MODULE FIVE PROBLEM SET

5-3 Problem Set

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MAT-230-15848-M01 Discrete Mathematics

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Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

Problem 1

Indicate whether the two functions are equal. If the two functions are not equal, then give an element of the domain on which the two functions have different values.

(a)

$$f : \mathbb{Z} \to \mathbb{Z}, \text{ where } f(x) = x^2.$$
$$g : \mathbb{Z} \to \mathbb{Z}, \text{ where } g(x) = |x|^2.$$

These two functions are **EQUAL**. As x^2, where x is an integer in $\mathbb{Z}$, will always produce an even number, evaluating the absolute value of integer x then squaring will also produce the same result. For example:

$$(-3)^2 = 9$$
$$|-3|^2 \equiv 3^2 = 9$$

(b)

$$f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \text{ where } f(x, y) = |x + y|.$$
$$g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \text{ where } g(x, y) = |x| + |y|.$$

These two funcion are **NOT EQUAL**. Assuming integers x and y in domain $\mathbb{Z}$, where $x \neq 0, y \neq 0$, if ($x < 0$ and $y > 0$) or ($y < 0$ and $x > 0$) then functions will not be equal. For example: if $x = -3$ and $y = 2$:

$$f(x, y) = |-3 + 2| \;=\; |-1| \;=\; 1$$
$$g(x, y) = |-3| + |2| \;=\; 3 + 2 \;=\; 5$$

Problem 2

The domain and target set of functions f and g is $\mathbb{R}$. The functions are defined as:

- $f(x) = 2x + 3$
- $g(x) = 5x + 7$

(a) $f \circ g$?

$f \circ g = 2(5x + 7) + 3 = (10x + 14) + 3 = 10x + 17$

(b) $g \circ f$?

$g \circ f = 5(2x + 3) + 7 = (10x + 15) + 7 = 10x + 22$

(c) $(f \circ g)^{-1}$?

$(f \circ g)^{-1}$ represents the inverse of $(f \circ g)$
Previously evaluated, $(f \circ G) = 10x + 17$
$(f \circ g)^{-1} \Rightarrow\ y = 10x + 17 \Rightarrow x = 10y + 17 \Rightarrow x - 17 = 10y \Rightarrow \frac{x-17}{10}$

Therefore, $(f \circ g)^{-1} = \frac{x-17}{10}$

(d) $f^{-1} \circ g^{-1}$?

$f = 2x + 3 \rightarrow$
$f^{-1} = y = 2x + 3\ \rightarrow\ x = 2y + 3\ \rightarrow\ x - 3 = 2y\ \Rightarrow\ y = \mathbf{\frac{x-3}{2}}$
$g = 5x + 7 \rightarrow$
$g^{-1} = y = 5x + 7\ \rightarrow\ x = 5y + 7\ \rightarrow\ x - 7 = 5y\ \Rightarrow\ y = \mathbf{\frac{x-7}{5}}$
Therefore:
$(f^{-1} \circ g^{-1}) = \frac{\frac{x-7}{5} - 3}{2} = \frac{(x-7)-15}{10} = \mathbf{\frac{x-22}{10}}$

(e) $g^{-1} \circ f^{-1}$?

Using the above evaluations:
$(g^{-1} \circ f^{-1}) = \frac{\frac{x-3}{2} - 7}{5} = \frac{(x-3)-14}{10} = \mathbf{\frac{x-17}{10}}$

Are any of the above equal?
YES. In the evaluations above, $(f \circ g)^{-1} \equiv (g^{-1} \circ f^{-1}) \equiv \frac{x-17}{10}$

Proof:
Let $h = (f \circ g)^{-1}$

Then: $(f \circ g) \circ h = x \rightarrow ((f \circ g) \circ (f \circ g)^{-1}) = x \equiv f(g(h(x))) = x$

$f^{-1}(f(g(h(x)))) = f^{-1}(x) \ \equiv \ g(h(x)) = f^{-1}(x)$

$g^{-1}(g(h(x))) = g^{-1}(f^{-1}(x)) \equiv h(x) = g^{-1}(f^{-1}(x))$

Substituting h and expanding the right hand side of equation:

$h(x) = g^{-1}(f^{-1}(x)) \ \equiv \ (f \circ g)^{-1} = (g^{-1} \circ f^{-1})$

Problem 3

(a) Give the matrix representation for the relation depicted in the arrow diagram. Then, express the relation as a set of ordered pairs.

