Compound Poisson Processes

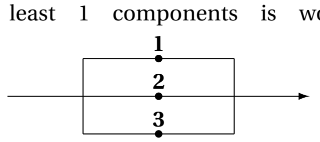
- A **compound** random variable S is define by $S = \sum_{i=1}^N X_i$ where N is the **primary** distribution and X the **secondary** distribution.
- If $N(t)$ is a Poisson process, then $S(t)$ is a compound Poisson process with :
  - $\mathrm{E}[S(t)] = \lambda t \mathrm{E}[X]$
  - $\mathrm{Var}(S(t)) = \lambda t \mathrm{E}\left[X^2\right]$
- If X_i is discrete, we can separate the process into a sum of subprocess view in Lesson 13 : Poisson Process Counting Special Type.
- **Sums of compound** homogeneous Poisson process is also a Poisson process with :
  - $N(t) \sim \text{Pois}(\sum \lambda_i)$
  - $F_X(x) = \sum_i w_i F_{X_i(t)}(t), \quad w_i = \frac{\lambda_i}{\sum \lambda_i}$

Lesson 17 : Reliability Structure Functions

- $\phi(\mathbf{x})$ is the **structure** function for a systeme. It equal 1 if the systeme function, 0 otherwise.
- A **series** system is define as a **minimal path set**. The system is working if all components are working.



The serie structure function is define as
$$\phi(\mathbf{x}) = \prod_{i=1}^n x_i$$
- A **parallel** system is define as a **minimal cut set**. The systeme is working if at least 1 components is working.



The parallel structure function is define as
$$\phi(\mathbf{x}) = 1 - \prod_{i=1}^n (1 - x_i)$$
- Tips : Minimal path set is all way for the system to work, and the minimal cut set is all the way for the system to not work.
- Tips : If set is $\{1,2,3\}$ and $\{1,2\}$, the *minimal* mean we only take $\{1,2\}$.
- Tips : *Minimal cut* is a serie of parallel structure and *minimal path* is a parallel of serie structure.

Lesson 18 : Reliability Probabilities

- $r(\mathbf{p})$ is the same polynomial as $\phi(\mathbf{x})$.
- Inclusion/exclusion bounds using minimal path :
$$r(\mathbf{p}) \le \sum A_i$$
$$r(\mathbf{p}) \ge \sum A_i - \sum A_i \cup A_j$$
$$r(\mathbf{p}) \le \sum A_i - \sum A_i \cup A_j + \sum A_i \cup A_j \cup A_k$$
where $A_i = \sum p_i$ is the probability of the i^{e} minimal path set work.
- Inclusion/exclusion bounds using minimal cut :
$$1 - r(\mathbf{p}) \le \sum A_i$$
$$1 - r(\mathbf{p}) \ge \sum A_i - \sum A_i \cup A_j$$
$$1 - r(\mathbf{p}) \le \sum A_i - \sum A_i \cup A_j + \sum A_i \cup A_j \cup A_k$$
where $A_i = \sum(1 - p_i)$ is the probability of the i^{e} minimal cut set work.
- Bounds using intersections :
$$\prod \phi(\mathbf{X})^{\textbf{min. cut}} \le r(\mathbf{p}) \le \prod \phi(\mathbf{X})^{\textbf{min. path}}$$
- **Random graph** :
$$1 - P_n = \sum_{k=1}^{n-1} \binom{n-1}{k-1} q^{k(n-k)} P_k$$
$$1 - P_n \le (n+1)q^{n-1}$$
$$P_1 = 1$$

Lesson 19 : Reliability Time to Failure

- Expected amound of time to failure :
$$\mathrm{E}\left[\textbf{system life}\right] = \int_0^\infty r(\bar{\mathbf{F}}(\mathbf{t}))\,\mathrm{d}t$$
where,
  - For serie system :
  $$r(\bar{\mathbf{F}}(\mathbf{t})) = \prod_{i=1}^n \bar{F}_i(t)$$
  - For parallel system :
  $$r(\bar{\mathbf{F}}(\mathbf{t})) = 1 - \prod_{i=1}^n F_i(t)$$
- *Shortcut* : k out of n system with exponentials(θ) : $\mathrm{E}[T] = \theta \sum_{i=k}^n \frac{1}{i}$
- **Hazard rate function** (*failure rate function*) :
$$h(t) = \frac{f(t)}{\bar{F}(t)}$$
and we say that the distribution
  - is an increasing failure rate if $h(t)$ is non-deacreasing function of t.
  - is an deacreasing failure rate if $h(t)$ is non-increasing function of t.
- **Cumulatice hazard function** :
$$H(t) = \int_0^t h(u)\,\mathrm{d}u = -\ln \bar{F}(t)$$
with $\frac{H(t)}{t}$ the average of the hazard rate.

