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Project 1

Preliminary Report

Integrity Statement

On my honor, I have neither given nor received assistance on this project.

Payton Glynn   Payton Glynn

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a.) $\vec{r}_3 = x\hat{b}_1 + y\hat{b}_2$

$$\vec{V}_3 = \dot{\vec{r}}_3 = \dot{x}\hat{b}_1 + x\dot{\hat{b}}_1 + \dot{y}\hat{b}_2 + y\dot{\hat{b}}_2$$

$$\dot{\hat{b}}_1 = \vec{\omega}_{B/N} \times \hat{b}_1 = \dot{\theta}\hat{b}_3 \times \hat{b}_1 = \dot{\theta}\hat{b}_2$$

where $\theta = \omega t$, so $\dot{\theta} = \omega$

$$\dot{\hat{b}}_2 = \vec{\omega}_{B/N} \times \hat{b}_2 = \dot{\theta}\hat{b}_3 \times \hat{b}_2 = -\dot{\theta}\hat{b}_1$$

Plug back into $\vec{V}_3$:

$$\vec{V}_3 = \dot{x}\hat{b}_1 + \dot{y}\hat{b}_2 + x(\omega)\hat{b}_2 + y(-\omega)\hat{b}_1 = (\dot{x} - \omega y)\hat{b}_1 + (\dot{y} + \omega x)\hat{b}_2$$

$$T = \frac{1}{2}m_3(\vec{V}_3 \cdot \vec{V}_3) = \frac{1}{2}m_3(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2$$

$$T = \frac{1}{2}m_3\left[(\dot{x}^2 - 2\dot{x}\omega y + \omega^2 y^2) + (\dot{y}^2 + 2\dot{y}\omega x + \omega^2 x^2)\right]$$

$$V = -G\frac{m_3 m_1}{p_1} - G\frac{m_3 m_2}{p_2}$$

$$\boxed{L = T - V = \frac{1}{2}m_3\left[(\dot{x}^2 - 2\dot{x}\omega y + \omega^2 y^2) + (\dot{y}^2 + 2\dot{y}\omega x + \omega^2 x^2)\right] + G\frac{m_3 m_1}{p_1} + G\frac{m_3 m_2}{p_2}}$$

where $p_1 = \sqrt{(r_1 + x)^2 + y^2}$ & $p_2 = \sqrt{(r_2 - x)^2 + y^2}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

Calculate each derivative:

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m_3\left[(2\dot{x} - 2\omega y)\right] \quad \Big| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{1}{2}m_3(2\ddot{x} - 2\omega\dot{y})$$

$$\frac{\partial L}{\partial x} = \frac{1}{2}m_3\left[2\dot{y}\omega + 2\omega^2 x\right] + Gm_3m_1\left(-\frac{1}{2}\right)\left[(r_1 + x)^2 + y^2\right]^{-3/2} \cdot 2(r_1 + x)$$

$$+ Gm_3m_2\left(-\frac{1}{2}\right)\left[(r_2 - x)^2 + y^2\right]^{-3/2} \cdot 2(r_2 - x) \cdot (-1)$$

$$\frac{1}{2}m_3(2\ddot{x}-2\omega\dot{y})-\Big[\frac{1}{2}m_3(2\dot{y}\omega+2\omega^2x)-Gm_3m_1\left[(r_1+x_1)^2+y^2\right]^{-3/2}(r_1+x)$$

$$+Gm_3m_2\left[(r_2-x)^2+y^2\right]^{-3/2}(r_2-x)\Big]=0$$

$$\frac{1}{2}m_3(2\ddot{x}-2\omega\dot{y})-\left[\frac{1}{2}m_3(2\dot{y}\omega+2\omega^2x)-\frac{Gm_3m_1(r_1+x)}{((r_1+x)^2+y^2)^{3/2}}+\frac{Gm_3m_2(r_2-x)}{((r_2-x)^2+y^2)^{3/2}}\right]=0$$

