Probabilistic Modeling and Reasoning

Homework — 1

Nikolaos Liouliakis (AID25001)
Vasileios-Efraim Tsavalia (AID25006)

MSc in Artificial Intelligence and Data Analytics
University of Macedonia

Supervisor: Professor Dimitris Christou-Varsakelis

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Problem 1

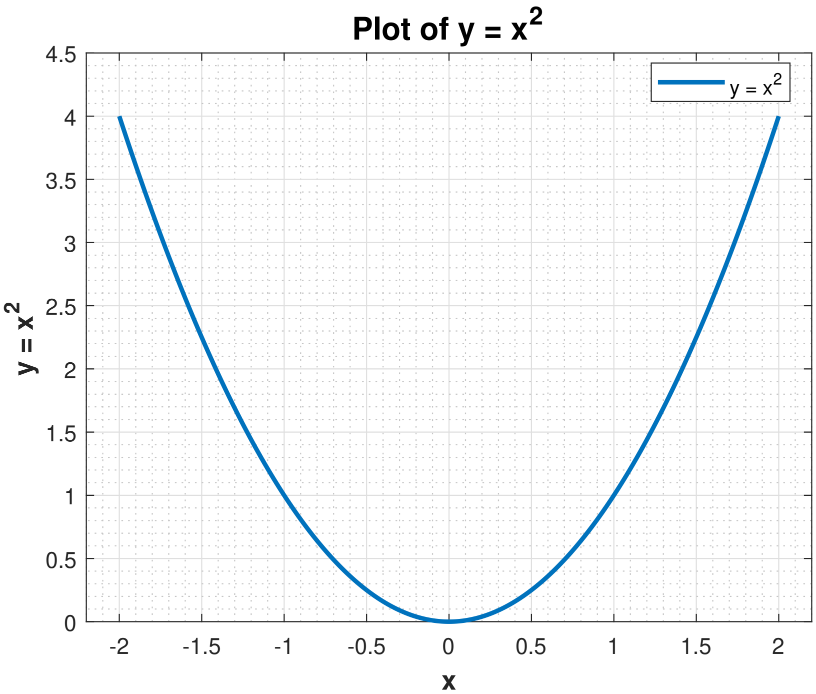

Figure 1: Plot of $y = x^2$.

Index	**Day 1**	**Day 2**	**Day 3**	**Day 4**	**Day 5**
DOW-JONES	42,080.37	41,954.24	42,352.75	42,011.59	42,196.52
NIKKEI	38,937.54	39,332.74	38,635.62	38,552.06	37,808.76
DAX	19,066.47	19,104.10	19,120.93	19,015.41	19,164.75

Table 1: Closing prices of major stock indices over five days

$$0 \leq P(A) \leq 1 \tag{1}$$

$$P(A^c) + P(A) = 1 \tag{2}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{3}$$

$$P(A \cap B) = 0 \tag{4}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \tag{5}$$

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)} \tag{6}$$

$$P(A \cap B) = P(A) \cdot P(B) \tag{7}$$

$$F_X(x) = P(X \leq x) \tag{8}$$

$$\sum_{i=1}^{n} P(X = x_i) = 1 \tag{9}$$

$$f_X(x) = \frac{dF_X(x)}{dx} \tag{10}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)\, dy \tag{11}$$

$$F_X(x) = \sum_{i=1}^{n} P(X = k) \tag{12}$$

$$P(a \leq X \leq b) = \int_{a}^{b} f_X(x)\, dx \tag{13}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \tag{14}$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] = \mathbb{E}[XY] - \mu_X \mu_Y \tag{15}$$

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} \tag{16}$$

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Problem 2

Part 1

Prove that

$$\Pr(x, y \mid z) = \Pr(x \mid z) \Pr(y \mid x, z) \tag{17}$$

Proof. Left-hand side:

$$\Pr(x, y \mid z) = \frac{\Pr(x, y, z)}{\Pr(z)}$$

Right-hand side:

$$\Pr(x \mid z) \Pr(y \mid x, z) = \frac{\Pr(x, z)}{\Pr(z)} \cdot \frac{\Pr(x, y, z)}{\Pr(x, z)} = \frac{\Pr(x, y, z)}{\Pr(z)}$$

