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Math 461
Assignment VIII

Exercise 1 (7.4)

$$f(x,y) = \begin{cases} \frac{1}{y} & \text{if } 0 < y < 1,\ 0 < x < y \\ 0 & \text{otherwise} \end{cases}$$

a) $\mathbb{E}(XY) = \int_0^1 \int_0^y xyf(x,y)dxdy$
$= \int_0^1 \int_0^y xdxdy$
$= \int_0^1 \frac{y^2}{2}dy$
$= \frac{y^3}{6}\Big|_0^1$
$= \frac{1}{6}.$

b) $\mathbb{E}(X) = \int_0^1 \int_0^y xf(x,y)dxdy$
$= \int_0^1 \int_0^y \frac{x}{y}dxdy$
$= \int_0^1 \frac{y^2}{2y}dy$
$= \int_0^1 \frac{y}{2}dy$
$= \frac{1}{4}$

c) $\mathbb{E}(Y) = \int_0^1 \int_0^y yf(x,y)dxdy$
$= \int_0^1 \int_0^y \frac{y}{y}dxdy$
$= \int_0^1 ydy$
$= \frac{1}{2}.$

Exercise 2 (7.8)

Let X_i be the event defined by $X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ arrival sits at an un-occuped table.} \\ 0 & \text{otherwise} \end{cases}$.

We know that the expected value of X_i is equal to $(1-p)^{i-1}$, since $1-p$ is the probability that an arrival has no seated friends, the events are independent, and there are $i-1$ previously seated people.

Just as we did in class, we can write the expected value of the number of occupied tables as $\mathbb{E}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N} \mathbb{E}\left(X_i\right) = \sum_{i=1}^{N} (1-p)^{i-1}$. This is a finite geometric sum, so we have that the expected value is $\frac{(1-p)^N - 1}{(1-p)-1} = \frac{1-(1-p)^N}{p}$.

Exercise 3 (7.38)

$$f(x,y) = \begin{cases} \frac{2e^{-2x}}{x} & \text{if } 0 \le x < \infty,\ 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$$

$\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

First we need to calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, and $\mathbb{E}(XY)$.

$$\begin{aligned} \mathbb{E}(X) &= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f(x,y)\,dy\,dx \\ &= \int_0^{\infty} x \int_0^x \frac{2e^{-2x}}{x}\,dy\,dx \\ &= \int_0^{\infty} x^2 \frac{2e^{-2x}}{x}\,dx \\ &= \int_0^{\infty} 2xe^{-2x}\,dx \end{aligned}$$

Let $u = x$, $du = dx$, $v = -e^{-2x}$, $dv = 2e^{-2x}dx$.

Now we have $$-xe^{-2x}\Big|_0^{\infty} + \int_0^{\infty} e^{-2x}dx = \int_0^{\infty} e^{-2x}dx = \frac{-e^{-2x}}{2}\Big|_0^{\infty} = \frac{1}{2}.$$

$$\begin{aligned} \mathbb{E}(Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)\,dy\,dx \\ &= \int_0^{\infty}\int_0^x y\frac{2e^{-2x}}{x}\,dy\,dx \\ &= \int_0^{\infty} \frac{2e^{-2x}}{x}\int_0^x y\,dy\,dx \\ &= \int_0^{\infty} x^2\frac{2e^{-2x}}{2x}\,dx \end{aligned}$$

$= \int_0^\infty xe^{-2x}dx$
$= \frac{1}{4}$, since this is twice the integral from $\mathbb{E}(X)$,

$\mathbb{E}(XY) = \int_{-\infty}^\infty \int_{-\infty}^\infty xyf(x,y)dydx$
$= \int_0^\infty \int_0^x xy\frac{2e^{-2x}}{x}dydx$
$= \int_0^\infty \int_0^x y2e^{-2x}dydx$
$= \int_0^\infty 2e^{-2x}\int_0^x ydydx$
$= \int_0^\infty 2e^{-2x}\frac{x^2}{2}dx$
$= \int_0^\infty x^2e^{-2x}dx$
$= \int_0^\infty x^2e^{-2x}dx$

Let $u = x^2,\ du = 2xdx,\ v = \frac{-e^{-2x}}{2},\ dv = e^{-2x}dx.$

So $\mathbb{E}(XY) = x^2\frac{-e^{-2x}}{2}\Big|_0^\infty + \int_0^\infty 2x\frac{e^{-2x}}{2}dx$
$= x^2\frac{-e^{-2x}}{2}\Big|_0^\infty + \int_0^\infty xe^{-2x}dx$

The first term is zero by L'Hôpital's Rule, and the integral is $\mathbb{E}(Y)$, which is $\frac{1}{4}$.

