P1.1.6

Suppose $W \in \mathbb{R}^{n\times n}$ is defined by

$$w_{ij} = \sum_{p=1}^{n} \sum_{q=1}^{n} x_{ip} y_{pq} z_{qj}$$

where $X, Y, Z \in \mathbb{R}^{n\times n}$.

If we use this formula we get $O(n^4)$ operations

On the other hand,

$$w_{ij} = \sum_{p=1}^{n} x_{ip} \left(\sum_{q=1}^{n} y_{pq} z_{qj} \right) = \sum_{p=1}^{n} x_{ip} u_{pj}$$

where $U = YZ$. Thus $W = XU = XYZ$ and only require $O(n^3)$ operations

Question

Use this methodology to develop a $O(n^3)$ procedure for computing the n-by-n matrix A defined by:

$$a_{ij} = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} E(k_1, i) F(k_1, i) G(k_2, k_1) H(k_2, k_3) F(k_2, k_3) G(k_3, j)$$

Hint: Transposes and Pointwise products

1. Pointwise products:
   1. Note: $E(k_1, i)F(k_1, i) = I(k_1, i)$, i.e. $I = E .* F$
   2. Note: $H(k_2, k_3)F(k_2, k_3) = J(k_2, k_3)$, i.e. $J = H .* F$

$$a_{ij} = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \sum_{k_3=1}^{n} I(k_1, i) G(k_2, k_1) J(k_2, k_3) G(k_3, j)$$

2. Multiply k_3:
   1. $K = JG$

$$\begin{aligned} a_{ij} &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} I(k_1, i) G(k_2, k_1) \sum_{k_3=1}^{n} J(k_2, k_3) G(k_3, j) \\ &= \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} I(k_1, i) G(k_2, k_1) K(k_2, j) \end{aligned}$$

3. Transpose G

$$a_{ij} = \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} I(k_1, i) G^T(k_1, k_2) K(k_2, j)$$

4. Multiply k_2
   1. $L = G^T K$

$$\begin{aligned} a_{ij} &= \sum_{k_1=1}^{n} I(k_1, i) \sum_{k_2=1}^{n} G^T(k_1, k_2) K(k_2, j) \\ &= \sum_{k_1=1}^{n} I(k_1, i) L(k_1, j) \end{aligned}$$

5. Transpose I and Multiply k_1

$$a_{ij} = \sum_{k_1=1}^{n} I^T(i, k_1) L(k_1, j)$$

In all we get:

$$\begin{aligned} A &= I^T L \\ &= (E. * F)^T G^T K \\ &= (E. * F)^T G^T J G \\ &= (E. * F)^T G^T (H. * F) G \end{aligned}$$

Operation count:

1. 2 pointwise products: $2 \times O(n^2)$
2. 2 transposes: $2 \times O(n^2)$
3. 3 matrix multiplication: $3 \times O(n^3)$

Dominated by $O(n^3)$