Lesson 20 : Survival Models

- ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}, \quad {}_tq_x = \frac{\ell_x - \ell_{x+t}}{\ell_x}$
- ${}_{t|u}q_x = \frac{\ell_{x+t} - \ell_{x+t+u}}{\ell_x}$
- ${}_{t+u}p_x = {}_up_x \cdot {}_tp_{x+u}$
- ${}_{t|u}q_x = {}_{t+u}q_x - {}_tq_x = {}_tp_x \cdot {}_uq_{x+t}$
- Let be N_x the number of life surviving to age x, then
$$(N_{x+t}|N_x = n) \sim \text{Bin}(n, {}_tp_x)$$
- **Force of mortality** :
$$\mu_{x+t} = \frac{f_{T_x}(t)}{{}_tp_x} = -\frac{\mathrm{d}}{\mathrm{d}t} \ln {}_tp_x$$

- **Linear interpolation(D.U.D) :**
$$\ell_{x+t} = (1-t)\,\ell_x + t\,\ell_{x+1}$$
Shortcut : $\forall\, t \in (0,1),\ \forall\, x \in \mathbb{N},\ x < x+t < x+1$:
  - ➡ ${}_tq_x = t \cdot q_x$
  - ➡ $\mu_{x+t} = \frac{q_x}{1-t\cdot q_x}$
- **Expected life time :** Let $k_x = \lfloor T_x \rfloor$, the *full years* until death. Then e_x is the **curtate life expectancy** and $\mathring{e}_x$ the **complete life expectancy**. ω is the age where $\ell_\omega = 0$ and $\omega = \infty$ by convention is nothing is said.
$$e_x = \mathrm{E}[K_x] = \sum_{k=1}^{\omega-x-1} {}_kp_x$$
$$\mathring{e}_x = \mathrm{E}[T_x] = \int_0^{\omega-x} {}_tp_x\,\mathrm{d}t \overset{\text{D.U.D}}{=} e_x + 0.5$$

Lesson 21 : Contingent Payments

The contract here are define with K_x to pay at the end of death year. All same contract can be define with T_x to pay at the moment of death. Then we use integral instead of sum and use
$$\Pr(K=k) = {}_kp_x\,q_{x+k} \Rightarrow f_{T_x}(t) = {}_tp_x\mu_{x+t}$$

- **Life Insurance :**
  - Whole Life insurance :
  $$A_x = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x\,q_{x+k}$$
  - Term Life insurance :
  $$A^{1}_{x:\overline{n}|} = \sum_{k=0}^{n} v^{k+1}\,{}_kp_x\,q_{x+k}$$
  - Deferred insurance :
  $${}_{m|}A_x = \sum_{k=m}^{\infty} v^{k+1}\,{}_kp_x\,q_{x+k}$$
  - Endowment insurance :
  $$A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + {}_nE_x$$
  - Pure Endowment :
  $${}_nE_x = v^n\,{}_np_x$$
- **Life Annuities :**
  - Whole Life annuity
  $$\ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x$$
  - Temporary Life annuity
  $$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n} v^k\,{}_kp_x$$
  - Deferred annuity
  $${}_{m|}\ddot{a}_x = \sum_{k=m}^{\infty} v^k\,{}_kp_x$$
  - Certain and life annuity
  $$\ddot{a}_{\overline{\overline{x:\overline{n}|}}} = \ddot{a}_{\overline{n}|} + {}_{m|}\ddot{a}_x$$
- **Illustrative Life Table :**
  - $A_x = v^n\,q_x + p_x\,A_{x+1}$
  - $\ddot{a}_x = 1 + v\,p_x\,\ddot{a}_{x+1}$
  - $A^{1}_{x:\overline{n}|} = A_x - {}_nE_x\,A_{x+n}$
  - $\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - {}_nE_x\,\ddot{a}_{x+n}$
  - ${}_{m|}A_x = {}_mE_x\,A_{x+m}$
  - ${}_{m|}\ddot{a}_x = {}_mE_x\,\ddot{a}_{x+m}$
  - $\ddot{a}_x = 1 + a_x$
  - $A_x = 1 - d\,\ddot{a}_x$

- **Joint life annuity**($\ddot{a}_{xy}$) make payments until the earliest death pf two lives.
- **Last survivor annuity**($\ddot{a}_{\overline{xy}}$) make payments until the last death of two lives.
$$\ddot{a}_x + \ddot{a}_y = \ddot{a}_{xy} + \ddot{a}_{\overline{xy}}$$
- **Premiums :**
$$M \cdot A_x = P\,\ddot{a}_x$$
$$P = \frac{M\cdot A_x}{\ddot{a}_x} = \frac{M}{\ddot{a}_x} - M\cdot d$$

