MODULE SIX PROBLEM SET

6-3 Problem Set

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MAT-230-15848-M01 Discrete Mathematics

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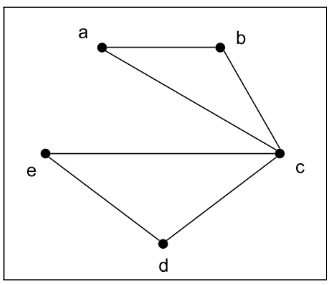
December 8, 2024

Directions: Type your solutions into this document and be sure to show all steps for arriving at your solution. Just giving a final number may not receive full credit.

Problem 1

For parts (a) and (b), indicate if each of the two graphs are equal. Justify your answer.

(a)



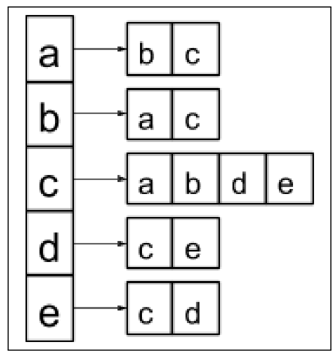
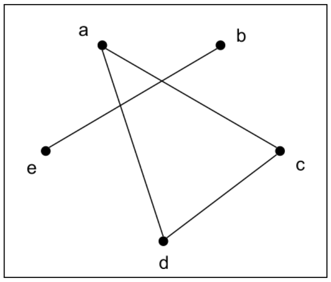


Figure 1: *Left: An undirected graph has 5 vertices. The vertices are arranged in the form of an inverted pentagon. From the top left vertex, moving clockwise, the vertices are labeled: a, b, c, d, and e. Undirected edges, line segments, are between the following vertices: a and b; a and c; b and c; c and d; e and d; and e and c.*

Figure 2: *Right: The adjacency list representation of a graph. The list shows all the vertices, a through e, in a column from top to bottom. The adjacent vertices for each vertex in the column are placed in a row to the right of the corresponding vertex's cell in the column. An arrow points from each cell in the column to its corresponding row on the right. Data from the list, as follows: Vertex a is adjacent to vertices b and c. Vertex b is adjacent to vertices a and c. Vertex c is adjacent to vertices a, b, d, and e. Vertex d is adjacent to vertices c and e. Vertex e is adjacent to vertices c and d.*

In review of the two graphs, it is clear that they are equal. Following the adjacency list, each vertice in the left column is listed with its neighboring vertices to the right. Comparing the adjacency list to the graph in figure 1, each vertice and listed neighbors are equal to the depicted graph.

(b) 

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Figure 3: *An undirected graph has 5 vertices. The vertices are arranged in the form of an inverted pentagon. Moving clockwise from the top left vertex a, the other vertices are, b, c, d, and e. Undirected edges, line segments, are between the following vertices: a and c; a and d; d and c; and e and b.*

Reviewing the graph and matrix, the two are not equal. The stand out relation in the undirected graph is (e, b). In the matrix representation, row 5 represents the relations of e. Though column 2 which represents vertex b is true, so is column 4, which represents the relationship (e, d) that does not exist in the graph.

(c) Prove that the two graphs below are isomorphic.

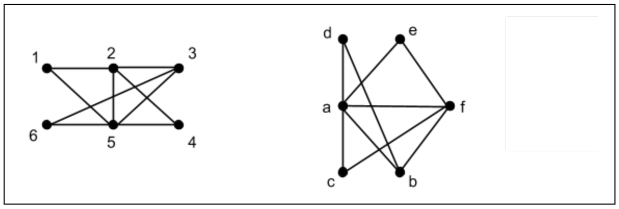


Figure 4: *Two undirected graphs. Each graph has 6 vertices. The vertices in the first graph are arranged in two rows and 3 columns. From left to right, the vertices in the top row are 1, 2, and 3. From left to right, the vertices in the bottom row are 6, 5, and 4. Undirected edges, line segments, are between the following vertices: 1 and 2; 2 and 3; 1 and 5; 2 and 5; 5 and 3; 2 and 4; 3 and 6; 6 and 5; and 5 and 4. The vertices in the second graph are a through f. Vertices d, a, and c, are vertically inline. Vertices e, f, and b, are horizontally to the right of vertices d, a, and c, respectively. Undirected edges, line segments, are between the following vertices: a and d; a and c; a and e; a and b; d and b; a and f; e and f; c and f; and b and f.*

