THE MULTI-LEVEL SPARSE GRID INTERPOLATION KERNEL COLLOCATION (MUSIK-C) ALGORITHM, APPLIED TO BASKET OPTIONS

NICHOLAS WILTON

1. Introduction

2. Theoretical Background

2.1. Radial Basis Functions.

2.2. Spectral Methods.

2.3. Finite Difference.

2.4. Method of Lines.

2.5. Sparse Grid Collocation.

3. Algorithm Details

3.1. SiK-c Algo.

3.2. MuSik-c Algo.

3.3. Computational Complexity Comparisons.

4. Implentation Details

4.1. Eigen API and MatLab comparison.

4.2. Eigen Expression Trees vs MatLab.

4.3. CUDA parallelisation vs Threading.

5. Appendices

5.1. Code Repositories.

5.2. Supporting Documents.

Numerical Experiments

6. Lagrange Functions in Buhmann Form

This note contains my view of the now classical Buhmann cardinal function theory, i.e. the Lagrange function theory, all of which is a footnote to the Poisson Summation Formula, in the best possible sense. All of this is probably somewhere in [1], albeit implicitly or in changed guise. We begin with the univariate theory for simplicity.

7. Lagrange Functions on $\mathbb{Z}$

We begin with the classical Poisson Summation Formula in one dimension, then develop the Buhmann form of the Lagrange function on $\mathbb{Z}$.

7.1. **The Poisson Summation Formula on $\mathbb{Z}$.** Let $f \in S(\mathbb{R})$, to avoid analytical inconvenience. We need the classical form of the Poisson Summation Formula. To this end, we define $\mathbb{T} := [-\pi, \pi]$ and define the $\mathbb{T}$-*periodization* of f by

$$P_{\mathbb{T}} f(x) \equiv Pf(x) := \sum_{j\in\mathbb{Z}} f(x + 2\pi j). \tag{1}$$

Theorem 7.1.

$$Pf(x) = \sum_{j\in\mathbb{Z}} f(x+2\pi j) = (2\pi)^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ikx}. \tag{2}$$

Proof. The smoothness and decay of f imply that the Fourier series

$$Pf(x) = \sum_{\ell\in\mathbb{Z}} c_\ell e^{i\ell x}$$

converges absolutely and uniformly. Further,

$$\begin{aligned} c_\ell &= (2\pi)^{-1} \int_{-\pi}^{\pi} Pf(x) e^{-i\ell x}\, dx \\ &= (2\pi)^{-1} \int_{-\pi}^{\pi} \left(\sum_{j\in\mathbb{Z}} f(x+2\pi j) \right) e^{-i\ell x}\, dx \\ &= (2\pi)^{-1} \int_{\mathbb{R}} f(x) e^{-i\ell x}\, dx \\ &= (2\pi)^{-1} \widehat{f}(\ell), \end{aligned}$$

using the Dominated Convergence Theorem to justify the interchange of summation and integration. □

If we replace f by $\widehat{f}$ in Theorem 7.1, then we obtain a *dual* Poisson Summation Formula, as it were.

Corollary 7.2. *We have*

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi k) = \sum_{j\in\mathbb{Z}} f(j) e^{-ij\xi}. \tag{3}$$

Proof. Replace f by $\widehat{f}$ in Theorem 7.1, recalling that $\widehat{\widehat{f}}(x) = 2\pi f(-x)$. □

7.2. **The Lagrange Function on $\mathbb{Z}$.** Let $\phi \in S(\mathbb{R})$, to avoid all analytic inconvenience. We also want to choose ϕ with interpolation in mind, so we shall also assume that its Fourier transform $\widehat{\phi}$ is strictly positive, which implies that f is a strictly positive definite function.

We want to construct a function $L \in \mathrm{Span}_{k\in\mathbb{Z}}\, \phi(\cdot - k)$ for which $L(j) = \delta_{oj}$, for $j \in \mathbb{Z}$. Such a function will be called the Lagrange function, by analogy with the Lagrange form of the interpolating polynomial. Thus, proceeding formally for the moment, we have

$$L(x) = \sum_{k\in\mathbb{Z}} \lambda_k \phi(x-k) \tag{4}$$

or, in the Fourier domain,

$$\widehat{L}(\xi) = \left(\sum_{k\in\mathbb{Z}} \lambda_k e^{-ik\xi} \right) \widehat{\phi}(\xi) \tag{5}$$

It is not obvious that (5) is well defined, but we shall soon show that all is well. We periodize both sides to form a 2π–periodic function and, using the Poisson Summation Formula in the form of Corollary 7.2, we find We obtain

$$1 \equiv \sum_{\ell\in\mathbb{Z}} L(\ell)e^{-i\ell\xi} = \sum_{j\in\mathbb{Z}} \widehat{L}(\xi+2\pi j) = \left(\sum_{k\in\mathbb{Z}} \lambda_k e^{-ik\xi}\right)\left(\sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi+2\pi j)\right). \tag{6}$$

Hence, recalling that $\widehat{\phi}(\xi) > 0$, for all $\xi \in \mathbb{R}$, (6) implies

$$\sum_{k\in\mathbb{Z}} \lambda_k \exp(-ik\xi) = \frac{1}{\sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi+2\pi j)}. \tag{7}$$

Substituting (7) in (5), we obtain the Buhmann form of the Fourier transform of the Lagrange function, that is,

$$\widehat{L}(\xi) = \frac{\widehat{\phi}(\xi)}{\sum_{j\in\mathbb{Z}} \widehat{\phi}(\xi+2\pi j)}. \tag{8}$$

8. Lagrange Functions on $h\mathbb{Z}$

We follow the same route as before.

