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Math 461
Assignment II

Exercise 1

A group of 2*N* men and 2*N* women is divided into two equal groups. Find the probability *p* that each group will be equally divided into men and women. Estimate *p*, using Stirling's Formula.

Divide 2N men and 2N women into two equal groups A and B. There are $\binom{4N}{2N}$ ways to divide 4N people into two equal groups, so this is the size of our sample space. For each of the two 2N people in each group, we can choose N, so there are $\binom{2N}{N}^2$ ways to divide the groups so that they're equally divided into men and women.

Stirling's formula: $n! \approx n^{n+\frac{1}{2}}e^{-n}\sqrt{2\pi} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

$$\frac{\binom{2N}{N}^2}{\binom{4N}{2N}} = \frac{\left(\frac{(2N)!}{N!^2}\right)^2}{\frac{(4N)!}{(2N)!^2}} = \frac{(2N)!^2(2N)!^2}{N!^4(4N)!} = \frac{(2N)!^4}{N!^4(4N)!}$$

$$\frac{(2N)!^4}{N!^4(4N)!} \approx \frac{\left(\sqrt{4\pi N}\left(\frac{2N}{e}\right)^{2N}\right)^4}{\left(\sqrt{2\pi N}\left(\frac{N}{e}\right)^N\right)^4\left(\sqrt{8\pi N}\left(\frac{4N}{e}\right)^{4N}\right)} = \frac{\sqrt{2}\sqrt{2}\pi N\left(\frac{2N}{e}\right)^{8N}}{\sqrt{2\pi N}\left(\frac{N}{e}\right)^{4N}2\sqrt{2\pi N}\left(\frac{4N}{e}\right)^{4N}}$$

$$= \frac{2\left(\frac{2N}{e}\right)^{8N}}{\sqrt{\pi N}\left(\frac{N}{e}\right)^{4N}\left(\frac{4N}{e}\right)^{4N}} = \frac{2^{4N+1}\left(\frac{N}{e}\right)^{4N}\left(\frac{2N}{e}\right)^{4N}}{\sqrt{\pi N}\left(\frac{N}{e}\right)^{4N}2^{4N}\left(\frac{2N}{e}\right)^{4N}} = \frac{2\left(\frac{N}{e}\right)^{4N}}{\sqrt{\pi N}\left(\frac{N}{e}\right)^{4N}}$$

$$= \frac{2\left(\frac{N}{e}\right)^{4N}}{\sqrt{\pi N}\left(\frac{N}{e}\right)^{4N}} = \frac{2}{\sqrt{\pi N}}.$$

Exercise 2

In an experiment, die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let E_n denote the event that n rolls are necessary to complete the experiment. What points of the sample space are contained in E_n? What is $\left(\bigcup_{n=1}^{\infty} E_n\right)^c$?

The sample space S is the set of all sequences (e_n) with the property that if (e_n) has N terms, then the first N-1 terms of (e_n) are in $\mathbb{N}_6 := \{1,2,3,4,5\}$ and the N$^{\text{th}}$ term is 6. Additionally (e_n) may have an infinite number of terms, in which case they are all in $\mathbb{N}_6$.

E_n contains all sequences of the form (e_n). i.e. E_n is the set of all sequences matching the previous construction, with the additional constraint that each sequence has finite length n. $\bigcup_{n=1}^{\infty} E_n$ is the union of all collections of sequences (e_n) whose length is in $\mathbb{N}$; it is the collection of all finite sequences (e_n) matching our first construction.

The complement, $\left(\bigcup_{n=1}^{\infty} E_n\right)^c$, is the set of all non-terminating sequences of the same form. In other words, $\left(\bigcup_{n=1}^{\infty} E_n\right)^c$ is the collection of all infinite sequences that can be constructed with elements of $\mathbb{N}_6 := \{1,2,3,4,5\}$.

Exercise 3

Suppose that A and B are mutually exclusive events for which $\mathbb{P}(A) = 0.3$ and $\mathbb{P}(B) = 0.5$.

a) What is the probability that either A or B occurs?

Since $A \cap B = \varnothing$, the probability of their union is obtained by the sum of the independent probabilities. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) = 0.3 + 0.5 = 0.8$.

b) What is the probability that A occurs but not B?

"But" is just another name for "and", so we want the probability that A occurs and B does not occur. A and B cannot both occur, since A and B are mutually exclusive, so the probability is simply $\mathbb{P}(A) = 0.3$.

c) What is the probability that both A and B occur?