$$\cancel{\tfrac{1}{2}}\cancel{m_3}(\cancel{2}\ddot{x}-\cancel{2}\omega\dot{y})-\cancel{\tfrac{1}{2}}\cancel{m_3}(\cancel{2}\dot{y}\omega+\cancel{2}\omega^2x)+\frac{G\cancel{m_3}m_1(r_1+x)}{((r_1+x)^2+y^2)^{3/2}}-\frac{G\cancel{m_3}m_2(r_2-x)}{((r_2-x)^2+y^2)^{3/2}}=0$$

$$\ddot{x}-2\omega\dot{y}-\omega^2x=\underbrace{-\frac{Gm_1(r_1+x)}{p_1^3}}+\underbrace{\frac{Gm_2(r_2-x)}{p_2^3}}$$

$$\frac{Gm_2(r_2-x)}{p_2^3}=\frac{-Gm_2(x-r_2)}{p_2^3}=\frac{-Gm_2(x-1+\mu)}{p_2^3}=-G\left(\frac{m_1+m_2}{m_1+m_2}\right)\frac{m_2(x-1+\mu)}{p_2^3}$$

$$\frac{-G(m_1+m_2)\mu(x-1+\mu)}{(r_1+r_2)^3\ p_2^3}=\frac{-\omega^2\mu(x-1+\mu)}{p_2^3}$$

$$\frac{-Gm_1(r_1+x)}{p_1^3}=\frac{-Gm_1(x+\mu)}{p_1^3}=-G\left(\frac{m_1+m_2}{m_1+m_2}\right)\frac{m_1(x+\mu)}{p_1^3}$$

$$=\frac{-G(m_1+m_2)\ m_1\ (x+\mu)}{(r_1+r_2)^3\cdot(m_1+m_2)\ p_1^3}=\frac{-\omega^2m_1(x+\mu)}{(m_1+m_2)\ p_1^3}=\frac{-\omega^2(1-\mu)(x+\mu)}{p_1^3}$$

$$\boxed{\ddot{x}-2\omega\dot{y}-\omega^2x=-\omega^2\frac{(1-\mu)(x+\mu)}{p_1^3}-\omega^2\frac{\mu(x-1+\mu)}{p_2^3}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \qquad p_1 = \sqrt{(r_1+x)^2+y^2} \quad \& \quad p_2 = \sqrt{(r_2-x)^2+y^2}$$

$$L = T - V = \tfrac{1}{2}m_3\left[(\dot{x}^2 - 2\dot{x}\omega y + \omega^2 y^2) + (\dot{y}^2 + 2\dot{y}\omega x + \omega^2 x^2)\right] + G\frac{m_3 m_1}{p_1} + G\frac{m_3 m_2}{p_2}$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}m_3\left[2\dot{y} + 2\omega x\right] \quad \Bigg| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{1}{2}m_3\left[2\ddot{y} + 2\omega\dot{x}\right]$$

$$\frac{\partial L}{\partial y} = \frac{1}{2}m_3\left[-2\dot{x}\omega + 2\omega^2 y\right] + Gm_1m_3\left(-\tfrac{1}{2}\right)\left[(r_1+x)^2+y^2\right]^{-3/2}\cdot 2y$$