Therefore $\text{Cov}(X,Y) = \frac{1}{4} - \frac{1}{2}\frac{1}{4} = \frac{1}{8}$.

Exercise 4 (7.39)

Let $X_1\ldots$ be independent with common mean μ and common variance σ^2, and set $Y_n = X_n + X_{n+1} + X_{n+2}$. For $j \geq 0$, find $\text{Cov}(Y_n, Y_{n+j})$.

First expand $\text{Cov}(Y_n, Y_{n+j}) = \mathbb{E}(Y_nY_{n+j}) - \mathbb{E}(Y_n)\mathbb{E}(Y_{n+j})$. So we need to find $\mathbb{E}(Y_nY_{n+j})$, $\mathbb{E}(Y_n)$, and $\mathbb{E}(Y_{n+j})$.

$\mathbb{E}(Y_n) = \mathbb{E}(X_n + X_{n+1} + X_{n+2}) = \mathbb{E}(X_n) + \mathbb{E}(X_{n+1}) + \mathbb{E}(X_{n+2}) = 3\mu$.

We can say the same of Y_{n+j}, since its expected value with clearly also result in a sum of expected values of three Xs. i.e. $\mathbb{E}(Y_n) = \mathbb{E}(Y_{n+j}) = 3\mu$. And therefore $\mathbb{E}(Y_n)\mathbb{E}(Y_{n+j}) = 9\mu^2$ for any $j \neq 0$.

If $j = 0$, we have

$$\begin{aligned}
\mathbb{E}\left(Y_n Y_{n+j}\right) &= \mathbb{E}\left(Y_n^2\right) \\
&= \mathbb{E}\left(\left(X_n + X_{n+1} + X_{n+2}\right)^2\right) \\
&= \mathbb{E}\left(X_n^2 + 2X_nX_{n+1} + X_{n+1}^2 + 2X_nX_{n+2} + 2X_{n+1}X_{n+2} + X_{n+2}^2\right) \\
&= \mathbb{E}(X_n^2) + \mathbb{E}(2X_nX_{n+1}) + \mathbb{E}(X_{n+1}^2) + \mathbb{E}(2X_nX_{n+2}) + \mathbb{E}(2X_{n+1}X_{n+2}) + \mathbb{E}(X_{n+2}^2)
\end{aligned}$$

And since the Xs are independent, we can replace every $\mathbb{E}\left(X_aX_b\right), a \neq b$ with the square of $\mathbb{E}\left(X_a\right) = \mu$. Likewise, we can replace $\mathbb{E}\left(X_aX_a\right)$ with $\text{Var}\left(X_a\right) + \mathbb{E}(X_a)^2 = \mu^2 + \sigma^2$.

By grouping the square terms and the remaining terms, we conclude that $\text{Cov}\left(Y_n, Y_{n+j}\right) = 3(\mu^2 + \sigma^2) + 6\mu^2 - 9\mu^2 = 3\sigma^2$ in this case.

If $j = 1$, we have

$$\begin{aligned}
\mathbb{E}\left(Y_n Y_{n+j}\right) &= \mathbb{E}\left(Y_n Y_{n+1}\right) \\
&= \mathbb{E}\left(\left(X_n + X_{n+1} + X_{n+2}\right)\left(X_{n+1} + X_{n+2} + X_{n+3}\right)\right) \\
&= \mathbb{E}\Big(X_nX_{n+1} + X_{n+1}^2 + X_nX_{n+2} + 2X_{n+1}X_{n+2} + \\
&\qquad X_{n+2}^2 + X_nX_{n+3} + X_{n+1}X_{n+3} + X_{n+2}X_{n+3}\Big)
\end{aligned}$$

Just as we did for $j = 0$, we group terms and find that $\text{Cov}\left(Y_n, Y_{n+j}\right) = 2(\mu^2 + \sigma^2) + 5\mu^2 + 2\mu^2 - 9\mu^2 = 2\sigma^2 + 9\mu^2 - 9\mu^2 = 2\sigma^2$ in this case.

