The Log-Rank Conjecture for Linear Threshold Functions composed with AND

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1 Notation and Definitions

We first set our notation for the relevant complexity measures. Consider an arbitrary boolean function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ and define $\mathsf{CC}(F)$ denote the communication complexity of F. For such an F, let M_F denote its communication matrix.

For each subset $S \subseteq [n]$ define the polynomial $\alpha_S : \{0,1\}^n \to \{0,1\}$ by $\alpha_S(x_1, \ldots, x_n) := \prod_{i \in S} x_i$ with $\alpha_\emptyset(x) = 1$. An AND decision tree is then a binary decision tree where each node makes an α_S query for some $S \subseteq [n]$. Such an AND decision tree T decides a boolean function $f : \{0,1\}^n \to \{0,1\}$ if T agrees with f on all inputs $x \in \{0,1\}^n$. The AND decision tree complexity $\mathsf{DT}_\wedge(f)$ is the minimum depth of an AND decision tree which decides f.

Given a boolean function f, we will frequently write f as a multilinear polynomial. It is shown in **[?]** that any boolean function f can uniquely written as $f(x) = \sum_{S \subseteq [n]} \hat{\alpha}[S]\alpha_S(x)$ where $\hat{\alpha}[S]$ is the coefficient of $\alpha_S(x)$. We denote by $\mathsf{mon}(f)$ be number of nonzero $\hat{\alpha}[S]$ for a function f. The support of a function f is defined as $\mathsf{sup}(f) := f^{-1}(\{1\})$.

We will only consider a subclass of all boolean functions $f : \{0,1\}^n \to \{0,1\}$ which we now define. Let

$$f(x_1, \ldots, x_n) := \begin{cases} 1 & w_1 w_1 + w_2 x_2 + \cdots + w_n x_n \geq w_0 \\ 0 & \text{otherwise} \end{cases}$$

be a linear threshold function (LTF). We will assume without loss of generality that $w_0 \geq 0$. We form our communication function $F(x_1, \ldots, x_n, y_1, \ldots, y_n) := f(x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$.

Finally, there are several relevant results to state. It is shown in **[?]** that $\mathsf{rank}(M_F) = \mathsf{mon}(f)$ (Note this result holds for a general f, not just LTFs). In addition, it is easy to see that $\mathsf{CC}(F) \leq 2\mathsf{DT}_\wedge(f)$. Thus, showing the following claim implies the Log-Rank Conjecture for our class of communication functions F.

Proposition 1.1. *If f is an LTF, $\mathsf{DF}_\wedge(f) \leq \log^c(\mathsf{mon}(f))$ for some $c \in \mathbb{R}^+$.*

2 ????

References