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Nov 11, 2019
Math 461
Assignment V

Exercise 1 (4.61)

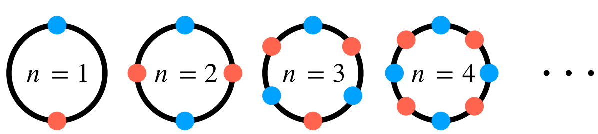
The probability of being dealt a full house in a standard deck is 0.0014. Assuming that each of the 1000 hands comes from a new deck, the event where we get at least 2 full houses is the complement of the event where we get no full houses or one full house. We can use the Poisson distribution as follows:

Let X be the event where k full houses are obtained from 1000 hands,
Let $\lambda = 1000(0.0014) = 1.4$.
Then $\mathscr{P}_X(k) = \mathbb{P}(X = k) = \frac{e^{\lambda}\lambda^k}{k!} = \frac{e^{1.4}1.4^k}{k!}$, $k \in \mathbb{W}$.
So the solution is

$$\begin{aligned} 1 - \left(\mathscr{P}_X(0) + \mathscr{P}_X(1)\right) &= 1 - \frac{e^{-1.4}1.4^0}{0!} - \frac{e^{-1.4}1.4^1}{1!} \\ &= 1 - e^{-1.4} - 1.4e^{-1.4} \\ &= 1 - 2.4e^{-1.4} \\ &\approx 0.4. \end{aligned}$$

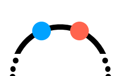
Exercise 2 (4.67)

(a) Observe the following figure:



It's easy to see that if $n = 1$ or $n = 2$, cells will always be sitting next to each other. In general, except for when $n = 1$, notice that each cell of a given color sits next to exactly two distinct cells of the opposite color. This, coupled with the fact that colors alternate, means that the probability of couple i being seated together is $\frac{2}{n}$ when $n > 1$. i.e. $\mathbb{P}(C_i) = \frac{2}{n}$.

(b) Given C_i, we know we have something that looks like the following figure:



Now we have to place $n - 1$ couples on the remaining segment, which is just a line.

So a person p of a particular gender can be seated in one of $(n-1)$ ways. p's spouse, q, can sit next to p in two spots if p is not seated at the end of the line, or one spot if p is seated at the end of the line (i.e. next to someone in the couple which is already seated). So q can be seated in one of $(n-1)$ seats. In $\frac{1}{n-1}$ cases, q's seat will be next to p, and in the remaining $\frac{n-2}{n-1}$ cases, q's seat will be next to p (and there are two seats next to q). Putting these pieces together, we obtain the solution: $\frac{1}{(n-1)^2}+2\frac{n-2}{(n-1)^2}=\frac{2n-3}{(n-1)^2}$.

(c) This is a Poisson random variable with parameter $\lambda = np = n\left(\frac{2}{n}\right) = 2$. So generally we have $\frac{e^{-2}2^k}{k!}$. And when k = 0, that's just e^{-2}.

Problem 3 (4.85)

Let X_i be a random variable, defined for all i in $\{1,2...k\}$, s.t. X_i is 1 if coupon i is in the set, and 0 otherwise. $\mathbb{P}(X_i=1)=1-\mathbb{P}(X_i=0)=1-(1-p_i)^n$, since $X_i=0$ means that the $S=\sum_{i=1}^n X_i$ set contains coupons only of types other than i. Now let $S=\sum_{i=1}^n X_i$ be a random variable representing the number of distinct coupon types in the set. The expected value of S is $S=\sum_{i=1}^n \mathbb{E}(X_i)=\sum_{i=1}^n \left(1-(1-p_i)^n\right)$.

Problem 4 (5.2)

$$f(x) = \begin{cases} Cxe^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

First we determine C by solving the equation $\int_{\mathbb{R}} f(x)dx = 1$ for C. i.e.

$$\begin{aligned} \int_{\mathbb{R}} f(x)dx &= \int_0^\infty Cxe^{-\frac{x}{2}}dx \\ &= C\int_0^\infty xe^{-\frac{x}{2}}dx \end{aligned}$$

The next step is to use integration by parts, with

$$\begin{aligned} u &= x & v &= -2e^{-\frac{x}{2}} \\ du &= dx & dv &= e^{-\frac{x}{2}}dx \end{aligned}$$

Which gives us $C\left(-2xe^{-\frac{x}{2}}\Big|_0^\infty + 2\int_0^\infty e^{-\frac{x}{2}}dx\right) = C\left(-2xe^{-\frac{x}{2}}\Big|_0^\infty - 4e^{-\frac{x}{2}}\Big|_0^\infty\right)$

$$= C\left(-2xe^{-\frac{x}{2}}\Big|_0^\infty - 4e^{-\frac{x}{2}}\Big|_0^\infty\right)$$
$$= -2C\left(xe^{-\frac{x}{2}}\Big|_0^\infty + 2e^{-\frac{x}{2}}\Big|_0^\infty\right)$$
$$= -2C\big((0-0)+(0-2)\big)$$

So $4C = 1 \longleftrightarrow C = \frac{1}{4}$.