Comparing the degree sequences of the graphs:

$1 \rightarrow 2, 5$
$2 \rightarrow 1, 3, 4, 5$
$3 \rightarrow 2, 5, 6$
$4 \rightarrow 2, 5$
$5 \rightarrow 1, 2, 3, 4, 6$
$6 \rightarrow 3, 5$
Degree Sequence: $2, 2, 2, 3, 4, 5$

$a \rightarrow b, c, d, e, f$
$b \rightarrow a, d, f$
$c \rightarrow a, f$
$d \rightarrow a, b$
$e \rightarrow a, f$
$f \rightarrow a, b, c, e$
Degree Sequence: $2, 2, 2, 3, 4, 5$

Using the theorem that degree sequence is preserved under isomorphism, the degree sequence is proven to be preserved under isomorphism and therefore of an isomorphic relationship.

(d) Show that the pair of graphs are not isomorphic by showing that there is a property that is preserved under isomorphism which one graph has and the other does not.

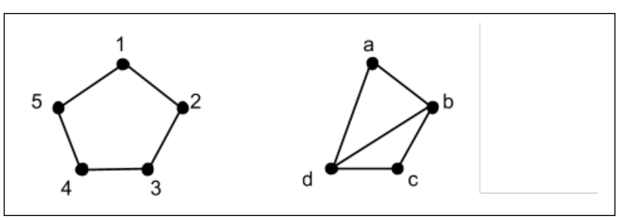


Figure 5: *Two undirected graphs. The first graph has 5 vertices, in the form of a regular pentagon. From the top vertex, moving clockwise, the vertices are labeled: 1, 2, 3, 4, and 5. Undirected edges, line segments, are between the following vertices: 1 and 2; 2 and 3; 3 and 4; 4 and 5; and 5 and 1. The second graph has 4 vertices, a through d. Vertices d and c are horizontally inline, where vertex d is to the left of vertex c. Vertex a is above and between vertices d and c. vertex b is to the right and below vertex a, but above the other two vertices. Undirected edges, line segments, are between the following vertices: a and b; b and c; a and d; d and c; d and b.*

Using the same theorem as part a, it is clearly proven that the two graphs are not isomorphic.

$1 \rightarrow 2, 5$	$a \rightarrow b, d$
$2 \rightarrow 1, 3$	$b \rightarrow a, c, d$
$3 \rightarrow 2, 4$	$c \rightarrow b, d$
$4 \rightarrow 3, 5$	$d \rightarrow a, b, c$
$5 \rightarrow 4, 1$	
Degree Sequence: $2, 2, 2, 2, 2$	Degree Sequence: $2, 2, 3, 3$

The left graph is clearly 2-regular, and the right graph has 2 vertices of degree 3. Since the number of edges and number of vertices are not the same so the graphs are **NOT** preserved under isomorphism and therefore **NOT** isomorphic.

Problem 2

Refer to the undirected graph provided below:

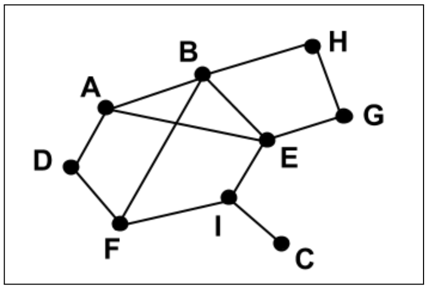


Figure 6: *An undirected graph has 9 vertices. 6 vertices form a hexagon, which is tilted upward to the right. Starting from the leftmost vertex, moving clockwise, the vertices forming the hexagon shape are: D, A, B, E, I, and F. Vertex H is above and to the right of vertex B. Vertex G is the rightmost vertex, below vertex H and above vertex E. Vertex C is the bottommost vertex, a little to the right of vertex E. Undirected edges, line segments, are between the following vertices: A and D; A and B; B and F; B and H; H and G; G and E; B and E; A and E; E and I; I and C; I and F; and F and D.*

(i) What is the maximum length of a path in the graph? Give an example of a path of that length.