The arrow diagram below represents a relation.

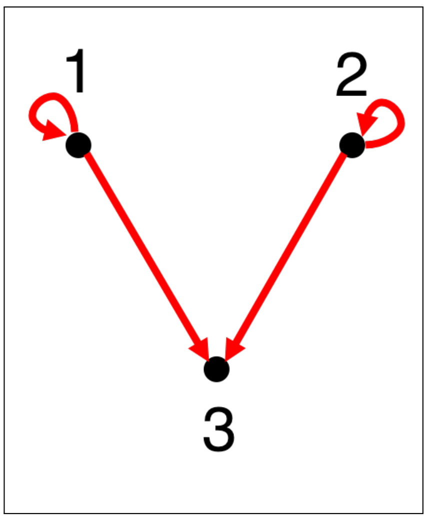


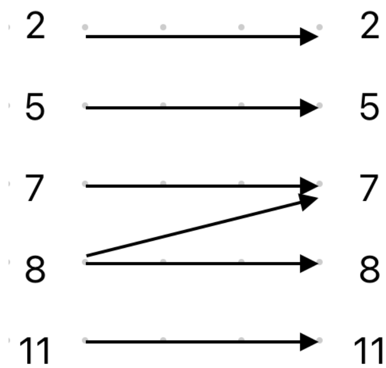
Figure 1: *An arrow diagram shows three vertices, 1, 2, and 3. An arrow from vertex 1 points to vertex 3, and another arrow from vertex 2 points to vertex 3. Two self loops are formed, one at vertex 1 and another at vertex 2.*

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A = \{(1,1), (1,3), (2,2), (2,3)\}$$

(b) Draw the arrow diagram for the relation.
The domain for the relation A is the set $\{2, 5, 7, 8, 11\}$. For x, y in the domain, xAy if $|x - y|$ is less than 2.

$$A \to A,\ where\ |x-y| < 2$$



Problem 4

For each relation, indicate whether the relation is:

- Reflexive, anti-reflexive, or neither
- Symmetric, anti-symmetric, or neither
- Transitive or not transitive

Justify your answer.

(a) The domain of the relation L is the set of all real numbers. For $x, y \in \mathbb{R}$, xLy if $x < y$.

Anti-reflexive:
If $(x = y)$, then does not satisfy the theorem $(x < y)$

Anti-symmetric:
If $(x < y)$, then y can not be less than x

Transitive:
If $(x < y)$ and $(y < z)$ then $(x < z)$

(b) The domain of the relation A is the set of all real numbers. xAy if $|x-y| \leq 2$

Reflexive:
$\forall x(x = y) \to |x - y| < 2$

Sometimes Symmetric:
When $y = (x + 1)$ or $y = (x - 1)$, then $|x - y| < 2$ and $|y - x| < 2$
(ie. $|3 - 2| = |2 - 3| = 1$)

Not Transitive:
For all instances of $|x - y|$, if $y < |x - 1|$ or $y > |x + 1|$ then theorem is not satisfied.
(ie. $3 - 2 < 2$ and $4 - 3 < 2$, however $4 - 2 \geq 2$)

(c) The domain of the relation Z is the set of all real numbers. xZy if $y = 2x$

Anti-reflexive:
For all x and y in R, no values exists where $x = 2x$

Anti-symmetric:
For every pair (x, y), $xRy \neq yRx$

Not Transitive:
If $y = 2x$ and $z = 2y$, then there does not exist any instance where $z = 2x$

Problem 5

The number of watermelons in a truck are all weighed on a scale. The scale rounds the weight of every watermelon to the nearest pound. The number of pounds read off the scale for each watermelon is called its measured weight. The domain for each of the following relations below is the set of watermelons on the truck. For each relation, indicate whether the relation is:

- Reflexive, anti-reflexive, or neither
- Symmetric, anti-symmetric, or neither
- Transitive or not transitive

Justify your answer.

