CSC/MAT-220: Discrete Structures
EFY 2

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Puzzle of the week: A clever (or not so clever) warden decides to play a game with 8 of his inmates. He tells them: "Later this evening, I am going to line you all up, with each of you facing the same direction, and place a hat (black or white) on each of your heads. One at a time, starting with the person whose hat none of the other inmates can see, each of you can say black or white. If any inmate guesses the color of their hat correctly, then they can go free."

Assuming the inmates can hear each answer, derive an algorithm for freeing as many inmates as possible during this game. Provide a formal proof for why your algorithm works, and give a brief discussion on why a better algorithm does not exist.

Solution: We can guarantee that all inmates, except for the first will go free. In the proof below, we will make use of the following definition.

Definition 1. We say that two integers are of the *same parity* if they are both even or odd. We say that two integers are of *different parity* if one is even and the other odd.

Proof. We can view the inmates standing in a line, all facing the same direction, as an 8-digit number represented in binary, where we associate the black hat with 1 and the white hat with 0. Consider the list below

$$[x_7,\ x_6,\ x_5,\ x_4,\ x_3,\ x_2,\ x_1,\ ?]\,,$$

where the ? denotes the first hat, and $x_1, \ldots, x_7$ denote the remaining hats. The first person sacrifices themselves in order to give essential information to everyone else; they say white, if $\sum_{i=1}^{7} x_i$ is even, and black if the sum is odd. The second person now knows their hat color, since

$$\begin{cases} x_1 = 0 & \text{if } \sum_{i=1}^{7} x_i \text{ and } \sum_{i=2}^{7} x_i \text{ are of the same parity,} \\ x_1 = 1 & \text{if } \sum_{i=1}^{7} x_i \text{ and } \sum_{i=2}^{7} x_i \text{ are of different parity.} \end{cases}$$

In a similar fashion, each person after the first knows their hat color, since

$$\begin{cases} x_k = 0 & \text{if } \left(\sum_{i=1}^{7} - (x_1 + \cdots + x_{k-1})\right) \text{ and } \sum_{i=k+1}^{7} x_i \text{ are of the same parity,} \\ x_k = 1 & \text{if } \left(\sum_{i=1}^{7} - (x_1 + \cdots + x_{k-1})\right) \text{ and } \sum_{i=k+1}^{7} x_i \text{ are of different parity.} \end{cases}$$

□

Note that freeing all except for the first inmate is the best possible scenario. Since no one can see the first person's hat, the ability to get all 8 people free is equivalent to knowing the value of an 8-digit binary number, without knowing if it is even or odd.