9 Sample Test Questions With Solutions

Problem 1. True or false? Let A and B be square matrices.

(a) $(A+B)^2 = (B+A)^2$

(b) If columns 1 and 3 of B are the same, then so are the columns 1 and 3 of BA.

(c) If rows 1 and 3 of A are the same, so are rows 1 and 3 of AB.

Problem 2. Find the products EFG and GFE if (upper triangular entries are all zeros)

$$E = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 2 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad G = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

Problem 3. There are 16 2-by-2 matrices $\begin{bmatrix} * & * \\ * & * \end{bmatrix}$ whose $*$ entries are 0s or 1s. How many of them are invertible?

Problem 4. Describe the column spaces of the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Clearly indicate whether the column space is a line or the $\mathbb{R}^2$ plane.

Problem 5. Write the following problem in matrix form and solve it: "Two points with coordinates (x, y) are $(2, 5)$ and $(3, 7)$. The two points lie on the line $y = mx + c$. Find m and c."

Problem 6. Compute A^3 and A^{2166} where $A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Problem 7. Suppose you solve $A\boldsymbol{x} = \boldsymbol{b}$ for the three right-hand sides shown below

$$A\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If the solutions $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are the columns of a matrix X, what is A times X? (*Hint:* Recall how the Gauss-Jordan way of inverting a matrix works.)

Problem 8. Carry out Gauss-Jordan elimination to find the inverse of $A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$. Then prove that A is invertible if $a \neq 0$ and $a \neq b$.

Problem 9. Consider the $A\boldsymbol{x} = \boldsymbol{b}$ system $\begin{cases} x_1 + 2x_2 + 3x_3 + \ \ 5x_4 = b_1 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 = b_2 \\ 3x_1 + 6x_2 + 7x_3 + 13x_4 = b_3 \end{cases}$.

(a) Reduce $[\,A\ \boldsymbol{b}\,]$ to $[\,U\ \boldsymbol{c}\,]$ so that $A\boldsymbol{x} = \boldsymbol{b}$ becomes a triangular system $U\boldsymbol{x} = \boldsymbol{c}$.

(b) Find the conditions on b_1, b_2, b_3 for $A\boldsymbol{x} = \boldsymbol{b}$ to have a solution.

(c) Describe the column space of A. Is it a line or is it a plane. Is it in $\mathbb{R}^3$?

(d) Describe the nullspace of A by writing down the special solutions. Is the nullspace in $\mathbb{R}^4$?

(e) Reduce $[\,U\ \boldsymbol{c}\,]$ to $[\,R\ \boldsymbol{d}\,]$ and determine the special solutions and the particular solution.

(f) Find a particular solution to $A\boldsymbol{x} = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$ and then the complete solution.

Problem 10. Suppose you know that the 3 by 4 matrix A has all the multiples of $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ as the only vectors in its nullspace.

(a) How many pivot variables does A have?

(b) Write down the complete solution to $A\boldsymbol{x} = \mathbf{0}$.

(c) What is the exact row reduced echelon form of A?

Problem 11. Recall that the allowed row operations of the Gaussian elimination algorithm correspond to the so-called elementary matrices. Elementary matrices E, F, G, and H left-multiply matrix

A in the order shown below. The result is the 4×5 matrix R on the right of the $=$ sign.

$$\underbrace{\begin{bmatrix}1&0&0&4\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{H}\underbrace{\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{G}\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&11&1\end{bmatrix}}_{F}\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&10&1&0\\0&0&0&1\end{bmatrix}}_{E}A=\underbrace{\begin{bmatrix}1&0&0&0&1\\0&1&0&0&2\\0&0&1&0&3\\0&0&0&1&4\end{bmatrix}}_{R}$$

(a) Which row operations do the four elimination matrices in the product correspond to? Please write down in words the operations corresponding to the matrices E, F, G, and H.

(b) What must be the upper left hand corner of A (the entry in the first row and the first column of A)? Why must it be zero?

(c) The equation implies that A factors as $A = LPUR$. Here R is the matrix on the right of the $=$ sign. The matrices U, P, and L are invertible 4×4 matrices: U is upper triangular, P is a permutation matrix, and L is lower triangular. Find U, P, and L without a calculator and explain how you got them.

Problem 12. Compute the eigenvalues and eigenvectors of A and represent it as a product $S\Lambda S^{-1}$ where Λ is a diagonal eigenvalue matrix and S the corresponding eigenvector matrix. Using that representation compute A^8, showing your steps.

$$A = \begin{bmatrix}0&2&2\\2&0&2\\2&2&0\end{bmatrix}$$

Problem 13. Suppose A and B have the same eigenvalues $\lambda_1, \ldots, \lambda_n$ and the same independent eigenvectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$. Is it true that then A must equal B? If it is true, prove it; if it is not, provide a counterexample with 2 by 2 matrices.

Problem 14. Let $\begin{bmatrix}.4 & 1-c\\ .6 & c\end{bmatrix}$ where c is some real number.

(a) Find the eigenvalues and eigenvectors of in terms of c.

(b) Show that A has just one line of eigenvectors when $c = 1.6$.

Problem 15. $B = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ has eigenvalues 1 and 9. Find matrix A such that $A^2 = B$.

Problem 16. Which of these matrices approaches the zero matrix as $n \to \infty$?

$$A_1 = \begin{bmatrix} .6 & .9 \\ .4 & .1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} .6 & .9 \\ .1 & .6 \end{bmatrix}$$

Hint: Try the $A = S\Lambda S^{-1}$ decomposition and look at the eigenvalues λ_1 and λ_2 on the diagonal of Λ.

Problem 17. Suppose A is a 4 by 4 upper triangular matrix with $1, 2, 3, 4$ on its main diagonal. (You can put all 1s above the diagonal.)

(a) For $A - 3I$, which columns have pivots? Which components of the eigenvector $\boldsymbol{x}_3$ (the special solution in the nullspace) are definitely zero?

(b) Using part (a), show that the eigenvector matrix is also upper triangular.

Problem 18. Choose the last row of A so that its eigenvalues are $1, 2, 3$:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix}.$$