$$+ Gm_3m_2\left(-\tfrac{1}{2}\right)\left[(r_2-x)^2+y^2\right]^{-3/2}\cdot 2y$$

$$\cancel{\tfrac{1}{2}}\cancel{m_3}\left[\cancel{2}\ddot{y} + \cancel{2}\omega\dot{x}\right] - \left[\cancel{\tfrac{1}{2}}\cancel{m_3}\left(-\cancel{2}\dot{x}\omega + \cancel{2}\omega^2 y\right) - \frac{Gm_1\cancel{m_3}y}{\left[(r_1+x)^2+y^2\right]^{3/2}} - \frac{Gm_2\cancel{m_3}y}{\left[(r_2-x)^2+y^2\right]^{3/2}}\right] = 0$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{Gm_1 y}{\underbrace{\left[(r_1+x)^2+y^2\right]^{3/2}}_{p_1^3}} - \frac{Gm_2 y}{\underbrace{\left[(r_2-x)^2+y^2\right]^{3/2}}_{p_2^3}}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{Gm_1 y}{p_1^3} - \frac{Gm_2 y}{p_2^3}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{\left[G(m_1+m_2) - Gm_2\right]y}{p_1^3} - \left[\frac{Gm_2(m_1+m_2)}{(r_1+r_2)^3(m_1+m_2)}\right]\frac{y}{p_2^3}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\left[\frac{G(m_1+m_2)}{(r_1+r_2)^3} - \frac{Gm_2}{(r_1+r_2)^3}\right]\frac{y}{p_1^3} - \left[\frac{G(m_1+m_2)}{(r_1+r_2)^2}\cdot\frac{m_2}{m_1+m_2}\right]\frac{y}{p_2^3}$$

$$\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\left[\frac{G(m_1+m_2)}{(r_1+r_2)^3}\left[1 - \frac{m_2}{m_1+m_2}\right]\right]\frac{y}{p_1^3} - \frac{\omega^2 \mu y}{p_2^3}$$

$$\boxed{\ddot{y} + 2\omega\dot{x} - \omega^2 y = -\frac{\omega^2(1-\mu)y}{p_1^3} - \frac{\omega^2 \mu y}{p_2^3}}$$

$t = \frac{1}{\omega}\tau$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{d\left(\frac{\tau}{\omega}\right)} = \omega \frac{dx}{d\tau} = \omega x'$$

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d^2x}{d\left(\frac{\tau}{\omega}\right)^2} = \omega^2 \frac{d^2x}{d\tau^2} = \omega^2 x''$$

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{d\left(\frac{\tau}{\omega}\right)} = \omega \frac{dy}{d\tau} = \omega y'$$

$$\ddot{y} = \frac{d^2y}{dt^2} = \frac{d^2y}{d\left(\frac{\tau}{\omega}\right)^2} = \omega^2 \frac{d^2y}{d\tau^2} = \omega^2 y''$$

Plug these into the $\ddot{x}$ & $\ddot{y}$ eqns above:

$$\cancel{\omega^2} x'' - 2\cancel{\omega}(\cancel{\omega} y') - \cancel{\omega^2} x = -\cancel{\omega^2}\frac{(1-\mu)(x+\mu)}{p_1^3} - \cancel{\omega^2}\frac{\mu(x-1+\mu)}{p_2^3}$$

$$\boxed{x'' - 2y' - x = -\frac{(1-\mu)(x+\mu)}{p_1^3} - \frac{\mu(x-1+\mu)}{p_2^3}}$$

$$\cancel{\omega^2} y'' + 2\cancel{\omega}(\cancel{\omega} x') - \cancel{\omega^2} y = -\frac{\cancel{\omega^2}(1-\mu)y}{p_1^3} - \cancel{\omega^2}\frac{\mu y}{p_2^3}$$

$$\boxed{y'' + 2x' - y = -\frac{(1-\mu)y}{p_1^3} - \frac{\mu y}{p_2^3}}$$

$$U = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{p_1} + \frac{\mu}{p_2}$$

$$p_1 = \sqrt{(r_1+x)^2+y^2}$$

$$\& \quad p_2 = \sqrt{(r_2-x)^2+y^2}$$

$$\frac{\partial U}{\partial x} = x + (1-\mu)\left[-\frac{1}{2}\left((r_1+x)^2+y^2\right)^{-3/2} 2(r_1+x)\right] + \mu\left[-\frac{1}{2}\left((r_2-x)^2+y^2\right)^{-3/2} 2(r_2-x)(-1)\right]$$

$$\frac{\partial U}{\partial x} = x + (1-\mu)\left[\frac{-(r_1+x)}{((r_1+x)^2+y^2)^{3/2}}\right] + \mu\left[\frac{r_2-x}{((r_2-x)^2+y^2)^{3/2}}\right]$$

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(r_1+x)}{p_1^3} + \frac{\mu(r_2-x)}{p_2^3}$$

now plug in $r_1 = \mu$ & $r_2 = (1-\mu)$

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{p_1^3} - \frac{\mu(x-1+\mu)}{p_2^3}$$