The maximum length of a path in the undirected graph shown is $k - 1$, where k represents the total number of vertices. In this case, the maximum length would be $9 - 1 = 8$.
An example of a maximum path would be:
$\{$(A, D, F, B, H, G, E, I, C)$\}$

(ii) What is the maximum length of a cycle in the graph? Give an example of a cycle of that length.

The maximum length of a cycle in the graph would be 8. Since vertex C is isolated, it can not be accessed in a cycle as vertex I would need to be repeated.
An example of this cycle would be:
$\{$(A, B, H, G, E, I, F, C, A)$\}$

(iii) Give an example of an open walk of length five in the graph that is a trail but not a path.

An example of an open walk of length 5 that is a trail but not a path would require 5 edges with no duplicate edges to maintain a trail, but having a duplicate vertice to remove the property of being a path.
An example of this would be:
$\{$(B, A, E, B, F, I)$\}$

(iv) Give an example of a closed walk of length four in the graph that is not a circuit.

A closed walk of length 4 that is not a circuit requires 4 edges, as well as no duplicate vertices.
An example would be:
$\{$(B, E, A, D, F)$\}$

(v) Give an example of a circuit of length zero in the graph.

As a circuit is a closed walk with no duplicate edges, any single vertex in the graph is concidered a circuit of length zero since there are no edges involved in traversing and each sequence starts and ends at the same vertex. Therefore, (A), (B), (C), (D), (E), (F), (G), (H), (I) are all circuits of length 0.

Problem 3

(a) Find the connected components of each graph.

(i) $G = (V, E)$. $V = \{a, b, c, d, e\}$. $E = \emptyset$

Since the graph of E has no edges, then each vertex is its own connector component:
Therefore, {a}, {b}, {c}, {d}, {e} are all connected components.

(ii) $G = (V, E)$. $V = \{a, b, c, d, e, f\}$. $E = \{\{c, f\}, \{a, b\}, \{d, a\}, \{e, c\}, \{b, f\}\}$

When listing out all the vertices and their connections, you can see:
$a \to b, d$
$b \to f$
$c \to f, e$
$d \to a$
$e \to c$
$f \to b, c$

Since every vertex is in some way connected to all the other vertices then the connected components of the graph E are {a, b, c, d, e, f}.

(b) Determine the edge connectivity and the vertex connectivity of each graph.

(i)

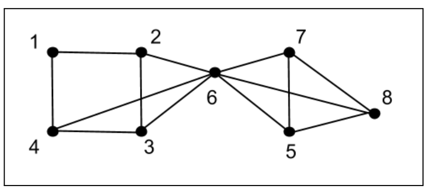


Figure 7: *An undirected graph has 8 vertices, 1 through 8. 4 vertices form a rectangular-shape on the left. Starting from the top left vertex and moving clockwise, the vertices of the rectangular shape are, 1, 2, 3, and 4. 3 vertices form a triangle on the right, with a vertical side on the left and the other vertex on the extreme right. Starting from the top vertex and moving clockwise, the vertices of the triangular shape are, 7, 8, and 5. Vertex 6 is between the rectangular shape and the triangular shape. Undirected edges, line segments, are between the following vertices: 1 and 2; 2 and 3; 3 and 4; 4 and 1; 2 and 6; 4 and 6; 3 and 6; 6 and 7; 6 and 8; 6 and 5; 7 and 5; 7 and 8; and 5 and 8.*

- The edge connectivity of the graph is 6 due to being able to remove a maximum of 6 edges and maintain connectivity. For example, creating walk (1, 4, 3, 2, 6, 5, 7, 8).
- The vertex connectivity of the graph is 1. This is due to fact that if vertex 6 is removed, the graph is splut into two and no longer connected.

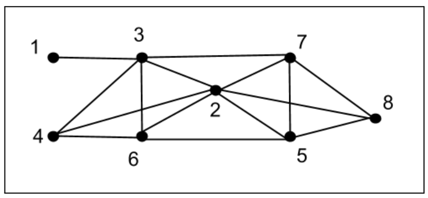
(ii) 

