CSC/MAT-220: Discrete Structures
EFY 15

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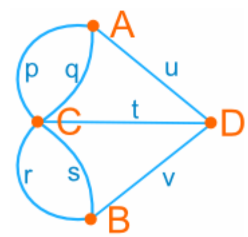
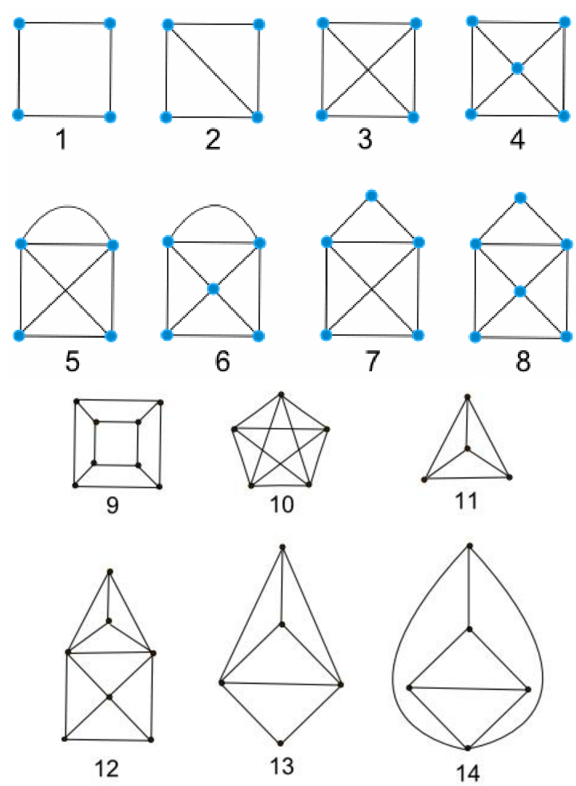
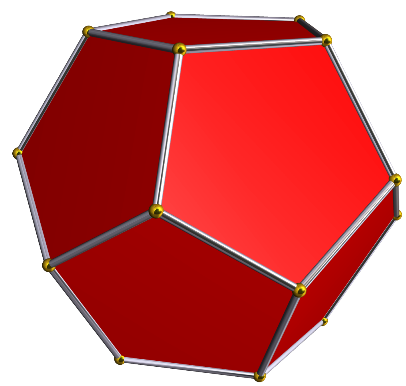
The Seven Bridges of Königsberg. Is there a walk in the following graph that contains each edge exactly once? Is there a circuit (vertices may repeat, edges cannot repeat, starts and ends at same vertex) that contains each edge? Provide explanation.



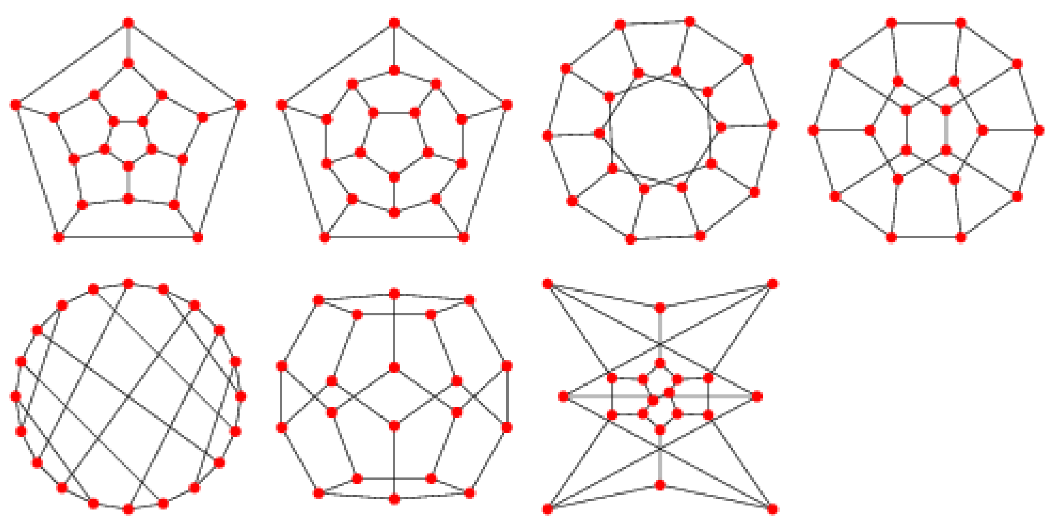
Such a walk is called an *Euler walk* and such a circuit is called an *Euler circuit*. Decide which of the following graphs have an Euler walk or an Euler circuit.



Around the World. In 1859, the great mathematician William Rowe Hamilton found himself short on drinking money. So, he marketed a game which he called "Around the World." It consists of a dodecahedron (polyhedron with twelve flat faces, see figure below) the vertices of which are labeled with names of major cities. The task is to discover a route along the edges that would pass through each city exactly once.

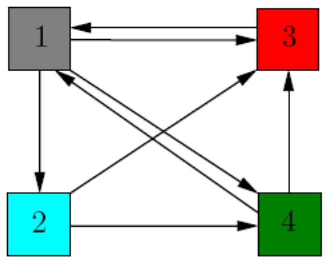


We can turn this into a graph theory problem by projecting the dodecahedron resulting in graphs like the ones below.

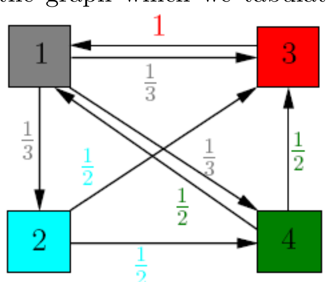


A path in a graph that contains each vertex is a *Hamilton Path.* A cycle that contains each vertex is called a *Hamilton cycle.* Which of the following graphs above have a Hamilton Path, which have a Hamilton Cycle?

The Google Page Rank. Suppose we have a small internet consisting of just 4 web sites, which we will denote by the directed graph below, where the edges denote links.



In our model, each page transfers its importance uniformly to the pages it links to. For instance, vertex 1 has 3 outgoing edges, so it passes $1/3$ of its importance to each of the other three vertices. We represent this by weights on the graph which we tabulate into the transition matrix A, as seen below.



$$\begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Suppose that the importance of each page is initially uniformly distributed among the 4 pages (vertices). Then our initial rank vector would be $v = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$.

We view the importance of a page as the probability that a random surfer of the internet will visit that page. Thus, to start a random surfer has a uniform distribution v. However, as this surfer begins clicking links their next state depends on their previous state, but no prior states and no future states. Therefore, we have a Markov process (thank you Andriy) as described below

$$v,\ Av,\ A^2v,\ A^3v,\ \cdots$$

This process tends toward a steady state described by the vector $p = \begin{bmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{bmatrix}$.

In a Google search that involved these 4 pages, the results would be returned in order: page 1, page 3, page 4, page 2.