Therefore:

$$\boxed{x'' - 2y' = \frac{\partial U}{\partial x} = U_x}$$

$$p_1 = \sqrt{(r_1+x)^2+y^2}$$

$$\& \quad p_2 = \sqrt{(r_2-x)^2+y^2}$$

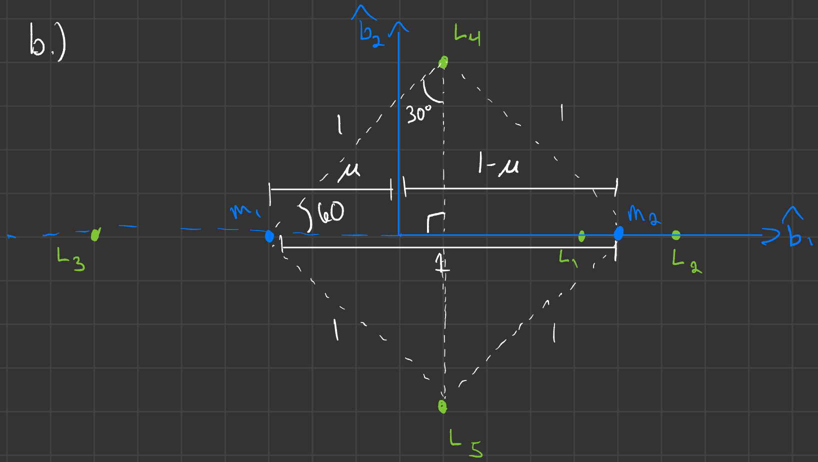
$$\frac{\partial U}{\partial y} = y + (1-\mu)\left[-\frac{1}{2}\left((r_1+x)^2+y^2\right)^{-3/2} \cdot 2y\right] + \mu\left[-\frac{1}{2}\left((r_1-x)^2+y^2\right)^{-3/2}(2y)\right]$$

$$\frac{\partial U}{\partial y} = y - \frac{(1-\mu)y}{p_1^3} - \frac{\mu y}{p_2^3}$$

Therefore:

$$\boxed{y'' + 2x' = \frac{\partial U}{\partial y} = U_y}$$

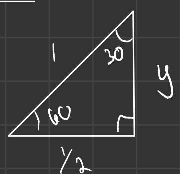
b.)



From part a, we know that the distance between m_1 and the origin is μ. The total distance between m_1 and m_2 is 1, so the distance between the origin and m_2 is $1-\mu$.

Therefore, we find that the x-coordinate of L_4 and L_5 is $\frac{1}{2}-\mu$.

For L_4:



$$y = \sin(60) \Rightarrow y = \frac{\sqrt{3}}{2}$$

- L_5 is the same y-coordinate, but negative

L_4: $\left(\frac{1}{2}-\mu, \frac{\sqrt{3}}{2}\right)$

L_5: $\left(\frac{1}{2}-\mu, -\frac{\sqrt{3}}{2}\right)$

Filling in the table:

- the y-coordinate for L_4 is always $\frac{\sqrt{3}}{2}$
- the y-coordinate for L_5 is always $-\frac{\sqrt{3}}{2}$

Sun-Earth: $x = 0.4999$

Earth-Moon: $x = 0.488$

Saturn-Titan: $x: 0.4998$

- x coordinate is the same for L_4 & L_5

Sun-Earth: $L_4 = (0.4999, \frac{\sqrt{3}}{2})$; $L_5 = (0.4999, -\frac{\sqrt{3}}{2})$

Earth-Moon: $L_4 = (0.488, \frac{\sqrt{3}}{2})$; $L_5 = (0.488, -\frac{\sqrt{3}}{2})$

Saturn-Titan: $L_4 = (0.4998, \frac{\sqrt{3}}{2})$; $L_5 = (0.4998, -\frac{\sqrt{3}}{2})$

c.) (7) $x'' - 2y' = \frac{\partial U}{\partial x} = U_x$

From Part A

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{((\mu+x)^2+y^2)^{3/2}} - \frac{\mu(x-1-\mu)}{((1-\mu-x)^2+y^2)^{3/2}}$$