Figure 8: *An undirected graph has 8 vertices, 1 through 8. 4 vertices form a rectangular shape in the center. Starting from the top left vertex and moving clockwise, the vertices of the rectangular shape are, 3, 7, 5, and 6. Vertex 2 is at about the center of the rectangular shape. Vertex 8 is to the right of the rectangular shape. Vertex 1 and 4 are to the left of the rectangular shape, horizontally in-line with vertices 3 and 6, respectively. Undirected edges, line segments, are between the following vertices: 1 and 3; 3 and 7; 3 and 4; 3 and 6; 3 and 2; 4 and 2; 4 and 6; 6 and 2; 6 and 5; 2 and 5; 2 and 7; 2 and 8; 7 and 5; 7 and 8; and 5 and 8.*

- The edge connectivity of the graph is 7 because 7 edges can be removed and maintain connection. An example would be walk (1, 3, 7, 8, 5, 2, 6, 4).
- The vertex connectivity of the graph is 1. This is due to being able to remove just vertex 3 and segregating the graph creating 2 non-connected components.

PROBLEM 4

For parts (a) and (b) below, find an Euler circuit in the graph or explain why the graph does not have an Euler circuit.

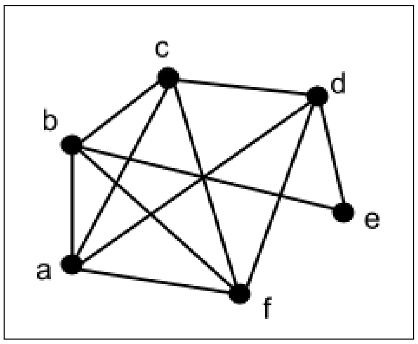
(a) 

Figure 9: *An undirected graph has 6 vertices, a through f. 5 vertices are in the form of a regular pentagon, rotated 90 degrees clockwise. Hence, the top vertex becomes the rightmost vertex. From the bottom left vertex, moving clockwise, the vertices in the pentagon shape are labeled: a, b, c, e, and f. Vertex d is above vertex e, below and to the right of vertex c. Undirected edges, line segments, are between the following vertices: a and b; a and c; a and d; a and f; b and f; b and c; b and e; c and d; d and e; and d and f. Edges c f, a d, and b e intersect at the same point.*

The graph consists of vertices of only even degrees. An example of an Euler circuit of the graph shown is the closed walk:
(a, c, f, b, e, d, a, b, c, d, f, a)

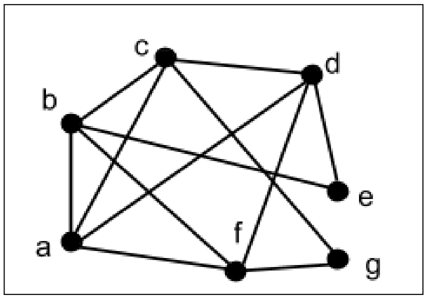
(b) 

Figure 10: *An undirected graph has 7 vertices, a through g. 5 vertices are in the form of a regular pentagon, rotated 90 degrees clockwise. Hence, the top vertex becomes the rightmost vertex. From the bottom left vertex, moving clockwise, the vertices in the pentagon shape are labeled: a, b, c, e, and f. Vertex d is above vertex e, below and to the right of vertex c. Vertex g is below vertex e, above and to the right of vertex f. Undirected edges, line segments, are between the following vertices: a and b; a and c; a and d; a and f; b and f; b and c; b and e; c and d; c and g; d and e; d and f; and f and g.*

The graph consists of vertices of only even degrees. An example of an Euler circuit of the graph shown is the closed walk:
(e, d, f, a, d, c, b, a, c, g, f, b, e)

(c) For each graph below, find an Euler trail in the graph or explain why the graph does not have an Euler trail.

(Hint: One way to find an Euler trail is to add an edge between two vertices with odd degree, find an Euler circuit in the resulting graph, and then delete the added edge from the circuit.)

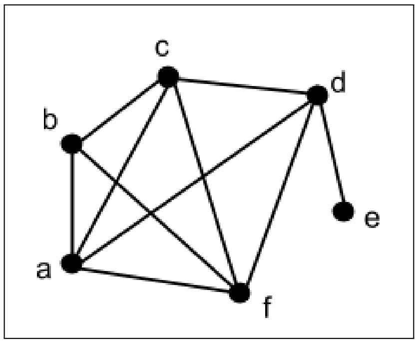
(i) 

