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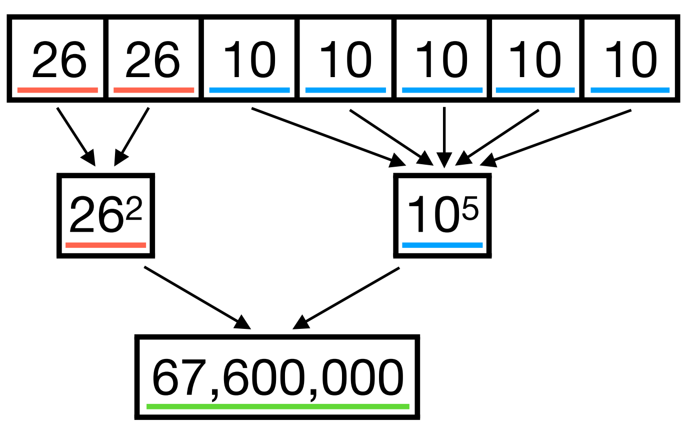
Math 461
Assignment I

Exercise 1

a) **How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?**

We will assume that letters must come from a twenty-six letter alphabet, and numbers must come from the decimal system.

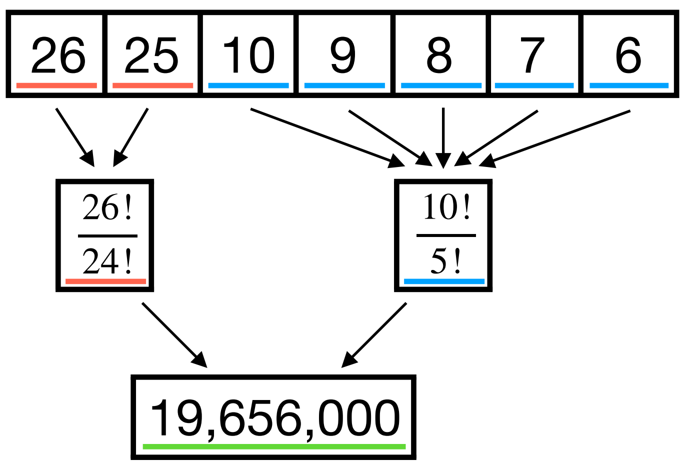
For the first entry in the license plate, we clearly have twenty-six choices. We are not prohibited from duplicate letters, so we also have twenty-six choices for the second letter. By the basic principle of counting, there are 26^2 possible choices for the first two entries. Next we pick the third, fourth, fifth, sixth, and seventh entries, which each contribute one of ten numbers. Just as we calculated the possible choices for the first two entries, we find that the next five entries produce 10^5 possible choices. Finally, since there is no restriction on which numbers follow a given pair of letters, we can follow any one of the 26^2 letter sequences with any one of the 10^5 number sequences. We obtain our final answer as the product of the quantities of letter and number sequences. i.e. It is possible to construct $26^2 \times 10^5 = 67{,}600{,}000$ different license plates under the given constraints.



b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

As in part a, we will assume that letters must come from a twenty-six letter alphabet, and numbers must come from the decimal system.

Initially, we do as we did in part a; choose one of twenty-six letters for the first entry. At this point, our solution differs from that of part a in that we must pick another letter for the second entry, but we cannot repeat the letters which occupy other positions in the license plate. Since we have used one letter, our choice for the second letter comes from a twenty-five letter alphabet. By the basic principle of counting, there are $26 * 25$ letter prefixes for a license plate. We count the numeric part by the same process, but for five characters. From the second step, we find that there are $10 * 9 * 8 * 7 * 6$ numeric suffixes for a license plate. Lastly, we use the basic principle of counting to combine these quantities, and obtain the fact that $19{,}656{,}000$ license plates can be constructed.



