Computer project 2.2

Wednesday, November 23, 2022 9:06 AM

(2) Temporally, discretize Eq. (9) using Crank-Nicolson scheme with a time step size of Δt. At the jth time step, the solution is denoted by $\mathbf{u}_j = \mathbf{u}(j\Delta t)$, $j = 0, 1, \cdots, N_t$, where $N_t = \frac{T}{\Delta t}$. Show that

$$\mathbf{T}\mathbf{u}_{n+1} = \mathbf{S}\mathbf{u}_n + \mathbf{f} \tag{17}$$

where the initial condition is $\mathbf{u}_0 = \mathbf{0}$ as given by Eq. (1), $\mathbf{f} = \Delta t\tilde{\mathbf{f}}$, and

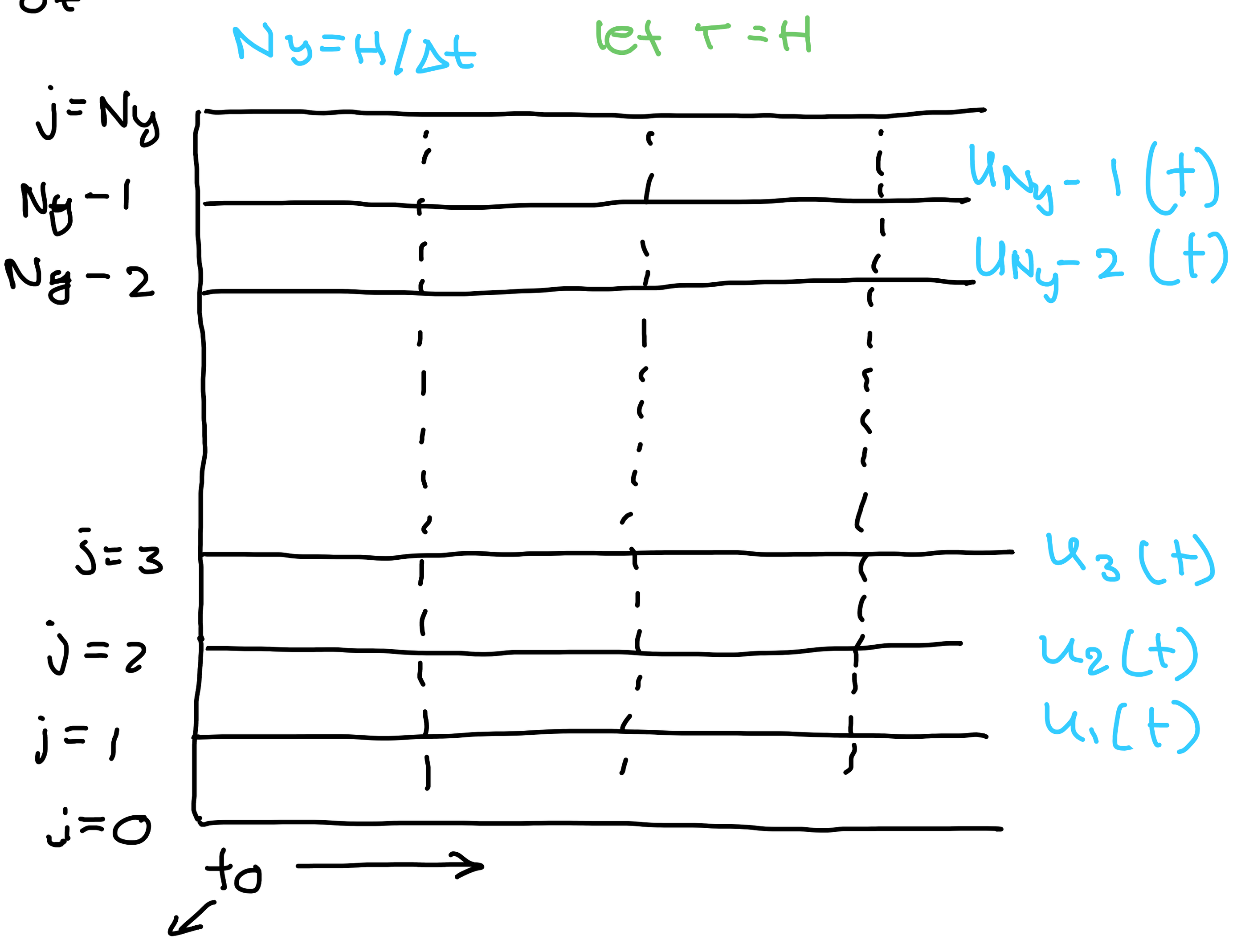
$$\mathbf{T} = \mathbf{I} - \frac{\Delta t}{2}\mathbf{A}, \quad \mathbf{S} = \mathbf{I} + \frac{\Delta t}{2}\mathbf{A} \tag{18}$$

eq. (9) $\Rightarrow$ $\dot{u} = Au + \tilde{f}$

Step size Δt $\quad u_j = u(j\Delta t)$, $j = 0, 1, \dots N_t$, where $N_t = \frac{T}{\Delta t}$