(a) Watermelon x is related to watermelon y if the measured weight of watermelon x is at least the measured weight of watermelon y. No two watermelons have the same measured weight.

Since it is stated that no two watermelons have the same weight, xRx will never be true. Therefore, the relation is **Anti-reflexive**

As $x > y$, then $y \not> x$ and relation cannot be symmetric. Additionally, as no two watermelons are the same weight, $x \neq y$ and the relationship cannot be anti-symmetric. Therefore the relation is **Neither**.

As each pair of watermellons is evaluated individually, then $x > y$ and $y > z$ which means that $x > z$, therefore the relation is **Transitive**

(b) Watermelon x is related to watermelon y if the measured weight of watermelon x is at least the measured weight of watermelon y. All watermelons have exactly the same measured weight.

Since every element is the same weight, then watermellon x weighs at least as much as itself. Therefore, the relation is **Reflexive**.

Again, since every watermelon has the same weight, then xRy and yRx are both valid, and therefore the relation is **Symmetric**.

Lastly, once again, since all elements of the relation are the same weight, then $x \geq y$ and $y \geq z$ implies $x \geq z$, therefore the relation is **Trasitive**.

Problem 6

Part 1. Give the adjacency matrix for the graph G as pictured below:

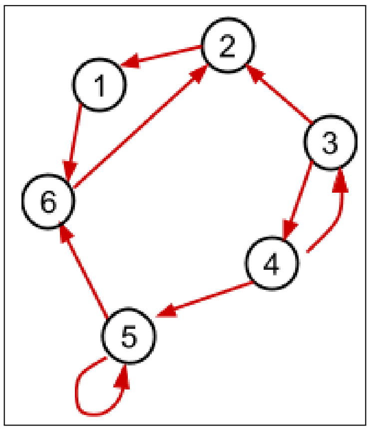


Figure 2: *A graph shows 6 vertices and 9 edges. The vertices are 1, 2, 3, 4, 5, and 6, represented by circles. The edges between the vertices are represented by arrows, as follows: 4 to 3; 3 to 2; 2 to 1; 1 to 6; 6 to 2; 3 to 4; 4 to 5; 5 to 6; and a self loop on vertex 5.*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Part 2. A directed graph G has 5 vertices, numbered 1 through 5. The 5×5 matrix A is the adjacency matrix for G. The matrices A^2 and A^3 are given below.

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Use the information given to answer the questions about the graph G.

(a) Which vertices can reach vertex 2 by a walk of length 3?

Observing the third power matrix A^3, column 2, there are values of 1 in rows 4 and 5 which indicate that vertices 4 and 5 can reach vertex 2 by a walk of length 3.

(b) Is there a walk of length 4 from vertex 4 to vertex 5 in G? (Hint: $A^4 = A^2 \cdot A^2$.)

To find adjacency matrix A^4, first need to multiple: $A^2 * A^2$:

$$A^4 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Observing column 4 of the matrix A^4 there is a 0 in row 5. This indicates that there is no walk of length 4 from vertex 4 to vertex 5.

Problem 7

Part 1. The drawing below shows a Hasse diagram for a partial order on the set $\{A, B, C, D, E, F, G, H, I, J\}$

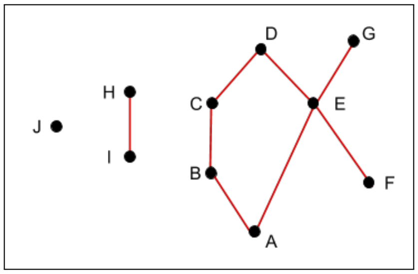


Figure 3: *A Hasse diagram shows 10 vertices and 8 edges. The vertices, represented by dots, are as follows: vertex J; vertices H and I are aligned vertically to the right of vertex J; vertices A, B, C, D, and E forms a closed loop, which is to the right of vertices H and I; vertex G is inclined upward to the right of vertex E; and vertex F is inclined downward to the right of vertex E. The edges, represented by line segments, between the vertices are as follows: Vertex J is connected to no vertex; a vertical edge connects vertices H and I; a vertical edge connects vertices B and C; and 6 inclined edges connect the following vertices, A and B, C and D, D and E, A and E, E and G, and E and F.*

(a) What are the minimal elements of the partial order?