If $j = 2$, we have

$$\begin{aligned}
\mathbb{E}\left(Y_n Y_{n+j}\right) &= \mathbb{E}\left(Y_n Y_{n+2}\right) \\
&= \mathbb{E}\left(\left(X_n + X_{n+1} + X_{n+2}\right)\left(X_{n+2} + X_{n+3} + X_{n+4}\right)\right) \\
&= \mathbb{E}\Big(X_nX_{n+2} + X_{n+1}X_{n+2} + X_{n+2}^2 + X_nX_{n+3} + X_{n+1}X_{n+3} + \\
&\qquad X_{n+2}X_{n+3} + X_nX_{n+4} + X_{n+1}X_{n+4} + X_{n+2}X_{n+4}\Big)
\end{aligned}$$

Just as we did for $j = 0$ and $j = 1$, we group terms and find that $\text{Cov}\left(Y_n, Y_{n+j}\right) = \mu^2 + \sigma^2 + 8\mu^2 - 9\mu^2 = \sigma^2$ in this case.

If $j \geq 3$, we have

$\mathbb{E}\left(Y_nY_{n+j}\right) = 9\mu^2 - 9\mu^2 = 0$, since the events are independent, so $\text{Cov}\left(Y_n, Y_{n+j}\right) = 0$ in this case.

We can now write the covariance in the following distilled form

$$\text{Cov}\left(Y_n, Y_{n+j}\right) = \begin{cases} (3-j)\sigma^2 & \text{if } 0 \le j < 3 \\ 0 & \text{if } j \ge 3 \end{cases}$$

Exercise 5 (7.43)

Let $X_1 \ldots X_n$ be independent random variables having an unknown continuous distribution function F, and let $Y_1 \ldots Y_m$ be independent random variables having an unknown continuous distribution function G. Now order those $n + m$ variables and let

$$I_i = \begin{cases} 1 & \text{if the } i^{th} \text{ smallest of the } n+m \text{ variables is from the } X's \\ 0 & \text{otherwise} \end{cases}$$

The random variable $R = \sum_{i=1}^{n+m} iI_i$ is the sum of the ranks of the X sample and is the basis of a standard statistical procedure for testing whether F and G are identical distributions. This test accepts the hypothesis that $F = G$ when R is neither too large or too small.

Under the assumption that the hypothesis of equality is correct, we want to compute the mean and variance of R.

We can begin by observing that iI_i is a negative hypergeometric random variable. In the language used in the distribution index in the back of he book, iI_i is the number of balls that need to be removed from an urn containing $n + m$ balls, n of which are white, until i balls have been removed.

Now the expected value of iI_i can be expressed as $\mathbb{E}(iI_i) = i\,\frac{m+n+1}{n+1}$, and the variance of iI_i can be expressed as $\text{Var}(iI_i) = \frac{mi(n+1-i)(n+m+1)}{(n+1)^2(n+2)}$.

First let's determine $\mathbb{E}(R)$.

$$\begin{aligned}
\mathbb{E}(R) &= \mathbb{E}\left(\textstyle\sum_{i=1}^{n+m} iI_i\right) \\
&= \textstyle\sum_{i=1}^{n+m} \mathbb{E}(iI_i) \\
&= \textstyle\sum_{i=1}^{n+m} i\,\frac{m+n+1}{n+1} \\
&= \frac{m+n+1}{n+1} \textstyle\sum_{i=1}^{n+m} i \\
&= \frac{m+n+1}{n+1} \frac{(n+m)(n+m+1)}{2} \\
&= \frac{(n+m)(n+m+1)^2}{2(n+1)}.
\end{aligned}$$

Finally let's determine Var(R).

$$\text{Var}(R) = \sum_{i=1}^{n+m} \frac{mi(n+1-i)(n+m+1)}{(n+1)^2(n+2)}$$
$$= \frac{(n+m+1)}{(n+1)^2(n+2)} \sum_{i=1}^{n+m} mi(n+1-i)$$

