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Math 461
Assignment VI

Exercise 1 (5.17)

(a) We want to determine the probability that a physician chosen at random makes less than \$200,000, i.e. $\mathbb{P}(X \leq 200{,}000)$. To do this, we first need to find μ and σ. Clearly $\mu = \frac{320{,}000 + 180{,}000}{2} = 250{,}000$. Now we need to find σ. We can do this by observing that $\mathbb{P}(X \leq 320{,}000) = \frac{3}{4}$, and inverting this quantity with the table on page 190. Doing so gives us the approximation $\sigma = \frac{250{,}000 - 180{,}000}{0.675} = \frac{70{,}000}{0.675}$. Finally

$$1 - \phi\left(\frac{x-u}{\sigma}\right) = 1 - \phi\left(\frac{200{,}000 - 250{,}000}{\frac{70{,}000}{0.675}}\right) \approx 1 - \phi(-0.48).$$

Looking up 0.48 in the table, we find that $1 - \phi(-0.48) \approx 1 - 0.6844 = 0.3156$.

(b) $0.75 - \phi\left(\frac{x-u}{\sigma}\right) = 0.75 - \phi\left(\frac{280{,}000 - 250{,}000}{\frac{70{,}000}{0.675}}\right) \approx 0.75 - \phi(0.29) \approx 0.75 - .6141 = 0.1359.$

Exercise 2 (5.20)

In each part, let X be the number of people in favor of a rise in school taxes out of a pool of 100 people. Note that $X \sim \text{Binomial}\big(100{,}0.65\big) \approx \mathcal{N}\big(65{,}\sqrt{100 * 0.65 * 0.35}\big) = \mathcal{N}\big(65{,}22.75\big)$, and that $SD(X) = \sqrt{22.75} \approx 4.77$.

(a) At least 50 who are in favor.

$$\begin{aligned}
\mathbb{P}(X \geq 50) &= 1 - \mathbb{P}(X < 50) \\
&= 1 - \mathbb{P}\left\{\frac{X-\mu}{\sigma} < \frac{50-65}{4.77}\right\} \\
&\approx 1 - \mathbb{P}\big(Y < -3.145\big) \\
&= \mathbb{P}\big(Y \geq 3.145\big) \\
&= 1 - \Big(1 - \mathbb{P}\big(Y \geq 3.145\big)\Big) \\
&= \mathbb{P}\big(Y \geq 3.145\big) \\
&= \Phi(3.145) \\
&= 0.9992.
\end{aligned}$$

(b) Between 60 and 70 inclusive who are in favor.

$$\begin{aligned}\mathbb{P}(60 \leq X \leq 70) &= \mathbb{P}\left\{\frac{59.5-65}{4.77} \leq \frac{X-\mu}{\sigma} \leq \frac{70.5-65}{4.77}\right\} \\ &\approx \left(-1.15 \leq Y \leq 1.15\right) \\ &= \Phi\left(1.15\right) - \Phi\left(-1.15\right) \\ &= \Phi\left(1.15\right) - \left(1 - \Phi(-1.15)\right) \\ &= .8749 - \left(1 - .8749\right) = 0.75\end{aligned}$$

(c) fewer than 75 in favor.

$$\begin{aligned}\mathbb{P}(X \leq 75) &= \mathbb{P}\left\{\frac{X-\mu}{\sigma} < \frac{75-65}{4.77}\right\} \\ &\approx \mathbb{P}\left(Y < 2.1\right) \\ &= \Phi(2.1) \\ &= 0.9821\end{aligned}$$

Exercise 3 (5.22)

Let X be the number of serves until Jo reaches 50 successful serves.
Note that $SD(X) = \sqrt{100 * 0.4 * 0.6} \approx 4.898$.

$$\begin{aligned}\mathbb{P}(X \geq 100) &= 1 - \mathbb{P}(X < 100) \\ &\approx 1 - \mathbb{P}\left\{\frac{X-\mu}{\sigma} < \frac{49.5-40}{4.898}\right\} \\ &\approx \mathbb{P}\left(Y \leq 1.94\right) \\ &= \Phi(1.94) \\ &= 0.9738.\end{aligned}$$

Exercise 4 (5.31)

(a) $\mathbb{E}\big(|X-a|\big) = \int_0^A \frac{1}{A}|t-a|dt$

$$= \frac{1}{A}\Big(\int_0^a (a-t)dt + \int_a^A (t-a)dt\Big)$$
$$= \frac{1}{A}\Big(\big(at - \tfrac{t^2}{2}\big)\Big|_0^a + \big(\tfrac{t^2}{2} - at\big)\Big|_a^A\Big)$$
$$= \frac{1}{A}\Big(\big(a^2 - \tfrac{a^2}{2}\big) + \big(\tfrac{A^2}{2} - aA\big) - \big(\tfrac{a^2}{2} - a^2\big)\Big)$$
$$= \frac{1}{A}\Big(\frac{A^2}{2} - aA + a^2\Big)$$
$$= \frac{A}{2} - a + \frac{a^2}{A}$$

Now take the derivative with respect to a, and set it equal to zero.