Figure 11: *An undirected graph has 6 vertices, a through f. 5 vertices are in the form of a regular pentagon, rotated 90 degrees clockwise. Hence, the top vertex becomes the rightmost vertex. From the bottom left vertex, moving clockwise, the vertices in the pentagon shape are labeled: a, b, c, e, and f. Vertex d is above vertex e, below and to the right of vertex c. Undirected edges, line segments, are between the following vertices: a and b; a and c; a and d; a and f; b and f; b and c; c and d; c and f; d and e; and d and f.*

The graph shown has 2 vertices (d and b) with odd degrees so an Euler trail exists. An example of an Euler trail in the graph is:
(e, d, a, f, d, c, b, f, c, a, b)

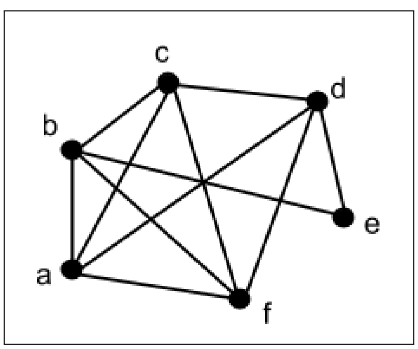
(ii) 

Figure 12: *An undirected graph has 6 vertices, a through f. 5 vertices are in the form of a regular pentagon, rotated 90 degrees clockwise. Hence, the top vertex becomes the rightmost vertex. From the bottom left vertex, moving clockwise, the vertices in the pentagon shape are labeled: a, b, c, e, and f. Vertex d is above vertex e, below and to the right of vertex c. Undirected edges, line segments, are between the following vertices: a and b; a and c; a and d; a and f; b and f; b and c; b and e; c and d; d and e; and d and f. Edges c f, a d, and b e intersect at the same point.*

As explained in the theorem of Euler trails, an undirected graph has an Euler trail ONLY IF the graph is connected and consists of exactly two vertices with odd degrees. Therefore, the graph shown does not have an Euler trail because every vertex has an even level of degree.

PROBLEM 5

Consider the following tree for a prefix code:

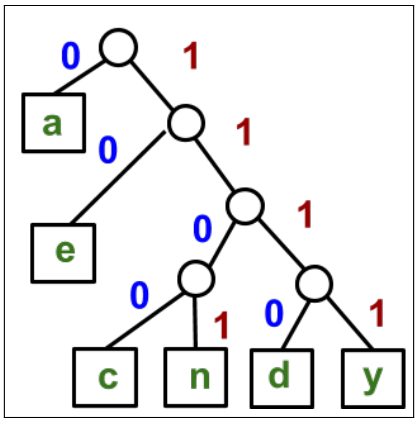


Figure 13: *A tree with 5 vertices. The top vertex branches into character, a, on the left, and a vertex on the right. The vertex in the second level branches into character, e, on the left, and a vertex on the right. The vertex in the third level branches into two vertices. The left vertex in the fourth level branches into character, c, on the left, and character, n, on the right. The right vertex in the fourth level branches into character, d, on the left, and character, y, on the right. The weight of each edge branching left from a vertex is 0. The weight of each edge branching right from a vertex is 1.*

(a) Use the tree to encode “day”.

”day” = 111001111

(b) Use the tree to encode “candy”.

”candy” = 110001101111101111

(c) Use the tree to decode “1110101101″.

1110101101 = ”den”

(d) Use the tree to decode “111001101110010″.

111001101110010 = ”dance”

Problem 6

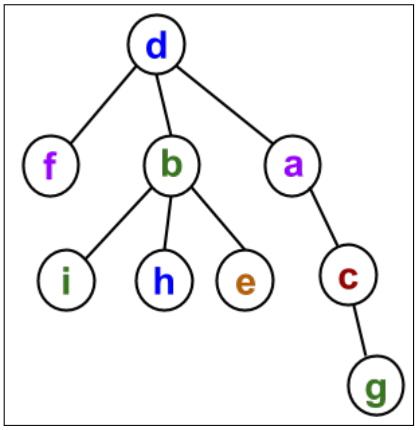


Figure 14: *A tree diagram has 9 vertices. The top vertex is d. Vertex d has three branches to vertices, f, b, and a. Vertex b branches to three vertices, i, h, and e. Vertex a branches to vertex c. Vertex c branches to vertex g.*

(a) Give the order in which the vertices of the tree are visited in a post-order traversal.

The post-order traversal of the diagram shown is: $f, i, h, e, b, g, c, a, d$

(b) Give the order in which the vertices of the tree are visited in a pre-order traversal.