Quickly observe that

$$\begin{aligned}\sum_{i=1}^{n+m} mi(n+1-i) &= \sum_{i=1}^{n+m} imn + \sum_{i=1}^{n+m} im - \sum_{i=1}^{n+m} i^2 m \\ &= mn\sum_{i=1}^{n+m} i + m\sum_{i=1}^{n+m} i - m\sum_{i=1}^{n+m} i^2 \\ &= m\big((n+1)\sum_{i=1}^{n+m} i - \sum_{i=1}^{n+m} i^2\big) \\ &= m\big((n+1)\frac{(n+m)(n+m+1)}{2} - \frac{(n+m)(n+m+1)(2(n+m)+1)}{6}\big) \\ &= \frac{m(n+m)(n+m+1)}{2}\big((n+1) - \frac{2(n+m)+1}{3}\big)\end{aligned}$$

So $\text{Var}(R) = \frac{m(n+m)(n+m+1)^2}{2(n+1)^2(n+2)}\big(n+1-\frac{2(n+m)+1}{3}\big)$.

Exercise 6 (7.45)

a) $\rho\big(X_1+X_2, X_2+X_3\big) = \frac{\text{Cov}\big(X_1+X_2, X_2+X_3\big)}{\sqrt{\text{Var}\big(X_1+X_2\big)\text{Var}\big(X_2+X_3\big)}}$.

So we need to determine $\text{Cov}\big(X_1+X_2, X_2+X_3\big)$, $\text{Var}\big(X_1+X_2\big)$, and $\text{Var}\big(X_2+X_3\big)$.

Since the Xs are uncorrelated, we have the following

$\text{Cov}\big(X_1+X_2, X_2+X_3\big) = \text{Var}\big(X_2\big) = 1$

$\text{Var}\big(X_1+X_2\big) = \text{Var}\big(X_1\big) + \text{Var}\big(X_2\big) = 1+1 = 2$

$\text{Var}\big(X_2+X_3\big) = \text{Var}\big(X_2\big) + \text{Var}\big(X_3\big) = 1+1 = 2$

Thus $\rho\big(X_1+X_2, X_2+X_3\big) = \frac{1}{\sqrt{2+2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$.

b) $\rho\big(X_1+X_2, X_3+X_4\big) = \frac{\text{Cov}\big(X_1+X_2, X_3+X_4\big)}{\sqrt{\text{Var}\big(X_1+X_2\big)\text{Var}\big(X_3+X_4\big)}}$.

Since the Xs are uncorrelated, and $\{X_1, X_2\} \cap \{X_3, X_4\} = \varnothing$, we have that $\text{Cov}\big(X_1+X_2, X_3+X_4\big) = 0$, which means $\rho\big(X_1+X_2, X_3+X_4\big) = 0$.

Extra Exercises

Exercise 8.13

a)

$$\mathbb{P}\left\{\frac{X-\mu n}{\sigma\sqrt{n}} > \frac{Y-\mu n}{\sigma\sqrt{n}}\right\} = 1-\mathbb{P}\left\{Z \leq \frac{80(25)-74(25)}{14\sqrt{25}}\right\}$$
$$= 1-\mathbb{P}\{Z \leq 2.14\}$$
$$= 1-\Phi(2.14)$$
$$= 1-0.983823$$
$$= 0.016177$$

b)

$$\mathbb{P}\left\{\frac{X-\mu n}{\sigma\sqrt{n}} > \frac{Y-\mu n}{\sigma\sqrt{n}}\right\} = 1-\mathbb{P}\left\{Z \leq \frac{80(64)-74(64)}{14\sqrt{64}}\right\}$$
$$= 1-\mathbb{P}\{Z \leq 3.43\}$$
$$= 1-\Phi(3.43)$$
$$= 1-0.999698$$
$$= 0.000302$$

Exercise 8.15

$$\mathbb{P}\left\{\frac{X-\mu n}{\sigma\sqrt{n}} > \frac{Y-\mu n}{\sigma\sqrt{n}}\right\} = \left\{\frac{X-10{,}000(240)}{800\sqrt{10{,}000}} > \frac{2{,}700{,}000-10{,}000(240)}{800\sqrt{10{,}000}}\right\}$$
$$= 1-\mathbb{P}\left\{Z \leq \frac{2{,}700{,}000-2{,}400{,}000}{80000}\right\}$$
$$= 1-\mathbb{P}\left\{Z \leq \frac{30}{8}\right\}$$
$$= 1-\mathbb{P}\left\{Z \leq 3.75\right\}$$
$$= 1-\Phi(3.75)$$
$$\approx 0$$