1 Multi-Layer Bending Model

In this section, we propose a mathematical model to describe the observed curvature. Our approach is based on the nonlinear multi-layer bending model originally used in [?] to predict the radius of curled nanotubes. We adapt this model for our setting.
In the presence of heterogeneous layers, sudden expansion or contraction of either layer induces a curving strain. Since the leaf thickness is assumed to be small, and the top and bottom layers are each homogeneous, [?] showed that the leaf attains an energetically stable state when its radius κ^* satisfies

$$\kappa^* \;=\; \arg\min_{\kappa\in\mathbb{R}^+} \overline{Q}_2\big(\kappa\cdot(1,0,0)^\top\big).$$

Here, $\overline{Q}_2$ is the dimensionally reduced linearized energy functional at the identity, which depends on the material properties, layer properties, and the mismatch between layers, as constructed below.
Because obtaining a precise material description for a heterogeneous leaf is challenging, we assume each layer is isotropic. Let $-\frac{1}{2} \le t < \frac{1}{2}$ be the rescaled height within the leaf, and let λ and μ be the Lamé parameters of each layer. We define

$$Q_2(t,G) \;=\; 2\mu\Big|\tfrac{G+G^\top}{2}\Big|^2 \;+\; \frac{2\mu\,\lambda}{2\mu+\lambda}\big(\operatorname{tr} G\big)^2.$$

In our two-layer setup, the bottom layer occupies $[-\frac{1}{2},\tau]$, and the top layer occupies $[\tau,\frac{1}{2}]$. We then set

$$\overline{Q}_2(F) \;=\; \min_{A\,\in\,\mathbb{R}^{2\times2}_{\rm sym}} \int_{-1/2}^{1/2} Q_2\Big(t,\ A + t\,F + B\Big)\,\mathrm{d}t,$$

where $B\in\mathbb{R}^{2\times2}_{\rm sym}$ represents the layer mismatch. To compute $\overline{Q}_2$ explicitly, let $\mathcal{M}(t)$ be the matrix representation of Q_2 to define the following "moments":

$$\mathcal{M}_1 \;=\; \int_{-1/2}^{1/2}\mathcal{M}(t)\,\mathrm{d}t,\quad \mathcal{M}_2 \;:=\; \int_{-1/2}^{1/2} t\,\mathcal{M}(t)\,\mathrm{d}t,\quad \mathcal{M}_3 \;=\; \int_{-1/2}^{1/2} t^2\,\mathcal{M}(t)\,\mathrm{d}t.$$

It can be shown that $\overline{Q}_2(f) \;=\; \big(f-f_0\big)^\top\mathcal{M}_0\,\big(f-f_0\big) \;+\; c$, where $\mathcal{M}_0 \;=\; \mathcal{M}_3 - \mathcal{M}_2\,\mathcal{M}_1^{-1}\,\mathcal{M}_2$ and $f_0 \;=\; \mathcal{M}_0^{-1}\Big(\mathcal{M}_2\,\mathcal{M}_1^{-1}\,b_1 \;-\; b_2\Big)$ as well as

$$b_1 = 0,\qquad b_2 = \left(\frac{1}{2}\tau^2 - \frac{1}{8}\right)\mathcal{M}_{\rm bot}b_{\rm bot} - \left(\frac{1}{8} - \frac{1}{2}\tau^2\right)\mathcal{M}_{\rm top}b_{\rm top}$$

Hence, minimizing $\overline{Q}_2(\kappa e_1)$ is equivalent to

$$\kappa^* \;=\; \arg\min_{\kappa\in\mathbb{R}}\big(\alpha^2\,\kappa \;+\; 2\beta\,\kappa \;+\; \gamma\big),\quad \text{where } \alpha \;=\; e_1^\top\mathcal{M}_0 e_1,\quad \beta \;=\; e_1^\top\mathcal{M}_0 f_0.$$

and a radius is obtained by $r=\frac{d}{h}\kappa$ where d refers to the leaf thickness and h the aspect ratio. Since the layer mismatch is unknown in our setting, we compute it via an explicit formula for a prescribed radius. We assume the mismatch is symmetric, i.e. $b_{\rm bot} = -\,b_{\rm top}$, and that the bottom and top layers share identical material parameters, implying $\mathcal{M}(t) = \mathcal{M}$. In this case,

$$b_{\rm bot} = -b_{\rm top} \;=\; \frac{d}{h}\,\frac{e_1^\top\mathcal{M}_0\,e_1}{e_1^\top\mathcal{M}\,(e_1+e_2)}\,\frac{1}{\frac{1}{4}-\tau^2}\,\frac{1}{r_{\rm measured}}.$$

A mismatch of b can be interpreted as the original layer expanding or contracting by a factor of

$$\frac{1}{1+h\,b},$$