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Math 461
Assignment III

Exercise 1 (3.2)

If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is i? Compute for all values of i between 2 and 12.

Each die takes on a value between 1 and 6, and since we are talking about a pair, we know that the sum will be bounded by 2 and 12. We know that the probability of the first die landing on 6 is zero if i is less than seven, since the restriction that the first die lands on six means that the sum of the die will be bounded below by the sum of 6 and the minimum of the second die. Now we consider only the values of i between 7 and 12. For each of these cases, we want to know how many ways there are to make i as a sum of two numbers between 1 and 6. Once that number is obtained, we take the reciprocal, since we know that $6 + x = i$ has exactly one solution, namely $x = i - 6$. There are $13 - i$ ways to make a pair of die sum to i, as seen in the following enumeration

7 = 6+1, 5+2, 4+3, 3+4, 2+5, 1+6;
8 = 6+2, 5+3, 4+4, 3+5, 2+6;
9 = 6+3, 5+4, 4+5, 3+6;
10 = 6+4, 5+5, 4+6;
11 = 6+5, 5+6;
12 = 6+6.

Finally, we can express the probability that the first die lands on 6 given that the sum of the dice is i by the following function

$$\mathbb{P}(\text{ First die lands on 6} \mid \text{The sum of the dice is i }) = \begin{cases} 0 & \text{if } i \leq 6 \\ \frac{1}{13 - i} & \text{if } i > 6 \end{cases}.$$

Exercise 2 (3.9)

Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

Drawing from A, we have $\mathbb{P}$(White) = 1/3, $\mathbb{P}$(Red) = $\mathbb{P}$(Not white) = 2/3.
Drawing from B, we have $\mathbb{P}$(White) = 2/3, $\mathbb{P}$(Red) = $\mathbb{P}$(Not white) = 1/3.
Drawing from C, we have $\mathbb{P}$(White) = 1/4, $\mathbb{P}$(Red) = $\mathbb{P}$(Not white) = 3/4.

Let E be the event that the ball chosen from urn A was white.
Let F be the event that exactly two white balls are selected.

We want to find $\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E)\mathbb{P}(F \mid E)}{\mathbb{P}(F)}$.

Note $\mathbb{P}(E) = \frac{2}{6} = \frac{1}{3}$. And $\mathbb{P}(F \mid E)$ is the probability that a white ball will be chosen from either Urn B or Urn C (not both). To calculate this, we add the probability of getting white from urn B and red from urn C to the probability of getting white from urn C and red from urn B. So $\mathbb{P}(F \mid E) = \frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{4} = \frac{6}{12} + \frac{1}{12} = \frac{7}{12}$. Next we need to calculate $\mathbb{P}(F)$, so we need to count how many ways we can pick exactly two whites. To do this, sum the products of the probabilities of getting some permutation of 2 white balls and a red ball.
$\mathbb{P}(F) = (\frac{1}{3}\frac{1}{3}\frac{1}{4}) + (\frac{2}{3}\frac{2}{3}\frac{1}{4}) + (\frac{1}{3}\frac{2}{3}\frac{3}{4}) = \frac{1+4+6}{36} = \frac{11}{36}$.

So $\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E)\mathbb{P}(F \mid E)}{\mathbb{P}(F)} = \frac{\frac{7}{12}}{\frac{33}{36}} = \frac{7(36)}{12(33)} = \frac{7(36)}{11(36)} = \frac{7}{11}$.

Exercise 3 (3.14)

An urn initially contains 5 white and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the urn along with 2 other balls of the same color.

(a) Compute the probability that the first 2 balls are black and the next 2 are white;

Let W denote the event where a white ball is chosen, and let B denote the event where a black ball is chosen. We want to find $\mathbb{P}(BBWW)$.