A and B are mutually exclusive, so $A \cap B = \varnothing$. Therefore $\mathbb{P}(A \cap B) = \mathbb{P}(\varnothing)$. And $\mathbb{P}(\varnothing) = 0$, since $\mathbb{P}(\varnothing^c) = 1$ by the second axiom of probability. So $\mathbb{P}(A \cap B) = 0$.

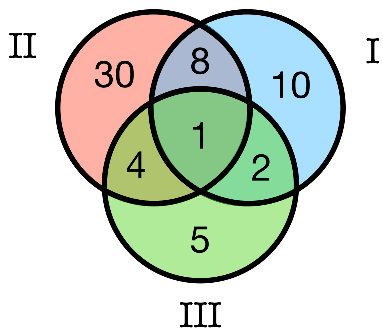
Exercise 4

A certain town with a population of 100,000 has 3 newspapers: I,II, and III. The proportion of townspeople who read these papers are as follows

I: 10 percent   II: 30 percent   III: 5 percent

I and II: 8 percent   I and III: 2 percent   II and III: 4 percent

I and II and III: 1 percent



a) Find the number of people who read only one newspaper.

The number of people who read only one newspaper is given by taking all those who read any newspaper, and removing those who read multiple newspapers.

Ten percent of people read newspaper I, eight percent of people read newspapers I and II, two percent of people read newspapers I and III, and one percent of people read all three. So of those who read newspaper I, $((8+2)-1)$ percent also read another newspaper. Therefore the number of people who read just newspaper I is $\frac{100{,}000(10-((8+2)-1))}{100} = 1{,}000$. Applying the same process to newspapers II and III, we find that people read only newspaper II, and $(5-((4+2)-1)) = 0$ people only read newspaper III. Finally, we take the sum and find that $1{,}000 + 19{,}000 + 0 = 20{,}000$ people read just one newspaper.

b) How many people read at least two newspapers?

In this case, we can simply take the difference between the number of people who read any number of newspapers, and those who read exactly one (which we obtained in part a). We obtain the number of people who read any number of newspapers by taking $((10+30+5)-(4+8+2)+1)\%$ = 32% of the total population. Therefore $100{,}000\frac{32}{100} - 20{,}000 = 32{,}000 - 20{,}000 = 12{,}000$ people read at least two newspapers.

c) If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?

Here, the solution is obtained by determining how many people read one morning paper as well as the evening paper, or both of the morning papers and the evening paper. In other words, we need to take the sum of people who read papers I and II, of people who read papers II and III, and of people who read all three. So 7,000 people + 3,000 people + 1,000 people, which is 11,000 people.

d) How many people do not read any newspapers?

This was can be deduced from part b. Since 32% of the population reads some newspaper, the remaining 68% do not read any newspaper. So 68,000 people do not read any newspaper.

e) How many people read only one morning paper and one evening paper?

In this last case, we want to add the people who read papers I and II to those who read papers II and III. So $100{,}000\frac{(8-1)+(4-1)}{100} = 1{,}000 * 10 = 10{,}000$ people read only one morning paper and one evening paper.

Exercise 5

If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of the following events?

(a) Being dealt a flush?

According to Wikipedia's List of poker hands, a flush is a hand where all cards have the same suit. So we want to determine the probability of being dealt five cards with the same suit. First we can break up the problem by noticing that because we have four suits which the cards into four disjoint sets of thirteen cards each, we can solve the problem for one suit, and multiply by four to account for the other suits. Considering only the one suit, there are obviously $\binom{13}{5} = \frac{13!}{5!8!}$ ways to get five cards from the set. There are $\binom{52}{5} = \frac{52!}{5!47!}$ ways to get 5 cards from the whole deck, so there are $4\frac{13!5!47!}{5!8!52!}$ ways to be dealt a flush.

(b) Being dealt one pair?

In this case, we choose one of thirteen denominations (A,2,…,Q,K) for the pair, as well as two suits from that denomination. The remaining three cards have any denomination except for the first denomination choice (so that we don't get more than one pair), and a suit for each. Finally, we divide by the total number of possible hands to obtain the solution

$$\frac{\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3}{\binom{52}{5}}.$$

(c) Being dealt two pairs?

In this case, we choose two of thirteen denominations (A,2,…,Q,K), as well as two suits for each chosen that denomination. The remaining card can have any denomination except for the first two denomination choices, and it can take any suit. Taking the product and dividing by the total number of possible hands, we obtain the solution

$$\frac{\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1}}{\binom{52}{5}}.$$

(d) Being dealt three of a kind?

