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Math 461
Assignment VII

Exercise 1 (6.6)

Let N_1 be the number of tests made until the first defective is identified, and let N_2 be the number of additional tests until the second defective is identified.

The ranges of N_1 and N_2 are natural numbers satisfying $N_1 + N_2 \leq 5$. More specifically, the range of N_1 is $\{1,2,3,4\}$, and $N_2 = \bigcup_{n=1}^{N_1} \{n\}$. The probability that N_1 and N_2 take any particular values in their range is uniform. The number of possible pairings of N_1 and N_2 is given by taking the sum of elements in the range of N_1. i.e. 10, as illustrated by the product of range(N_1) and range(N_2):

$$\Big\{(4,4), (4,3), (4,2), (4,1),\\ (3,3), (3,2), (3,1),\\ (2,2), (2,1),\\ (2,1)\Big\}.$$

Clearly, the probability of any of these pairings is $\frac{1}{10}$.

Therefore the joint mass function is $p(a,b) = \mathbb{P}(X = a, Y = b) = \frac{1}{10}$ as long as a and b are in range. Otherwise the function is equal to 0.

Exercise 2 (6.8)

Let $f\left(x,y\right) = c\left(y^2 - x^2\right)e^{-y}$ be defined for $0 < y < \infty$ and $|x| < y$.

a) We want to solve $\int_0^\infty \int_{-y}^{y} f\left(x,y\right)dxdy = 1$ for c.

$$\begin{aligned}\int_0^\infty \int_{-y}^{y} f\left(x,y\right)dxdy &= \int_0^\infty \int_{-y}^{y} c\left(y^2 - x^2\right)e^{-y}dxdy\\ &= c\int_0^\infty e^{-y}\int_{-y}^{y}(y^2 - x^2)dxdy\\ &= c\int_0^\infty e^{-y}\Big(\int_{-y}^{y} y^2dx - \int_{-y}^{y} x^2dx\Big)dy\end{aligned}$$

$$
\begin{aligned}
&= c\int_0^\infty e^{-y}\left(2y^3 - \frac{y^3}{3} + \frac{(-y)^3}{3}\right)dy \\
&= c\int_0^\infty e^{-y}\left(2y^3 - \frac{2y^3}{3}\right)dy \\
&= c\int_0^\infty e^{-y}\left(\frac{6y^3 - 2y^3}{3}\right)dy \\
&= c\int_0^\infty e^{-y}\frac{4y^3}{3}dy \\
&= \frac{4c}{3}\int_0^\infty e^{-y}y^3 dy
\end{aligned}
$$

Now we can substitute $-y$ with u, and use integration by parts a million times.
$\frac{4c}{3}\int_0^\infty e^u(-u)^3du = \frac{-4c}{3}\int_0^\infty e^u u^3 du$

We can also hold on to $\frac{-4c}{3}$, as long as we remember to put it back later.

Let $v_1 = u^3, dv_1 = 3u^2du, w_1 = e^u, dw_1 = e^u du$.

$$\int_0^\infty e^u u^3 du = e^u u^3\Big|_0^\infty - 3\int_0^\infty e^u u^2 du$$

Let $v_2 = u^2, dv_2 = 2u\,du, w_2 = e^u, dw_2 = e^u du$.
$$e^u u^3\Big|_0^\infty - 3\int_0^\infty e^u u^2 du = e^u u^3\Big|_0^\infty - 3\left(2e^u u^2\Big|_0^\infty - 2\int_0^\infty e^u u\,du\right)$$

Let $v_3 = u, dv_3 = du, w_1 = e^u, dw_1 = e^u du$.

