CSC/MAT-220: Discrete Structures
Solution 3

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Book Problems

Problem 17.18

Consider an $m \times n$ chessboard, which has $m+1$ horizontal lines and $n+1$ vertical lines. In order to make a rectangle we need to select 2 horizontal lines and 2 vertical lines. There are $\binom{m+1}{2}$ ways to select 2 horizontal lines and $\binom{n+1}{2}$ ways to select 2 vertical lines, and it follows that there are $\binom{m+1}{2}\binom{n+1}{2}$ different rectangles that can be formed from an $m \times n$ chessboard.

Problem 17.25

For all natural numbers $n \geq 1$,

$$1 + 2 + \cdots + (n-1) + n + (n-1) + \cdots + 2 + 1 = n^2.$$

Algebraic Proof. Let $n \geq 1$ be a natural number and define

$$\begin{aligned} s(n) &= 1 + 2 + \cdots + (n-1) \\ &= (n-1) + (n-2) + \cdots + 1. \end{aligned}$$

It is clear from the above equation that $2s(n) = n(n-1)$. Finally, it is also clear that $n + 2s(n) = n + n(n-1) = n^2$. $\square$

Geometric Proof. Consider an $n \times n$ chess board, where each square is considered to have area 1 unit^2. Then, the chess board clearly has area n^2 unit^2. To prove the identity, consider deconstructing the $n \times n$ chessboard into $2(n-1)+1$ pieces as follows:

- Remove the first row of area n unit^2, leaving a $(n-1) \times n$ chessboard.
- Remove the first column of area $(n-1)$ unit^2, leaving a $(n-1) \times (n-1)$ chessboard.
- Repeat until you are left with a 2×2 chessboard with area equal to $2 + 1 + 1$ unit^2.

Then, the sum of the area of the deconstructed pieces is $n + 2(n-1) + \cdots + 2(2) + 2(1)$ unit^2. Since this sum of areas must be equal to the area of the whole, the result follows. $\square$

Problem 20.7

If the sum of two primes is prime, then one of the primes must be 2.

Proof. The following is a proof by contradiction. Suppose that p and q are prime numbers whose sum is prime, but neither is 2. By definition of a prime number, only even prime number is 2. Therefore, both p and q are odd prime numbers. Since the sum of two odd numbers is even this implies that $p+q$ must be even. But this is absurd, since $p+q$ must be prime, and $p+q > 2$. Therefore, either p or q must be 2. $\square$

Problem 21.8

For $n \in \mathbb{N}$, let F_n denote the nth Fibonacci number and let $S_n = F_0+F_1+\cdots+F_n$ denote the sum of the first n Fibonacci numbers. Then,

$$S_n = F_{n+2} - 1$$

for all $n \geq 0$.

Proof. Let us first establish that this result holds for $n = 0$:

$$S_0 = F_0 = 1 \text{ and } F_2 - 1 = 2 - 1 = 1.$$

What follows is a proof by contradiction (lack of counterexample).

Suppose there exists a $k \in \mathbb{N}$ such that $S_k \neq F_{k+2} - 1$ and define the set of all counterexamples by

$$X = \{n \in \mathbb{N} \colon \; S_n \neq F_{n+2} - 1\}.$$

Since $k \in X$, we know that X is non-empty, and by the Well Ordering Principle it follows that there exists a least element of X, which we denote by l. Note that $l > 0$, since $S_0 = F_2 - 1$. Therefore, $l - 1 \geq 0$ and satisfies

$$S_{l-1} = F_{l+1} - 1.$$

$\square$

Adding F_l to both sides of the above equation, and noting that $S_l = S_{l-1}+F_l$ and $F_{l+2} = F_l + F_{l+1}$, gives

$$S_l = F_{l+2} - 1,$$

which contradict $l \in X$. Therefore, X is empty, and the result therefore holds.

Problem 22.13

a. The given proof shows that for every $n \in \mathbb{N}$, any subset of $\mathbb{N}$ of size n has a least element. But the Well Ordering Principle is a statement about all subsets of $\mathbb{N}$, regardless of size.

b. Below is a correct proof by induction of the Well Ordering Principle.

Proof. Let $T \subseteq \mathbb{N}$ and suppose that T has no least element. We will show by induction that the set

$$S = \{n \in \mathbb{N}\colon\ n < t,\ \forall t \in T\}$$

is equal to $\mathbb{N}$. To this end, note that $0 \in S$, since otherwise $0 \in T$ which would contradict T having no least element. Next, suppose that $k \in S$, for some $k \in \mathbb{N}$, and suppose that $(k+1) \notin S$. Then, there exists a $t_1 \in T$ such that $t_1 \leq (k+1)$. Furthermore, since T has no least element, there exists a $t_2 \in T$ such that $t_2 < t_1$ and therefore satisfies $t_2 \leq k$. This clearly contradicts the fact that $k \in S$, and it follows that $k \in S \implies (k+1) \in S$. Therefore, by the principle of mathematical induction $S = \mathbb{N}$, which implies that $T = \emptyset$. Thus, the only set with no least element is the empty set, which proves the Well Ordering Principle. $\square$

Problem 22.14

Let $A_1, A_2, \ldots, A_n$ be sets, where $n \geq 2$. If for any two sets A_i and A_j either $A_i \subseteq A_j$ or $A_j \subseteq A_i$, then one of these n sets is a subset of all of them.