Since both sides are equal:

$$\Pr(x, y \mid z) = \Pr(x \mid z) \Pr(y \mid x, z)$$

□

Part 2

Prove that

$$\Pr(x \mid y, z) = \frac{\Pr(y \mid x, z) \Pr(x \mid z)}{\Pr(y \mid z)} \tag{18}$$

Proof. Left-hand side:

$$\Pr(x \mid y, z) = \frac{\Pr(x, y, z)}{\Pr(y, z)}$$

Right-hand side:

$$\frac{\Pr(y \mid x, z) \Pr(x \mid z)}{\Pr(y \mid z)} = \frac{\frac{\Pr(x,y,z)}{\Pr(x,z)} \frac{\Pr(x,z)}{\Pr(z)}}{\frac{\Pr(y,z)}{\Pr(z)}} = \frac{\Pr(x, y, z)}{\Pr(y, z)}$$

Since both sides are equal:

$$\Pr(x, y \mid z) = \Pr(x \mid z) \Pr(y \mid x, z)$$

$\square$

Problem 3

We are given:

- Box 1 contains 3 red and 5 white balls.
- Box 2 contains 2 red and 5 white balls.
- A box is chosen at random, so Pr(box 1) = Pr(box 2) = 0.5.
- A red ball is chosen, and we want to find the Probability that it came from box 1.

We apply Bayes' theorem:

$$\Pr(\text{box 1} \mid \text{red}) = \frac{\Pr(\text{red} \mid \text{box 1}) \cdot \Pr(\text{box 1})}{\Pr(\text{red})}$$

Where:

$$\Pr(\text{red} \mid \text{box 1}) = \frac{3}{3+5} = \frac{3}{8}, \quad \Pr(\text{red} \mid \text{box 2}) = \frac{2}{2+5} = \frac{2}{7}$$

and

$$\Pr(\text{red}) = \Pr(\text{red} \mid \text{box 1}) \cdot \Pr(\text{box 1}) + \Pr(\text{red} \mid \text{box 2}) \cdot \Pr(\text{box 2})$$

Substitute the known values:

$$\Pr(\text{red}) = \left(\frac{3}{8} \cdot 0.5\right) + \left(\frac{2}{7} \cdot 0.5\right) = \frac{3}{16} + \frac{2}{14} = \frac{37}{112}$$

Now, applying Bayes' theorem:

$$\Pr(\text{box 1} \mid \text{red}) = \frac{\frac{3}{8} \cdot 0.5}{\frac{37}{112}} = 0.567$$

Thus, the posterior probability that the red ball came from box 1 is approximately 0.568 or 56.8%.

Problem 4

We are provided with the following information:

- There is exactly one terrorist on the plane, and there are 100 passengers.
- The scanner has a true positive rate of 95% (i.e., it correctly identifies terrorists 95% of the time).

- The scanner has a true negative rate of 95% (i.e., it correctly identifies upstanding citizens 95% of the time).
- We scan the passengers one by one until we find the first positive result.

The idea is to consider all the possible cases where the terrorist is seated in the k-th seat (for k=1,2,...,100), and sum over these cases. In particular, for the terrorist to be the person identified as the terrorist when sitting in seat k, the following must happen:

1. The first $k-1$ people must test negative (since they are all upstanding citizens).
2. The person in the k-th seat (the terrorist) must test positive.

Let $P(K=k)$ be the event that the terrorist is in the k-th seat.

$$P(K=k) = \frac{1}{100}, \quad k = 1, 2, \dots, 100$$

Now, let $P(FP=1, K=k)$ be the joint probability that the terrorist is in the k-th seat, is the first to test positive and is correctly identified.

$$P(FP=1, K=k) = P(FP=1 \mid K=k)P(K=k)$$

$$P(FP=1 \mid K=k) = 0.95^{k-1} \cdot 0.95 = 0.95^{k}$$

Using the law of total probability $P(FP=1)$ is expressed as:

$$P(FP=1) = \sum_{k=1}^{100} P(FP=1 \mid K=k) \cdot P(K=k)$$

$$P(FP=1) = \sum_{k=1}^{100} 0.95^{k} \frac{1}{100}$$

$$P(FP=1) \simeq 0.1889$$

Therefore, the probability that the first person to test positive is, indeed, a terrorist is 18.89%