$\mathbb{P}(B)$ is clearly $\frac{7}{12}$. And getting a second black ball means we must further choose any one of now-seven black balls from a total of fourteen balls. So $\mathbb{P}(BB) = \frac{7}{12}\frac{9}{14}$. Next, we want to pick a white ball from the sum of now-eleven black balls and still-five white balls, so $\mathbb{P}(BBW) = \frac{7}{12}\frac{9}{14}\frac{5}{16}$. And lastly, we want to pick another white ball from the sum of eleven black balls and now-seven white balls, so $\mathbb{P}(BBWW) = \frac{7}{12}\frac{9}{14}\frac{5}{16}\frac{7}{18} = \frac{35}{768}$.

(b) Compute the probability that of the first 4 balls selected, exactly 2 are black.

Note that the solution in part a, $\mathbb{P}(BBWW)$, is equal to the solutions of $\mathbb{P}(WWBB)$, $\mathbb{P}(BWBW)$, $\mathbb{P}(WBWB)$, $\mathbb{P}(BWWB)$, and $\mathbb{P}(WBBW)$. And these six cases account for the whole event space where two black balls are drawn. Therefore, the solution is six times the answer obtained in a. i.e. $\frac{6(35)}{768} = \frac{210}{768} = \frac{35}{128}$.

Exercise 4 (3.26)

Suppose that 5 percent of men and 0.25 percent of women are color blind. A color blind person is chosen at random. What is the probability of this person being male? Assume there are an equal number of males and females. What if the population consisted of twice as many males as females?

Let M be the event where a random color blind person is male, and let F be the event where a random color blind person is female.

A male is twenty times more likely to be color blind than a female, since $\frac{5}{0.25} = 20$. So in a group of twenty-one color blind people, we can expect that twenty of those people are men. Therefore the probability of a random color blind person being male is $\frac{20}{21}$.

If males are twice as common among color blind people as females, then a random color blind person would be twice as likely as before to be male. So a male is forty times more likely to be color blind than a female. Therefore the probability of a random color blind person being male is $\frac{40}{41}$.

Exercise 5 (3.31)

Ms. Aquina has just had a biopsy on a possibly cancerous tumor. Not wanting to spoil a weekend family event, she does not want to hear any bad news in the next few days. But if she tells the doctor to call only if the the news is good, then if the doctor does not call, Ms. Aquina can conclude that the news is bad. So, being a student of probability, Ms. Aquina instructs the doctor to flop a coin. If it comes up heads, the doctor is to call if the news is good and not call if the news is bad. If the coin comes up tails, the doctor is not to call. In this way, even if the doctor doesn't call, the news is not necessarily bad. Let α be the probability that the tumor is cancerous; let β be the conditional probability that the tumor is cancerous given that the doctor does not call.

(a) Which should be larger, α or β?

$\beta \geq \alpha$. Ordinarily I wouldn't answer a question with an expression, but I think the proof in b is sufficient justification of the claim. Still, I felt compelled to write this justification.

(b) Find β in terms of α, and prove your answer to (a).

Proof (By contradiction):

Let $\alpha = \mathbb{P}(\text{Tumor is cancerous})$,
and let $\beta = \mathbb{P}(\text{Tumor is cancerous} \mid \text{Doctor does not call})$.

Assume for contradiction that $\beta < \alpha$. (1)

We will show inequality (1) leads to a contradiction.

$$\text{So } \beta = \frac{\mathbb{P}(\text{Tumor is cancerous})\mathbb{P}(\text{Doctor does not call} \mid \text{Tumor is cancerous})}{\mathbb{P}(\text{Doctor does not call})}$$

$$= \alpha \frac{\mathbb{P}(\text{Doctor does not call} \mid \text{Tumor is cancerous})}{\mathbb{P}(\text{Doctor does not call})}$$

And $\mathbb{P}(\text{Doctor does not call} \mid \text{Tumor is cancerous}) = 1$, as stated in the exercise.

So our sequence of equalities can be continued by $\dfrac{\alpha}{\mathbb{P}(\text{Doctor does not call})}$.

Now, by (1), we have $\dfrac{\alpha}{\mathbb{P}(\text{Doctor does not call})} < \alpha$.

