

Computational Photography Assignment 3

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Task 1

Def(Linear) Given two measurable function f and g and $\alpha, \beta \in \mathbb{C}$. An operator \mathcal{F} is called *linear* if the following identity holds true:

$$\mathcal{F}\{\alpha f + \beta g\}(x) = \alpha \mathcal{F}\{f\}(x) + \beta \mathcal{F}\{g\}(x) \quad (1)$$

Def(Shift Invariant) Given a measurable function f . An operator \mathcal{F} is called *shift invariant* if the following identity holds true:

$$\mathcal{F}\{S_\delta f\}(x) = S_\delta(\mathcal{F}\{f\})(x) \quad (2)$$

where the shift operator S_δ is defined as the following:

$$S_\delta(f)(x) = f(x + \delta) \quad (3)$$

For any given function f and small number $\delta \in \mathbb{R}$.

Note that notational convention the operation notion for any operator F , we write $F(f)(x)$ instead of $F(f(x))$. In words we apply the operator F (i.e. the functional F) to the function f . Thus it would even be possible to omit the argument x . However to make the reasoning clear, I will keep it.

Therefore, directly applying the definition from equation 3 to equation 2 we can also define the property *shift invariant* as the following:

$$\mathcal{F}\{S_\delta(f(x))\} := \mathcal{F}\{f(x + \delta)\} = S_\delta(\mathcal{F}\{f\})(x) \quad (4)$$

Relying on the definitions of the equations before, equation 4 and equation 1 let I solved this task,

(a) $\mathcal{F}\{f\}(x) = e^{f(x)}$ is *not linear* and is *shift invariant*.

Show: \mathcal{F} is not *linear*

Proof.

$$\begin{aligned} \mathcal{F}\{\alpha f + \beta g\}(x) &= e^{\alpha f(x) + \beta g(x)} \\ &= e^{\alpha f(x)} e^{\beta g(x)} \\ &\neq \alpha e^{f(x)} + \beta e^{g(x)} \\ &= \alpha \mathcal{F}\{f\}(x) + \beta \mathcal{F}\{g\}(x) \end{aligned}$$

□

Show \mathcal{F} is *shift invariant*

Proof.

$$\begin{aligned} \mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x + \delta)\} \\ &= e^{f(x + \delta)} \\ &= S_\delta(e^{f(x)}) \\ &= S_\delta(\mathcal{F}\{f\})(x) \end{aligned}$$

□

(b) $\mathcal{F}\{f\}(x) = f(x)f(x - 1)$ is *not linear* and is *shift invariant*.

Show \mathcal{F} is not *linear*

Proof.

$$\begin{aligned}\mathcal{F}\{\alpha f + \beta g\}(x) &= (\alpha f(x) + \beta g(x))(\alpha f(x-1) + \beta g(x-1)) \\ &= \alpha^2 f(x)f(x-1) + \beta^2 g(x)g(x-1) + \alpha\beta(f(x)g(x-1) + f(x-1)g(x)) \\ &\neq \alpha f(x)f(x-1) + \beta g(x)g(x-1) \\ &= \alpha \mathcal{F}\{f\}(x) + \beta \mathcal{F}\{g\}(x)\end{aligned}$$

□

Show \mathcal{F} is *shift invariant*

Proof.

$$\begin{aligned}\mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x+\delta)\} \\ &= f(x+\delta)f(x-1+\delta) \\ &= S_\delta(f(x)f(x-1)) \\ &= S_\delta(\mathcal{F}\{f\}(x))\end{aligned}$$

□

(c) $\mathcal{F}\{f\}(x) = \sum_{k=x-4}^{x+2} f(k)$ is *linear* and is *shift invariant*.