(8) $y'' + 2x' = \frac{\partial U}{\partial y} = U_y$

$$\frac{\partial U}{\partial y} = y - \frac{(1-\mu)y}{((\mu+x)^2+y^2)^{3/2}} - \frac{\mu y}{((1-\mu-x)^2+y^2)^{3/2}}$$

$x'' = U_x + 2y'$ and $y'' = U_y - 2x'$

$$x'' = \cancel{f(x_0, y_0, y'_0)} + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y'}\delta y' = \cancel{x''_0} + \delta x''$$

$\frac{\partial f}{\partial x} = U_{xx}$ $\quad \frac{\partial f}{\partial y} = U_{xy}$ $\quad \frac{\partial f}{\partial x'} = 0$ $\quad \frac{\partial f}{\partial y'} = 2$

$$\delta x'' = U_{xx}\delta x + U_{xy}\delta y + (0)\delta x' + 2\delta y'$$

$$y'' = \cancel{f(x_0, y_0, y'_0)} + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial x'}\delta x' + \frac{\partial f}{\partial y'}\delta y' = \cancel{y''_0} + \delta y''$$

$\frac{\partial f}{\partial x} = U_{xy}$ $\quad \frac{\partial f}{\partial y} = U_{yy}$ $\quad \frac{\partial f}{\partial x'} = -2$ $\quad \frac{\partial f}{\partial y'} = 0$

$$\delta y'' = U_{xy}\delta x + U_{yy}\delta y - 2\delta x' + (0)\delta y'$$

$x' = 0 + \delta x'$ & $y' = 0 + \delta y'$

$$\delta\bar{x} = \begin{bmatrix} \delta x \\ \delta y \\ \delta x' \\ \delta y' \end{bmatrix}, \quad \delta\bar{x}' = \begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix} = \begin{bmatrix} 0\,\delta x + 0\,\delta y + 1\,\delta x' + 0\,\delta y' \\ 0\,\delta x + 0\,\delta y + 0\,\delta x' + 1\,\delta y' \\ U_{xx}\delta x + U_{xy}\delta y + 0\,\delta x' + 2\,\delta y' \\ U_{xy}\delta x + U_{yy}\,\delta y - 2\,\delta x' + 0\,\delta y' \end{bmatrix}$$

Therefore

$$\underbrace{\begin{bmatrix} \delta x' \\ \delta y' \\ \delta x'' \\ \delta y'' \end{bmatrix}}_{\delta\bar{x}'} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ U_{xx} & U_{xy} & 0 & 2 \\ U_{xy} & U_{yy} & -2 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \delta x \\ \delta y \\ \delta x' \\ \delta y' \end{bmatrix}}_{\delta\bar{x}}$$

$$\downarrow$$

$$\boxed{\delta\bar{x}' = A\,\delta\bar{x}}$$

Find U_{xx}, U_{yy}, U_{xy}:

$$\frac{\partial U}{\partial x} = x - \frac{(1-\mu)(x+\mu)}{((\mu+x)^2+y^2)^{3/2}} - \mu\frac{(x-1+\mu)}{((1-\mu-x)^2+y^2)^{3/2}}$$

$$\frac{\partial U}{\partial y} = y - \frac{(1-\mu)y}{((\mu+x)^2+y^2)^{3/2}} - \frac{\mu y}{((1-\mu-x)^2+y^2)^{3/2}}$$

$$U_{xx} = 1-(1-\mu)\left[\frac{\partial}{\partial x}\left(\frac{(x+\mu)}{((x+\mu)^2+y^2)^{3/2}}\right)\right] - \mu\left[\frac{\partial}{\partial x}\left(\frac{x-1+\mu}{((1-\mu-x)^2+y^2)^{3/2}}\right)\right]$$