Problem 5

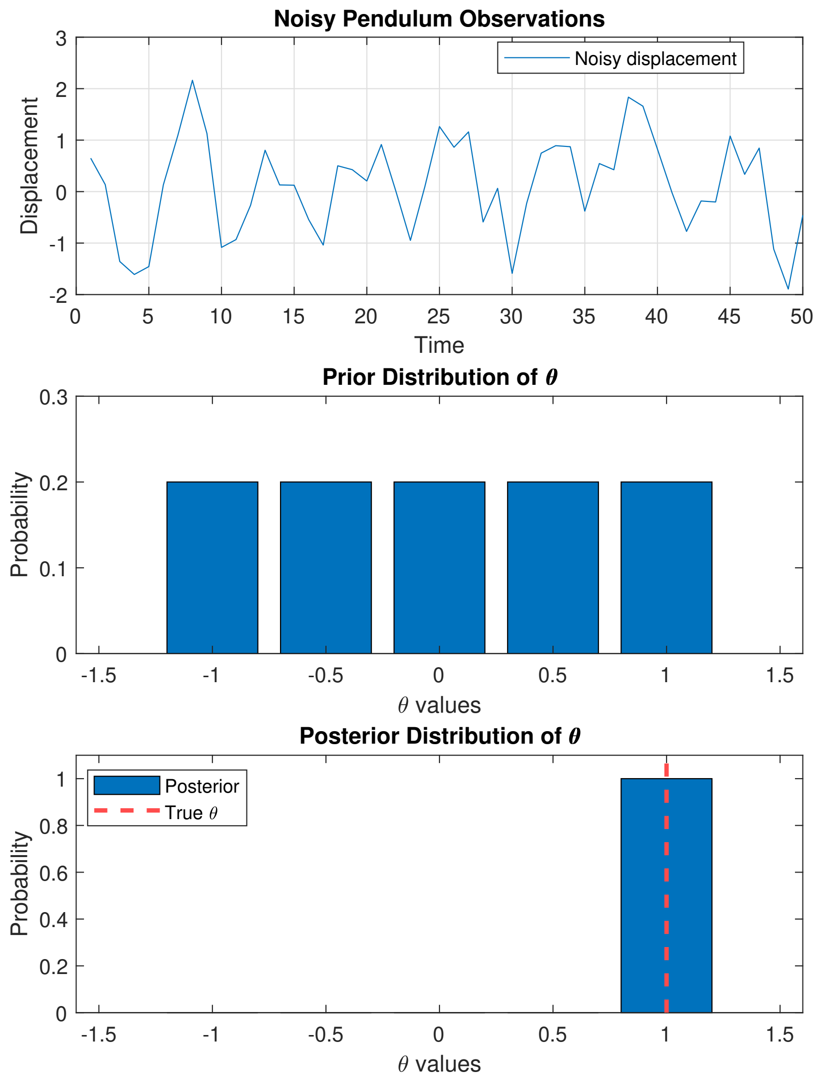

Figure 2: Diagram produced by code

Problem 6

Prior distribution of θ:

$$\theta \sim \mathcal{N}(\mu_{\text{prior}}, \sigma^2_{\text{prior}})$$

Observations y_i are generated as:

$$y_i = \theta + \sigma_y \nu_i, \quad \nu_i \sim \mathcal{N}(0, 1)$$

$$\mathbb{E}[Y] = \mathbb{E}[\theta + \sigma_y \nu_i] = \theta + \sigma_y \mathbb{E}[\nu_i] = \theta$$

$$\mathrm{Var}(Y) = \mathrm{Var}(\theta + \sigma_y \nu_i) = \mathrm{Var}(\sigma_y \nu_i) = \sigma_y^2 \mathrm{Var}(\nu_i) = \sigma_y^2$$

Therefore

$$Y \sim \mathcal{N}(\theta, \sigma_y^2)$$

Assuming that the measurements are independent, the likelihood for a sequence of observations $y_1, y_2, \ldots, y_N$ is:

$$p(y_1, \ldots, y_N|\theta) = \prod_{i=1}^{N} p(y_i|\theta)$$

Each observation follows a normal distribution, so:

$$p(y_i|\theta) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left(-\frac{(y_i - \theta)^2}{2\sigma_y^2}\right)$$