Lesson 22 : Simulation Inverse Method

- **Linear congruential generators :**
$$x_k = (a x_{k-1} + c) \bmod m$$
$$x_k = b - \left\lfloor \frac{b}{m} \right\rfloor m$$
where $b = (a x_{i-k} + c)$ and $x_0 \equiv$ seed
- **Inverse transformation method :**
$\Pr\left(F^{-1}(u) \le x\right) = \Pr(u \le F(x)) = F(x)$
then $x = F^{-1}(u)$ where $U \sim \mathrm{Unif}(0,1)$
  - Normal Case : $x = \mu + \sigma z$
  - Log-Normal Case : $x = e^{\mu+\sigma z}$

  where $z = \Phi^{-1}(u)$, with linear interpolation.
- Tips : Discrete Cumulative Function
- Tips : if $\Uparrow U \equiv \Downarrow X$ then $(1-u_i) \Rightarrow u_i$

Lesson 23 : Simulation Application

- $\Pr(X \le x) \simeq \frac{1}{m}\sum_{j=1}^{m} \mathbb{1}_{\{x^{(j)} \le x\}}$
- $\mathrm{E}[X^k] \simeq \frac{1}{m}\sum_{j=1}^{m} [x^{(j)}]^k$
- $\mathrm{VaR}_k(X) \simeq X^{[j]}$
- $$\mathrm{TVaR}_k(X) \simeq \frac{1}{m(1-k)} \sum_{j=j_0+1}^{m} X^{(j)}\mathbb{1}_{\{X^{(j)} > X^{[j_0]}\}} \simeq \frac{1}{m-j_0}\sum_{j=j_0+1}^{m} X^{[j]}$$
where
  - $j_0 = \lfloor m \cdot k \rfloor$
  - m is the number of simulations.
  - $X^{(j)}$ is the j^th simulations.
  - $X^{[j]}$ is the j^th simulations in order statistics.

Lesson 24 : Simulation Rejection Method

- **General method :** Let $f(x)$ be the density function of variable to simulate, and let $g(x)$ be the **base distribution**, a random density function that is easy-to-simulate with nonzero wherever $f(x) \neq 0$.
$$c = \max \frac{f(x)}{g(x)}$$
Generate two uniform number u_1, u_2. Let $x = G^{-1}(u_i)$. Accept x_1 only if
$$u_2 \le \frac{f(x_1)}{c\cdot g(x_1)}$$
- **Simulating gamma distribution :** Use $\mathrm{Exp}(\alpha\cdot\theta)$ as the base distribution and $x = \alpha\cdot\theta$ that maximize c.
- **Simulating standard normal distribution :** Generate 3 uniform u_1, u_2, u_3. Let $y_1 = -\ln u_2$ and $y_2 = -\ln u_2$. Accept y_1 if
$$y_2 \ge \frac{(y_1-1)^2}{2}$$
and add $(-)$ if $u_3 \ge 0.5$
- The **Number of iteration** is a Ross-geometric distribution with mean c. Let be β the mean of a geometric distribution given in the exam appendix :
$$\mathrm{E}[N] = 1 + \beta = c$$
$$\mathrm{Var}(N) = \beta(1+\beta)$$

Lesson 25 : Estimator Quality

- **Bias :** This quality measures if, on average, the estimator is on the expected value of the parameter.
$$\mathrm{E}[\hat{\theta}] = \theta + \mathrm{bias}_{\hat{\theta}}(\theta)$$
  - If $\mathrm{bias}_{\hat{\theta}}(\theta) = 0$, then $\hat{\theta}$ is **unbiased**.
  - If $\lim_{n\to\infty} \mathrm{bias}_{\hat{\theta}}(\theta) = 0$, then $\hat{\theta}$ is **asymptotically unbiased**.
  - If $\mathrm{bias}_{\hat{\theta}}(\theta) \neq 0$, then $\hat{\theta}$ is **biased**.
- **Consistency :** This quality measures if the probability that the estimator is different from the parameter by more than ε goes to 0 as n goes to infinity.
$$\lim_{n\to\infty} \Pr\left(|\hat{\theta}-\theta| > \varepsilon\right) \to 0,\ \forall \varepsilon > 0$$
In other word, as $n \to \infty$, $\mathrm{E}[\hat{\theta}] \to \theta$, $\mathrm{Var}(\hat{\theta}) \to 0$
- **Efficiency :** This quality measures the variance of the estimator. Lower the variance is, more efficient is the estimator.
$$\text{Efficiency of } \hat{\theta} = \frac{\mathrm{Var}(\hat{\theta})^{\text{rao}}}{\mathrm{Var}(\hat{\theta})}$$
$$\text{Relative efficiency of } \hat{\theta}_1 \text{ to } \hat{\theta}_2 = \frac{\mathrm{Var}(\hat{\theta}_2)}{\mathrm{Var}(\hat{\theta}_1)}$$
See the rao-cramer lower bound.
- **Mean Square Error :** This quality measures the expected value of the square difference between the estimator and the parameter.
$$\mathrm{MSE}_{\hat{\theta}}(\theta) = \mathrm{E}\left[(\hat{\theta}-\theta)^2\right] = \left(\mathrm{bias}_{\hat{\theta}}(\theta)\right)^2 + \mathrm{Var}(\hat{\theta})$$
- An estimator is called a **uniformly minimum variance unbiased estimator(UMVUE)** if it's unbiased and if there is no other unbiased estimator with a smaller variance for any true value θ.
- Some estimator :
  - $\bar{x} = \frac{1}{n}\sum x_i$ is a unbiased estimator of the mean μ. $\mathrm{Var}(\bar{x}) = \frac{1}{n}\mathrm{Var}(x)$