We begin as we did in part b, choosing one denomination for the three-of-a-kind, but now taking three suits from the four, since each of the three will have a different suit. We choose the remaining two cards from any of the remaining twelve denominations, and for each of those we can choose one from four suits. Taking the product and dividing by the total number of possible hands, we obtain the solution

$$\frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}.$$

(e) Being dealt four of a kind?

Similarly to parts (b) and (d), choose one denomination for the four-of-a-kind, and this time we must choose all four suits. For the remaining card, we pick one from the twelve remaining denominations, and one from four suits. Taking the product and dividing by the total number of possible hands, we obtain the solution

$$\frac{\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1}}{\binom{52}{5}}.$$

Exercise 6

A pair of dice are rolled, what is the probability that the second die lands on a higher value than the first?

Let P be the property of an ordered pair of dice that the value of the second die exceeds that of the first. We can solve this problem by enumerating all possible rolls, and taking the ratio of pairs satisfying P to all possible rolls. To make it especially easy, make pairs satisfying P red, and other pairs black, then count and take the ratio of red to black pairs.

(⚀, ⚀), (⚀, ⚁), (⚀, ⚂), (⚀, ⚃), (⚀, ⚄), (⚀, ⚅)
(⚁, ⚀), (⚁, ⚁), (⚁, ⚂), (⚁, ⚃), (⚁, ⚄), (⚁, ⚅)
(⚂, ⚀), (⚂, ⚁), (⚂, ⚂), (⚂, ⚃), (⚂, ⚄), (⚂, ⚅)
(⚃, ⚀), (⚃, ⚁), (⚃, ⚂), (⚃, ⚃), (⚃, ⚄), (⚃, ⚅)
(⚄, ⚀), (⚄, ⚁), (⚄, ⚂), (⚄, ⚃), (⚄, ⚄), (⚄, ⚅)
(⚅, ⚀), (⚅, ⚁), (⚅, ⚂), (⚅, ⚃), (⚅, ⚄), (⚅, ⚅)

There are 36 pairs total, and 15 red pairs, so the solution is $\frac{15}{36} = \frac{5}{12}$.

Exercise 7

A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first.

There are four ways to get a sum of 5 from a pair of dice, and six ways to get a sum of seven. In particular, $5 = (1 + 4) = (2 + 3)$ and $7 = (1 + 6) = (2 + 5) = (3 + 4)$. Note that we count each of the sums twice, since $\mathbb{N}(+)$ is commutative.

$\mathbb{P}(\text{ Roll 5 }) = \frac{1}{9}$, and $\mathbb{P}(\text{ Roll 7 }) = \frac{1}{6}$.

$\mathbb{P}(\text{ Roll 5} \cup \text{ Roll 7}) = \mathbb{P}(\text{ Roll 5 }) + \mathbb{P}(\text{ Roll 7 }) = \frac{1}{9} + \frac{1}{6} = \frac{15}{54} = \frac{5}{18}$.

$$\mathbb{P}(\text{ Roll 5 } \mid \text{ Roll 5 or Roll 7 }) = \frac{\mathbb{P}(\text{ Roll 5 })\mathbb{P}(\text{ Roll 5 or Roll 7 } \mid \text{ Roll 7 })}{\mathbb{P}(\text{ Roll 5 or Roll 7 })} = \frac{\frac{1}{9}}{\frac{5}{18}} = \frac{18}{9(5)} = \frac{2}{5}.$$

So the probability that a five occurs first is $\frac{2}{5}$.

Exercise 8

An urn contains n white and m black balls, where n and m are positive numbers.

a) If two balls are randomly withdrawn, what is the probability that they are the same color?

There are $\binom{m}{2}$ ways to choose two black balls, and $\binom{n}{2}$ ways to choose two white balls. Since these are disjoint, we just need to take their sum and divide by the size of the sample space, which is $\binom{m+n}{2}$, since we are choosing two balls from the sum of m and n. So the solution is

$$\frac{\binom{n}{2}+\binom{m}{2}}{\binom{n+m}{2}}.$$

b) If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?