The minimal elements of the partial order are J, I, A, F

(b) What are the maximal elements of the partial order?

The maximal elements of the partial order are J, H, D, G

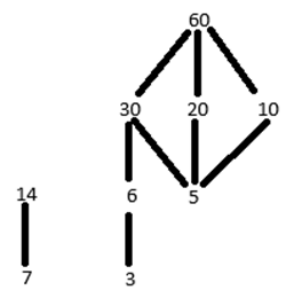
(c) Which of the following pairs are comparable?

(A, D), (J, F), (B, E), (G, F), (D, B), (C, F), (H, I), (C, E)

Reviewing the listed pairs, the following have a path between them in a constant upward or downward progression, which means they are comparable. $(A, D), (G, F), (D, B), (H, I)$

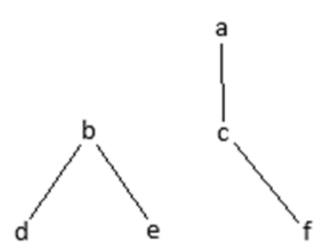
Part 2. Each relation given below is a partial order. Draw the Hasse diagram for the partial order.

(a) The domain is $\{3, 5, 6, 7, 10, 14, 20, 30, 60\}$. $x \leq y$ if x evenly divides y.



(b) The domain is $\{a, b, c, d, e, f\}$. The relation is the set:

$\{(b, e), (b, d), (c, a), (c, f), (a, f), (a, a), (b, b), (c, c), (d, d), (e, e), (f, f)\}$



PROBLEM 8

Determine whether each relation is an equivalence relation. Justify your answer. If the relation is an equivalence relation, then describe the partition defined by the equivalence classes.

(a) The domain is a group of people. Person x is related to person y under relation M if x and y have the same favorite color. You can assume that there is at least one pair in the group, x and y, such that xMy.

Since everyones favorite color is the same as their own favorite color, xRx is valid meaning the relation is **Reflexive**.

Since the relationships are determined solely on a person's favorite color, then xMy then yMx which shows the relation is **Symmetric**.

Again, as the relationships are determined solely on the favorite colors, then if person x has the same favorite color as person y, and person y has the same favorite color as person z then inheritantly person x has the same favorite color as person z. $xMy \land yMz \rightarrow xMz$, which means that the relation is **Transitive**.

Since the relation is **Reflexive, Symmetric, and Transitive**, then the relation is equivalence relation. The partition classes are each group of all individuals sharing a matching favorite colors. As each individual can only have one favorite color, the union of all groups is all the people in the group.

(b) The domain is the set of all integers. xEy if $x + y$ is even. An integer z is even if $z = 2k$ for some integer k.

Whether even or odd, each integer satisfies xEx:
$x + x = 2x$ which is even.
$2x + 1$ is odd, whereas if y is odd, $y + y = (2x + 1) + (2x + 1) = 4x + 2 \Rightarrow 2(2x + 1)$, as a product of 2, 2 times any integer is even. Therefore, any integer added to itself is even, and xEx is valid for all relations which means the relation is **Reflexive**.

For any pair of integers, if $x + y$ is even, then $y + x$ is even, therefore the relation is **Symmetric**.

Let $x + y$ be an even integer $2m$ then $x = 2m - y$, and $y + z$ be an even integer $2n$ then $z = 2n - y$. This means $x + z = 2n - y + 2m - y \Rightarrow 2n + 2m - 2y \rightarrow 2(m + n - y)$. Any product of an integer times an even number is even, therefore $x + z$ is even and the relation is **Transitive**.

Since the relation is **Reflexive, Symmetric, and Transitive**, then the relation is equivalence relation. The partition classes are all even integers, and all odd integers. Since every integer can only be even or odd, the union of both partition classes is all integers.