THE MULTI-LEVEL SPARSE GRID INTERPOLATION KERNEL COLLOCATION (MUSIK-C) ALGORITHM, APPLIED TO BASKET OPTIONS

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1. Introduction

Numerical Analysis is the study and use of computation algorithms in approximation within the field of mathematical analysis of which (and in particular interest to this paper), partial differential equations (PDEs) are common throughout many areas of science. Whilst analytic solutions exist for some PDEs using such techniques as 'seperation of variables', 'integral transform' and 'change of variables', often either no solution exists orthe analytic tools are currently insufficient to find it.

The known algorithms developed for solving PDEs are generally classified into nine main methods:

·Finite Difference
Where functions whose known values at certain points are approximated by differences between these values. The grid on which these points (aka. nodes) reside is often referred to as a mesh there each node in a finite difference scheme is connected to it's neighbour in either a forward or backwards manner by the algorithm employed

·Method of Lines
PDE's in multiple dimensions are discretised into Ordinary Differential Equations (ODEs), in all but one dimension allowing for any of the vast number of numerical integration solvers to be used in these dimensions, meanwhile the final dimension is solved by ...

·Finite Element
Used for approximating boundary value problems where by the solution is approximated by the composition of many linear elements again using a mesh of nodes connected to each other.

·Gradient Discretisation

·Finite Volume Similar to both Finite Element and Finite Difference methods, a mesh is created on which values are calculcated at discrete points. Using the diveregence theorem to convert volume integrals into surface integrals around each point...

·Spectral
Where approximation is done by combining a series of basis functions by superposition and then choosing the co-efficients of the series which minimise the error of the result. For example a Fourier series of sinusoidal waveforms or Radial Basis Functions (RBF).

·Meshfree
In contrast to the previous methods, which all require connections between nodes of a grid a mesh free method requires no connection between nodes.
·Domain Decomposition
Whereby a boundary value problem is split into smaller problems on sub-domains each of which is independent from the others allowing for parallelisation of the overall global problem.
·Multigrid
Using a heirachy of descritised grids with different levels of coarseness between the nodes. The idea being that the convergence of an iterative method can be accelerated by solving on a coarser grid to make a global correction to the finer grid.
Recent work by

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6. Lagrange Functions in Buhmann Form

This note contains my view of the now classical Buhmann cardinal function theory, i.e. the Lagrange function theory, all of which is a footnote to the Poisson Summation Formula, in the best possible sense. All of this is probably somewhere in [**?**], albeit implicitly or in changed guise. We begin with the univariate theory for simplicity.

7. Lagrange Functions on $\mathbb{Z}$

We begin with the classical Poisson Summation Formula in one dimension, then develop the Buhmann form of the Lagrange function on $\mathbb{Z}$.

7.1. **The Poisson Summation Formula on $\mathbb{Z}$.** Let $f \in S(\mathbb{R})$, to avoid analytical inconvenience. We need the classical form of the Poisson Summation Formula. To this end, we define $\mathbb{T} := [-\pi, \pi]$ and define the $\mathbb{T}$-*periodization* of f by

$$P_{\mathbb{T}} f(x) \equiv Pf(x) := \sum_{j\in\mathbb{Z}} f(x + 2\pi j). \tag{1}$$

Theorem 7.1.

$$Pf(x) = \sum_{j\in\mathbb{Z}} f(x + 2\pi j) = (2\pi)^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ikx}. \tag{2}$$

Proof. The smoothness and decay of f imply that the Fourier series

$$Pf(x) = \sum_{\ell\in\mathbb{Z}} c_\ell e^{i\ell x}$$

converges absolutely and uniformly. Further,

$$\begin{aligned} c_\ell &= (2\pi)^{-1} \int_{-\pi}^{\pi} Pf(x) e^{-i\ell x}\, dx \\ &= (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\sum_{j\in\mathbb{Z}} f(x + 2\pi j) \right) e^{-i\ell x}\, dx \\ &= (2\pi)^{-1} \int_{\mathbb{R}} f(x) e^{-i\ell x}\, dx \\ &= (2\pi)^{-1} \widehat{f}(\ell), \end{aligned}$$

using the Dominated Convergence Theorem to justify the interchange of summation and integration. □

If we replace f by $\widehat{f}$ in Theorem 7.1, then we obtain a *dual* Poisson Summation Formula, as it were.