Now we solve can determine the probability by integrating the original function with $C = \frac{1}{4}$ over the real numbers greater than or equal to 5. i.e. $\int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}}dx$. We can finish by using the antiderivative which we obtained while finding C.

$$\int_5^\infty \frac{1}{4}xe^{-\frac{x}{2}}dx = -\frac{2}{4}\left(xe^{-\frac{x}{2}}\Big|_5^\infty + 2e^{-\frac{x}{2}}\Big|_5^\infty\right)$$
$$= -\frac{1}{2}\left((0-5e^{-\frac{5}{2}}) + (0-2e^{-\frac{5}{2}})\right)$$
$$= \frac{1}{2}\left(7e^{-\frac{5}{2}}\right)$$
$$= \frac{7}{2}e^{-\frac{5}{2}}$$
$$\approx 0.29.$$

Exercise 5 (5.4)

$$f(x) = \begin{cases} \frac{10}{x^2} & \text{if } x > 10 \\ 0 & \text{otherwise} \end{cases}$$

(a) Integrate f over $(20,\infty)$. i.e.

$$\int_{20}^\infty f(x)dx = \int_{20}^\infty \frac{10}{x^2}dx$$
$$= 10\int_{20}^\infty \frac{dx}{x^2}$$
$$= 10\left(\frac{-1}{x}\Big|_{20}^\infty\right)$$
$$= 10\left(-0 - \frac{-1}{20}\right)$$
$$= \frac{1}{2}.$$

(b) $\int_{10}^{t} f(x)dx = 10\int_{10}^{t} \frac{dx}{x^2} = \frac{-10}{x}\Big|_{10}^{t} = \frac{-10}{t} - \frac{-10}{10} = \frac{t-10}{t}$

So the cdf of X is $\begin{cases} \frac{t-10}{t} & \text{if } x > 10 \\ 0 & \text{otherwise} \end{cases}$.

(c) I'm going to assume that "years" is a typo, and that the intended time is at least 15 hours.

First calculate the probability that a single device lasts for 15 hours:
$\mathbb{P}(X \geq 15) = \int_{15}^{\infty} \frac{10}{x^2} dx = 10\big(-0 - \frac{-1}{15}\big) = \frac{2}{3}$.

We now have a binomial random variable B with parameters $n = 6$ and $p = \frac{2}{3}$.

So $\mathbb{P}(B = 3) = \sum_{k=3}^{6} \binom{6}{k} \big(\frac{2}{3}\big)^k \big(\frac{1}{3}\big)^{6-k} \approx \frac{9}{10}$.

Exercise 6 (5.6)

(a) $f(x) = \begin{cases} \frac{1}{4}xe^{-\frac{x}{2}} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned} \int_{\mathbb{R}} xf(x)dx &= \int_0^{\infty} \frac{1}{4}x^2e^{-\frac{x}{2}}dx \\ &= \frac{1}{4}\int_0^{\infty} x^2e^{-\frac{x}{2}}dx \end{aligned}$$

Now use integration by parts with

$u = x^2 \qquad v = -2e^{-\frac{x}{2}}$

$du = 2xdx \qquad dv = e^{-\frac{x}{2}}dx.$

So we have $\frac{1}{4}\big(-2x^2e^{-\frac{x}{2}}\Big|_0^{\infty} + 4\int_0^{\infty} xe^{-\frac{x}{2}}dx\big)$

Use integration by parts again with

$u = 2x \qquad v = -2e^{-\frac{x}{2}}$

$du = 2dx \qquad dv = e^{-\frac{x}{2}}dx.$

So we have $\frac{1}{4}\big(-2x^2e^{-\frac{x}{2}}\Big|_0^{\infty} + 4xe^{-\frac{x}{2}}\Big|_0^{\infty} + 8\int_0^{\infty} e^{-\frac{x}{2}}dx\big)$

And $8\int_0^{\infty} e^{-\frac{x}{2}}dx = -16e^{-\frac{x}{2}}\Big|_0^{\infty} = 16.$

And the first two terms converge to zero by L'Hôpital's rule, so we have $\frac{1}{4}(0+0+16)=4.$

(b) $f(x)=\begin{cases} C(1-x^2) & \text{if } x\in(-1,1) \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned}\int_{\mathbb{R}} xf(x)dx &= \int_{\mathbb{R}} Cx(1-x^2)dx \\ &= C\int_{-1}^{1}(x-x^3)dx \\ &= C\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_{-1}^{1} \\ &= C\left(\frac{1}{2}-\frac{1}{4}-\frac{1}{2}+\frac{1}{4}\right) \\ &= C(0) \\ &= 0.\end{aligned}$$

So the expected value is 0.

