1D, Parabolic PDE

$$U_t = K(u) U_{yy}$$

$$u(y,0) = 0$$

$$U_y(0,t) = -Q_1(u(0,t))$$

$$U_y(H,t) = -Q_2$$

- u: temp increase in structure
- K: coefficient of structural properties
- Q_1: aerodynamic heating on surface $(y=0)$
- Q_2: heat flux absorbed by the cooling system $(y=h)$

☆ Q_1 is a function of vehicle surface temperature

☆ K also a function of temperature

Goal: choose Q_2 such that max temp increase of structure is less than prescribed value, u_{max}, during entire mission $0 < t \leq T$

Analytical Solution

Non-Homogeneous System: $U(y,t) = U^{H}(y,t) + U^{NH}(y,t)$

- Non-Homogeneous boundary conditions are a function of y since $U_y(0,t) = -Q_1$ & $U_y(H,t) = -Q_2$

$$U_y^{NH} = \frac{-(Q_2 - Q_1)}{H} y - Q_1 \xrightarrow[\text{Integrate}]{} U^{NH} = \underbrace{\frac{(Q_1 - Q_2)}{2H} y^2 - Q_1 y}_{w(y)}$$

$$U_t^{NH} = K U_{yy}^{NH} \longrightarrow U_t^{NH} = K \underbrace{\left[\frac{(Q_1 - Q_2)}{H}\right]}_{U_{yy}}$$

Now integrate both sides:

$$U^{NH} = \underbrace{K\left[\frac{(Q_1 - Q_2)}{H}\right] t}_{g(t)}$$

Homogeneous Solution

$V_t = K\, U_{yy}$

$V(y,0) = -w(y)$

$V_y(0,t) = 0$

$V_y(H,t) = 0$

$$U^{Total} = U^{H} + U^{NH}$$

where $U^{H} = V(t,y)$ & $U^{NH} = w(y) + g(t)$

$$U^{Total}(y,t) = \underbrace{\left[w(y) + g(t)\right]}_{\text{Non-Homogeneous}} + \underbrace{v(t,y)}_{\text{Homogeneous}}$$

Now solve for $v(y,t)$:

$V_t = K V_{yy}$

From equation sheet: $F''(y) + K F(y) = 0$

$G'(t) + K c^2 G(t) = 0$

BC's: $V(y,t) = F(y)\,G(t)$

$$0 = F'(0)\,G(t) \rightarrow F'(0) = 0$$

$$0 = F'(H)\,G(t) \longrightarrow F'(H = 0$$

From table: $K_n = P_n^2 = \left(\frac{n\pi}{H}\right)^2, n = 0, 1, 2, 3 \ldots$

$$F(y) = A_n \cos(p_n y)$$

when $n=0 \longrightarrow F(y) = A_0$

When $n = 1, 2, 3 \ldots \rightarrow F(y) = A_n \cos(p_n y)$

Now find $G(t)$: $G'(t) + K c^2 G(t) = 0$

$$G'(t) + p_n K^2 G(t) = 0$$

$$G'(t) = -p_n K^2 G(t)$$

$$G(t) = \exp(-p_n K^2 t)$$

$$V_n(y,t) = A_0 + A_n \cos(p_n y) \exp(-p_n k^2 t)$$

$$V(y,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(p_n y) \exp(-p_n k^2 t)$$

Apply IC: $V(y,0) = -w(y)$

$$\underbrace{V(y,0) = -w(y) = A_0 + \sum_{n=1}^{\infty} A_n \cos(p_n y)}_{\text{Fourier Series}}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{H}\int_0^H -w(y)\,dy$$

$$A_0 = \frac{1}{H}\int_0^H \left[\left(\frac{Q_2 - Q_1}{2H}\right) y^2 + Q_1 y\right] dy$$

$$A_0 = \frac{1}{H}\left[\frac{Q_2 - Q_1}{6H} y^3 \Big|_0^H + \frac{Q_1 y^2}{2}\Big|_0^H\right]$$

$$A_0 = \frac{1}{H}\left[\frac{(Q_2 - Q_1)}{6} H^2 + \frac{Q_1}{2} H^2\right]$$

$$A_0 = \frac{Q_2 H}{6} - \frac{Q_1 H}{6} + \frac{Q_1 H}{2}$$

$$A_0 = \frac{Q_2 H}{6} + \frac{Q_1 H}{3} = \frac{H(Q_2 + 2Q_1)}{6} \rightarrow \boxed{A_0 = \frac{H}{6}(2Q_1 + Q_2)}$$

Now find A_n:

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{H}\int_{-H}^{H} -w(y)\cos(p_n y)\,dy$$

$$A_n = \frac{2}{H}\int_0^H \left[\left(\frac{Q_2 - Q_1}{2H}y^2 + Q_1 y\right)\cos(p_n y)\right]dy$$

$$A_n = \frac{2}{H}\left[\frac{((H^2 Q_2 + H^2 Q_1)p_n^2 - 2Q_2 + 2Q_1)\sin(Hp_n) + 2HQ_2 p_n \cos(Hp_n)}{2Hp_n^3} - \frac{Q_1}{p_n^2}\right]$$