Therefore $\dfrac{1}{\mathbb{P}(\text{Doctor does not call})} < 1$, and finally $\mathbb{P}(\text{Doctor does not call}) > 1$.

The last inequality clearly violates the very first axiom of probability spaces, and thus we have a contradiction, and we cannot have $\beta < \alpha$.

Lastly, we know that its possible to have $\alpha = \beta$, since $\mathbb{P}(\text{Doctor does not call})$ could be any number in $[0,1]$ by the very second axiom of probability spaces.

Thence $\beta \geq \alpha$.

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Exercise 6 (3.37)

(a) A gambler has a fair coin and a two-headed coin in his pocket. He selects one of the the coins at random; when he flips it, it shows heads. What is the probability that it is the fair coin?

The outcomes are: choose the fair coin and get heads, choose the fair coin and get tails, or choose the unfair coin and get heads. The first two outcomes are equally likely, and the third outcome is twice as likely as either of the first two. Therefore the probability of getting heads is $\frac{3}{4}$. The numerator is determined by adding a head for the fair coin, and two heads for the other coin. In other words, of the three ways to get heads, and only one corresponds to the fair coin. Therefore the probability that the fair coin was selected is $\frac{1}{3}$.

(b) Suppose that he flips the same coin a second time, and it shows heads again. Now what is the probability that it is the fair coin?

Let E be the event where the coin is fair, and let H_1H_2 be the event where two heads are obtained in two flips of the same coin. We want to determine $\mathbb{P}(E \mid H_1H_2)$. To begin, expand the conditional probability to obtain $\mathbb{P}(E \mid H_1H_2) = \frac{\mathbb{P}(E)\mathbb{P}(H_1H_2 \mid E)}{\mathbb{P}(H_1H_2)}$. We can now use Bayes formula. i.e. $\frac{\mathbb{P}(E)\mathbb{P}(H_1H_2 \mid E)}{\mathbb{P}(H_1H_2)} = \frac{\mathbb{P}(E)\mathbb{P}(H_1H_2 \mid E)}{\mathbb{P}(H_1H_2 \mid E)\mathbb{P}(E) + \mathbb{P}(H_1H_2 \mid E^c)\mathbb{P}(E^c)}$. Clearly $\mathbb{P}(E) = \mathbb{P}(E^c) = \frac{1}{2}$, $\mathbb{P}(H_1H_2 \mid E) = \frac{1}{2^2} = \frac{1}{4}$, and $\mathbb{P}(H_1H_2 \mid E^c) = 1$, so plug these values in to obtain the final result

$$\mathbb{P}(E \mid H_1H_2) = \frac{\mathbb{P}(E)\mathbb{P}(H_1H_2 \mid E)}{\mathbb{P}(H_1H_2 \mid E)\mathbb{P}(E) + \mathbb{P}(H_1H_2 \mid E^c)\mathbb{P}(E^c)} = \frac{\frac{1}{2}\frac{1}{4}}{\frac{1}{4}\frac{1}{2} + (1)\frac{1}{2}} = \frac{\frac{1}{8}}{\frac{5}{8}} = \frac{1}{5}.$$

(c) Suppose that he flips the same coin a third time and it shows tails. Now what is the probability that it is the fair coin?

Let E be the event where the coin is fair, and let HHT be the event where two heads and then a tail are obtained in three flips (same coin).

The probability of getting tails with the double headed coin is clearly zero, and we got tails, so it must be true that we have not flipped the double headed coin. And since we are considering only two coins, and no exchanges are taking place, it must also be true that we flipped the fair coin.

This result is trivial to prove using Bayes formula, and is left as an exercise for the reader.

But in case I would otherwise be marked down for sass, here is a quick result:

$$\mathbb{P}(E \mid H_1H_2T) = \frac{\mathbb{P}(E)\mathbb{P}(H_1H_2T \mid E)}{\mathbb{P}(H_1H_2T \mid E)\mathbb{P}(E) + \mathbb{P}(H_1H_2T \mid E^c)\mathbb{P}(E^c)} = \frac{\frac{1}{2}\frac{1}{8}}{\frac{1}{8}\frac{1}{2} + 0} = 1.$$

Exercise 8 (3.53)

A parallel system is functioning whenever at least one of its components works. Consider a parallel system of n components, and suppose that each component works independently with probability $\frac{1}{2}$. Find the conditional probability that component 1 works given that the system is functioning.

