CSC/MAT-220: Discrete Structures

EFY 9

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Alice and Bob. Alice and Bob are flipping fair coins. Alice may stop as soon as she gets a head immediately followed by a tails, and Bob may stop as soon as he gets two heads in a row. Who should expect to make fewer flips?

Please do each of the following.

i. Provide a brief justification for who can expect to make fewer flips.

ii. Provide a formula for the probability that Bob stops on the nth flip.

*iii. Provide an answer and justification to the following question:
Who (if anyone) has a better probability of finishing first?*

Solution.

i. After two flips, they both have the same number of ways in which they could win. However, if they have not won after two flips, then Alice starts to gain an advantage. To see this, note that failing to win after 2 flips means that Alice has a 2 ways in which she could have a Head and Bob has only 1 way in which he could have a head. Therefore, at this point, Alice has twice as many paths to victory as Bob. This pattern continues, and this is the intuition for why Alice can expect to make fewer flips.

ii. Consider the table below which shows the number of ways in which Bob can win after $n \geq 2$ flips:

n	number of ways for Bob to win on nth flip
2	1
3	1
4	2
5	3
6	5

Let $B(n)$ denote the number ways for Bob to win on the nth flip. Note that in order to win on the nth flip, Bob must have had a head on the $(n-1)$ flip and a tails on the $(n-2)$ flip, and never two heads in a row prior. Furthermore, there are $B(n-1)$ ways in which Bob could win on the $(n-1)$ flip and $B(n-2)$ ways in which Bob could win on the $(n-2)$ flip. For each outcome in this set, we are replacing the $(n-2)$ flip with a tails to ensure that Bob does not win until the nth flip. It follows that $B(n) = B(n-1) + B(n-2)$ for $n \geq 4$, where $B(2) = 1$ and $B(3) = 1$. Let $F(n)$ denote the nth Fibonacci number, as defined in Definition 21.12 of our book, then it is clear that $B(n) = F(n-2)$. Therefore, the probability that Bob wins on the nth flip is

$$\frac{F(n-2)}{2^n},$$

where 2^n is the total number of possibilities after n flips.

iii. The game really does not start until either person gets their first head, which they have equal probability of getting. Furthermore, once they get their first head they each have the same probability of following that up with a heads or a tails. Therefore, while it is true that Alice can expect to win in fewer flips, they actually have equal probability of winning.