The Log-Rank Conjecture for Linear Threshold Functions composed with AND

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Abstract

In this paper, we show that the Log-rank conjecture holds for linear threshold functions (LTFs) of the form $f \circ \wedge$. The result relies on conjecture holding for montone functions of the same form. We show that any LTF can be recursively broken down into montone functions. We further explore the multilinear polynomial of functions of the form $f \circ \wedge$ and give characterizations of its the monomial complexity in terms of the support of the function.

1 Introduction

2 Notation and Definitions

We first set our notation for the relevant complexity measures. Consider an arbitrary boolean function $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ and define $\mathsf{CC}(F)$ denote the *communication complexity* of F. For such an F, let M_F denote its communication matrix.

For each subset $S \subseteq [n]$ define the polynomial $\alpha_S : \{0,1\}^n \to \{0,1\}$ by $\alpha_S(x_1, \ldots, x_n) := \prod_{i \in S} x_i$ with $\alpha_\emptyset(x) = 1$. An *AND decision tree* is then a binary decision tree where each node makes an α_S query for some $S \subseteq [n]$. Such an AND decision tree T decides a boolean function $f : \{0,1\}^n \to \{0,1\}$ if T agrees with f on all inputs $x \in \{0,1\}^n$. The *AND decision tree complexity* $\mathsf{DT}_\wedge(f)$ is the minimum depth of an AND decision tree which decides f.

Given a boolean function f, we will frequently write f as a multilinear polynomial. It is shown in [?] that any boolean function f can uniquely written as $f(x) = \sum_{S \subseteq [n]} \hat{\alpha}[S]\alpha_S(x)$ where $\hat{\alpha}[S]$ is the coefficient of $\alpha_S(x)$. We denote by $\mathsf{mon}(f)$ be number of nonzero $\hat{\alpha}[S]$ for a function f. The *support* of a function f is defined as $\mathsf{sup}(f) := f^{-1}(\{1\})$.

We will only consider a subclass of all boolean functions $f : \{0,1\}^n \to \{0,1\}$ which we now define. Let

$$f(x_1, \ldots, x_n) := \begin{cases} 1 & w_1 w_1 + w_2 x_2 + \cdots + w_n x_n \geq w_0 \\ 0 & \text{otherwise} \end{cases}$$

be a *linear threshold function* (LTF). We will assume without loss of generality that $w_0 \geq 0$. We form our communication function $F(x_1, \ldots, x_n, y_1, \ldots, y_n) := f(x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$.

Finally, there are several relevant results to state. It is shown in [?] that $\mathsf{rank}(M_F) = \mathsf{mon}(f)$ (Note this result holds for a general f, not just LTFs). In addition, it is easy to see that $\mathsf{CC}(F) \leq 2\mathsf{DT}_\wedge(f)$. Thus, showing the following claim implies the Log-Rank Conjecture for our class of communication functions F.

Proposition 2.1. *If f is an LTF, $\mathsf{DF}_\wedge(f) \leq \log^c(\mathsf{mon}(f))$ for some $c \in \mathbb{R}^+$.*

3 Properties of Monomials

4 Breaking Down LTFs

5 Conclusion

Using subroutines for monotone functions in an iteration for which the log-rank conjecture holds, we have inductively shown that the log-rank conjecture holds for all LTFs of the form $f \circ \wedge$. Although this result heavily uses a previous result, we conjecture that this algorithm is the optimum AND Decision Tree algorithm. Our result uses [?] to show the log-rank conjecture for the communication complexity. An interesting problem would be to show it holds for the AND Decision Tree complexity also.

References