Euler Equations for 2-D Compressible Flow

(Differential Eqn. Form)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = 0 \qquad (1)$$

$$\frac{\partial (\rho u)}{\partial t} + \vec{\nabla} \cdot (\rho u \vec{V}) = -\frac{\partial p}{\partial x} \qquad (2)$$

$$\frac{\partial (\rho v)}{\partial t} + \vec{\nabla} \cdot (\rho v \vec{V}) = -\frac{\partial p}{\partial y} \qquad (3)$$

$$\frac{\partial (\rho e_0)}{\partial t} + \vec{\nabla} \cdot (\rho h_0 \vec{V}) = 0 \qquad (4)$$

* No visous/heat transfer terms in Euler

$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$ ($\vec{\nabla}\cdot$ is "divergence" operator)

Recall $\boxed{h_0 = h + \frac{1}{2}V^2}$ where $V^2 = \sqrt{u^2 + v^2}$

and $\boxed{h \equiv \underbrace{e}_{\text{internal energy}} + \frac{p}{\rho}}$

so $h_0 = \underbrace{e + \frac{1}{2}V^2}_{e_0} + \frac{p}{\rho} = e_0 + \frac{p}{\rho}$

$$\boxed{e_0 = h_0 - p/\rho}$$

Euler equations contain 6 unknowns at each location within a flowfield:

$$\rho, u, v, p, e, T$$

Need two more equations to supplement (1)–(4) and above boxed equations:

$\boxed{p = \rho R T}$ (ideal gas law)

$\boxed{e = c_v T}$ (calorically perfect gas)

Euler Equations in Integral Form

Use Gauss Divergence Theorem to convert from Differential Form to Integral Form:

$$\oint_{S} \vec{F} \cdot d\vec{S} = \int_{\forall} \vec{\nabla} \cdot \vec{F} \, d\forall \qquad \text{(for any differentiable function } \vec{F}\text{)}$$

S ~ SURFACE CONTOUR OF CONTROL VOLUME

$\forall$ ~ CONTROL VOLUME

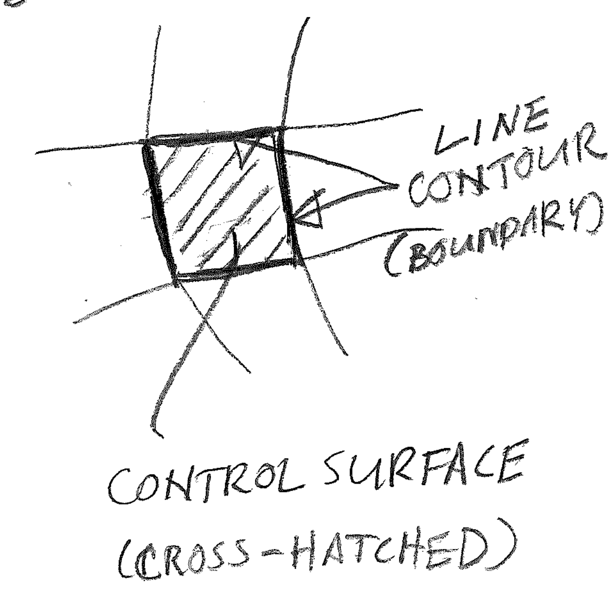
For 2-D:

$$\oint_{C} \vec{F} \cdot d\vec{C} = \int_{S} \vec{\nabla} \cdot \vec{F} \, dS$$

C ~ CURVE (LINE) CONTOUR OF CONTROL SURFACE

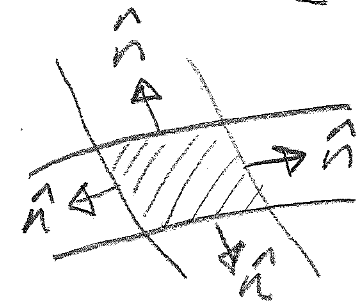
S ~ CONTROL SURFACE

2-D:



Applied to continuity eqn:

Let $\vec{F} = \varphi \vec{V}$ $\Rightarrow$ $$\int_S \vec{\nabla} \cdot \overbrace{(\varphi \vec{V})}^{\vec{F}} \, ds = \oint_C \overbrace{(\varphi \vec{V})}^{\vec{F}} \cdot d\vec{s} = \oint_C (\varphi \vec{V}) \cdot \hat{n} \, ds$$



$\hat{n}$: outward unit vector normal (along contour)

$(\hat{n} = n_x \hat{i} + n_y \hat{j})$

Continuity becomes:

$$\underbrace{\int_S \frac{\partial \varphi}{\partial t} \, ds}_{\text{time rate of change of density within control surface}} + \oint_C \underbrace{(\varphi \vec{V} \cdot \hat{n}) \, d\vec{c}}_{\text{mass flux rate thru boundary of control surface}} = 0$$

Apply to all 4 equations and condense:

