9.1 Solutions

1 (a) True (because $A + B = B + A$).
(b) False.
(c) True.

2 $EFG = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ and $GFE = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 8 & 4 & 2 & 1 \end{bmatrix}$.

3 All such matrices other than $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have columns or rows that are multiples (-1 times or 1 times) of each other. So these are the only invertible ones.

4 A's column space is a *line* — all vectors of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$ (the x-axis). The column space of B is also a *line* generated by the $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ vector; that is all vectors of the form $\begin{bmatrix} x \\ 2x \end{bmatrix}$. For C, the column space is the entire $\mathbb{R}^2$ *plane.*

5 Since $(x, y) = (2, 5)$ lies on the $y = mx + c$ line, we have $5 = m \cdot 2 + c$. Similarly, for $(x, y) = (3, 7)$ we have $7 = m \cdot 3 + c$. This corresponds to the system $\begin{cases} 2m + c = 5 \\ 3m + c = 7 \end{cases}$, whose matrix form is $\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ and whose solution is $m = 2$ and $c = 1$ (a line with slope 2).

6 Note how with successive powers of A the 2s turn to $2^2, 2^3$, etc. and move up north-east:

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \; A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdots$$

All higher powers of A give a 4 by 4 matrix of zeros.

7 AX will be the identity matrix $I = [A\boldsymbol{x}_1 \; A\boldsymbol{x}_2 \; A\boldsymbol{x}_3]$ when $X = [\boldsymbol{x}_1 \; \boldsymbol{x}_2 \; \boldsymbol{x}_3]$.

8 $A^{-1} = \dfrac{1}{a(a-b)} \begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$. As long as $a \neq 0$ *and* $a - b \neq 0$, A^{-1} exists.

9 (a)

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 2 & 4 & 8 & 12 & b_2 \\ 3 & 6 & 7 & 13 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & -2 & -2 & b_3 - 3b_1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & b_1 \\ 0 & 0 & 2 & 2 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + b_2 - 5b_1 \end{array}\right] \tag{9.1}$$

(b) From (a) we get the solvability condition $b_3 + b_2 - 5b_1 = 0$, which makes the last row $0 = 0$.

(c) The column space is the plane in $\mathbb{R}^3$ containing all combinations of the pivot columns $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 8 \\ 7 \end{bmatrix}$.
The pivots are in columns 1 and 3.

(d) The special solutions come from $\mathbb{R}^4$ and have free variables $x_2 = 1, x_4 = 0$ and then $x_2 = 0, x_4 = 1$:

$$\text{All combinations of the two special solutions:} \quad \boldsymbol{x}_n = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(the nullspace $N(A)$ in $\mathbb{R}^4$)

(e) Continue row reductions upward to find the reduced form $[\,R\ \boldsymbol{d}\,]$. Starting from from 9.1 above, the right-hand side is $\begin{bmatrix} b_1 \\ b_2 - b_1 \\ b_3 + b_2 - 5b_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 - 0 \\ (-6) + 6 - 5 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$ (because the original vector $\boldsymbol{b}$ was equal to $\begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$):

$$[\,U\ \boldsymbol{c}\,] = \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [\,R\ \boldsymbol{d}\,].$$

So the right-hand side becomes $\boldsymbol{d} = \begin{bmatrix} -9 \\ 3 \\ 0 \end{bmatrix}$, showing -9 and 3 in the particular solution $\boldsymbol{x}_p$ (shown in part (f)).

(f) The particular solution is obtained by setting all free variables to zero and back substituting in $[\,R\ \boldsymbol{d}\,]$ to find the pivots is thus $\boldsymbol{x}_p = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix}$ The complete solution to $A\boldsymbol{x} = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$ is then $\boldsymbol{x}_p + \boldsymbol{x}_n$:

$$\text{The complete solution:} \quad \boldsymbol{x} = \begin{bmatrix} -9 \\ 0 \\ 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

10 (a) The dimension of the nullspace is 1, so A must have $4 - 1 = 3$ pivots.

(b) The complete solution to $A\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ (for any real number x_3).

(c) The row reduced echelon form has 3 pivots and the special solution $\boldsymbol{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$, so

$$R = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11 (a) The row operations are:

- E adds 10 times row 2 to row 3
- F adds 11 times row 3 to row 4
- G swaps rows 1 and 2
- H adds 4 times row 4 to row 1

(b) The upper right corner of A is zero because if it were anything other than zero we would never have to swap rows 1 and 2 (which is what G does).