- $s^2 = \sum \frac{(x_i-\bar{x})^2}{n-1}$ is a unbiased estimator of the variance σ^2.
- $\hat{\sigma}^2 = \sum \frac{(x_i-\bar{x})^2}{n}$ is an asymptotically unbiased of the variance σ^2.
- $\hat{\mu}'_k = \frac{1}{n}\sum x_i^k$, where $\hat{\mu}'_1 = \bar{x}$ and $\hat{\mu}_k = \frac{1}{n}\sum(x_i-\bar{x})^k$, where $\hat{\mu}_1 = 0$ and $\hat{\mu}_2 = \hat{\sigma}^2$.

Lesson 26 : Kernel Density Estimation

- **Empirical distribution :** All data is assigning a probability of $\frac{1}{n}$. This is the same method used for simulation, see Lesson 23 : Simulation Application.

$$F_e(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{x_i \le x\}}$$
$$f_e(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{x_i = x\}} = F_e(x) - F_e(x_{i-1})$$

- **Kernel Density** is a empirical distribution smoothed with a base fonction. Let define the scaling factor b called **bandwith**.
  - The kernel-density estimate of the density function is : $\hat{f}(x) = \frac{1}{n}\sum k\left(\frac{x-x_i}{b}\right)$ $\Leftrightarrow \sum f_e(x) k\left(\frac{x-x_i}{b}\right)$
  - The kernel-density estimate of the distribution is : $\hat{F}(x) = \frac{1}{n}\sum K\left(\frac{x-x_i}{b}\right)$
- **Rectangular(uniform, box) kermel :**

$$k(x) = \begin{cases} \frac{1}{2b}, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$
$$K(x) = \begin{cases} 0, & x < -1 \\ 0.5(x+1), & -1 \le x \le 1 \\ 1, & x > 1 \end{cases}$$
$$\hat{f}(x) = \frac{F_e(x+b) - F_e(x-b^-)}{2b}$$

- **Triangular kernel :**

$$k(x) = \begin{cases} \frac{1-|x|}{b}, & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$
$$K(x) = \begin{cases} 0, & x < -1 \\ \frac{(1+x)^2}{2}, & -1 \le x \le 0 \\ 1 - \frac{(1-x)^2}{2}, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

- **Gaussian kernel :** The distribution become normal with $\mu = x_i$ and $\sigma = b$.

$$k(x) = \frac{e^{-x^2/2}}{b\sqrt{2\pi}}$$
$$K(x) = \Phi(x)$$

- Other kernel : $k(x) = \beta(x)$ and $k(x) = B(x)$
- **kernel moments :** Let X be the kernel density estimate and x_i the empirical estimate. We then condition on x_i.

$$\mathrm{E}[X] = \mathrm{E}\big[\mathrm{E}[X|x_i]\big] = \mathrm{E}[x_i]$$
$$\mathrm{Var}(X_R) = \mathrm{Var}(x_i) + \frac{b^2}{3}$$
$$\mathrm{Var}(X_T) = \mathrm{Var}(x_i) + \frac{b^2}{6}$$
$$\mathrm{Var}(X_G) = \mathrm{Var}(x_i) + b^2$$

- Tips : For rectangular kernel, $\mathrm{E}[x|x_i]$ is a uniform$(x_i - b, x_i + b)$.

Lesson 27 : Method of Moments

- **Types of data :**
  - Complete data : Data is complete if we are given the exact value of each observation.
  - Grouped data : Set of interval and we know how many observation are in each.
  - Censored data : Value that are in a interval, but we don't know the exact value. Like limits ($\min(X,u)$).
  - Truncated data : We have data only when it in certain range, otherwise we don't know. Like deductible ($X|X>d$).
- **Method of Moments :** We match $\hat{\mu}'_k = \mathrm{E}\left[X^k\right]$ and find the parameters. If data is Censored or Truncated, we need to match the censored or truncated moment : $\hat{\mu}'_k = \mathrm{E}\left[\min(X,u)^k\right]$ or $\hat{\mu}'_k = \mathrm{E}\left[X^k|X>d\right]$.
- For pareto distribution, if $\hat{\mu}'_2 = \hat{\sigma}^2 + \bar{x}^2 \le 2\bar{x}^2$, the method of moment is unstable and can't be used.