$$U_{xx} = 1-(1-\mu)\left[\frac{(1)((x+\mu)^2+y^2)^{3/2} - (x+\mu)\cdot\frac{3}{\cancel{2}}((x+\mu)^2+y^2)^{1/2}\cdot\cancel{2}(x+\mu)}{((x+\mu)^2+y^2)^3}\right]$$

$$-\mu\left[\frac{(1)((1-\mu-x)^2+y^2)^{3/2} - (x-1+\mu)\cdot\frac{3}{\cancel{2}}((1-\mu-x)^2+y^2)^{1/2}\cdot\cancel{2}(1-\mu-x)(-1)}{((1-\mu-x)^2+y^2)^3}\right]$$

$$U_{xx} = 1-(1-\mu)\left[\frac{1}{((x+\mu)^2+y^2)^{3/2}} - \frac{3(x+\mu)^2}{((x+\mu)^2+y^2)^{5/2}}\right] - \mu\left[\frac{1}{((1-\mu-x)^2+y^2)^{3/2}} + \frac{3(x-1+\mu)(1-\mu-x)}{((1-\mu-x)^2+y^2)^{5/2}}\right]$$

$$\boxed{U_{xx} = 1 - \frac{(1-\mu)}{((x+\mu)^2+y^2)^{3/2}} + \frac{3(1-\mu)(x+\mu)^2}{((x+\mu)^2+y^2)^{5/2}} - \frac{\mu}{((1-\mu-x)^2+y^2)^{3/2}} + \frac{3\mu(1-\mu-x)^2}{((1-x-\mu)^2+y^2)^{5/2}}}$$

$$U_{yy} = 1-(1-\mu)\left[\frac{\partial}{\partial y}\left(\frac{y}{((x+\mu)^2+y^2)^{3/2}}\right)\right] - \mu\left[\frac{\partial}{\partial y}\left(\frac{y}{((1-\mu-x)^2+y^2)^{3/2}}\right)\right]$$

$$U_{yy} = 1-(1-\mu)\left[\frac{(1)((x+\mu)^2+y^2)^{3/2} - (y)\cdot\frac{3}{\cancel{2}}((x+\mu)^2+y^2)^{1/2}\cdot\cancel{2}y}{((x+\mu)^2+y^2)^3}\right]$$

$$-\mu\left[\frac{(1)((1-\mu-x)^2+y^2)^{3/2} - (y)\cdot\frac{3}{\cancel{2}}((1-\mu-x)^2+y^2)^{1/2}\cdot\cancel{2}y}{((1-\mu-x)^2+y^2)^3}\right]$$

$$\boxed{U_{yy} = 1 - \frac{(1-\mu)}{((x+\mu)^2+y^2)^{3/2}} + \frac{3(1-\mu)y^2}{((x+\mu)^2+y^2)^{5/2}} - \frac{\mu}{((1-\mu-x)^2+y^2)^{3/2}} + \frac{3\mu y^2}{((1-\mu-x)^2+y^2)^{5/2}}}$$

$$U_{xy} = \frac{\partial}{\partial x}\left[y - \frac{(1-\mu)y}{((\mu+x)^2+y^2)^{3/2}} - \frac{\mu y}{((1-\mu-x)^2+y^2)^{3/2}} \right]$$

$$U_{xy} = \left[0 - (1-\mu)y\left(\frac{\partial}{\partial x}\left[\frac{1}{((\mu+x)^2+y^2)^{3/2}}\right]\right) - \mu y\left(\frac{\partial}{\partial x}\left[\frac{1}{((1-\mu-x)^2+y^2)^{3/2}}\right]\right)\right]$$

$$U_{xy} = (1-\mu)y\left[-\frac{3}{\cancel{2}}((x+\mu)^2+y^2)^{-5/2}\cdot \cancel{2}(x+\mu)\right] - \mu y\left[-\frac{3}{\cancel{2}}((1-\mu-x)^2+y^2)^{-5/2}\cdot \cancel{2}(1-\mu-x)(-1)\right]$$

$$\boxed{U_{xy} = \frac{3(1-\mu)(x+\mu)y}{((x+\mu)^2+y^2)^{5/2}} - \frac{3\mu y(1-\mu-x)}{((1-\mu-x)^2+y^2)^{5/2}}}$$