Show \mathcal{F} is *linear*

Proof.

$$\begin{aligned}\mathcal{F}\{\alpha f + \beta g\}(x) &= \sum_{k=x-4}^{x+2} (\alpha f(k) + \beta g(k)) \\ &= \alpha \sum_{k=x-4}^{x+2} f(k) + \beta \sum_{k=x-4}^{x+2} g(k) \\ &= \alpha \mathcal{F}\{f\}(x) + \beta \mathcal{F}\{g\}(x)\end{aligned}$$

□

Show \mathcal{F} is *shift invariant*

Proof.

$$\begin{aligned}
 \mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x + \delta)\} \\
 &= \sum_{k=x-4+\delta}^{x+2+\delta} f(k) \\
 &= \sum_{k=x-4}^{x+2} f(k + \delta) \\
 &= S_\delta\left(\sum_{k=x-4}^{x+2} f(k)\right) \\
 &= S_\delta(\mathcal{F}\{f\})(x)
 \end{aligned}$$

□

(d) $\mathcal{F}\{f\}(x) = f(2x)$ is *linear* and is *not shift invariant*.

Show \mathcal{F} is *linear*

Proof.

$$\begin{aligned}
 \mathcal{F}\{\alpha f + \beta g\}(x) &= \alpha f(2x) + \beta g(2x) \\
 &= \alpha \mathcal{F}\{f\}(x) + \beta \mathcal{F}\{g\}(x)
 \end{aligned}$$

□

Show \mathcal{F} is *not shift invariant*

Proof.

$$\begin{aligned}
 \mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x + \delta)\} \\
 &= f(2(x + \delta)) \\
 &= f(2x + 2\delta) \\
 &\neq f(2x + \delta) \\
 &= S_\delta(f(2x)) \\
 &= S_\delta(\mathcal{F}\{f\})(x)
 \end{aligned}$$

□

(e) $\mathcal{F}\{f\}(x) = \sin(2x)f(x)$ is *linear* and is *not shift invariant*.

Show \mathcal{F} is *linear*

Proof.

$$\begin{aligned}\mathcal{F}\{\alpha f + \beta g\}(x) &= \sin(2x)(\alpha f(x) + \beta g(x)) \\ &= \sin(2x)\alpha f(x) + \sin(2x)\beta g(x) \\ &= \alpha \sin(2x)f(x) + \beta \sin(2x)g(x) \\ &= \alpha \mathcal{F}\{f\}(x) + \beta \mathcal{F}\{g\}(x)\end{aligned}$$

□

Show \mathcal{F} is not *shift invariant*

Proof.

$$\begin{aligned}\mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x + \delta)\} \\ &= \sin(2x)f(x + \delta) \\ &\neq \sin(2x + \delta)f(x + \delta) \\ &= S_\delta(\sin(2x)f(x)) \\ &= S_\delta(\mathcal{F}\{f\}(x))\end{aligned}$$

□

(f) $\mathcal{F}\{f\}(x) = xf(x)$ is *linear* and is *not shift invariant*.

Show \mathcal{F} is *linear*

Proof.

$$\begin{aligned}\mathcal{F}\{\alpha f + \beta g\}(x) &= x(\alpha f(x) + \beta f(x)) \\ &= x\alpha f(x) + x\beta f(x) \\ &= \alpha xf(x) + \beta xf(x) \\ &= \alpha \mathcal{F}\{f\}(x) + \beta \mathcal{F}\{g\}(x)\end{aligned}$$

□

Show \mathcal{F} is not *shift invariant*

Proof.

$$\begin{aligned}
 \mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x+\delta)\} \\
 &= xf(x+\delta) \\
 &\neq (x+\delta)f(x+\delta) \\
 &= S_\delta(xf(x)) \\
 &= S_\delta(\mathcal{F}\{f\})(x)
 \end{aligned}$$

□

(g) $\mathcal{F}\{f\}(x) = f(x) - f(x-5)$ is *linear* and is *shift invariant*.