Lesson 28 : Percentile Matching

- **Percentile Matching :** We match $F_e(\hat{\pi}_p) = p$ and find the parameters.
  - For censored data, we need select percentiles within the range of the uncensored portion of the data.
  - For truncated data, we need to match the percentiles of the conditional distribution :
    $F(x|X>d) = \frac{\Pr(d<X\le x)}{\Pr(X>d)} = \frac{F(x)-F(d)}{1-F(d)}$
    $S(x|X>d) = \frac{S(x)}{S(d)}$
- **Smoothed empirical percentile :**

$$\hat{\pi}_p = (1-h)X^{[j]} + hX^{[j+1]}$$

where
  - $j = \lfloor (n+1)p \rfloor$
  - $h = (n+1)p - j$
  - $X^{[j]}$ is the j[th] order statistics.

Lesson 29 : Maximum Likehood Estimators

- **Maximum Likehood Estimators :** We maximize the probability of observing the data.

$$L(\theta) = \prod g(x_i;\theta)$$
$$l(\theta) = \sum \ln g(x_i;\theta)$$

  - Individual data : $g(x_i;\theta) = f(x_i)$
  - Grouped data : $g(x_i;\theta) = F(x_i) - F(x_{i-1})$
  - Censored data : $g(x_i;\theta) = S(x_i)$
  - Truncated data : $g(x_i;\theta) = \frac{f(x)}{s(x)}$

Lesson 30 : MLE Special Techniques

- Case MLE equals MME
  - For Exponential, $\hat{\theta}^{\mathrm{MLE}} = \bar{x}$
  - For Gamma with fixed α, $\hat{\theta}^{\mathrm{MLE}} = \hat{\theta}^{\mathrm{MME}}$
  - For Normal, $\hat{\mu}^{\mathrm{MLE}} = \bar{x}$ and $(\hat{\sigma}^2)^{\mathrm{MLE}} = \frac{1}{n}\sum(x_i - \hat{\mu})^2$
  - For Binomial, $mq = \bar{x}$ then given m, $\hat{q}^{\mathrm{MLE}} = \frac{\bar{x}}{m}$
  - For Poisson, $\hat{\lambda}^{\mathrm{MLE}} = \hat{\lambda}^{\mathrm{MME}}$
  - For Binomial Negative, given r or β, $(r\beta)^{\mathrm{MLE}} = \bar{x}$
- Parametrization and Shifting :
  - Parametrization : MLE's are independent of parametrization
    $\lambda = \frac{1}{\theta} \Leftrightarrow \hat{\lambda}^{\mathrm{MLE}} = \frac{1}{\hat{\theta}^{\mathrm{MLE}}}$
  - Shifting the distribution is equivalent of shifting the data.
- Transformations : MLE's are invariant under one-to-one transformation. Then if we have a transformed variable, we can untransform the data and find the parameter of the untransform distribution.
  Tips : Transformations of distribution
- Weibull distribution : If the data is censored(u) or truncated(d), then

$$\left(\hat{\theta}^{\mathrm{MLE}}\right)^{\tau} = \frac{\sum (x_i - d_i)^{\tau}}{\sum \mathbb{1}_{\{x_i \le u\}}}$$

if $\tau = 1$, then the distribution is Exponential.
- Pareto distribution with fixed θ : $\hat{\alpha} = \frac{n}{K}$

$$K = \sum_{i=1}^{n+c} \ln(\theta + d_i) - \sum_{i=1}^{n+c} \ln(\theta + x_i)$$

where $n \equiv$ number of non-censored(c) data.
- Single-parameter Pareto : $\hat{\alpha} = \frac{n}{K}$

$$K = \sum_{i=1}^{n+c} \ln\max(\theta, d_i) - \sum_{i=1}^{n+c} \ln x_i$$

where $n \equiv$ number of non-censored(c) data.
- Uniform$(0,\theta)$: We take the smalest θ possible, $\hat{\theta}^{\mathrm{MLE}} = \max(x_1, \ldots, x_n)$
  - Censored(u) : $\hat{\theta}^{\mathrm{MLE}} = \frac{nd}{\sum \mathbb{1}_{\{x_i < d\}}}$

  - Grouped : We take the heighest interval(L, U). $\hat{\theta}^{\text{MLE}} = \min(U, \hat{\theta}^{\text{MLE}}_{\text{Censored(L)}})$
- Bernouilli : Let have a random variable that can take 2 values, n and m. Then
$$\hat{p} = \frac{n}{n+m}$$
- Tips : If $L(\theta)$ look like a density distribution, $\hat{\theta}^{\text{MLE}} \equiv$ mode of this distribution.

