MAT-150 **Name:** ____________________

Fall 2017

Midterm Solution

Handout: 10/6, Due: 10/16 **Pledge:** ____________________

Each question topic and point value is recorded in the tables below. You may review these topics from any resource at your leisure. Once you decide to start an exam problem, you are on the clock and you must work without any external resources. Each problem can be done one at a time, but must be finished in a single sitting. Answer each question in the space provided, if you run out of room, then you may continue on the back of the page. It is your responsibility to plan out your time to ensure that you can finish all problems within the 3.5 hours allotted. By writing your name and signing the pledge you are stating that your work adheres to these terms and the Davidson honor code.

Scoring Table

Question	Points	Score
1	12	
2	10	
3	8	
4	10	
5	10	
6	10	
7	16	
8	14	
Total:	90	

Topics Table

Question	Topic
1	Matrix Equations and Solution Sets
2	Linear Independence
3	Linear Transformations
4	Matrix Representation of Linear Transformations
5	The Determinant's Properties
6	Computing the Determinant
7	Subspaces, Basis, and Dimension
8	Change of Basis

Start Time: **End Time:**

1. Let A be an $m \times n$ matrix and consider the matrix equation $Ax = b$.

   (a) (2 points) If the matrix equation has a solution, what can you say about the vector b in terms of the column vectors of A?
   Solution: The vector b is a linear combination of the column vectors of A; that is, $b \in \mathrm{Col}(A)$.

   (b) (2 points) If $m < n$, then what can you say about the solution set to the matrix equation? Is the system consistent, are the solutions unique: always, sometimes, never?
   Solution: We know that there is at most m pivots. Since $m < n$, not every column of A has a pivot. Therefore, if the system is consistent, then the solution is not unique.

   (c) (2 points) If $m > n$, then what can you say about the solution set to the matrix equation? Is the system consistent, are the solutions unique: always, sometimes, never?
   Solution: We know that there is at most n pivots. Since $n < m$, not every row of A has a pivot. Therefore, the system is not always consistent.

   (d) (2 points) If $m = n$, then what is a necessary and sufficient condition for the matrix equation to always be consistent. If consistent, is the solution always unique?
   Solution: The matrix equation is always consistent if and only if the number of pivots is equal to m. Since $m = n$, it follows that the solution is always unique.

   (e) (4 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & 6 \\ -5 & 6 & 15 \end{bmatrix}$$

   Find a vector b for which $Ax = b$ has a solution, and compute the general solution vector.
   Solution: Consider the following augmented matrix and row operations

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ -3 & 2 & 6 & b_2 \\ -5 & 6 & 15 & b_3 \end{bmatrix} \xrightarrow[5r_1+r_3 \to r_3]{3r_1+r_2 \to r_2} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 8 & 15 & b_2 + 3b_1 \\ 0 & 16 & 30 & b_3 + 5b_1 \end{bmatrix}$$

$$\xrightarrow{-2r_2+r_3 \to r_3} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 8 & 15 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 - b_1 \end{bmatrix}$$

   At this point, it is clear that the system has a solution if and only if $b_3 - 2b_2 - b_1$. For instance, let $b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Then the solution can be formed as follows

$$\begin{aligned} x_3 &= t \quad \text{(free variable)} \\ x_2 &= \frac{1}{2} - \frac{15}{8}t \\ x_3 &= 1 - 3t - 2x_2 = \frac{3}{4}t \end{aligned}$$

Start Time: **End Time:**

2. Let $S = \{v_1, \dots, v_p\}$ be a set of vectors in $\mathbb{R}^n$.

   (a) (2 points) State the definition of S being linearly independent and dependent.
   Solution: The set S is linearly independent if the vector equation

$$x_1 v_1 + \cdots + x_p v_p$$

   has only the trivial solution $x_1 = \cdots = x_p = 0$. Otherwise, the set S is linearly dependent.

   (b) (2 points) If $p > n$ what can you say about the linear independence of the set S?
   Solution: If $p > n$, then the matrix $A = [v_1 \cdots v_p]$ has more columns than rows. Therefore, not every column of A has a pivot and it follows that the vectors are linearly dependent.

   (c) (2 points) If $v_i = 0$ for any $i \in \{1, \dots, p\}$ what can you say about the linear independence of the set S?
   Solution: Suppose $v_i = 0$. Then,

$$0v_1 + \cdots + 0v_{i-1} + v_i + 0v_{i+1} + \cdots + 0v_n = 0,$$

   and it follows that the vectors are linearly dependent.

   (d) (4 points) Let

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}.$$

   Determine if the given vectors are linearly independent or dependent.
   Solution: Consider the following matrix and row operations

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -5 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow[-2r_1+r_3 \to r_3]{1r_1+r_2 \to r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

   At this point, it is clear that not every column has a pivot. Therefore, the vectors are linearly dependent.

3. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a transformation.