8.1. **The Poisson Summation Formula on $h\mathbb{Z}$.** We could deduce the scaled version of the Poisson Summation Formula directly from Theorem 7.1, but I prefer to begin *ab initio*. We shall now periodize f over $h^{-1}\mathbb{T}$, for $h > 0$, i.e. we define

$$P_{h^{-1}\mathbb{T}} f(x) = \sum_{j\in\mathbb{Z}} f(x + 2\pi h^{-1} j). \tag{9}$$

The *scaled exponentials* are the $2\pi h^{-1}$-periodic functions defined by

$$e_j^h(x) := e^{ihjx}, \qquad j \in \mathbb{Z}, \tag{10}$$

and form a complete orthonormal set with respect to the inner product

$$\langle F, G\rangle = \frac{1}{2\pi h^{-1}} \int_{-\pi h^{-1}}^{\pi h^{-1}} F(s)G(s)^*\, ds. \tag{11}$$

In other words, $\{e_j^h : j \in \mathbb{Z}\}$ forms a complete orthonormal set for $L^2(h^{-1}\mathbb{T})$ endowed with the normalized inner product

$$\langle F, G\rangle = \frac{1}{\mathrm{Vol}_1\, h^{-1}\mathbb{T}} \int_{h^{-1}\mathbb{T}} FG^*.$$

Theorem 8.1. *We have*

$$P_{h^{-1}\mathbb{T}} f(x) = \sum_{j\in\mathbb{Z}} f(x + 2\pi h^{-1} j) = \left(2\pi h^{-1}\right)^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}(kh) e^{ihkx}, \tag{12}$$

i.e.

$$P_{h^{-1}\mathbb{T}} f = \left(\mathrm{Vol}_1\, h^{-1}\mathbb{T}\right)^{-1} \sum_{k\in\mathbb{Z}} \widehat{f}(kh) e_k^{h^{-1}}. \tag{13}$$

Proof. As in Theorem 7.1, the Fourier series

$$P_{h^{-1}\mathbb{T}} f(x) = \left(\mathrm{Vol}_1\, h^{-1}\mathbb{T}\right)^{-1} \sum_{\ell\in\mathbb{Z}} c_\ell^h e^{ih\ell x}$$

converges absolutely and uniformly, and

$$\begin{aligned} c_\ell^h &= \left(2\pi h^{-1}\right)^{-1} \int_{-\pi/h}^{\pi/h} P_h f(x) e^{-ih\ell x}\, dx \\ &= \left(2\pi h^{-1}\right)^{-1} \int_{-\pi h^{-1}}^{\pi h^{-1}} \left(\sum_{j\in\mathbb{Z}} f(x + 2\pi h^{-1} j) \right) e^{-ih\ell x}\, dx \\ &= \left(2\pi h^{-1}\right)^{-1} \int_{\mathbb{R}} f(x) e^{-ih\ell x}\, dx \\ &= \left(2\pi h^{-1}\right)^{-1} \widehat{f}(h\ell), \end{aligned}$$

using the Dominated Convergence Theorem to justify the interchange of summation and integration. □

The analogous form of Corollary 7.2 is now fairly clear.

Corollary 8.2.

$$\sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi h^{-1} k) = h \sum_{\ell\in\mathbb{Z}} f(h\ell) e^{-ih\ell\xi}. \tag{14}$$

8.2. **The Lagrange Function on $h\mathbb{Z}$.** We now consider the Lagrange function when interpolating on the scaled integer grid $h\mathbb{Z}$, for $h > 0$. Thus we define

$$\phi_h(x) := \phi(h^{-1}x) \tag{15}$$

and we now want to construct a function $L^h \in \operatorname{Span}_{k\in\mathbb{Z}} \phi_h(\cdot - kh)$ for which $L^h(jh) = \delta_{oj}$, for $j \in \mathbb{Z}$. It is almost obvious that $L^h(x) = L(h^{-1}x)$, but the Fourier analysis is satisfying. Thus we consider

$$L^h(x) = \sum_{k\in\mathbb{Z}} \lambda_k^h \phi_h(x - kh) \tag{16}$$

or, in the Fourier domain,

$$\widehat{L^h}(\xi) = \left(\sum_{k\in\mathbb{Z}} \lambda_k^h e^{-ihk\xi} \right) \widehat{\phi^h}(\xi) \tag{17}$$