<u>Euler Eqns in Integral Form</u>

$$\int_S \frac{d\vec{u}}{dt}\,ds \quad \oint_C \left(\underbrace{\vec{F}_{INV}}_{\text{inviscid flux vector}} \cdot \hat{n}\right) = 0$$

where:

$$\vec{u} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho e_0 \end{bmatrix}, \quad \vec{F}_{INV} = \begin{bmatrix} \rho(\vec{V}\cdot\hat{n}) \\ \rho u(\vec{V}\cdot\hat{n}) + n_x P \\ \rho v(\vec{V}\cdot\hat{n}) + n_y P \\ \rho h_0(\vec{V}\cdot\hat{n}) \end{bmatrix}$$

("state vector") ("flux vector")

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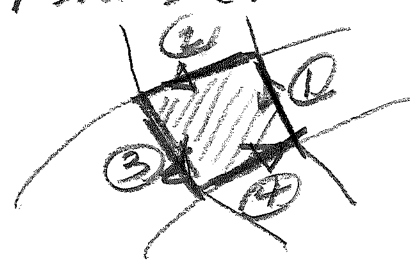
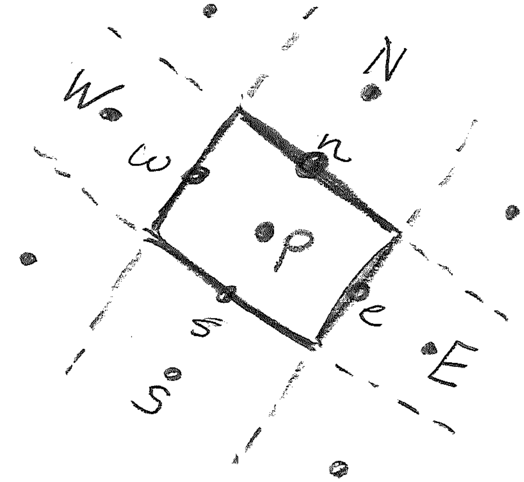
* can evaluate contour integral using piecewise integration of 4 sides (for a quadrilateral cell)



Finite-Volume Method

* approximate $\int_S \frac{\partial \vec{u}}{\partial t} ds \tilde{=} S \frac{d\vec{u}_p}{dt} \cong S \frac{\Delta \vec{u}_p}{\Delta t}$ where $\vec{u}_p$ is state vector at cell center "p"

OK for small Δt's



* replace contour integral with sum over all cell sides and using "face center" state vectors (N-E-S-W)

Euler equations become:

$$S \frac{\Delta \vec{u}_p}{\Delta t} = -\sum_{\substack{i=1 \\ \text{(faces)}}}^{4} \left(\vec{F}_{INV,i} \cdot \hat{n}_i \right)$$

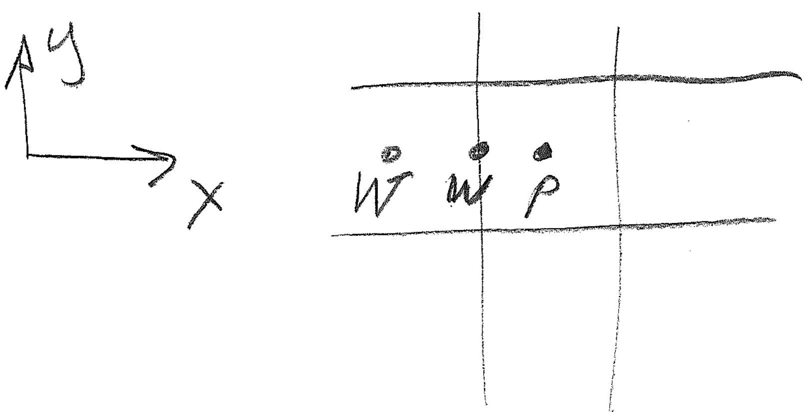
* $\vec{F}_{INV}$ at each face is estimated based on nearby cell states

* Can solve for $\frac{d\vec{u}_p}{dt}$ with $\vec{F}_{INV}$ $\Rightarrow$ get change in $\vec{u}_p$

Upwinding

Using convective and/or acoustic info to estimate cell face states

Here is simple method used by Fluent's pressure based solver:



$$\left.\begin{array}{ll} \vec{u}_w = \vec{u}_W \text{ if } u>0 & \text{(flow to left)} \\ \vec{u}_w = \vec{u}_P \text{ if } u<0 & \text{(flow to right)} \end{array}\right\} \text{based on convection alone}$$

* this is "1st order upwinding" (uses only nearest neighbor cells)

CFD (Fluent) Terminology

pressure-based scheme: solves Euler equations separately ("decoupled") during each time step (iteration)

(very efficient for low speed flows when ρ variations are weak)

density-based scheme: solves Euler equations together ("coupled") during each time step (iteration)

steady method: time step applied at each cell is different and as large as possible to converge quickly

transient method: time step at each cell is identical so that solution develops in time-accurate manner

Solution controls for density-based scheme:

AUSM ~ robust method for estimating cell face states
(captures strong shocks well)

Roe ~ precise (but less robust) method for
estimating cell face states
(not recommended for supersonic flows)

Boundary conditions needed for high speed flow over airfoil:

1. Pressure farfield at boundaries "far" from airfoil
(enforces freestream conditions with minimal "reflections")

2. Wall at airfoil surface boundaries
(enforces "no-slip" condition)