   (a) (2 points) State the definition of T being a linear transformation.
   Solution: The transformation T is linear provided that the following holds for all $u, v \in \mathbb{R}^n$ and for all $c \in \mathbb{R}$

   - $T(u + v) = T(u) + T(v)$
   - $T(cu) = cT(u)$

   (b) (6 points) Prove the following statement.

   The linear transformation T is one-to-one if and only if $T(x) = 0$ has only the trivial solution.

   Solution: Suppose that T is one-to-one. Then, the equation $T(x) = 0$ has at most one solution, and it follows that the only solution is the trivial solution. Conversely, suppose that $T(x) = 0$ has only the trivial solution and for some $b \in \mathbb{R}^n$, there exists two vectors x and $\hat{x}$ such that $T(x) = b$ and $T(\hat{x}) = b$. Since T is a linear, it follows that $T(x - \hat{x}) = 0$. But, since $T(x) = 0$ has only the trivial solution, this implies that $x = \hat{x}$. Therefore, at most one solution exists to the equation $T(x) = b$, and it follows that T is one-to-one.

Start Time: **End Time:**

4. For each part below, the action of a linear transformation will be described. Use that description to find the matrix representation of the linear transformation, and answer the given question.

   (a) (5 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by a projection on the x_2-axis followed by a reflection about the x_1-axis. Is this transformation one-to-one, is it onto?
   Solution: The standard matrix representation of T is defined by

$$\begin{aligned} A &= [T(e_1)\ T(e_2)] \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

   Since T projects onto x_2-axis, it cannot be onto. Furthermore, since $T(e_1) = 0$, it follows that T is not one-to-one.

   (b) (5 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by a counter-clockwise rotation about the x_3-axis followed by a reflection about the x_1x_3-plane. Is this transformation invertible?
   Solution: The standard matrix representation of T is defined by

$$\begin{aligned} A &= [T(e_1)\ T(e_2)\ T(e_3)] \\ &= \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ -\sin(\theta) & -\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

   Since both rotations and reflections are invertible transformations, it follows that T is invertible.

Start Time: **End Time:**

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Denote by $\{a_1, \ldots, a_n\}$ a basis for $\mathbb{R}^n$, and by $\mathcal{P}$ the parallelepiped determined by these vectors.

   (a) (4 points) State the definition of $\det(T)$ in terms of volume magnification and orientation change.
   Solution: The determinant of T is the product of the volume magnification and orientation change under T, which we denote by

$$\begin{aligned}\det(T) &= V_T \cdot O_T \\ &= \frac{V(T(\mathcal{P}))}{V(\mathcal{P})} \cdot \frac{O(T(\mathcal{P}))}{O(\mathcal{P})}\end{aligned}$$

   (b) (6 points) Provide a sketch and brief justification for each of the following properties of the determinant.

      i. $\det(T) = 0$ if and only if T is not invertible.
      Solution: Let $\{a_1, \ldots, a_n\}$ be any basis for $\mathbb{R}^n$. Then T is invertible if and only if $\{T(a_1), \ldots, T(a_n)\}$ is also a basis for $\mathbb{R}^n$. This is true if and only if $\det(T) \neq 0$.

      ii. If T is invertible, then $\det(T^{-1}) = \frac{1}{\det(T)}$
      Solution: Suppose the linear transformation T maps a parallelepiped $\mathcal{P}_1$ in $\mathbb{R}^n$ of dimension n to a parallelepiped $\mathcal{P}_2$ in $\mathbb{R}^n$ of dimension n. Then $\det(T) = \frac{O(\mathcal{P}_2)}{O(\mathcal{P}_1)} \cdot \frac{V(\mathcal{P}_2)}{V(\mathcal{P}_1)}$, and the inverse transformation maps $\mathcal{P}_2$ back to $\mathcal{P}_1$. Therefore,

$$\det(T^{-1}) = \frac{O(\mathcal{P}_1)}{O(\mathcal{P}_2)} \cdot \frac{V(\mathcal{P}_1)}{V(\mathcal{P}_2)} = \frac{1}{\det(T)}.$$

      iii. If $U\colon \mathbb{R}^n \to \mathbb{R}^n$ is also a linear transformation, then the composition $TU(x) = T(U(x))$ satisfies $\det(TU) = \det(T)\det(U)$.
      Solution: Let $\{a_1, \ldots, a_n\}$ be any basis for $\mathbb{R}^n$, and denote by $\mathcal{P}_1$ the parallelepiped of dimension n that it determines. Then $\{U(a_1), \ldots, U(a_n)\}$ is also a basis for $\mathbb{R}^n$ if and only if U is an invertible transformation. If U is not invertible, then TU is not invertible and the result follows from part (i).
      If U is invertible, then the vectors $\{U(a_1), \ldots, U(a_n)\}$ determine a parallelepiped of dimension n, which we denote by $\mathcal{P}_2$. By definition, we have

$$\det(TU) = \frac{O(TU(\mathcal{P}_1))}{O(\mathcal{P}_1)} \cdot \frac{V(TU(\mathcal{P}_1))}{V(\mathcal{P}_1)}.$$

      Furthermore, it is easy to see that

$$\begin{aligned}\frac{O(TU(\mathcal{P}_1))}{O(\mathcal{P}_1)} &= \frac{O(T(\mathcal{P}_2))}{O(\mathcal{P}_2)} \cdot \frac{O(U(\mathcal{P}_1))}{O(\mathcal{P}_1)}, \\ \frac{V(TU(\mathcal{P}_1))}{V(\mathcal{P}_1)} &= \frac{V(T(\mathcal{P}_2))}{V(\mathcal{P}_2)} \cdot \frac{V(U(\mathcal{P}_1))}{V(\mathcal{P}_1)},\end{aligned}$$

      and the result follows.