Proof. The base case when $n = 2$ is clear: if either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$, then either A_1 or A_2 is a subset of both of them. Suppose that the result holds for k subsets, where $k \geq 2$, and consider the $(k+1)$ subsets: $A_1, \ldots, A_k, A_{k+1}$. By the induction hypothesis, there exists an integer p such that $1 \leq p \leq k$ and A_p is a subset of all the sets $A_1, \ldots, A_k$. Furthermore, either $A_p \subset A_{k+1}$ or $A_{k+1} \subset A_p$. In the former case it follows that A_p is a subset of all $(k+1)$ sets, and in the later case it follows that A_{k+1} is a subset of all $(k+1)$ sets. In either case, it follows that the result holds for $(k+1)$ sets, if it holds for k sets. Therefore, by the principle of mathematical induction the result holds for all $n \geq 2$. $\square$

Problem 22.15

If the grid has $a+1$ vertical lines and $b+1$ horizontal lines, there are $\binom{a+b}{a}$ lattice paths from the lower left to the upper right corner.

Proof. The following is a proof by strong induction on all grids that have a path of length n, where $n \in \mathbb{N}$. The base case is for a path of the length zero. The only grid construction that is possible is a single point, where both $a = 0$ and $b = 0$. In this case, the result clearly holds since $\binom{0}{0} = 1$, which denotes the 1 path of length zero.

Now, consider a path of length $n \in \mathbb{N}$ and suppose the result holds for all $a, b \in \mathbb{N}$ such that $a + b = n$ (i.e. for all grids that have a lattice path from the lower left to upper right corner of length n). To get a path of length $n+1$, we can either add 1 to a (add a vertical line), or we can add 1 to b (add a horizontal line). Consider the former, then we have a grid with $(a+1)+1$ vertical lines and $b+1$ horizontal lines. To get to the point $(a+1, b)$, the path must pass through the point (a, b) or $(a+1, b-1)$, both of which are the right corner of a grid (grid 1 and grid 2, respectively) with a lattice path of length n. Therefore, by the induction hypothesis, there are $\binom{a+b}{a}$ lattice paths in grid 1, and $\binom{a+b}{a+1}$ lattice paths in grid 2. Adding these combinations together and using Pascals identity gives

$$\binom{a+b}{a} + \binom{a+b}{a+1} = \binom{a+1+b}{a+1},$$

and it follows that there are $\binom{a+1+b}{a+1}$ lattice paths on a grid with $(a+1)+1$ vertical lines and $b+1$ horizontal lines. In a similar fashion, we can show that there are $\binom{a+b+1}{a}$ lattice paths on a grid with $a+1$ vertical lines $(b+1)+1$ horizontal lines. Therefore, the result follows. $\square$

Other Problems

Problem 1

Theorem. *Let $m \in \mathbb{N}$ and let $P(n)$ be a statement that is either true or false for each $n \geq m$. Then $P(n)$ is true for all $n \geq m$, provided that*

i. $P(m)$ is true, and

ii. for each $k \geq m$, if $P(k)$ is true, then $P(k+1)$ is true.

Proof. The following is a proof by contradiction. Suppose that both (i) and (ii) hold true, but there exists a natural number $k \geq m$ such that $P(k)$ is false. Define the set of all counterexamples by

$$X = \{n \in \mathbb{N}: \ P(n) \text{ is false}\}.$$

Since $P(k)$ is false, we know that X is non-empty. Furthermore, by the Well Ordering Principle, it follows that there exists a least element of X, which we denote by l. Note that $l > m$, since $P(m)$ is true due to (i), and therefore, $l - 1 \geq m$. It follows from (ii) that since $P(l-1)$ must be true, so to must $P(l)$ be true. But this contradicts l being an element of X. Therefore, X is in fact empty, and the result follows. $\square$

Problem 2

Theorem. *Let A be a subset of the natural numbers. If*

i. $0 \in A$, and

ii. for all $k \in \mathbb{N}$*, if* $0, 1, \ldots, k \in A$*, then* $k+1 \in A$*.*

then $A = \mathbb{N}$

Proof. The following is a proof by contradiction. Suppose that both (i) and (ii) hold true, but there exists a natural number $k \in \mathbb{N}$ such that $k \notin A$. Define the set of all elements not in A by

$$X = \{n \in \mathbb{N}\colon\ n \notin A\}.$$

Since $k \notin A$, we know that X is non-empty. Furthermore, by the Well Ordering Principle, it follows that there exists a least element of X, which we denote by l. Note that $l > 0$, since $0 \in A$ due to (i). Therefore $l - 1 \geq 0$, and $0, \ldots, l-1 \in A$. It follows from (ii) that $l \in A$ which contradicts l being an element of X. Therefore, X is in fact empty, and the result follows. $\square$