We must now periodize (17) to obtain a $2\pi h^{-1}$-periodic function, using the scaled Poisson Summation Formula, i.e.

$$\sum_{k\in\mathbb{Z}} \widehat{L^h}(\xi + 2\pi h^{-1} k) = h \sum_{\ell\in\mathbb{Z}} L^h(h\ell) e^{-ih\ell\xi} \equiv h.$$

Hence (17) becomes

$$1 \equiv h^{-1} \sum_{k\in\mathbb{Z}} \widehat{L^h}(\xi + 2\pi h^{-1} k) = h^{-1} \left(\sum_{k\in\mathbb{Z}} \lambda_k^h e^{-ihk\xi} \right) \sum_{m\in\mathbb{Z}} \widehat{\phi^h}(\xi + 2\pi h^{-1} m) \tag{18}$$

Eliminating $\sum \lambda_k^h \exp(-ihk\xi)$ from (17) and (18), we obtain

$$\widehat{L^h}(\xi) = \frac{\widehat{\phi^h}(\xi)}{h^{-1} \sum_{m\in\mathbb{Z}} \widehat{\phi^h}(\xi + 2\pi h^{-1} m)} = h \left(\frac{\widehat{\phi}(h\xi)}{\sum_{m\in\mathbb{Z}} \widehat{\phi}(h\xi + 2\pi m)} \right) = h \widehat{L}(h\xi). \tag{19}$$

Hence $L^h(x) = L(h^{-1}x)$, as expected.

9. Lagrange functions on $A\mathbb{Z}^d$ for $A \in \mathrm{GL}(\mathbb{R}^d)$

Let $A \in \mathrm{GL}(\mathbb{R}^d)$ and $f \in S(\mathbb{R}^d)$. shall be using the normalized inner product on $L^2(A^{-1}\mathbb{T}^d)$, that is,

$$\langle F, G \rangle = \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} F(x)G(x)^*\, dx \tag{20}$$

and $\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d = (2\pi)^d |A|^{-1}$. The $A^{-1}\mathbb{T}^d$-periodic exponentials providing our complete orthonormal sequence are given by

$$e_k^A(x) = e^{i\langle k, Ax\rangle}, \qquad k \in \mathbb{Z}^d. \tag{21}$$

9.1. The Poisson Summation Formula on $A\mathbb{Z}^d$.

We define the $A^{-1}\mathbb{T}^d$-periodization $P_{A^{-1}\mathbb{T}}f$ by

$$P_{A^{-1}\mathbb{T}^d} f(x) = \sum_{j \in \mathbb{Z}^2} f(x + 2\pi A^{-1} j). \tag{22}$$

Theorem 9.1.

$$P_{A^{-1}\mathbb{T}^d} f(x) = \left(\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d\right)^{-1} \sum_{\ell \in \mathbb{Z}^d} \widehat{f}(A^T \ell) e_{ell}^A(xi). \tag{23}$$

Proof. We have the Fourier series

$$P_{A^{-1}\mathbb{T}^d} f(x) = \sum_{k \in \mathbb{Z}^d} c_k^A e_k^A(x),$$

where

$$\begin{aligned} c_k^A &= \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} P_{A^{-1}\mathbb{T}^d} f(x) e_{-k}^A(x)\, dx \\ &= \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} \int_{\mathbb{R}^d} f(x) e_{-k}^A(x)\, dx \\ &= \frac{1}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} \int_{\mathbb{R}^d} f(x) e^{-i\langle A^T k, x\rangle}\, dx \\ &= \frac{\widehat{f}(A^T k)}{\mathrm{Vol}_d\, A^{-1}\mathbb{T}^d} \int_{A^{-1}\mathbb{T}^d} \end{aligned}$$

□

The dual form takes a similar form.

Corollary 9.2.

$$\sum_{k \in \mathbb{Z}^d} \widehat{f}(\xi + 2\pi A^{-T} k) = |A| \sum_{\ell \in \mathbb{Z}^d} f(A\ell) e^{-i\langle \ell, A^T \xi\rangle}. \tag{24}$$

9.2. The Lagrange Function on $A\mathbb{Z}$.

We define $\phi_A(x) = \phi(A^{-1}x)$, for $x \in \mathbb{R}^d$. Then $\widehat{\phi^A}(\xi) = |A| \widehat{\phi}(A^T \xi)$, and it is again almost obvious that $L^A(x) = L(A^{-1}x)$, so that $\widehat{L^A}(\xi) = |A| \widehat{L}(A^T \xi)$. For completeness, we again provide the full Fourier derivation.

Theorem 9.3. *The Fourier transform of the Lagrange function L^A is given by*

$$\widehat{L^A}(\xi) = \frac{\widehat{\phi^A}(\xi)}{|A|^{-1} \sum_{k \in \mathbb{Z}^d} \widehat{\phi^A}(\xi + 2\pi A^{-T} k)}. \tag{25}$$

References

[1] Buhmann (2003), *Radial Basis Functions*, CUP.