(c) The upper triangular U corresponds to the upward elimination, so we have

$$U = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The permutation matrix P corresponds to the permutation of the first two rows, so we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The lower triangular L corresponds to the first two elimination steps combined (the product FE):

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 110 & 11 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -10 & 1 & 0 \\ 0 & 0 & -11 & 1 \end{bmatrix}$$

12 A's characteristic equation: $\det \begin{bmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{bmatrix} = 0 \quad \Rightarrow \quad (\lambda + 2)^2(\lambda - 4) = 0$ gives $\lambda_{1,2} = -2$ (a repeated eigenvalue of algebraic multiplicity 2) and $\lambda_3 = 4$.

- *Eigenvectors for $\lambda_3 = 4$:* $\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

  Gaussian elimination gives $\begin{bmatrix} -4 & 2 & 2 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and we pick the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

- *Eigenvectors for $\lambda_{1,2} = -2$:* $\begin{bmatrix} 0-(-2) & 2 & 2 \\ 2 & 0-(-2) & 2 \\ 2 & 2 & 0-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Gaussian elimination of which gives $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Here we have 2 free variables (x_3 and x_2) and one pivot (x_1). Setting the two free variables as usual (setting one free variable to 1, the other to 0, and determining the pivot), we see that the space of eigenvectors corresponding to $\lambda_2 = -2$ *is two-dimensional*: $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

So the matrix S is $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $S^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. Thus A can be diagonalized as

$$A = S\Lambda S^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

And A^8 can be computed as

$$A^8 = S\Lambda^8 S^{-1} = \frac{1}{3} \cdot \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4^8 & 0 & 0 \\ 0 & (-2)^8 & 0 \\ 0 & 0 & (-2)^8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix},$$

which works out to $A^8 = \begin{bmatrix} 22016 & 21760 & 21760 \\ 21760 & 22016 & 21760 \\ 21760 & 21760 & 22016 \end{bmatrix}$.

13 There's more than one way to argue why $A = B$, but one way is to note that the decompositions of $A = S\Lambda S^{-1}$ and $B = S\Lambda S^{-1}$ are identical because their S matrices are made up of the eigenvectors that are the same and their Λ matrices have the same entries because their eigenvalues are the same.

14 (a) Because both columns add up to 1, A has an eigenvalue $\lambda_1 = 1$. To find the other eigenvalue, recall that A's eigenvalues always add up to A's trace (which is the sum of A's diagonal entries). Checking the trace, $\texttt{trace}(A) = .4 + c$, we see that the other eigenvalue must be $\lambda_2 = .4 + c - \lambda_1 = .4 + c - 1 = c - .6$.

(b) If $c = 1.6$, then both eigenvalues of A are 1. The eigenvectors of $A = \begin{bmatrix} .4 & -.6 \\ .6 & 1.6 \end{bmatrix}$ are all multiples of $\boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (that is, $\lambda = 1$ has geometric multiplicity equal to 1).

15 You want to make $A = S\Lambda^{1/2}S^{-1}$, where S is the eigenvector matrix shared by both A and B (recall that the eigenvalues of A^2 are the squares of the eigenvalues of A, and the eigenvectors of A^2 are the same as the eigenvectors of A). So, find the eigenvectors and put them in S to get

$$A = S\Lambda^{1/2}S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \cdot \left(\frac{1}{2}\right) \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{has } A^2 = B.$$

16 $A = S\Lambda S^{-1}$ approaches zero if and only if every $|\lambda| < 1$. A_1 is column stochastic and has an eigenvalue $= 1$, so it will *not* approach the zero matrix. But A_2 will because it has $\lambda = .9 < 1$ and $\lambda = .3 < 1$.

17 (a)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Columns $1, 2$ and 4 have pivots. Because of the nonzero bottom-right element in $A - 3I$, the fourth component of $\boldsymbol{x}_3$ is definitely zero.

(b) In the same way as above, the special solutions for the matrices $A-1I, A-2I, A-3I$, and $A-4I$ must have $3, 2, 1$, and 0 zeros as the last components. The eigenvector matrix S is then upper triangular.

18 $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$. Notice that $6 = 1 + 2 + 3$ (the sum of eigenvalues) and determinant is $6 = 1 \cdot 2 \cdot 3$ (the product of eigenvalues).