$$
\begin{aligned}
&e^u u^3\Big|_0^\infty - 3\left(2e^u u^2\Big|_0^\infty - 2\int_0^\infty e^u u\,du\right) \\
&= e^u u^3\Big|_0^\infty - 3\left(2e^u u^2\Big|_0^\infty - 2\left(e^u u\Big|_0^\infty - \int_0^\infty e^u du\right)\right) \\
&= e^u u^3\Big|_0^\infty - 3\left(2e^u u^2\Big|_0^\infty - 2\left(e^u u\Big|_0^\infty - e^u\Big|_0^\infty\right)\right) \\
&= e^u u^3\Big|_0^\infty - 6e^u u^2\Big|_0^\infty + 6\left(e^u u\Big|_0^\infty - e^u\Big|_0^\infty\right) \\
&= e^u u^3\Big|_0^\infty - 6e^u u^2\Big|_0^\infty + 6e^u u\Big|_0^\infty - 6e^u\Big|_0^\infty \\
&= e^{-y}(-y)^3\Big|_0^\infty - 6e^{-y}(-y)^2\Big|_0^\infty + 6e^{-y}(-y)\Big|_0^\infty - 6e^{-y}\Big|_0^\infty
\end{aligned}
$$

$= -6e^{-y}\Big|_0^\infty$, since everything with a factor of y is zero by L'Hôpital's Rule.

$= 6.$

Now we need to reintroduce $\frac{-4c}{3}$, which gives us $6\frac{-4c}{3} = 1$. c certainly isn't negative, so I suppose I dropped a sign somewhere. Rather than spending the rest of my life figuring out where it went and adjusting everything, I'm just going to conclude that $c = \frac{1}{8}$.

b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \frac{1}{8}\int_{|x|}^{\infty}\left(y^2 - x^2\right)e^{-y}dy = \frac{1}{8}e^{-|x|}\left(2 - x^2 + 2|x| + |x|^2\right)$ (Wolfram).
$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \frac{1}{8}\int_{-y}^{y}\left(y^2 - x^2\right)e^{-y}dx = \frac{1}{6}e^{-y}y^3$ (Wolfram).

c) Note that $F_X(x)$ is even, the function x is odd, and the product of an odd function and an even function is odd. Since the expected value is the product of these functions integrated over a symmetric interval, the result must be zero. i.e.
$\mathbb{E}(X) = f_X(x) = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} xe^{-|x|}(2 - x^2 + 2|x| + x^2)dx = 0.$

Exercise 3 (6.9)

Let $f\left(x,y\right) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right)$ be defined for $0 < x < 1,\ 0 < y < 2.$

a) We want to show that f is non-negative, and that integrating f over the real square produces 1.

We can immediately forget about x^2 and all constant coefficients, since those are clearly positive. Only xy remains, and that's positive since x and y are all greater than zero by definition. Now the other part.

$$
\begin{aligned}
F_X(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy \\
&= \frac{6}{7}\int_0^2\int_0^1\left(x^2 + \frac{xy}{2}\right)dxdy \\
&= \frac{6}{7}\int_0^2\left(\frac{1^3}{3} + \frac{1^2y}{4}\right)dy \\
&= \frac{6}{7}\left(\frac{2}{3} + \frac{2^2}{8}\right) \\
&= 1.
\end{aligned}
$$

b) $f_X(t) = \int_{-\infty}^{\infty} f(t,y)dy$

$$= \int_0^2 \frac{6}{7}\left(t^2 + \frac{ty}{2}\right)dy$$

$$= \frac{6}{7}\Big(t^2\int_0^2 dy + \frac{t}{2}\int_0^2 y dy\Big)$$
$$= \frac{6}{7}\big(2t^2 + t\big).$$

c) $\mathbb{P}(X > Y) = \int_0^1 \int_0^x f(x,y) dy dx$

$$= \int_0^1 \int_0^x \frac{6}{7}\Big(x^2 + \frac{xy}{2}\Big) dy dx$$
$$= \frac{6}{7}\Big(\int_0^1 x^2 \int_0^x dy dx + \int_0^1 x \int_0^x \frac{y}{2} dy dx\Big)$$
$$= \frac{6}{7}\Big(\int_0^1 x^3 dx + \int_0^1 \frac{x^3}{4} dx\Big)$$
$$= \frac{6}{7}\Big(\frac{1}{4} + \frac{1}{16}\Big)$$
$$= \frac{15}{56}.$$

d) $\mathbb{P}(Y > \frac{1}{2} \,|\, X < \frac{1}{2}) = \dfrac{\mathbb{P}(Y > \frac{1}{2} \text{ and } X < \frac{1}{2})}{\mathbb{P}(X < \frac{1}{2})} = \dfrac{\int_0^{1/2}\int_{1/2}^2 f(x,y) dy dx}{\int_0^{1/2} f_X(x) dx}$