Let C_k be the event where the k[th] component is functioning. The system is functioning if and only if C_1 or C_2 or $\ldots$ or C_n, so we want to calculate $\mathbb{P}\left(C_1 \mid C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n\right)$.

Expanding this conditional probability, we obtain $\dfrac{\mathbb{P}(C_1)\mathbb{P}\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n \mid C_1\right)}{\mathbb{P}\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n\right)}$.
Clearly $\mathbb{P}(C_1) = \frac{1}{2}$, and $\mathbb{P}\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n \mid C_1\right) = 1$. So all we need to do is calculate $\mathbb{P}\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n\right)$. We can do this

$$\begin{aligned}\mathbb{P}\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n\right) &= 1 - \mathbb{P}\left(\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n\right)^c\right)\\ &= 1 - \mathbb{P}\left(C_1^c \text{ and } C_2^c \text{ and } \ldots \text{ and } C_n^c\right) \text{ (By De Morgan's laws)}\\ &= 1 - \frac{1}{2^n}.\end{aligned}$$

So $\dfrac{\mathbb{P}(C_1)\mathbb{P}\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n \mid C_1\right)}{\mathbb{P}\left(C_1 \text{ or } C_2 \text{ or } \ldots \text{ or } C_n\right)} = \dfrac{\frac{1}{2}}{1 - \frac{1}{2^n}} = \dfrac{\frac{1}{2}}{\frac{2^n - 1}{2^n}} = \dfrac{\frac{2^n}{2}}{2^n - 1} = \dfrac{2^{n-1}}{2^n - 1}.$

And now we are free to sleep.

Exercise 9 (3.58)

Suppose that we want to generate the outcome of the flip of a fair coin, but that all we have at our disposal is a biased coin that lands on heads with some unknown probability p that need not be equal to $\frac{1}{2}$. Consider the following procedure for accomplishing our task:

(1) Flip the coin.
(2) Flip the coin again.
(3) If both flips land on heads or both land on tails, return to step 1.
(4) Let the result of the last flip be the result of the experiment.

(a) Show that the result is equally likely to be either heads or tails.

Because it complements my prose, note that the first two steps can be combined into an equivalent step to obtain the following equivalent procedure:

1. Flip the coin twice, and record the results in an ordered pair (F_1, F_2).
2. If $F_1 = F_2$, return to step 1.
3. Let F_2 be the result of the experiment.

We will show that the probability of the procedure terminating with $F_2 = H$ is balanced by the probability of the procedure terminating with $F_2 = T$.

Clearly, since the coin flips are independent, we have

$\mathbb{P}(F_1 = H) = \mathbb{P}(F_2 = H) = p$, and
$\mathbb{P}(F_1 = T) = \mathbb{P}(F_2 = T) = 1 - p$.

Therefore

$\mathbb{P}(HH) = \mathbb{P}(H)\mathbb{P}(H) = p^2$,
$\mathbb{P}(TT) = \mathbb{P}(T)\mathbb{P}(T) = (1-p)^2$, and
$\mathbb{P}(HT) = \mathbb{P}(H)\mathbb{P}(T) = p(1-p) = \mathbb{P}(T)\mathbb{P}(H) = \mathbb{P}(TH)$.

We can assume the procedure terminates. Even if it went on forever, that would indicate nothing about the equality of the probability of getting heads or tails.

Since the procedure terminates, we know that we have either $(F_1, F_2) = (H, T)$ or $(F_1, F_2) = (T, H)$. These two events occur with the same probability, namely $p(1-p)$, and each corresponds to a distinct face value, which is what we wanted to show.

(b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different and then lets the result be the outcome of the final flip?