Corollary 7.2. *We have*

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi k) = \sum_{j\in\mathbb{Z}} f(j) e^{-ij\xi}. \tag{3}$$

Proof. Replace f by $\widehat{f}$ in Theorem 7.1, recalling that $\widehat{\widehat{f}}(x) = 2\pi f(-x)$. □

7.2. **The Lagrange Function on $\mathbb{Z}$.** Let $\phi \in S(\mathbb{R})$, to avoid all analytic inconvenience. We also want to choose ϕ with interpolation in mind, so we shall also assume that its Fourier transform $\widehat{\phi}$ is strictly positive, which implies that f is a strictly positive definite function.

We want to construct a function $L \in \operatorname{Span}_{k\in\mathbb{Z}} \phi(\cdot - k)$ for which $L(j) = \delta_{oj}$, for $j \in \mathbb{Z}$. Such a function will be called the Lagrange function, by analogy with the Lagrange form of the interpolating polynomial. Thus, proceeding formally for the moment, we have

$$L(x) = \sum_{k\in\mathbb{Z}} \lambda_k \phi(x - k) \tag{4}$$

or, in the Fourier domain,

$$\widehat{L}(\xi) = \left(\sum_{k\in\mathbb{Z}} \lambda_k e^{-ik\xi} \right) \widehat{\phi}(\xi) \tag{5}$$

It is not obvious that (5) is well defined, but we shall soon show that all is well. We periodize both sides to form a 2π–periodic function and, using the Poisson Summation Formula in the form of Corollary 7.2, we find We obtain

$$1 \equiv \sum_{\ell\in\mathbb{Z}} L(\ell)e^{-i\ell\xi} = \sum_{j\in\mathbb{Z}} \widehat{L}(\xi+2\pi j) = \left(\sum_{k\in\mathbb{Z}} \lambda_k e^{-ik\xi}\right)\left(\sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi+2\pi j)\right). \tag{6}$$

Hence, recalling that $\widehat{\phi}(\xi) > 0$, for all $\xi \in \mathbb{R}$, (6) implies

$$\sum_{k\in\mathbb{Z}} \lambda_k \exp(-ik\xi) = \frac{1}{\sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi+2\pi j)}. \tag{7}$$

Substituting (7) in (5), we obtain the Buhmann form of the Fourier transform of the Lagrange function, that is,

$$\widehat{L}(\xi) = \frac{\widehat{\phi}(\xi)}{\sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi+2\pi j)}. \tag{8}$$

8. Lagrange Functions on $h\mathbb{Z}$

We follow the same route as before.

8.1. **The Poisson Summation Formula on $h\mathbb{Z}$.** We could deduce the scaled version of the Poisson Summation Formula directly from Theorem 7.1, but I prefer to begin *ab initio*. We shall now periodize f over $h^{-1}\mathbb{T}$, for $h > 0$, i.e. we define

$$P_{h^{-1}\mathbb{T}} f(x) = \sum_{j\in\mathbb{Z}} f(x + 2\pi h^{-1} j). \tag{9}$$

The *scaled exponentials* are the $2\pi h^{-1}$-periodic functions defined by

$$e_j^h(x) := e^{ihjx}, \qquad j \in \mathbb{Z}, \tag{10}$$

and form a complete orthonormal set with respect to the inner product

$$\langle F, G\rangle = \frac{1}{2\pi h^{-1}} \int_{-\pi h^{-1}}^{\pi h^{-1}} F(s)G(s)^*\, ds. \tag{11}$$

In other words, $\{e_j^h : j \in \mathbb{Z}\}$ forms a complete orthonormal set for $L^2(h^{-1}\mathbb{T})$ endowed with the normalized inner product

$$\langle F, G\rangle = \frac{1}{\mathrm{Vol}_1\, h^{-1}\mathbb{T}} \int_{h^{-1}\mathbb{T}} FG^*.$$