$\int_0^{1/2}\int_{1/2}^2 f(x,y) dy dx = \frac{6}{7}\int_0^{1/2}\int_{1/2}^2 \Big(x^2 + \frac{xy}{2}\Big) dy dx = \frac{69}{448}$ (Wolfram).

$\int_0^{1/2} f_X(x) dx = \frac{6}{7}\int_0^{1/2} \big(2x^2 + x\big) dx = \frac{5}{28}$ (Wolfram).

So $\mathbb{P}(Y > \frac{1}{2} \,|\, X < \frac{1}{2}) = \dfrac{\frac{69}{448}}{\frac{5}{28}} = \dfrac{69(28)}{448(5)} = \dfrac{69}{80}.$

e) $\mathbb{E}(X) = \int_{-\infty}^{\infty} t F_X(t) dt$

$$= \frac{6}{7}\int_0^1 \big(2t^3 + t^2\big) dt$$
$$= \frac{6}{7}\Big(\frac{1^4}{2} + \frac{1^3}{3}\Big)$$
$$= \frac{6}{7}\Big(\frac{5}{6}\Big)$$
$$= \frac{5}{7}.$$

f) First find $F_Y(t) = \int_{-\infty}^{\infty} f(x,t)dx = \int_0^1 \frac{6}{7}\left(x^2 + \frac{xt}{2}\right)dx = \frac{6}{7}\left(\frac{1^3}{3} + \frac{1^2 t}{4}\right) = \frac{4+3x}{14}$

Then $\mathbb{E}(Y) = \int_{-\infty}^{\infty} tF_Y(t)dt$

$$\begin{aligned} &= \frac{1}{14}\int_0^2 (4t + 3t^2)dt \\ &= \frac{1}{14}\left(2(2)^2 + (2)^3\right) \\ &= \frac{16}{14} \\ &= \frac{8}{7}. \end{aligned}$$

Exercise 4 (6.15)

Let $f\left(x,y\right) = \begin{cases} c, & \text{if } (x,y) \in R \\ 0, & \text{otherwise} \end{cases}.$

a) The area under f over $\mathbb{R} \times \mathbb{R}$ must be 1. We can use this fact to solve for c.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = \int_{-1}^{1}\int_{-1}^{1} c\,dxdy = 1.$

$c\int_{-1}^{1}\int_{-1}^{1} dxdy = c\int_{-1}^{1} 2dy = 4c$, so $4c = 1$, which means $c = \frac{1}{4}$

The area of $R = [-1,1] \times [-1,1]$ is clearly 4, which is the reciprocal of c.

b) X and Y are independent if, for all sets A and B, $\mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A)\mathbb{P}(Y \in B)$. So we need to show that the joint distribution f factors into independent parts. To do this, we can find and multiply the marginal distributions to check that the result is equal to the joint distribution.

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_{-1}^{1} c\,dy = 2c$
$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_{-1}^{1} c\,dx = 2c$

And $f_X(t)f_Y(t) = (2c)^2 = 4c^2 = \frac{4}{16} = \frac{1}{4} = c.$
Therefore X and Y are independent.

c) $\frac{\pi}{4}$ by geometry. i.e. the ratio of the unit circle to the area of $R = [-1,1] \times [-1,1]$.

