The following definitions and theorems are adapted from James R. Munkres' Topology (second edition)

Definition$_{12.1}$**:** A topology $\mathcal{T}$ on a set $\mathbf{X}$ is a collection of subsets of $\mathbf{X}$, called open sets,
(Page 76) satisfying the following properties.

$i)$ $\varnothing \in \mathcal{T}$ and $\mathbf{X} \in \mathcal{T}$.

$ii)$ If $\mathbf{Y} \subseteq \mathcal{T}$, then $\bigcup_{t \in \mathbf{Y}} t \in \mathcal{T}$.

$iii)$ If $\mathbf{Y} \subseteq \mathcal{T}$ is finite, then $\bigcap_{t \in \mathbf{Y}} t \in \mathcal{T}$.

The ordered pair $(\mathbf{X}, \mathcal{T})$ is called a topological space.

Definition$_{13.1}$**:** A basis $\mathscr{B}$ for a topology $\mathcal{T}$ on $\mathbf{X}$ is a collection of subsets of $\mathbf{X}$ satisfying the
(Page 78) following properties.

$i)$ If $x \in \mathbf{X}$, then we can find $B \in \mathscr{B}$ such that $x \in B$.

$ii)$ If $x \in B_1 \cap B_2$ for some sets B_1 and B_2 in $\mathscr{B}$, then we can find a set $B_3 \in \mathscr{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Lemma$_{13.1}$**:** If $\mathscr{B}$ is a basis for a topology $\mathcal{T}$ on $\mathbf{X}$, then $\mathcal{T} = \left\{ \bigcup_{b \in B} b \,:\, B \subseteq \mathscr{B} \right\}$.
(Page 80)

Definition$_{17.1}$**:** Let $(\mathbf{X}, \mathcal{T})$ be a topological space. A set $\mathbf{S} \subseteq \mathbf{X}$ is closed if $\mathbf{S}^c \in \mathcal{T}$.
(Page 93)

Theorem$_{17.1(3)}$**:** The union of any finite number of closed sets in a topological space is closed.
(Page 94)

Theorem:

There are infinitely many prime numbers.

Proof(By Contradiction):

Part I – The evenly spaced integer topology.

Suppose $\mathscr{B}$ is the set of all arithmetic progressions $\mathbb{Z}(a,m) := \{a + nm : n \in \mathbb{Z}\}$ where $m \neq 0$. We must first show that $\mathscr{B}$ is a basis for a topology $\boldsymbol{\tau}$ on $\mathbb{Z}$. To do this, we must show that $\mathscr{B}$ satisfies both criteria from definition$_{13.1}$.

$i)$ If $k \in \mathbb{Z}$, then clearly $k \in \mathbb{Z}(0,k) \in \mathscr{B}$.

$ii)$ If $k \in \mathbb{Z}(a,b) \cap \mathbb{Z}(c,d)$, then $k \in \mathbb{Z}(k,bd)$, since we have $k = k + bd(0)$.
Now suppose we have $j \in \mathbb{Z}(k,bd)$. Then $j = k + bdn$.
We can write k as $a + bm$ for some integer m, so $j = a + bm + bdn = a + b(m + dn)$, and therefore $j \in \mathbb{Z}(a,b)$. Likewise, we can write k as $c + dn$ for some integer n, so $j = c + dn + bdn = c + d(n + bn)$, and therefore $j \in \mathbb{Z}(c,d)$.
So we have $\mathbb{Z}(k,bd) \subseteq \mathbb{Z}(a,b) \cap \mathbb{Z}(c,d)$.

We have shown that $\mathscr{B}$ is a basis for a topology on $\mathbb{Z}$, so lemma$_1$ gives us a topology $\boldsymbol{\tau}$ on $\mathbb{Z}$ whose elements are the unions of arithmetic progressions in $\mathscr{B}$. In other words, if B is any subset of $\mathscr{B}$, then $\bigcup_{b \in B} b$ is contained in $\boldsymbol{\tau}$.

Part II – Finite unions of arithmetic progressions of the form $\mathbb{Z}(0,p)$ are closed.

Now suppose p is prime and observe that $\bigcup_{k=1}^{p-1} \mathbb{Z}(k,p)$ is the set of all integers which are congruent to k modulo p for all values of k satisfying $0 < k < p$. Equivalently, this set can be described as the set of integers which are not divisible by p, which is precisely what $\mathbb{Z}(0,p)^c$ is. So $\mathbb{Z}(0,p)^c = \bigcup_{k=1}^{p-1} \mathbb{Z}(k,p)$. And since $\mathbb{Z}(0,p)^c$ can be written as a union of arithmetic progressions, i.e. open sets, it must be open by property (ii) of definition$_{12.1}$. Furthermore, since $\mathbb{Z}(0,p)^c$ is open, definition$_{17.1}$ tells us that $\left(\mathbb{Z}(0,p)^c\right)^c = \mathbb{Z}(0,p)$ must be closed. It follows by theorem$_{17.1(3)}$ that finite unions of arithmetic progressions of the form $\mathbb{Z}(0,p)$ are also closed.

Part III – If $\{p_1, p_2, \ldots, p_r\}$ is the set of all primes, then $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c$ can't be open.

Finally, suppose for contradiction that there are finitely many prime numbers $p_1, p_2, \ldots, p_r$. Since we have shown that finite unions of arithmetic progressions of the form $\mathbb{Z}(0,p)$ are closed, we know that $\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)$ must be closed. And this tells us that the complement of this union must be open. But $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c = \{-1,1\}$ is not an arithmetic progression nor the empty set. Therefore $\{-1,1\}$ cannot be open. So we have a contradiction. Thus we can conclude that there are infinitely many prime numbers. ∎

How we can conclude that $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c = \{-1,1\}$?

Proof:

We will show that $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c = \{-1,1\}$ by showing that

$i)$ $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c$ contains everything in $\{-1,1\}$, and

$ii)$ $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c$ and nothing that isn't part of $\{-1,1\}$.

$i)$ Observe that if p is prime, then $\mathbb{Z}(0,p)$ contains neither -1 nor 1, since neither of these is a multiple of p. Therefore $\{-1,1\} \cap \bigcup_{k=1}^{r} \mathbb{Z}(0,p_k) = \varnothing$. So $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c$ contains everything in $\{-1,1\}$.

$ii)$ Suppose k isn't part of $\{-1,1\}$. i.e. $k \in \{-1,1\}^c = \mathbb{Z} \backslash \{-1,1\}$.
There are two cases to consider.

Case 1) $k = 0$.
Let p be prime. Then $k \in \mathbb{Z}(0,p)$ since $k = 0 \cdot p$. So $k \in \bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)$, and therefore $k \notin \left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c$.

Case 2) $k \geq 2$.
Then k has a prime divisor q by the fundamental theorem of arithmetic. And since we have assumed that $\{p_1, p_2, \ldots, p_r\}$ contains all of the primes, we must have $q = p_i$ for some $1 \leq i \leq r$. Therefore $k \in \mathbb{Z}(0,q) = \mathbb{Z}(0,p_i)$. So $k \in \bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)$ and thus $k \notin \left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c$.

It follows that $\left(\bigcup_{k=1}^{r} \mathbb{Z}(0,p_k)\right)^c = \{-1,1\}$, so we are done.

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