$$\frac{\partial u}{\partial t} = Au + \tilde{f} \qquad r = K/\Delta t^2$$

$N_y = H/\Delta t$ $\quad$ let $T = H$



$y = 2\Delta y$

$U_t = K U_{yy}$

$U_2' = K/\Delta y^2 \left(u_1(t) - 2U_2(t) + U_3(t)\right)$

$U_3' = K/\Delta y^2 \left(U_2(t) - 2U_3(t) + U_4(t)\right)$

LU decomp. of A $\qquad \dot{u} = Au + \tilde{f}$

$$\underbrace{\frac{d}{dt}\begin{bmatrix} U_{Ny-1} \\ U_{Ny-2} \\ \vdots \\ u_3 \\ u_2 \\ u_1 \end{bmatrix}}_{\dot{u}} = \frac{K}{\Delta y^2}\underbrace{\begin{bmatrix} A_3-2 & A_4+1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & 1 & -2 & 1 \\ & & A_2+1 & A_1-2 \end{bmatrix}}_{A}\overbrace{\begin{bmatrix} U_{Ny-1} \\ U_{Ny-2} \\ \vdots \\ u_3 \\ u_2 \\ u_1 \end{bmatrix}}^{u} + \overbrace{\begin{bmatrix} \tilde{f}_2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \tilde{f}_1 \end{bmatrix}}^{\tilde{f}}$$

Top corner of $\underline{A}$: u_0, u_1, u_2

+ Bottom

$A_3 = A_1 \quad A_4 = A_2$

solve u_1, u_2

$$U_y(0,t) = \frac{-3u_0(t) + 4u_1(t) - u_2(t)}{2\Delta y} = -Q_1 \longrightarrow$$

solve for u_0 $\quad -3u_0 = -Q_1 2\Delta y - 4u_1 + u_2 \rightarrow u_0 = -Q_1 2\Delta y + \frac{4}{3}u_1 - \frac{1}{3}u_2$

$u_0 = \underbrace{4/3}_{A_1} u_1 - \underbrace{1/3}_{A_2} u_2 - Q_1 2\Delta y$

$A_1 - 2 = -2/3$ $\quad = A_3 - 2$

$A_2 + 1 = 2/3$ $\quad = A_4 + 1$

$$u_1' = \frac{K}{\Delta y^2}\left(u_0(t) - 2u_1(t) + u_2(t)\right)$$

$$u_0(t) = \frac{2}{3}\Delta y Q_1 + 4/3\, u_1(t) - \frac{1}{3}u_2(t)$$

top part $\quad N_y-2 \quad N_y-1 \quad N_y$

solve for U_{Ny}

$u_1 \rightsquigarrow$ last term in front of a 0

then use $U_y(H,t)$

$$-Q_2 = \frac{U_{Ny-2} - 4U_{Ny-1} + 3U_{Ny}}{2\Delta y}$$

$$-\frac{2\Delta y \boxed{Q_2}}{3} + \frac{4}{3}U_{Ny-1} - \frac{1}{3}U_{Ny-2} = U_{Ny}$$

$\tilde{f} \rightarrow \quad Q_1 \rightarrow Q_2$

$$U_y(H,t) = \frac{U_{Ny-2}(t) - 4U_{Ny-1}(t) + 3U_{Ny}(t)}{2\Delta y} = -Q_2$$

$f_1 \rightarrow \frac{2}{3}\Delta y\,(Q_1) \leftarrow [u_1(t)$

$$\tilde{f} = \frac{2K}{3\Delta y}\begin{bmatrix} Q_1 \\ \vdots \\ 0 \\ -Q_2 \end{bmatrix}$$

← common multiple in front of Q_1 + $-Q_2$ in $\tilde{f}$ vector

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ U_{Ny-2} \\ U_{Ny-1} \end{bmatrix} = \frac{K}{\Delta y^2}\begin{bmatrix} -2/3 & 2/3 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ U_{Ny-2} \\ U_{Ny-1} \end{bmatrix} + \frac{2K}{3\Delta y}\begin{bmatrix} Q_1 \\ 0 \\ \vdots \\ 0 \\ -Q_2 \end{bmatrix}$$

Proof of $TU_{K+1} = SU_K + f$

For crank Nicolson: $K + 1/2$

$K \quad K+1/2 \quad K+1$

$$\dot{u}_{K+1/2} = AU_{K+1/2} + f$$

$$\dot{u}_{K+1/2} = \frac{U_{K+1} - U_K}{\Delta t} \qquad U_{K+1/2} = \frac{1}{2}(U_{K+1} + U_K)$$

$$\frac{U_{K+1} - U_K}{\Delta t} = A\left[\frac{1}{2}(U_{K+1} + U_K)\right] + \tilde{f}$$

$$IU_{K+1} - IU_K = \frac{\Delta t}{2}AU_{K+1} + \frac{\Delta t}{2}AU_K + \tilde{f}\cdot\Delta t$$

$$= \underbrace{\left(I - \frac{\Delta t}{2}A\right)}_{T}U_{K+1} = \underbrace{\left(I + \frac{\Delta t}{2}A\right)}_{S}U_{K} + \overbrace{\tilde{f}\Delta t}^{f = \Delta t\tilde{f}}$$

$$\rightarrow \quad \boxed{TU_{K+1} = SU_K + f}$$