Exercise 2

Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

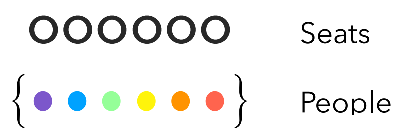
This problem is intuitively equivalent to the problem of counting the number of vertices in a bipartite graph constructed by the disjoint union of two sets of ten vertices in 1-1 correspondence. In this representation is clearly 20!.

On the other hand, I'm not sure that it's ok for me to introduce solutions which fall outside of the domain of our textbook. Hence, my official solution is as follows. First assign a worker to any of twenty jobs. Next, you will need to assign one of the remaining nineteen workers to any of nineteen jobs. This continues in the obvious way until no workers nor solutions are remaining. Since the assignments are chosen, it's clear that the solution is given by collecting the product of the cardinality of either workers or jobs with each iteration; $20 * 19 * \ldots * 1 = 20!$.

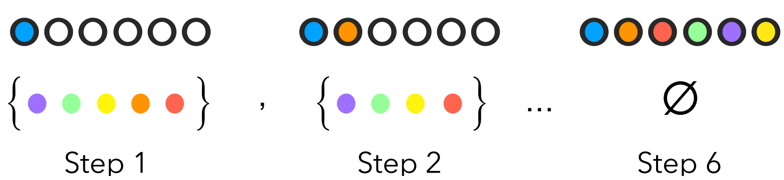
Exercise 3

a) **In how many ways can 3 boys and 3 girls sit in a row?**

Since we are not restricting the arrangement by gender, we simply want to determine the number of possible arrangements for six objects. Suppose we have six empty seats and a set of six distinct people, as in the following figure.



Now we need to fill the seats, so we first choose one of the six people to take the first seat. Our second step will be selecting someone for the second seat, and so on until all six seats are full and no people are remaining. The difference between each of these steps is that when we pick someone to take a seat, we must pick someone who has not already been chosen; nobody is allowed to occupy more than one seat. So for each each step n, we can choose from a set of $7 - n$ people. One possible outcome is illustrated in the following figure.



Since each step n produces $7 - n$ new potential outcomes, and we are counting for steps 1 through 6, we can use the basic principle of counting to conclude that there are $\prod_{k=1}^{6}(7-k) = \prod_{k=1}^{6} k = 6!$ possible arrangements.

b) **In how many ways can 3 boys and 3 girls sit in a row if the boys and girls are each to sit together?**

We have no restriction on the arrangement within the set of girls, nor within the set of boys, nor with the set containing the sets of boys and girls. From these facts, we can break the problem up and obtain our result by combining our subproblem-solutions. In particular, for each of the boys-set and the girls-set, we want to determine the number of distinct ways that set can be arranged. Then we want to determine the number of ways the collection of those sets can be arranged.

Note that since the set of boys is the same size as the set of girls, so these subproblems share a solution. In either case, we are permuting a set of three distinct objects. Following the same method used in part a, we see that each set has $3!$ distinct permutations. Next we have the collection of these two sets. Intuitively, we are just joining ends of the boys-set and girls-set, and there are clearly just two ways to do this.

Finally, we must combine the solutions from our subproblems. By invoking the basic principle of counting once more, we obtain the solution $2(3!)^2 = 72$.

c) In how many ways if only the boys must sit together?

Since the boys are always siting together, we can consider an abstraction of the set of boys. In particular, we can ask how many ways there are to arrange a set of with four elements (three girls and the collection of boys). We can do this for any permutation of boys, so by the basic principle of counting, we need to multiply the number from the previous step number by the permutations of just boys. A set of four elements can be arranged in $4!$ ways, and there are $3!$ ways to arrange just the boys, therefore so the solution is $4!3! = 144$.

d) In how many ways if no two people of the same sex are allowed to sit together?

In this case, we can permute the boys and girls, independently, however we like. So we can introduce a factor of $3!$ for each of these arrangements. Since the boys and girls cannot be mixed, we can arrange their coupling in two ways. From these facts and the basic principle of counting, it follows that the solution is $2(3!)^2 = 72$.