Lesson 31 : Variance of MLE

- **Fisher information matrix** :
$$I(\theta) = n\mathrm{E}\left[\left(\frac{\mathrm{d}\ln f(x;\theta)}{\mathrm{d}\theta}\right)^2\right]$$
$$= -n\mathrm{E}\left[\frac{\mathrm{d}^2 \ln f(x;\theta)}{\mathrm{d}\theta^2}\right]$$
using the loglikehood function
$$I(\theta) = \mathrm{E}\left[\left(\frac{\mathrm{d}l(x_1,\ldots,x_n;\theta)}{\mathrm{d}\theta}\right)^2\right]$$
$$= -\mathrm{E}\left[\frac{\mathrm{d}^2 l(x_1,\ldots,x_n;\theta)}{\mathrm{d}\theta^2}\right]$$
- **Rao-Cramer lower bound** is the lowest possible variance for a unbiased estimator $\hat{\theta}$ of θ. Then $\hat{\theta} \sim \text{Normal}(0, \text{Var}(\hat{\theta})^{\text{rao}})$
$$\text{Var}(\hat{\theta})^{\text{rao}} \geq \frac{1}{I(\theta)}$$
under certains regularity conditions
  - The seconde derivative of the loglikehood exist.
  - The support of the density function is not function of θ.

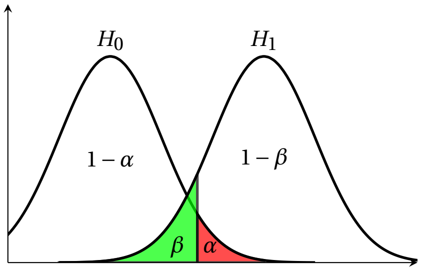
Lesson 32 : Sufficient Statistics

- A **sufficient statistics** are statistics that capture all the information about the parameter we are estimating that the sample as to offer.
- A statistics is sufficient when the distribution of a sample given a statistics does not depend on the parameter. Y is a sufficient statistics for a parameter θ if and only if
$$L(x_1,\ldots,x_n;\theta|Y) = h(x_1,\ldots,x_n)$$
$$L(x_1,\ldots,x_n;\theta) = g(y)h(x_1,\ldots,x_n)$$
where $h(x_1,\ldots,x_n)$ is a function that does not involve θ.
- **Rao-Blackwell** Theorem : For any unbiased estimator $\hat{\theta}$ and sufficient statistic Y, the estimator $\mathrm{E}[\hat{\theta}|Y]$ is unbiased and has variance less than or equal to $\text{Var}(\hat{\theta})$.
- The maximum likehood estimator is a function of a sufficient statistic.

Lesson 33 : Hypothesis Testing

- Let be H_0 the **null hypothesis** and H_1 the **alternative hypothesis**.

	Accept H_0	Reject H_0
H_0 True	$1-\alpha$	α
H_1 True	β	$1-\beta$



- The α value is usuly name :
  - Probability of Type I error
  - Size of critical region
  - signifiance level

  The β value is usuly name :
  - Probability of Type II error

  The $(1-\beta)$ value is usuly name :
  - The power of test.
- We will reject H_0 in favor of H_1 if a certain condition occured $(X > \gamma)$, named the **critical region**. Then the probability of rejecting H_0 is giving by
$$\Pr\left(X > \gamma | H_0 \equiv \text{true}\right) = \alpha$$
- Lowering the probability of type I error came at the cost of raising the probability of type II error. One way to lower both is to increase sample size.
- The **p-value** is the probability of being greater or equal to the observation if H_0 is true. H_0 is rejected if and only if the p-value is less then the significance level.
$$P_{\text{value}} < \alpha$$

Lesson 34 : Confidence Interval and Sample Size

- Let be c the **confidence coefficient**. Then we can say the we're $100c\%$ confident that the parameter is between (a, b), called the **confidence interval**. $\alpha = 1 - c$
$$\theta \in \hat{\theta} \pm z_{\frac{1+c}{2}} \sqrt{\text{Var}(\hat{\theta})}$$
- We can found the probability that the half-width of the interval is less then k.
$$\Pr\left(|\hat{\theta} - \theta| \leq k\right) \geq \frac{1+c}{2}$$
$$\Phi\left(\frac{k}{\sqrt{\sigma^2/n}}\right) \geq \frac{1+c}{2}$$
- To find the sample size needed to have a certain (α) and $(1-\beta)$, we resolve
$$\Pr(\bar{x} > k | H_0) = 1 - \Phi\left(\frac{k - \mu_0}{\sqrt{\sigma^2/n}}\right) = \alpha$$
$$\Pr(\bar{x} > k | H_1) = 1 - \Phi\left(\frac{k - \mu_1}{\sqrt{\sigma^2/n}}\right) = 1 - \beta$$