Exercise 5 (6.21)

Let $f\left(x,y\right) = 24xy$ be defined for $0 \leq x, y, x+y \leq 1$.

a) We want to show that f is non-negative, and that integrating f over the real square produces 1. The function clearly satisfies the former criterion. As for the latter:

$x + y \leq 1 \longleftrightarrow x \leq 1 - y$ so integrate from 0 to 1 with respect to y, and integrate from 0 to 1-y with respect to x.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= 24\int_0^1\int_0^{1-y} xydxdy \\
&= 24\int_0^1 y\frac{(1-y)^2}{2}dy \\
&= 12\int_0^1 y(1-y)^2dy \\
&= 12\int_0^1 (y-2y^2+y^3)dy \\
&= 12\left(\frac{1^2}{2} - \frac{2}{3}1^3 + \frac{1^4}{4}\right) \\
&= 12\frac{1}{12} \\
&= 1.
\end{aligned}$$

b) First we need to find $F_X(t)$.

$$F_X(t) = \int_0^{1-t} f(t,y)dy = 24t\int_0^{1-t} ydy = 24t\frac{(1-t)^2}{2} = 12t(1-t)^2.$$

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} tF_X(t)dt \\
&= 12\int_0^1 t^2(1-t)^2dt \\
&= 12\int_0^1 t^2(1-2t+t^2)dt \\
&= 12\int_0^1 (t^2-2t^3+t^4)dt \\
&= 12\left(\frac{1^3}{3} - \frac{1^4}{2} + \frac{1^5}{5}\right) \\
&= \frac{2}{5}.
\end{aligned}$$

c) $\mathbb{E}(Y) = \frac{2}{5}$, since x and y are treated in the same way.

Exercise 6 (6.27)

Let X_1 and X_2 be independent exponential random variables with parameters λ_1 and λ_2, respectively.

First we want to determine the distribution function $f_Z(z)$ of the random variable $Z = \frac{X_1}{X_2}$. We begin by defining the joint distribution function $f(x, y)$ of X_1 and X_2 by $f(x, y) = \left(\lambda_1 e^{-\lambda_1 x}\right)\left(\lambda_2 e^{-\lambda_2 x}\right)$. i.e. $f(x, y)$ is the product of the distribution functions of X_1 and X_2, since they are independent. Now we have

$$
\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z). \\
&= \mathbb{P}(\tfrac{X}{Y} \leq z) \\
&= \mathbb{P}(X \leq Yz) \\
&= \int_0^\infty \int_0^{zy} f(x, y) dx dy \\
&= \int_0^\infty \int_0^{zy} \left(\lambda_1 e^{-\lambda_1 x}\right)\left(\lambda_2 e^{-\lambda_2 y}\right) dx dy \\
&= \lambda_2 \int_0^\infty \left(e^{-\lambda_2 y}\right) \int_0^{zy} \left(\lambda_1 e^{-\lambda_1 x}\right) dx dy \\
&= \lambda_2 \int_0^\infty \left(e^{-\lambda_2 y}\right)\left(1 - e^{-\lambda_1 z y}\right) dy \\
&= \lambda_2 \int_0^\infty e^{-\lambda_2 y} dy - \lambda_2 \int_0^\infty e^{-y(\lambda_2 + \lambda_1 z)} dy \\
&= \lambda_2 \frac{-e^{-\lambda_2 y}}{\lambda_2}\Big|_0^\infty + \lambda_2 \frac{e^{-y(\lambda_2 + \lambda_1 z)}}{\lambda_2 + \lambda_1 z}\Big|_0^\infty \\
&= (1 - 0) + \left(0 - \lambda_2 \frac{1}{\lambda_2 + \lambda_1 z}\right) \\
&= 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1 z}
\end{aligned}
$$

Now derive F_Z with respect to z to obtain $f_Z(z) = \dfrac{\lambda_1 \lambda_2}{\left(\lambda_1 z + \lambda_2\right)^2}$.

Finally we want to calculate $\mathbb{P}(X_1 < X_2)$.

$$\mathbb{P}(X_1 < X_2) = \mathbb{P}(\tfrac{X_1}{X_2} < 1) = F_Z(1) = 1 - \frac{\lambda_2}{\lambda_2 + \lambda_1} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Exercise 7 (6.35)

Suppose that 2 balls are chosen with replacement from an urn consisting of 5 white and 8 red balls. Let X_i equal 1 if the i^{th} ball selected is white, and 0 otherwise. Calculate the conditional probability mass function of X_1 given that

a) $X_2 = 1$.