Thus, the likelihood for the sequence is:

$$p(y_1, \ldots, y_N|\theta) = \left(\frac{1}{\sqrt{2\pi\sigma_y^2}}\right)^N \exp\left(-\frac{1}{2\sigma_y^2} \sum_{i=1}^{N} (y_i - \theta)^2\right)$$

The posterior distribution is:

$$p(\theta|y_1, \ldots, y_N) \propto p(y_1, \ldots, y_N|\theta) p(\theta)$$

Substituting the likelihood and prior:

$$p(\theta|y_1, \ldots, y_N) \propto \left(\frac{1}{\sqrt{2\pi\sigma_y^2}}\right)^N \exp\left(-\frac{1}{2\sigma_y^2} \sum_{i=1}^{N} (y_i - \theta)^2\right) \frac{1}{\sqrt{2\pi\sigma_{\text{prior}}^2}} \exp\left(-\frac{(\mu_{\text{prior}} - \theta)^2}{2\sigma_{\text{prior}}^2}\right)$$

$$p(\theta|y_1, \ldots, y_N) \propto \exp\left(-\frac{1}{2\sigma_y^2} \sum_{i=1}^{N} (y_i - \theta)^2 - \frac{(\mu_{\text{prior}} - \theta)^2}{2\sigma_{\text{prior}}^2}\right)$$

$$p(\theta|y_1, \ldots, y_N) \propto \exp\left(-\frac{1}{2}\left[\left(\frac{1}{\sigma_{\text{prior}}^2} + \frac{N}{\sigma_y^2}\right)\theta^2 - 2\left(\frac{\mu_{\text{prior}}}{\sigma_{\text{prior}}^2} + \frac{\sum_{i=1}^{N} y_i}{\sigma_y^2}\right)\theta + \frac{\mu_{\text{prior}}^2}{\sigma_{\text{prior}}^2} + \frac{\sum_{i=1}^{N} y_i^2}{\sigma_y^2}\right]\right) \tag{19}$$

Since we know

$$p(\theta|y_1,\ldots,y_N) \sim \mathcal{N}(\mu_{\text{post}}, \sigma^2_{\text{post}})$$

$$\begin{aligned} p(\theta|y_1,\ldots,y_N) &= \frac{1}{\sqrt{2\pi\sigma^2_{\text{post}}}} \exp\left(-\frac{(\mu_{\text{post}}-\theta)^2}{2\sigma^2_{\text{post}}}\right) && (20) \\ &= \frac{1}{\sqrt{2\pi\sigma^2_{\text{post}}}} \exp\left(-\frac{1}{2}\left[\frac{1}{\sigma^2_{\text{post}}}\theta^2 - 2\frac{\mu_{\text{post}}}{\sigma^2_{\text{post}}}\theta + \frac{\mu^2_{\text{post}}}{\sigma^2_{\text{post}}}\right]\right) && (21) \end{aligned}$$

By equating the polynomial coefficients of the terms involving θ in the exponent, we can calculate the posterior mean and variance for θ after N observations.

$$\mu_{\text{post}} = \sigma^2_{\text{post}}\left(\frac{\mu_{\text{prior}}}{\sigma^2_{\text{prior}}} + \frac{\sum_{i=1}^N y_i}{\sigma^2_y}\right) \tag{22}$$

$$\sigma^2_{\text{post}} = \left(\frac{1}{\sigma^2_{\text{prior}}} + \frac{N}{\sigma^2_y}\right)^{-1} \tag{23}$$

With $\mu_{\text{prior}} = 1$, $\sigma^2_{\text{prior}} = 2$, $\sigma_y = 0.5$ and only one observation $y_1 = -1$, we have:

$$\sigma^2_{\text{post}} = \left(\frac{1}{2^2} + \frac{1}{0.5^2}\right)^{-1} = \frac{4}{17} \approx 0.2353$$

$$\mu_{\text{post}} = \frac{4}{17}\left(\frac{1}{2^2} + \frac{-1}{0.5^2}\right) = -\frac{15}{17} \approx -0.882$$