Lesson 35 : Confidence Intervals for Means

- The **chi-sqare** is a one-parameter family distribution. In definition, it a gamma with $\alpha = \frac{n}{2}$ and $\theta = 2$. The only parameter n is called **degree of freedom**.
  - Let $X_i, i = i, \ldots, n$ be normal random variable with mean μ and varianve σ^2.
$$Y = \sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2_{(n)}$$
  - Let $x_i, i = i, \ldots, n, n \geq 2$ be random sample from normal distribution with variance σ^2.
$$W = \sum_{i=1}^{n} \frac{(x_i - \bar{x})^2}{\sigma^2} \sim \chi^2_{(n-1)}$$
- The **strudent** is a one-parameter family distribution. We define it as
$$T_{(n)} = \frac{Z}{\sqrt{W/n}}$$
where $Z \sim \mathrm{N}(0,1)$ and $W \sim \chi^2_{(n)}$.
Note that as $n \to \infty$, $T_{(n)} \to N(0,1)$
- When the variance is unknow, we need to estimate it with the unbiased estimator S^2.
$$T_{(n-1)} = \frac{\bar{x} - \mu}{\sqrt{S^2/n}}$$
- Testing the difference of means from two population.
$$x_1, \ldots, x_n \sim N(\mu_x, \sigma_x^2)$$
$$y_1, \ldots, y_m \sim N(\mu_y, \sigma_y^2)$$
$$T_{(n+m-2)} = \frac{(\bar{x} + \bar{y}) - (\mu_x - \mu_y)}{S\sqrt{\frac{1}{n} + \frac{1}{m}}}$$
where $S^2 = \frac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}$.
- Testing for mean of bernouilli population. Let p_0 the probability on H_0.
$$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$

Lesson 36 : Kolmogorov-Smirnov Tests

- The **Kolmogorov-Smirnov test** is one methode for determining how well a parametric model fits its data. This test is only appropriate for continuous distribution.
$$D = \max|F_e(x) - F^*(x;\hat{\theta})|$$
where $d \leq x \leq u$ and $F^*(x) = \frac{F(x) - F(u)}{S(d)}$.

x_i	$F^*(x_i)$	$F_e(x_i^-)$	$F_e(x_i)$	max
x_1	0.5	0.2	0.6	0.3
⋮	⋮	⋮	⋮	⋮

Lesson 37 : Chi Square Test

- The **Chi Square** look for equality of means between k group. Let O_i be the observation and $E_i = np_i$ the expected on each group.
  $$H_0 : \mu_1 = ... = \mu_k$$
  $$Q = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} = \sum_{i=1}^{k} \left(\frac{O_i^2}{E_i} \right) - n \sim \chi^2_{(k-1-\theta')}$$
  Note : This test can be use to test the fit of as parametric model. $\theta^{'}$ is the number of parameter fited with the same data as the test.
- **Two-dimensional chi-square :**
  $$Q = \sum_{i=1}^{k} \sum_{j=1}^{c} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2_{(k-1)(c-1)}$$

Lesson 38 : Confience Interval for Variances

- To find a confidence interval for the variance, we need the following statistic.
  $$W = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$
- The **Fisher** distribution is define as
  $$F_{(r_1,r_2)} = \frac{W_1/r_1}{W_2/r_2}$$
  where r_1 and r_2 are the degree of freedom.
- To find a confidence interval for variance ratio, we need the following statistic.
  $$F_{(r_1,r_2)} = \frac{S_y^2/\sigma_y^2}{S_x^2/\sigma_x^2}$$

Appendix

Inverting a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ajouter pour une matrice 3x3

Synthetic Division

Exemple : Factorize $x^3 - 12x^2 - 81$

	1	-12	0	-81
3		3	-27	-81
	1	-9	-27	0

then, $x^3 - 12x^2 - 81 = (x-3)(x^2 - 9x - 27)$

Deductible and Limite

$$\begin{aligned} X &= \min(X;d) + \max(0; X-d) \\ \mathrm{E}[X] &= \mathrm{E}[\min(X;d)] + \mathrm{E}[\max(0;X-d)] \\ &= \mathrm{E}[(X \wedge d)] + \mathrm{E}[(x-d)_+] \\ &= \mathrm{E}[(X \wedge d)] + e_X(d) \cdot S_X(d) \end{aligned}$$

Statistic Order

- $Y_1 = \min(X_1, ..., X_n)$
$$f_{Y_1}(y) = n f(y)[S(y)]^{n-1}$$
$$S_{Y_1}(y) = \prod_{i=1}^{n} \Pr\left(X_i > x\right)$$
- $Y_n = \max(X_1, ..., X_n)$
$$f_{Y_n}(y) = n f(y)[F(y)]^{n-1}$$
$$F_{Y_n}(y) = \prod_{i=1}^{n} \Pr\left(X_i \leq x\right)$$
- $Y_k \in (Y_1, ..., Y_k, ..., Y_n)$
$$f_{Y_k}(y) = \frac{n! \cdot f(y)[F(y)]^{k-1}[S(y)]^{n-k}}{(k-1)!(n-k)!}$$
$$\begin{aligned} F_{Y_k}(y) &= \Pr\left(\text{at least k of n } X_i \text{ are} \leq \text{y}\right) \\ &= \sum_{i=k}^{n} \binom{n}{i} [F(y)]^i [S(y)]^{n-j} \end{aligned}$$
- $x + y = \min(x, y) + \max(x, y)$, since one is for sure the max and the other the min.