If the first toss results in heads, then the last flip can only be tails, since we know any flips in between must be heads. Therefore, the result of the procedure is immediately determined by the first toss. Now we have

$\mathbb{P}(F_1 = H) \iff \mathbb{P}(F_N = T)$, so $\mathbb{P}(T \mid \text{ procedure 2 }) = \mathbb{P}(H \mid \text{ procedure 1 }) = p$, and
$\mathbb{P}(F_1 = T) \iff \mathbb{P}(F_N = H)$, so $\mathbb{P}(H \mid \text{ procedure 2 }) = \mathbb{P}(T \mid \text{ procedure 1 }) = 1 - p$.

Exercise 10 (3.59)

Independent flips of a coin that lands on heads with probability p are made.

(a) What is the probability that the first four outcomes are H,H,H,H?

The probability that a single coin lands on heads is p. Repeating this experiment four times, we simply raise p to the fourth power, just as we would for a fair coin with $p = \frac{1}{2}$. i.e. $\mathbb{P}(H, H, H, H) = \mathbb{P}(H)^4 = p^4$. Note the independence of flips.

(b) What is the probability that the first four outcomes are T,H,H,H?

In this count, we use the same approach as in part a, but take a factor of (1-P) for tails. i.e. $\mathbb{P}(T, H, H, H) = \mathbb{P}(T)\mathbb{P}(H)^3 = (1 - p)p^3$. Note the independence of flips.

(c) What is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H?
Hint for part (c): How can the pattern H, H, H, H occur first?

Clearly H, H, H, H occurs before T,H,H,H if the sequence begins with four Heads.
Now suppose the sequence begins with at least one H, and less than four. Then we can throw out all of those Hs, since we are not concerned with sequences of the form HHHT.

What we are left with is a sequence of coin tosses starting with T.

Now notice that in order to have H,H,H,H as a subsequence, it must be preceded by a T. We can see this clearly by supposing that H,H,H,H were preceded by an H, instead. In that case, we would have H,H,H,H,H. But then we are in the same position! All that has changed is that we've shifted our index backwards. We must continue this process until we reach a T, which we know will happen because the sequence starts with T.

Therefore, the probability that T,H,H,H occurs before H,H,H,H is the probability of the complement of H,H,H,H occurring first. In part a, we obtained the probability of getting heads on the first four outcomes; the solution is $1 - p^4$.

Exercise 11 (3.61)

Genes relating to albinism are denoted by A and a. Only those people who receive the a gene from both parents will be albino. Persons having the gene pair (A, a) are normal in appearance and, because they can pass on the trait to their offspring, are called carriers. Suppose that a normal couple has two children, exactly one of whom is an albino. Suppose that the nonalbino child mates with a person who is known to be a carrier for albinism.

(a) What is the probability that their first offspring is an albino?

Let Alex be the non albino child, and let y be their partner. Then x has one of the following gene pairs: (A, A), (A, a), (a, A). Since y is a carrier, we know y has one of the following gene pairs: (A, a) or (a, A). If x has the pair (A, A), then x is not a carrier. Otherwise x is a carrier. So x is a carrier in $\frac{2}{3}$ cases. The probability that each carrier passes on a gene is $\frac{1}{2}$, and since both parents must pass on the gene in order to produce an albino child, the probability of both parents passing on the necessary genes for albinism is $\frac{1}{4}$. Finally, the probability that the first offspring is an albino is the probability that x is a carrier multiplied by the probability that both parents pass on their genes. So the solution is $\frac{1}{6}$.

(b) What is the conditional probability that their second offspring is an albino given that their firstborn is not?

The probability that the second offspring is albino where the first is non albino is the probability that the first is non albino and the second is not, divided by the probability that the first offspring is not albino. The first offspring is non albino with probability $1 - \frac{1}{6} = \frac{5}{6}$. Now, since the second offspring is albino, the parents are carriers.

Now the probability that the first child is non albino but the second is, is equal to the probability that the parents have the gene pairs (A, a) or (a, A).