Theorem 8.1. *We have*

$$P_{h^{-1}\mathbb{T}} f(x) = \sum_{j\in\mathbb{Z}} f(x + 2\pi h^{-1} j) = \left(2\pi h^{-1}\right)^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}(kh)e^{ihkx}, \tag{12}$$

i.e.

$$P_{h^{-1}\mathbb{T}} f = \left(\mathrm{Vol}_1\, h^{-1}\mathbb{T}\right)^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}(kh) e_k^{h^{-1}}. \tag{13}$$

Proof. As in Theorem 7.1, the Fourier series

$$P_{h^{-1}\mathbb{T}} f(x) = \left(\mathrm{Vol}_1\, h^{-1}\mathbb{T}\right)^{-1} \sum_{\ell\in\mathbb{Z}} c_\ell^h e^{ih\ell x}$$

converges absolutely and uniformly, and

$$\begin{aligned} c_\ell^h &= \left(2\pi h^{-1}\right)^{-1} \int_{-\pi/h}^{\pi/h} P_h f(x) e^{-ih\ell x}\, dx \\ &= \left(2\pi h^{-1}\right)^{-1} \int_{-\pi h^{-1}}^{\pi h^{-1}} \left(\sum_{j\in\mathbb{Z}} f(x + 2\pi h^{-1} j) \right) e^{-ih\ell x}\, dx \\ &= \left(2\pi h^{-1}\right)^{-1} \int_{\mathbb{R}} f(x) e^{-ih\ell x}\, dx \\ &= \left(2\pi h^{-1}\right)^{-1} \widehat{f}(h\ell), \end{aligned}$$

using the Dominated Convergence Theorem to justify the interchange of summation and integration. □

The analogous form of Corollary 7.2 is now fairly clear.

Corollary 8.2.

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi h^{-1} k) = h \sum_{\ell\in\mathbb{Z}} f(h\ell) e^{-ih\ell\xi}. \tag{14}$$

8.2. **The Lagrange Function on $h\mathbb{Z}$.** We now consider the Lagrange function when interpolating on the scaled integer grid $h\mathbb{Z}$, for $h > 0$. Thus we define

$$\phi_h(x) := \phi(h^{-1}x) \tag{15}$$

and we now want to construct a function $L^h \in \operatorname{Span}_{k\in\mathbb{Z}} \phi_h(\cdot - kh)$ for which $L^h(jh) = \delta_{oj}$, for $j \in \mathbb{Z}$. It is almost obvious that $L^h(x) = L(h^{-1}x)$, but the Fourier analysis is satisfying. Thus we consider

$$L^h(x) = \sum_{k\in\mathbb{Z}} \lambda_k^h \phi_h(x - kh) \tag{16}$$

or, in the Fourier domain,

$$\widehat{L^h}(\xi) = \left(\sum_{k\in\mathbb{Z}} \lambda_k^h e^{-ihk\xi} \right) \widehat{\phi^h}(\xi) \tag{17}$$

We must now periodize (17) to obtain a $2\pi h^{-1}$-periodic function, using the scaled Poisson Summation Formula, i.e.

$$\sum_{k\in\mathbb{Z}} \widehat{L^h}(\xi + 2\pi h^{-1} k) = h \sum_{\ell\in\mathbb{Z}} L^h(h\ell) e^{-ih\ell\xi} \equiv h.$$

Hence (17) becomes

$$1 \equiv h^{-1} \sum_{k\in\mathbb{Z}} \widehat{L^h}(\xi + 2\pi h^{-1} k) = h^{-1} \left(\sum_{k\in\mathbb{Z}} \lambda_k^h e^{-ihk\xi} \right) \sum_{m\in\mathbb{Z}} \widehat{\phi^h}(\xi + 2\pi h^{-1} m) \tag{18}$$

Eliminating $\sum \lambda_k^h \exp(-ihk\xi)$ from (17) and (18), we obtain

$$\widehat{L^h}(\xi) = \frac{\widehat{\phi^h}(\xi)}{h^{-1} \sum_{m\in\mathbb{Z}} \widehat{\phi^h}(\xi + 2\pi h^{-1} m)} = h \left(\frac{\widehat{\phi}(h\xi)}{\sum_{m\in\mathbb{Z}} \widehat{\phi}(h\xi + 2\pi m)} \right) = h\widehat{L}(h\xi). \tag{19}$$

Hence $L^h(x) = L(h^{-1}x)$, as expected.