$\partial_a\big(\frac{A}{2} - a + \frac{a^2}{A}\big) = \frac{2a}{A} - 1 = 0 \longrightarrow a = \frac{A}{2}$.

(b) $\mathbb{E}\big(|X-a|\big) = \int_0^a \lambda e^{-\lambda t}(a-t)dt + \int_a^A \lambda e^{-\lambda t}(t-a)dt$

$$= a + \frac{1}{\lambda} + \frac{2}{\lambda}e^{-a\lambda}$$

Now take the derivative with respect to a, and set it equal to zero.

$\partial_a\big(a + \frac{1}{\lambda} + \frac{2}{\lambda}e^{-a\lambda}\big) = 1 - 2e^{-a\lambda} = 0 \longrightarrow a = \frac{\ln 2}{\lambda}$.

Exercise 5 (5.32)

(a) $\mathbb{P}(X \geq 2)$ is equivalent to $1 - \mathbb{P}(X \leq 2)$, so we need to calculate $1 - \int_{-\infty}^2 \frac{1}{2}e^{-\frac{x}{2}}dx$.

$$\int_{-\infty}^2 \frac{1}{2}e^{-\frac{x}{2}}dx = \frac{1}{2}\int_{-\infty}^2 e^{-\frac{x}{2}}dx$$
$$= -e^{-\frac{x}{2}}\Big|_{-\infty}^2$$
$$= -e^{-\frac{2}{2}} + \lim_{x\to\infty} e^{-\frac{x}{2}}$$
$$= e^0 - e^{-1}$$
$$= 1 - e^{-1}$$

So $\mathbb{P}(X \geq 2) = 1 - \big(1 - e^{-1}\big) = e^{-1}$.

(b) In this case, we need to calculate $\mathbb{P}(X \geq 10 | X \geq 9) = \frac{\mathbb{P}(X \geq 10)\mathbb{P}(X \geq 9 | X \geq 10)}{\mathbb{P}(X \geq 9)}$.
We can reduce this to $\frac{\mathbb{P}(X \geq 10)}{\mathbb{P}(X \geq 9)}$, since $\mathbb{P}(X \geq 9 | X \geq 10)$ is clearly 1.

Using the antiderivative obtained in part a, we have

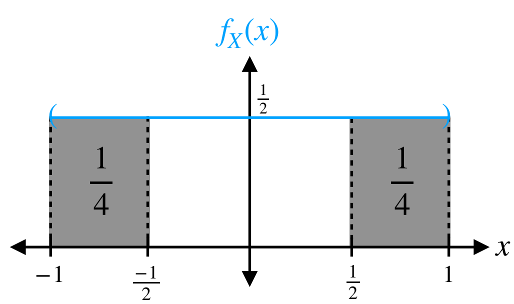
$\mathbb{P}(X \geq 10) = 1 - \mathbb{P}(X \leq 10) = 1 - \left(e^0 - e^{-\frac{10}{2}}\right) = e^{-5}$

and $\mathbb{P}(X \geq 9) = 1 - \mathbb{P}(X \leq 9) = 1 - \left(e^0 - e^{-\frac{9}{2}}\right) = e^{-4.5}$.

Therefore $\mathbb{P}(X \geq 10 | X \geq 9) = \frac{e^{-5}}{e^{-4.5}} = e^{-5+4.5} = e^{-0.5}$.

Exercise 6 (5.37)

a) Assuming that the distribution of X is defined strictly over $(-1,1)$, we know that the height of X's graph must be $\frac{1}{2}$ everywhere, since the total area under the graph must be 1. Then $\mathbb{P}(|X| > \frac{1}{2})$ is the area under the distribution graph for the subdomain $\left(-1,\frac{-1}{2}\right) \cup \left(\frac{1}{2},1\right)$. Since the distribution is uniform, $\mathbb{P}(|X| > \frac{1}{2})$ is equal to the sum of the rectangles $\left(-1,\frac{-1}{2}\right) \times \left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right) \times \left(0,\frac{1}{2}\right)$. Each of these rectangles has area $\frac{1}{2}\frac{1}{2} = \frac{1}{4}$, so the solution is $\frac{1}{2}$. This derivation is illustrated by the following figure:



b) $f_{|X|}(x) = f_X(x) + f_X(-x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$.

Exercise 7 (5.39)

$$\begin{aligned} Y &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(\ln X \leq y) \\ &= \mathbb{P}(X \leq e^y) \end{aligned}$$

And $X \sim F_X(t) = \int_0^t e^{-x} dx$

$$\begin{aligned} &= -e^{-x}\Big|_0^t \\ &= -e^{-t} + e^0 \\ &= 1 - e^{-t}. \end{aligned}$$

So $\mathbb{P}(X \leq e^t) = F_X(e^t)$

$$= 1 - e^{-e^y}.$$

Finally, $f_Y(t) = \partial_t\left(1 - e^{-e^t}\right)$

$$\begin{aligned} &= -\partial_t e^{-e^t} \\ &= \left(-e^{-e^t}\right)\left(-e^t\right) \\ &= e^{-e^t} e^t \\ &= e^{t-e^t}. \end{aligned}$$