Exercise 4

How many different letter arrangements can be made from the letters:

a) FLUKE

Notice that all of the letters in "FLUKE" are distinct. Therefore, since "FLUKE" has five letters, there are $5! = 120$ permutations of its letters.

b) PROPOSE

In this case, our word has two ‘P’s and two ‘O’s. In each word we construct, the existence of n-times-repeated letters means that those n letters can be permuted n! times and produce equivalent words. Consequently will solve the problem as we did in part a, and then divide by n! for each n-times-repeated letter. Since “PROPOSE” has seven letters, we divide the number of (all) permutations of “PROPOSE” by the product of the repeated character permutations. i.e. $\frac{7!}{2!2!} = 1{,}260$.

c) MISSISSIPPI

We can solve this problem as we solved part b.
“MISSISSIPPI” has eleven letters, four ’S’s, four ’I’s, two ‘P’s, and one remaining character. As with (b), the solution is the number of (all) permutations of “MISSISSIPPI”, divided by the product of the permutations of the repeating letters. i.e. $\frac{11!}{4!4!2!} = 34{,}650$.

d) ARRANGE

We can solve this problem as we solved parts b and c.
“ARRANGE” has seven letters, two ’R’s, two ‘A’s, and the remaining characters are unique. So, once again, take (all of) the permutations of “ARRANGE”, and divide by the product of the permutations of the repeating letters. i.e. $\frac{7!}{2!2!} = 1{,}260$.

Exercise 5

Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes are possible?

Intuitively, this problem is equivalent to counting the number of vertices of a complete 20-vertex graph. Thus the solution is given by $\frac{20(20-1)}{2} = 190$, since a complete n-graph has $\frac{n(n-1)}{2}$ vertices.

Alternatively, we can solve the problem in a more purely combinatorial way. Begin by assuming that a handshake takes place between two people. Therefore, the question can be asked in the more intuitive form: How many ways can two people, from a group of twenty, shake hands? Even more generally, we are asking how many ways we can choose two objects from a set of twenty. Asked in this way, the answer is clearly 20 choose 2, which is 190.

Exercise 6

A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents. How many committees are possible?

Assuming that Republicans, Democrats, and Independents are all mutually disjoint, we are just choosing two elements from a set of five, another two from a set of six, and three from a set of four. By the basic counting principle, the number of committees which can be constructed is obtained by the following expression

$$\binom{5}{2}\binom{6}{2}\binom{4}{3} = \frac{5!}{2!3!}\frac{6!}{2!4!}\frac{4!}{1!3!} = \frac{120}{12}\frac{720}{48}\frac{24}{6} = 40\frac{720}{48} = \frac{1800}{3} = 600.$$

Exercise 7

A person has 8 friends, of whom 5 will be invited to a party.

a) How many choices are there if 2 of the friends are feuding and will not attend together?

Without this extra criterion, there are clearly $\binom{8}{5}$ possible choices of friends to invite. In order to determine the number of ways to construct a civil party, we can calculate the number of combinations which include the feuding friends, and subtract it from the original number. Assuming we force the two feuding friends to attend, there are 6 people left to choose from, and 3 people left to choose. Therefore, the solution is

$$\binom{8}{5} - \binom{6}{3} = 36.$$

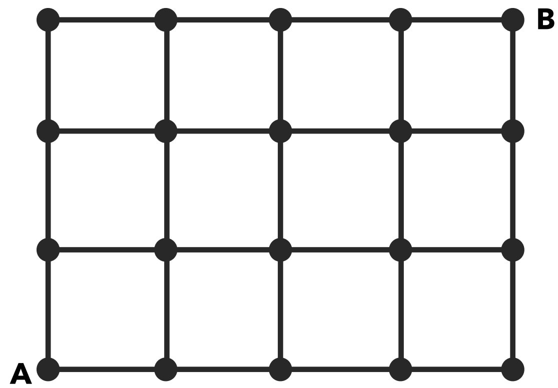
b) How many choices if 2 of the friends will only attend together?