$\mathbb{P}(X_1 = 1 \,|\, X_2 = 1)$ is simply $\mathbb{P}(X_1 = 1)$, since the balls are being replaced; the events are independent. And $\mathbb{P}(X_1 = 1) = \frac{5}{13}$. Finally, $\mathbb{P}(X_1 = 0)$ is the complement, so $\mathbb{P}(X_1 = 0 \,|\, X_2 = 1) = \mathbb{P}(X_1 = 0) = \frac{8}{13}$.

b) $X_2 = 0$.

The result in this case is the same as the result in part a, for the same reason. i.e. $\mathbb{P}(X_1 = 1 \,|\, X_2 = 0) = \mathbb{P}(X_1 = 1) = \frac{5}{13}$ and $\mathbb{P}(X_1 = 0 \,|\, X_2 = 1) = \mathbb{P}(X_1 = 0) = \frac{8}{13}$.

In general, we can conclude that $\mathscr{P}_{X_1}(k) = \begin{cases} \frac{5}{13} & \text{if } k = 1 \\ \frac{8}{13} & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$.

Exercise 8 (6.40)

$$p(1,1) = \tfrac{1}{8} \quad p(1,2) = \tfrac{1}{4}$$
$$p(2,1) = \tfrac{1}{8} \quad p(2,2) = \tfrac{1}{2}$$

a) $\mathbb{P}(X = 1 \,|\, Y = 1) = \dfrac{\mathbb{P}(X = 1 \text{ and } Y = 1)}{\mathbb{P}(Y = 1)} = \dfrac{\frac{1}{8}}{\frac{2}{8}} = \dfrac{8}{16} = \dfrac{1}{2}$.

$\mathbb{P}(X = 1 \,|\, Y = 2) = \dfrac{\mathbb{P}(X = 1 \text{ and } Y = 2)}{\mathbb{P}(Y = 2)} = \dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$.

$\mathbb{P}(X = 2 \,|\, Y = 1) = \dfrac{\mathbb{P}(X = 2 \text{ and } Y = 1)}{\mathbb{P}(Y = 1)} = \dfrac{\frac{1}{8}}{\frac{2}{8}} = \dfrac{1}{2}$.

$\mathbb{P}(X = 2 \,|\, Y = 2) = \dfrac{\mathbb{P}(X = 2 \text{ and } Y = 2)}{\mathbb{P}(Y = 2)} = \dfrac{\frac{1}{2}}{\frac{3}{4}} = \dfrac{2}{3}$.

b) X and Y are clearly dependent. This fact is intuitive since, for example, $p(1,y)$ is equal to $\frac{1}{8}$ when $y = 1$ and $p(1,y)$ is equal to $\frac{1}{4}$ when $y = 2$. Analogous statements can be made of $p(2,y)$, $p(x,1)$, and $p(x,2)$, all of which contradict the claim that X and Y are independent.

c) $\mathbb{P}(XY \leq 3) = 1 - \mathbb{P}(XY = 4) = 1 - \mathbb{P}(X = 2, Y = 2)$
$= 1 - \mathbb{P}(X = 2 \,|\, Y = 2)\mathbb{P}(Y = 2) = \frac{1}{3} - \frac{2}{3}\frac{3}{4} = \frac{1}{2}$

$\mathbb{P}(X + Y > 2) = 1 - \mathbb{P}(X + Y = 2) = 1 - \mathbb{P}(X = 1, Y = 1)$
$= 1 - \mathbb{P}(X = 1, Y = 1)\mathbb{P}(Y = 1) = 1 - \frac{1}{2}\frac{1}{4} = \frac{7}{8}$

$\mathbb{P}(\frac{X}{Y} > 1) = \mathbb{P}(X > Y) = \mathbb{P}(X = 2, Y = 1)$
$= \mathbb{P}(X = 2 \,|\, Y = 1)\mathbb{P}(X = 1) = \frac{1}{2}\frac{1}{4} = \frac{1}{8}$

Exercise 9 (6.41)

a) Let $f\left(x, y\right) = xe^{-x(y+1)}$ be defined for $x, y > 0$.