Show \mathcal{F} is *linear*

Proof.

$$\begin{aligned}
 \mathcal{F}\{\alpha f + \beta g\}(x) &= (\alpha f(x) + \beta g(x)) - (\alpha f(x-5) + \beta g(x-5)) \\
 &= (\alpha f(x) - \alpha f(x-5)) + (\beta g(x) - \beta g(x-5)) \\
 &= \alpha(f(x) - f(x-5)) + \beta(g(x) - g(x-5)) \\
 &= \alpha\mathcal{F}\{f\}(x) + \beta\mathcal{F}\{g\}(x)
 \end{aligned}$$

□

Show \mathcal{F} is *shift invariant*

Proof.

$$\begin{aligned}
 \mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x+\delta)\} \\
 &= f(x+\delta) - f(x-5+\delta) \\
 &= S_\delta(f(x) - f(x-5)) \\
 &= S_\delta(\mathcal{F}\{f\})(x)
 \end{aligned}$$

□

(h) $\mathcal{F}\{f\}(x) = f(x-3) - 2f(x-12)$ is *linear* and is *shift invariant*.

Show \mathcal{F} is *linear*

Proof.

$$\begin{aligned}
 \mathcal{F}\{\alpha f + \beta g\}(x) &= (\alpha f(x-3) + \beta g(x-3)) - 2(\alpha f(x-12) + \beta g(x-12)) \\
 &= (\alpha f(x-3) - 2\alpha f(x-12)) + (\beta g(x-3) - 2\beta g(x-12)) \\
 &= \alpha(f(x-3) - 2f(x-12)) + \beta(g(x-3) - 2g(x-12)) \\
 &= \alpha\mathcal{F}\{f\}(x) + \beta\mathcal{F}\{g\}(x)
 \end{aligned}$$

□

Show \mathcal{F} is *shift invariant*

Proof.

$$\begin{aligned}
 \mathcal{F}\{S_\delta(f(x))\} &= \mathcal{F}\{f(x+\delta)\} \\
 &= f(x-3+\delta) - 2f(x-12+\delta) \\
 &= S_\delta(f(x-3) - 2f(x-12)) \\
 &= S_\delta(\mathcal{F}\{f\}(x))
 \end{aligned}$$

□

Task 2

Given an $m \times n$ monochromatic (i.e. there is only one color-channel) Image I . Give an algorithm how to apply box-filtering on this image. Furthermore analyse the asymptotic complexity of this algorithm.

Algorithm 1 Moving Average box filter

Input: Grayscale *Image* I with resolution $m \times n$

Output: Box filtered Image *Image* \hat{I}

Procedures: $getDimensions(Image)$, $zeros(height, width)$

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1:  $[h, w] = getDimensions(I)$ 
2:  $\hat{I} = zeros(h, w)$ 
3:  $r = \lceil \frac{w-1}{2} \rceil$ 
4: Foreach Pixel  $p \in Image\ I$  do
5:    $contribution = 0$ 
6:   Foreach Pixel  $p_n \in r - Neighborhood\ \mathcal{N}_r(p)$  do
7:      $contribution = contribution + I(p + p_n)$ 
8:   end for
9:    $\hat{I}(p) = \frac{contribution}{m \cdot n}$ 
10: end for

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Remarks:

- By pixels in the Algorithm 1 we are referring to the coordinates of the pixel in the image. Therefore p corresponds to the x and y coordinates of pixel p in the Image I .
- $I(p)$ denotes accessing the pixel-(color)-values in the images at the position of the pixel p in the image I .
- $\mathcal{N}_r(p)$ denotes the neighborhood with radius r around a given pixel p . In the context of pixel-coordinates, think of it as a box-grid, centred at the pixel coordinates of p . This grid has a radius of r . This means there are r neighbors (pixel-coordinates in the grid) below, on top, on the left and on the right of p .
- Our algorithm can easily be extended for color Images by simply applying the same algorithm to each color-channel separately.
- The assumption of being provided by a m by n can easily be extended for the case when $n \neq m$. This only will affect the computation of the radius r in algorithm 1. Computing $\lceil 0.5 \cdot (\lceil \frac{m-1}{2} \rceil + \lceil \frac{n-1}{2} \rceil) \rceil$ would be a valid option in order to compute r .
- If w (i.e. n) is odd, then $\lceil \frac{w-1}{2} \rceil$ is equal to $\frac{w-1}{2}$.
- The procedure $getDimensions$ returns the width-and height resolution of a provided Image.
- The procedure $zeros$ creates a new image with the provided resolutions.

Aysmptotic Complexity

Task 3

Task 4

Task 5

Task 6