9. Lagrange functions on $A\mathbb{Z}^d$ for $A \in \mathrm{GL}(\mathbb{R}^d)$

Let $A \in \mathrm{GL}(\mathbb{R}^d)$ and $f \in S(\mathbb{R}^d)$. shall be using the normalized inner product on $L^2(A^{-1}\mathbb{T}^d)$, that is,

$$\langle F, G\rangle = \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} F(x)G(x)^*\, dx \tag{20}$$

and $\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d = (2\pi)^d|A|^{-1}$. The $A^{-1}\mathbb{T}^d$-periodic exponentials providing our complete orthonormal sequence are given by

$$e^A_k(x) = e^{i\langle k, Ax\rangle}, \qquad k \in \mathbb{Z}^d. \tag{21}$$

9.1. The Poisson Summation Formula on $A\mathbb{Z}^d$.

We define the $A^{-1}\mathbb{T}^d$-periodization $P_{A^{-1}\mathbb{T}}f$ by

$$P_{A^{-1}\mathbb{T}^d} f(x) = \sum_{j\in\mathbb{Z}^2} f(x + 2\pi A^{-1}j). \tag{22}$$

Theorem 9.1.

$$P_{A^{-1}\mathbb{T}^d} f(x) = \left(\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d\right)^{-1} \sum_{\ell\in\mathbb{Z}^d} \widehat{f}(A^T\ell) e^A_{ell}(xi). \tag{23}$$

Proof. We have the Fourier series

$$P_{A^{-1}\mathbb{T}^d} f(x) = \sum_{k\in\mathbb{Z}^d} c^A_k e^A_k(x),$$

where

$$\begin{aligned}
c^A_k &= \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} P_{A^{-1}\mathbb{T}^d} f(x) e^A_{-k}(x)\, dx \\
&= \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} \int_{\mathbb{R}^d} f(x) e^A_{-k}(x)\, dx \\
&= \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} \int_{\mathbb{R}^d} f(x) e^{-i\langle A^T k, x\rangle}\, dx \\
&= \frac{\widehat{f}(A^T k)}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d}
\end{aligned}$$

□

The dual form takes a similar form.

Corollary 9.2.

$$\sum_{k\in\mathbb{Z}^d} \widehat{f}(\xi + 2\pi A^{-T}k) = |A| \sum_{\ell\in\mathbb{Z}^d} f(A\ell) e^{-i\langle \ell, A^T\xi\rangle}. \tag{24}$$

9.2. The Lagrange Function on $A\mathbb{Z}$.

We define $\phi_A(x) = \phi(A^{-1}x)$, for $x \in \mathbb{R}^d$. Then $\widehat{\phi^A}(\xi) = |A|\widehat{\phi}(A^T\xi)$, and it is again almost obvious that $L^A(x) = L(A^{-1}x)$, so that $\widehat{L^A}(\xi) = |A|\widehat{L}(A^T\xi)$. For completeness, we again provide the full Fourier derivation.

Theorem 9.3. *The Fourier transform of the Lagrange function L^A is given by*

$$\widehat{L^A}(\xi) = \frac{\widehat{\phi^A}(\xi)}{|A|^{-1}\sum_{k\in\mathbb{Z}^d} \widehat{\phi^A}(\xi + 2\pi A^{-T}k)}. \tag{25}$$

References

[1] Georgoulis, E.h., Levesley, J., Subhan, F. (2013), *Multilevel sparse kernel-based interpolation*, SIAM Journal on Scientific Computing, 2013, 35(2):A815-A831.
[2] Subhan, F. (2011), *Multilevel sparse kernel-based interpolation*, Ph.D. Thesis, University of Leicester, 2011.