The pre-order traversal of the diagram shown is: $d, f, b, i, h, e, a, c, g$

PROBLEM 7

Consider the following tree. Assume that the neighbors of a vertex are considered in alphabetical order.

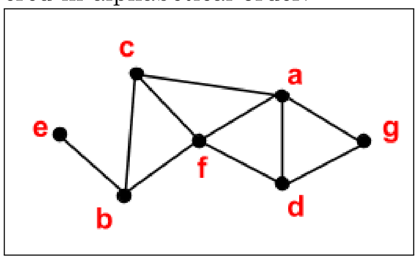


Figure 15: *A graph has 7 vertices, a through g, and 10 edges. Vertex e on the left end is horizontally inline with vertex g on the right end. Vertex b is below and to the right of vertex e. Vertex c is above vertex e and to the right of vertex b. Vertex f is between and to the right of vertices c and b. Vertex f is horizontally inline with vertices e and g. Vertex a is above and to the right of vertex f. Vertex d is below and to the right of vertex f. Vertex a is vertically inline with vertex d. Vertex g is between and to the right of vertices a and d. The edges between the vertices are as follows: e and b; b and c; c and f; c and a; a and d; b and f; f and a; f and d; a and g; and d and g.*

(a) Give the tree resulting from a traversal of the graph below starting at vertex a using BFS.

a
$| \; | \; | \; |$
$c \; d \; f \; g$
$|$
b
$|$
e

The order in which the vertices are visited in bfs would be: a, c, d, f, g, b, e

(b) Give the tree resulting from a traversal of the graph below starting at vertex a using DFS.

$$
\begin{array}{ccc}
 & a & \\
 & | & \\
 & c & \\
 & | & \\
 & b & \\
| & & | \\
e & & f \\
 & & | \\
 & & d \\
 & & | \\
 & & g
\end{array}
$$

The order in which the vertices are visited in DFS would be: a, c, b, e, f, d, g

PROBLEM 8

An undirected weighted graph G is given below:

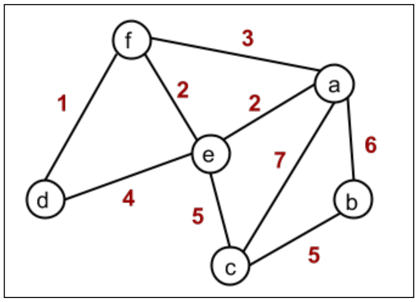


Figure 16: *An undirected weighted graph has 6 vertices, a through f, and 9 edges. Vertex d is on the left. Vertex f is above and to the right of vertex d. Vertex e is below and to the right of vertex f, but above vertex d. Vertex c is below and to the right of vertex e. Vertex a is above vertex e and to the right of vertex c. Vertex b is below and to the right of vertex a, but above vertex c. The edges between the vertices and their weight are as follows: d and f, 1; d and e, 4; f and e, 2; e and a, 2; f and a, 3; e and c, 5; c and a, 7; c and b, 5; and a and b, 6.*

(a) Use Prim's algorithm to compute the minimum spanning tree for the weighted graph. Start the algorithm at vertex a. Show the order in which the edges are added to the tree.

The minimum spanning tree of the graph is 15. The order in which the edges are added are:
$\{a, e\}, \{e, f\}, \{f, d\}, \{e, c\}, \{c, b\}$

(b) What is the minimum weight spanning tree for the weighted graph in the previous question subject to the condition that edge $\{d, e\}$ is in the spanning tree?

The minimum spanning tree of the graph subject to the condition that edge $\{d, e\}$ is in the tree is 18. The order in which the edges are added are:
$\{a, f\}, \{f, d\}, \{d, e\}, \{e, c\}, \{c, b\}$

(c) How would you generalize this idea? Suppose you are given a graph G and a particular edge $\{u, v\}$ in the graph. How would you alter Prim's algorithm to find the minimum spanning tree subject to the condition that $\{u, v\}$ is in the tree?

Given the graph G, the easiest way to alter Prim's algorithm to find the minimum spanning tree subject to the condition that {u, v} is in the tree would be the start the algorithm from vertex u and edge {u, v}. This would ensure that the required edge is included then the rest of the algorithm can be completed to find the minimum spanning tree.