(c) $f(x)=\begin{cases} \frac{5}{x^2} & \text{if } x>5 \\ 0 & \text{otherwise} \end{cases}$

In this case the expected value is undefined, since we would end up integrating $x\frac{5}{x^2}=\frac{5}{x}$ with an upper limit of ∞, and the antiderivative of $\frac{5}{x}$, $5\ln(x)+c$, diverges as $x\to\infty$.

Problem 7 (5.8)

$f(x)=xe^{-x}, x\geq 0$. We can calculate the expected value as usual.

$$\begin{aligned}\int_{\mathbb{R}} xf(x)dx &= \int_0^\infty x^2e^{-x}dx \\ &= \int_0^\infty x^2e^{-x}dx\end{aligned}$$

Now use integration by parts with

$$\begin{aligned} u &= x^2 & v &= -e^{-x} \\ du &= 2xdx & dv &= e^{-x}dx\end{aligned}$$

$$\int_0^\infty x^2e^{-x}dx = -x^2e^{-x}\Big|_0^\infty + \int_0^\infty 2xe^{-x}dx$$

Use integration by parts again with

$$u = 2x \qquad v = -e^{-x}$$
$$du = 2dx \qquad dv = e^{-x}dx$$

$$-x^2e^{-x}\Big|_0^\infty + \int_0^\infty 2xe^{-x}dx = -x^2e^{-x}\Big|_0^\infty - 2xe^{-x}\Big|_0^\infty + \int_0^\infty 2e^{-x}dx$$

And evaluate to obtain the final solution, as follows:

$$-x^2e^{-x}\Big|_0^\infty - 2xe^{-x}\Big|_0^\infty + \int_0^\infty 2e^{-x}dx = \left(-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right)\Big|_0^\infty$$

By applying L'Hôpital's rule, the first two terms evaluate to zero.

We are left with $-2e^{-x}\Big|_0^\infty = \lim_{x\to\infty}\left(-2e^{-x}\right) + 2e^0 = 2.$

So the expected lifetime of the tube is 2 hours.

Problem 8 (5.11)

Let R be a random point chosen with uniform probability from $[0,L]$. The following figure illustrates the situation.

It's clear by looking at the figure that we need to consider the relationship of two quantities, namely, the ratio of $\min(L-R, R)$ and $\max(L-R, R)$.

Now there are two cases we must consider.

Case 1: $\dfrac{\min(L-R, R)}{\max(L-R, R)} < \frac{1}{4}$, where $\min(L-R, R) = L-R$.

Since $\min(L-R, R) = L-R$, we know that $\max(L-R, R) = R$, so we want $\frac{L-R}{R} < \frac{1}{4} \longleftrightarrow 4L - 4R < R \longleftrightarrow R > \frac{4}{5}L$

Case 2: $\dfrac{\min(L-R,R)}{\max(L-R,R)} < \frac{1}{4}$, where $\min(L-R,R) = R$

Since $\min(L-R,R) = L-R$, we know that $\max(L-R,R) = R$, so we want $\frac{R}{L-R} < \frac{1}{4} \longleftrightarrow 5R < L-R \longleftrightarrow R < \frac{L}{5}$.

We need to integrate the probability density function, $\frac{1}{L}$, on the intervals in each case, and take the sum. $\int_0^{\frac{L}{5}} \frac{dx}{L} = \frac{1}{L}\frac{L}{5} = \frac{1}{5}$, and $\int_{\frac{4L}{5}}^{L} \frac{dx}{L} = \frac{1}{L}\left(L - \frac{4L}{5}\right) = 1 - \frac{4}{5} = \frac{1}{5}$, so the solution is $\frac{1}{5} + \frac{1}{5} = \frac{2}{5}$.

Problem 9 (5.17)

(a) We want to determine the probability that a physician chosen at random makes less than \$200,000, i.e. $\mathbb{P}(X \leq 200{,}000)$. To do this, we first need to find μ and σ. Clearly $\mu = \dfrac{320{,}000 + 180{,}000}{2} = 250{,}000$. Now we need to find σ. We can do this by observing that $\mathbb{P}(X \leq 320{,}000) = \frac{3}{4}$, and inverting this quantity with the table on page 190. Doing so gives us the approximation $\sigma = \dfrac{250{,}000 - 180{,}000}{0.675} = \dfrac{70{,}000}{0.675}$. Fucking finally:

$$1 - \phi\left(\frac{x-u}{\sigma}\right) = 1 - \phi\left(\frac{200{,}000 - 250{,}000}{\frac{70{,}000}{0.675}}\right) \approx 1 - \phi(-0.48).$$

Looking up 0.48 in the table, we find that $1 - \phi(-0.48) \approx 1 - 0.6844 = 0.3156$.

(b) $0.75 - \phi\left(\frac{x-u}{\sigma}\right) = 0.75 - \phi\left(\dfrac{280{,}000 - 250{,}000}{\frac{70{,}000}{0.675}}\right) \approx 0.75 - \phi(0.29) \approx 0.75 - .6141 = 0.1359.$