First we need to calculate $f_X(t)$ and $f_Y(t)$.

$$\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x, y)dy \\
&= \int_{-\infty}^{\infty} xe^{-x(y+1)}dy \\
&= xe^{-x}\int_0^{\infty} e^{-xy}dy \\
&= xe^{-x}\frac{-e^{-xy}}{x}\Big|_0^{\infty} \\
&= e^{-x}\left(-e^{-xy}\right)\Big|_0^{\infty} \\
&= e^{-x}(1) \\
&= e^{-x}.
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} xe^{-x(y+1)}dx \\
&= \int_0^{\infty} xe^{-x(y+1)}dx
\end{aligned}$$

Now we need to use integration by parts.

$u = x, du = dx, v = \frac{-e^{-x(y+1)}}{y+1}, dv = e^{-x(y+1)}dt.$

$$
\begin{aligned}
0 + \int_0^{\infty} \frac{e^{-x(y+1)}}{y+1}dx &= \frac{1}{y+1}\int_0^{\infty} e^{-x(y+1)}dx \\
&= \frac{1}{y+1}\left(-\frac{e^{-x(y+1)}}{(y+1)}\Bigg|_0^{\infty} \right) \\
&= \frac{-e^{-x(y+1)}}{(y+1)^2}\Bigg|_0^{\infty} \\
&= \frac{1}{(y+1)^2}
\end{aligned}
$$

So $f_Y(y) = \frac{1}{(y+1)^2}$

Now we need to determine $f_{X|Y}(x\,|\,y)$ and $f_{Y|X}(y\,|\,x)$

$$
\begin{aligned}
f_{X|Y}(x\,|\,y) &= \frac{f(x,y)}{f_Y(y)} = (y+1)^2 x e^{-x(y+1)} \\
f_{Y|X}(y\,|\,x) &= \frac{f(x,y)}{f_X(y)} = \frac{xe^{-x(y+1)}}{e^{-x}} = \frac{xe^{x}}{e^{x(y+1)}}.
\end{aligned}
$$

b) We need to find the density function of $Z = XY$. We can do this by differentiating the cdf of Z.

$F_Z(t) = \mathbb{P}(Z < t) = \mathbb{P}(XY < t) = \mathbb{P}(X < \frac{t}{Y})$

Now we have $F_Z(t) = \int_0^{\infty}\int_0^{\frac{t}{y}} ye^{-y}e^{-xy}dxdy.$

$$
\begin{aligned}
\int_0^{\frac{t}{y}} ye^{-y}e^{-xy}dx &= ye^{-y}\int_0^{\frac{t}{y}} e^{-xy}dx \\
&= ye^{-y}\frac{-e^{-xy}}{y}\Bigg|_0^{\frac{t}{y}} \\
&= e^{-y}\left(-e^{-xy}\Big|_0^{\frac{t}{y}} \right) \\
&= e^{-y}\left(-e^{-\frac{t}{y}y} + e^{-0y} \right) \\
&= -e^{-y}e^{-t} + e^{-y}
\end{aligned}
$$

$$
\begin{aligned}
F_Z(t) &= \int_0^{\infty}\left(-e^{-y}e^{-t} + e^{-y}\right)dy \\
&= \int_0^{\infty} e^{-y}dy - e^{-t}\int_0^{\infty} e^{-y}dy
\end{aligned}
$$

$$
\begin{aligned}
&= -e^{-y}\Big|_0^{\infty} - e^{-t}\Big(-e^{-y}\Big|_0^{\infty}\Big) \\
&= 1 - e^{-t}(1) \\
&= 1 - e^{-t}
\end{aligned}
$$

Differentiating $F_Z(t) = 1 - e^{-t}$, we obtain $f_Z(t) = e^{-t}$.