Computing Eigenvalues: (Used MATLAB eigenvalue function)

- Complex eigenvalues: $\lambda = \eta \pm i\omega$

For stability, $\eta \leq 0$ for all eigenvalues, which means $Re(\lambda) \leq 0$

Sun-Earth System

	L_1	L_2	L_3	L_4	L_5
λ_1	$-2.5193 + 0i$	$-2.5077 + 0i$	$0 + 1i$	$0 + 1i$	$0 + 1i$
λ_2	$2.5193 + 0i$	$2.5077 + 0i$	$0 - 1i$	$0 - 1i$	$0 - 1i$
λ_3	$0 + 2.0783i$	$0 + 2.0712i$	$0 + 0.0094i$	$0 + 0.0014i$	$0 + 0.0014i$
λ_4	$0 - 2.0783i$	$0 - 2.0712i$	$0 - 0.0094i$	$0 - 0.0014i$	$0 - 0.0014i$

Earth-Moon System

	L_1	L_2	L_3	L_4	L_5
λ_1	$-2.9321 + 0i$	$2.1588 + 0i$	$0 + 1.0104i$	$0 + 0.9545i$	$0 + 0.9545i$
λ_2	$2.9321 + 0i$	$-2.1588 + 0i$	$0 - 1.0104i$	$0 - 0.9545i$	$0 - 0.9545i$
λ_3	$0 + 2.3344i$	$0 + 1.8627i$	$-0.1779 + 0i$	$0 + 0.2982i$	$0 + 0.2982i$
λ_4	$0 - 2.3344i$	$0 - 1.8627i$	$0.1779 + 0i$	$0 - 0.2982i$	$0 - 0.2982i$

Saturn-Titan System

	L_1	L_2	L_3	L_4	L_5
λ_1	$-2.6151 + 0i$	$-2.4731 + 0i$	$0 + 1.0002i$	$0 + 0.9992i$	$0 + 0.9992i$
λ_2	$2.6151 + 0i$	$2.4731 + 0i$	$0 - 1.0002i$	$0 - 0.9992i$	$0 - 0.9992i$
λ_3	$0 + 2.1370i$	$0 + 2.0502i$	$-0.0247 + 0i$	$0 + 0.400i$	$0 + 0.0400i$
λ_4	$0 - 2.1370i$	$0 - 2.0502i$	$0.0247 + 0i$	$0 - 0.400i$	$0 - 0.0400i$

- Unstable

- Stable

- Not only are all points highlighted in green stable, but they are statically stable since the real part of the eigenvalues is 0. This means the mass will oscillate around the lagrange point, but never stop moving.

d.) <u>Nominal Orbit Solution</u>

- use state-space representation to break down equations (7) & (8) into simpler differential equations

$x'' - 2y' = U_x$ $\qquad\qquad$ $y'' + 2x' = U_y$

<u>call</u>

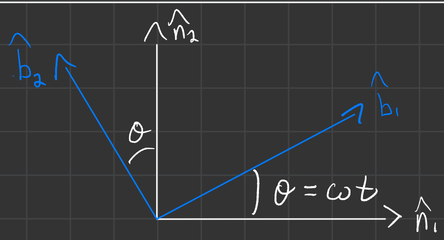
$$\begin{aligned} x_1 &= x \\ x_2 &= y \\ x_3 &= x' \\ x_4 &= y' \end{aligned} \longrightarrow \begin{aligned} x_1' &= x' = x_3 \\ x_2' &= y' = x_4 \\ x_3' &= x'' = 2x_4 + U_x \\ x_4' &= y'' = -2x_3 + U_y \end{aligned}$$

$$\bar{x} = \begin{bmatrix} x \\ y \\ x' \\ y' \end{bmatrix} \longrightarrow \bar{x}' = \begin{bmatrix} x_3 \\ x_4 \\ 2x_4 + U_x \\ -2x_3 + U_y \end{bmatrix}$$

- the initial conditions for x, y, x', and y' were given to us.
- our final time was given to us. we found a small timestep by using linspace in MATLAB.