Mode : Most likely probability

- $g(x) = f(x)$ or some time $g(x) = \ln f(x)$
- **Mode** is the x that respects : $g'(x) = 0$

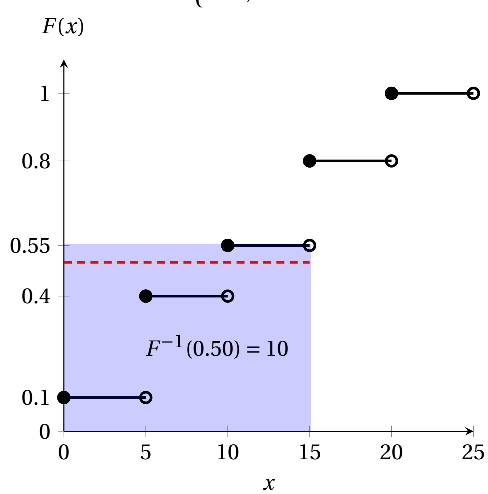
Normal Approximation

- $F_X(x) = \Phi\left(\frac{X - \mathrm{E}[X]}{\sqrt{\mathrm{Var}(X)}}\right)$
- **Continuity correction** is necessary when X is discrete. $F_X(x) = \Phi\left(\frac{(X \pm k) - \mathrm{E}[X]}{\sqrt{\mathrm{Var}(X)}}\right)$ where k is the mid-point of the discrete value.

Discrete Cumulative Function

$$\Pr(X = x) = \begin{cases} 0.10, & x = 0 \\ 0.30, & x = 5 \\ 0.15, & x = 10 \\ 0.25, & x = 15 \\ 0.20, & x = 20 \end{cases}$$

$$\Pr(X \leq x) = \begin{cases} 0.10, & 0 \leq x < 5 \\ 0.40, & 5 \leq x < 10 \\ 0.55, & 10 \leq x < 15 \\ 0.80, & 15 \leq x < 20 \\ 1, & x \geq 20 \end{cases}$$



Contract

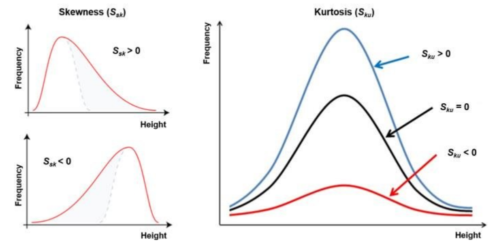
- **Deductible(d)**
- **Maximum(u)**
- **Inflation(r)**
- **Coinsurance(α)**

$$Y = \begin{cases} 0 & x \leq \frac{d}{1+r} \\ \alpha[(1+r)x - d] & \frac{d}{1+r} < x < \frac{u}{1+r} \\ \alpha[u-d] & x \geq \frac{u}{1+r} \end{cases}$$

Warning : The maximal don't include the deductible.

Moments

- kᵉ moment about the origin. $\mu'_k = \mathrm{E}\left[X^k\right]$
- kᵉ moment about the mean. $\mu_k = \mathrm{E}\left[(X-\mu)^k\right]$
- The **Skewness** moment give infomation about the asymmetry of the distribution. If $S_{sk} = 0$, the distribution is normal.
$$S_{sk} = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma^2}\right)^3\right]$$



- The **kurtosis** moment give infomation about the flattening of the distribution. If $S_{ku} = 0$, the distribution is normal.
$$S_{ku} = \mathrm{E}\left[\left(\frac{X-\mu}{\sigma^2}\right)^4\right]$$
- The **coefficient of variation** give information about the dispersion of the distribution.
$$\mathrm{CV} = \frac{\sigma}{\mathrm{E}[X]}$$

Transformations of distribution

- Lognormal : $Y = e^X$, where
$Y \sim \text{Lognormal}(\mu, \sigma)$
$X \sim \text{Normal}(\mu, \sigma)$
- Inverse Exponential : $Y = \frac{1}{X}$, where
$Y \sim \text{Inverse Exponential}(1/\theta)$
$X \sim \text{Exponential}(\theta)$
- Weibull : $Y = X^{1/\tau}$, where
$Y \sim \text{Weibull}(\sqrt[\tau]{\theta})$
$X \sim \text{Exponential}(\theta)$

Parameter interpretation

- **Scale parameter (θ, β, σ) :** Affect the spread of the distribution.
- **Rate parameter (λ) :** Affect the rate of data at mean. (1/scale)
- **Shape parameter (α, τ, γ) :** Affect the shape rather then simply shift the distribution.

Produit de convolution

The convolution of 2 random variable is difine as the sum of the two.

$$f_{X_1+X_2}(x) = \int_{-\infty}^{\infty} f_{X_1}(x-s) f_{X_2}(s)\,\mathrm{d}s$$
$$F_{X_1+X_2}(x) = \int_{-\infty}^{x} F_{X_1}(x-s) f_{X_2}(s)\,\mathrm{d}s$$