We can solve this problem by taking the sum of the probabilities of withdrawing balls of white and black, individually. Let u be a placeholder for n and m. Then there are $\binom{u}{1}$ ways of getting two of the same color (corresponding to u), and we must raise that to the second power since we are drawing two balls. It may be worth noting that if we were not replacing balls, this quantity would be $\binom{u}{1}\binom{u-1}{1}$. Anyway, we need to do this for both m and n, and then divide by the size of the sample space. In this case, the size of the sample space is $\binom{n+m}{1}^2$, since we are choosing one from n+m, twice.

So the probability is

$$\frac{\binom{m}{1}^2+\binom{n}{1}^2}{\binom{n+m}{1}^2}.$$

c) Show that the probability in part (b) is always larger than the one in part (a).

Note $\dfrac{\binom{n}{2}+\binom{m}{2}}{\binom{n+m}{2}} = \dfrac{n^2+m^2-n-m}{(n+m)^2-n-m} = \dfrac{n^2+m^2-n-m}{(n+m)(n+m-1)}$ and $\dfrac{\binom{m}{1}^2+\binom{n}{1}^2}{\binom{n+m}{1}^2} = \dfrac{m^2+n^2}{(n+m)^2}$.

$$\frac{n^2+m^2-n-m}{(n+m)(n+m-1)} < \frac{m^2+n^2}{(n+m)^2} \iff \frac{n^2+m^2-n-m}{(n+m-1)} < \frac{m^2+n^2}{(n+m)}$$

$$\iff (n^2+m^2-n-m)(n+m) < (m^2+n^2)(n+m-1)$$

$$\iff n^2(n+m)+m^2(n+m)-n(n+m)-m(n+m) < m^2(n+m-1)+n^2(n+m-1)$$

$$\iff m^3+n^3+mn^2+nm^2-m^2-n^2-2nm < m^3+n^3+mn^2+nm^2-m^2-n^2$$

$$\iff -2nm < 0$$

Which is true, since n and m are positive. So our inequality holds. i.e.

$$\frac{\binom{n}{2}+\binom{m}{2}}{\binom{n+m}{2}} < \frac{\binom{m}{1}^2+\binom{n}{1}^2}{\binom{n+m}{1}^2}$$

Exercise 9

Seven balls are randomly withdrawn from an urn that contains 12 red, 16 blue, and 18 green balls. Find the probability of the following events.

a) 3 red, 2 blue, and 2 green balls are withdrawn.

There are $\binom{12}{3}$ ways to choose three red balls from twelve, $\binom{16}{2}$ ways to choose two blue balls from sixteen, and $\binom{18}{2}$ ways to choose two green balls from eighteen. These choices are made in any order, so we obtain our solution by taking their product, and dividing by the total number of ways to choose seven balls from the sum of 46 balls. i.e.

$$\frac{\binom{12}{3}\binom{16}{2}\binom{18}{2}}{\binom{46}{7}}.$$

b) At least 2 red balls are withdrawn.

There are 34 choose 7 ways to withdraw all non-red balls, and we
The complement of the event space E where at least 2 red balls are drawn is the event

space E^c where no more than 1 red ball is drawn. We can determine $\mathbb{P}(E^c)$ by taking the sum of $\mathbb{P}$(0 red balls are drawn) and $\mathbb{P}$(1 red ball is drawn).

$\mathbb{P}$(0 red balls are drawn) = $\frac{\binom{34}{7}}{\binom{46}{7}}$, since we can choose any seven non-red balls, of which there are 34. And $\mathbb{P}$(1 red ball is drawn) = $12\frac{\binom{34}{6}}{\binom{46}{7}}$, since we are assuming one red ball, and the remaining six from the non-red balls. The factor of 12 comes from the fact that there are 12 red balls; we can choose the remaining balls for each red ball. So

$$\mathbb{P}(E^c) = \frac{\binom{34}{7}}{\binom{46}{7}} + 12\frac{\binom{34}{6}}{\binom{46}{7}}.$$

From this point, we can use the formula $\mathbb{P}(E) = 1 - \mathbb{P}(E^c)$ to obtain the result

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - \frac{\binom{34}{7}}{\binom{46}{7}} - 12\frac{\binom{34}{6}}{\binom{46}{7}}.$$

c) All withdrawn balls are the same color.

In this case, all we need to do is divide the sum of the probabilities of getting all red balls, all blue balls, or all green balls by the size of the sample space. Note that these events are disjoint, and each probability can be obtained by choosing seven from the number of balls of of each color. So the probability of all balls being the same color is

$$\frac{\binom{12}{7} + \binom{16}{7} + \binom{18}{7}}{\binom{46}{7}}.$$

d) Either exactly 3 red balls or exactly 3 blue balls are withdrawn.