- input all of this information into the ODE45 function

Inertial Frame Solution



- make a DCM from these coordinates

$$C_{BN} = \begin{bmatrix} c\theta & s\theta & 0 \\ -s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ where } \quad \theta = \omega t$$

$$C_{BN} = \begin{bmatrix} c(\omega t) & s(\omega t) & 0 \\ -s(\omega t) & c(\omega t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- but $\omega = 1$ since everything is normalized!

$$C_{BN} = \begin{bmatrix} c(t) & s(t) & 0 \\ -s(t) & c(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- But we want $C_{NB} = C_{BN}^T$

$$C_{NB} = \begin{bmatrix} c(t) & -s(t) & 0 \\ s(t) & c(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{use} \quad C_{NB} = \begin{bmatrix} c(t) & -s(t) \\ s(t) & c(t) \end{bmatrix}$$

because we only care about position (x & y)

$$\underbrace{\begin{bmatrix} x_I \\ y_I \end{bmatrix}}_{\text{inertial}} = \begin{bmatrix} c(t) & -s(t) \\ s(t) & c(t) \end{bmatrix} \underbrace{\begin{bmatrix} x_b \\ y_b \end{bmatrix}}_{\text{body}} \longrightarrow$$

$$x_I = \cos(t)\, x_b\, \hat{n}_1 - \sin(t)\, y_b\, \hat{n}_2$$

$$y_I = \sin(t)\, x_b\, \hat{n}_1 + \cos(t)\, y_b\, \hat{n}_2$$

- now we can use this DCM in MATLAB to convert the body frame to the inertial frame

Perturbation and Departure Motion

- we are given an initial perturbation, $\delta \bar{x}(0)$

$$\delta \bar{x}(0) = \begin{bmatrix} \delta x(0) \\ \delta y(0) \\ \delta x'(0) \\ \delta y'(0) \end{bmatrix}$$

- to find the perturbed initial condition, we must add the initial perturbation to our initial condition

Perturbed IC: $\bar{x}(0) + \delta \bar{x}(0)$

- utilize the same ODE45 function from the nominal solution, just change the initial condition to the perturbed initial condition

- use the perturbed motion found above, and subtract the nominal motion to find the departure motion ($\delta \bar{x}(t)$)

$$\underbrace{\delta \bar{x}(t)}_{\text{departure motion}} = \underbrace{\bar{x}(t)}_{\text{perturbed motion}} - \underbrace{\bar{x}_N(t)}_{\text{nominal motion}}$$

Linearization

- to solve the linearized solution, use this equation:

$$\bar{x}(t) = \sum_{i=1}^{n} c_i \bar{v}_i e^{\lambda_i t}$$

where c_i are constants, $\bar{v}_i$ are the eigenvectors, and λ_i are the eigenvalues

- we are given the A matrix, so we can use eig(A) in MATLAB to solve for the eigenvalues and eigenvectors

- to solve for the constants, we need to input the initial condition, $\delta \bar{x}(0)$

$$\delta \bar{x}(0) = \sum_{i=1}^{n} c_i \bar{v}_i \qquad (e^{\lambda_i t} \text{ becomes } 1 \text{ since } t=0)$$

$$\delta \bar{x}(0) = c_1 \bar{v}_1 + c_2 \bar{v}_2 + c_3 \bar{v}_3 + c_4 \bar{v}_4$$

$$\begin{bmatrix} x(0) \\ y(0) \\ x'(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} \begin{bmatrix} \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \bar{v}_4 \end{bmatrix}$$

- we can now solve for constants c_1, c_2, c_3, and c_4.

- now we can find the linearized solution

$$\bar{x}(t) = c_1 \bar{v}_1 e^{\lambda_1 t} + c_2 \bar{v}_2 e^{\lambda_2 t} + c_3 \bar{v}_3 e^{\lambda_3 t} + c_4 \bar{v}_4 e^{\lambda_4 t}$$