Start Time: **End Time:**

6. Let

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 3 & 5 & -2 \\ -4 & 4 & -6 \end{bmatrix}.$$

(a) (6 points) Compute $\det(A)$.

Solution: Consider the following row operations

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 3 & 5 & -2 \\ -4 & 4 & -6 \end{bmatrix} \xrightarrow[4r_1+r_3\to r_3]{-3r_1+r_2\to r_2} \begin{bmatrix} 1 & 3 & -3 \\ 0 & -4 & 7 \\ 0 & 16 & -18 \end{bmatrix} \xrightarrow{4r_2+r_3\to r_3} \begin{bmatrix} 1 & 3 & -3 \\ 0 & -4 & 7 \\ 0 & 0 & 10 \end{bmatrix}$$

The product of the diagonal entries in the echelon matrix is equal to -40. Furthermore, since we have not swapped any rows we have not changed the sign of the determinant. Therefore, $\det(A) = -40$.

(b) (4 points) Answer each of the following questions regarding the matrix A.

i. Is A invertible?

Solution: Yes, since $\det(A) \neq 0$.

ii. If the 3×3 matrix B satisfies $\det(B) = 0.5$, then what is the value of $\det(AB)$?

Solution: $\det(AB) = \det(A)\det(B) = -20$.

iii. Let $T(x) = Ax$ and $\mathcal{P}$ be a parallelepiped in $\mathbb{R}^3$ with volume 3 units^3 and orientation $+1$. What is the volume and orientation of the image parallelepiped $T(\mathcal{P})$?

Solution: $T(\mathcal{P})$ has volume equal to 120 units^3 and has orientation -1.

Start Time: **End Time:**

7. Let S be a set of vectors in $\mathbb{R}^n$.

   (a) (4 points) State the definition of S being a subspace.
   Solution: S is a subspace of $\mathbb{R}^n$ provided that

   - $0 \in S$,
   - $u + v \in S$ for all $u, v \in S$,
   - $cu \in S$ for all $u \in S$ and scalars c.

   (b) (4 points) State the definition of a basis and the dimension of the subspace S.
   Solution: A basis is any linearly independent set of vectors from S that also spans S. The dimension of S is the number of vectors in any basis for S.

   (c) (8 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

   Compute a basis for the Null Space and Column Space of A and note the dimension of both spaces.
   Solution: Using row operations we find that the echelon form of A is as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

   Therefore, the $1st$ and $3rd$ columns have pivots and a basis for the Column Space of A is

$$\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \}.$$

   Furthermore, we can solve the associated homogeneous equation as follows

$$\begin{aligned} x_3 &= 0 \\ x_2 &= t \\ x_1 &= -2t \end{aligned}$$

   Therefore, a basis for the Null Space of A is

$$\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \}.$$

   We conclude by noting the the dimension of the Column Space of A is 2 and the dimension of the Null Space of A is 1.

Start Time: **End Time:**

8. Let $\mathbb{P}_3$ denote the set of polynomials of degree 3 or less.

(a) (4 points) Find the change-of-coordinates matrix from the basis $\beta = \{1, x, 2x^2 - 1, 4x^3 - 3x\}$ to the standard basis $\mathcal{E} = \{1, x, x^2, x^3\}$. Denote this matrix by $P_{\mathcal{E}\leftarrow\beta}$.

Solution: The change-of-coordinate matrix is

$$P_{\mathcal{E}\leftarrow\beta} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(b) (4 points) Let $[p]_\beta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find $[p]_{\mathcal{E}}$.

Solution:

$$\begin{aligned} [p]_{\mathcal{E}} &= P_{\mathcal{E}\leftarrow\beta}[p]_\beta \\ &= \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} \end{aligned}$$

(c) (6 points) Find $(P_{\mathcal{E}\leftarrow\beta})^{-1}$ and use it to show that you can get $[p]_\beta$ from $[p]_{\mathcal{E}}$, as defined in part (b).

Solution: Since $P_{\mathcal{E}\leftarrow\beta}$ is already in echelon form, we can find its inverse by further transforming it to fully reduced echelon form and keeping track of row operations used. It follows that the inverse is

$$(P_{\mathcal{E}\leftarrow\beta})^{-1} = \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}.$$

Furthermore,

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$