In this case, we first want to calculate the number of ways we can get exactly three reds, and how many ways we can get exactly three blues. The former quantity can be obtained by choosing three reds from the twelve, and taking the remaining four from the blues and greens. The latter quantity can be obtained by choosing three blues from the sixteen, and choosing the remaining four from the reds and greens. In each of these cases, we divide by $\binom{46}{7}$ to obtain the probability. We must also consider the ways of of getting three reds and three blues simultaneously, which is done by choosing three reds, three blues, and one green, and dividing by the size of the sample space. Finally, add the first two quantities and subtract the last to obtain the solution

$$\frac{\binom{12}{3}\binom{34}{4}}{\binom{46}{7}} + \frac{\binom{16}{3}\binom{30}{4}}{\binom{46}{7}} - \frac{\binom{12}{3}\binom{16}{3}\binom{18}{1}}{\binom{46}{7}} = \frac{\binom{12}{3}\binom{34}{4} + \binom{16}{3}\binom{30}{4} - \binom{12}{3}\binom{16}{3}\binom{18}{1}}{\binom{46}{7}}.$$

Exercise 10

An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability of the following events?

(a) She correctly answers all 5 problems?

In this case, the size of the sample space is $\binom{10}{5}$, since the instructor is choosing five problems from ten. There are $\binom{7}{5}$ ways in which the exam can be constructed with five questions of the known seven. Therefore, the solution is

$$\frac{\binom{7}{5}}{\binom{10}{5}} = \frac{1}{12}.$$

(b) She correctly answers at least 4 of the problems?

We need to determine the probability that she correctly answers exactly four of the problems, as well as the probability that she correctly answers exactly five. We already did the second part in (a), so we just need to calculate the first part, and add the results. The probability that she correctly answers exactly four of the problems is obtained by taking four problems from the known seven, multiplying by the number of ways to choose the remaining three problems, and dividing by the size of the sample space, i.e.

$$\frac{\binom{7}{5} + \binom{7}{4}\binom{3}{1}}{\binom{10}{5}} = \frac{1}{2}.$$

Exercise 11

If N people, including A and B, are randomly arranged in a circle, what is the probability that A and B are next to each other?

For each person A, there are exactly two spots on the circle where another particular person B can be next to A. So B can be in two spots out of N-1, since B≠A. Therefore the solution is $\frac{2}{N-1}$.

Exercise 12

Suppose that n balls are randomly distributed into N compartments. Find the probability that m balls will fall into the first compartment. Assume that all N^n arrangements are equally likely.

We have $\binom{n}{m}$ ways to distribute m balls into the first compartment, and the remaining (n-m) balls are placed into the remaining N-1 compartments. So there are a total of $\binom{n}{m}(N-1)^{n-m}$ ways for exactly m balls to fall into the first compartment. We divide this number by the total number of arrangements, N^n, obtaining $\frac{\binom{n}{m}(N-1)^{n-m}}{N^n}$.

Exercise 13

Compute the probability that a bridge hand is void in at least one suit.

Let A, B, C, and D be events where a bridge hand is void in the suit corresponding to the event label. We want to find ℙ(A ∪ B ∪ C ∪ D). To do this, we can use the inclusion-exclusion identity.

ℙ(A ∪ B ∪ C ∪ D) = $v_1 - v_2 + v_3 - v_4$, where v_k is the probability that the bridge hand is void for k suits. v_k is equal to $\binom{4}{k}\frac{\binom{52-13k}{13}}{\binom{52}{13}}$. From this, we arrive at the formula

$$\mathbb{P}(A \cup B \cup C \cup D) = \sum_{k=1}^{4} (-1)^{k+1} v_k = \sum_{k=1}^{4} \binom{4}{k} (-1)^{k+1} \frac{\binom{52-13k}{13}}{\binom{52}{13}}.$$

Note that $\binom{4}{4}(-1)^5\frac{\binom{0}{13}}{\binom{52}{13}}$ is zero, so we actually just need the first three terms of the sum.

Finally, the probability that a bridge hand is void in at least one suit is

$$\sum_{k=1}^{3} \binom{4}{k} (-1)^{k+1} \frac{\binom{52-13k}{13}}{\binom{52}{13}} = \frac{4\binom{39}{13} - 6\binom{26}{13} + 4}{\binom{52}{13}}.$$