Consider two cases. In the first case, we choose to invite the pair, and in the second case we do not. Then in the first case, we have 6 people remaining, and we need to choose 3 more. And in the second case, we have 6 people remaining, and we still need to choose 5. Intuitively, we can take the sum of these quantities to obtain our solution

$$\binom{6}{3} + \binom{6}{5} = 26.$$

Exercise 8

Consider the grid of points shown below:

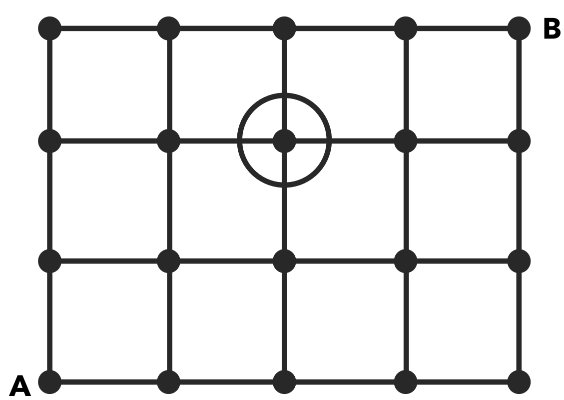


Suppose that, starting at the point labeled A, you can go one step up or one step to the right at each move. This procedure is continued until the point labeled B is reached. How many different paths from A to B are possible.

Observe that when you traverse from A to B by moving only up and right, you will always end up moving up three edges and right four edges. All you get to choose is when you make each move (unless you have no moves left). Therefore, we want to determine the number of ways we can choose to move up, and assume we'll go right everywhere else. Or we could choose the number of ways to move right, and assume we'll go up everywhere else. In either case, there are seven moves total, and thus there are $\binom{7}{3} = \binom{7}{4} = 35$ ways to choose a path through the lattice.

Exercise 9

In Problem 21 (Exercise 8), how many different paths are there from A to B that go through the point circled in the following lattice?



In this case, we can break the problem up by labeling the circled path with C, and asking how many paths can we take from A to C, and from C to B. i.e. there are 4 choose 2 paths from A to C, and 3 paths from C to B. We can combine these paths however we like (in order, of course), so take the product to obtain the solution

$$3\binom{4}{2} = \frac{72}{4} = 18.$$

Exercise 10

a) If 8 new teachers are to be divided among 4 schools how many divisions are possible?

Assume that the teachers are choosing the school where they want to teach. Then each of the eight teachers has four choices. There is no restriction on the number of teachers which can or must work at any school, so we simply need to multiply these individual choices. Specifically, 4 choose 1 eight times, so our solution is $4^8 = 2^{16} = 65{,}536.$

b) What if each school must receive 2 teachers?

There are $\binom{8}{2}$ ways for the first school to pick two teachers. Then the second school must pick two from the remaining six, then the third school picks two from four, and the last school gets the remaining two. We can multiply these quantities together to obtain our solution

$$\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 2520.$$

Exercise 11

a) If 8 identical blackboards are to be divided among 4 schools, how many divisions are possible?

Denote the four schools by $s_1, s_2, s_3,$ and s_4. Each school can have between 0 and 8 blackboards, so we want to count the number of solutions to the following equation

$$s_1 + s_2 + s_3 + s_4 = 8.$$

By the formula in proposition 6.2, the solution is

$$\binom{8+4-1}{4-1} = \binom{11}{3} = 165.$$

b) How many if each school must receive at least 1 blackboard?

In this case, we are solving the same problem as in part a, but each school can have between 1 and 8 blackboards. By the formula in proposition 6.1, the solution is

$$\binom{8-1}{4-1} = \binom{7}{3} = 35.$$