MAT – 450: Advanced Linear Algebra

Solution 1

Thomas R. Cameron

1/27/2018

Other Problems

Problem 1. Let V be a vector space over a field F.

Theorem 1. *The zero vector and additive inverse are unique.*

Proof. Suppose there exists two zero vectors 0 and $0'$. Then, by [1, VS 3], for all $x \in V$ we have

$$x + 0 = x + 0'.$$

It follows from [1, VS 1] that $0' = 0' + 0 = 0 + 0' = 0$. Therefore, $0' = 0$ and it follows that the zero vector is unique. Throughout this course we will denote the zero vector by 0.

Similarly, let $x \in V$ and suppose there exists two additive inverses y and y'. Then, by [1, VS 4], we have

$$x + y = 0 \quad \text{and} \quad x + y' = 0.$$

It follows from [1, VS 1 and VS 2] that

$$y = y + 0 = y + (x + y') = (y + x) + y' = 0 + y' = y' + 0 = y'.$$

Therefore, $y' = y$ and it follows that the additive inverse is unique. Throughout this course we will denote the additive inverse by $-x$. □

Theorem 2. *For $0 \in F$, we have $0x = 0$ (zero vector) for all $x \in V$.*

Proof. Let $x \in V$. It follows from [1, VS 8] that $x + 0x = (1 + 0)x = x$. So, by [1, VS 3] and Theorem 1, it follows that $0x = 0$ (zero vector). □

Theorem 3. *For $(-1) \in F$, we have $(-1)x = -x$ (additive inverse) for all $x \in V$.*

Proof. Let $x \in V$. It follows from [1, VS 8] that $x + (-1)x = (1 - 1)x = 0x$. Therefore, by [1, VS 4] and Theorem 2, it follows that $(-1)x = -x$ (additive inverse). □

Problem 2. Let $P_n(\mathbb{R})$ denote the vector space of polynomials of degree n over the field $\mathbb{R}$. Let S denote the set of polynomials that are zero at $t_1, \ldots, t_j \in \mathbb{R}$, where $j \le n$.

Theorem 4. *The set S is a subspace of $P_n(\mathbb{R})$.*

Proof. Let $x, y \in P_n(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then, $x(\lambda) = \hat{x}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j)$ and $y(\lambda) = \hat{y}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j)$, where $\hat{x}, \hat{y} \in P_{(n-j)}(\mathbb{R})$ and λ is a variable. Note that

$$x(\lambda) + y(\lambda) = (\hat{x}(\lambda) + \hat{y}(\lambda))(\lambda - t_1)\cdots(\lambda - t_j) \in S$$

and

$$\alpha x(\lambda) = \alpha\hat{x}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j) \in S.$$

It follows that S is closed under addition of vectors and scalar multiplication, and is therefore a subspace of $P_n(\mathbb{R})$. $\square$

Since each element of S can be represented by $\hat{x}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j)$, where $\hat{x} \in P_{(n-j)}(\mathbb{R})$, it follows that $\dim S = \dim P_{(n-j)}(\mathbb{R}) = (n - j) + 1$. Furthermore, as a corollary of [1, §1.6: Problem 35], we know that $\dim P_n(\mathbb{R})/S = \dim P_n(\mathbb{R}) - \dim S = j$.

Problem 3. Let X be a finite-dimensional vector space, and let U and V be two subspaces of X such that $X = U + V$.

Theorem 5. *Denote by W the intersection of U and V. Then*

$$\dim X = \dim U + \dim V - \dim W.$$

Proof. Let $\{w_1, \ldots, w_k\}$ denote a basis for the subspace W. Then the intersection of U and V forms a k-dimensional subspace of X. Furthermore, we can extend this basis to form a basis $\{w_1, \ldots, w_k, u_{k+1}, \ldots, u_m\}$ for U and a basis $\{w_1, \ldots, w_k, v_{k+1}, \ldots, v_n\}$ for V.

By definition of the sum, $X = \{u + v \colon\ u \in U \ \text{ and } \ v \in V\}$, it follows that the set

$$S = \{w_1, \ldots, w_k, u_{k+1}, \ldots, u_m, w_1, \ldots, w_k, v_{k+1}, \ldots, v_n\}$$

spans X. Furthermore, if we remove the repeated vectors $w_1, \ldots, w_k$ we can form a linearly independent set

$$S' = \{w_1, \ldots, w_k, u_{k+1}, \ldots, u_m, v_{k+1}, \ldots, v_n\}.$$

Thus, S' forms a basis for X and it follows that

$$\begin{aligned}\dim X &= m + n - k \\ &= \dim U + \dim V - \dim W.\end{aligned}$$

$\square$

Problem 4. Any subset of a vector space V which is equal to $\{v\} + U$ for some vector $v \in V$ and some subspace U of V is called an *affine space* associated with the subspace U in V.

Theorem 6. *Let S be a nonempty subset of a vector space V. Then, the following are equivalent:*

(i) S is an affine space in V.

(ii) If $x, y \in S$, then $\alpha x + (1-\alpha)y \in S$ for all $\alpha \in F$.

(iii) For any $n \in \mathbb{N}$, if $v_1, \dots, v_n \in S$ and $\alpha_1, \dots, \alpha_n \in F$ with $\sum_{j=1}^n \alpha_j = 1$, then $\sum_{j=1}^n \alpha_j v_j \in S$.

Proof. Assume that S is an affine space in V. Then $S = \{v\} + U$ for some $v \in V$ and for some subspace U of V. Let $v_1, \dots, v_n \in S$ and $\alpha_1, \dots, a_n \in F$ with $\sum_{j=1}^n \alpha_j = 1$. Then $v_1 = v + u_1, \dots, v_n = v + u_n$ for some $u_1, \dots, u_n \in U$. Hence

$$\sum_{j=1}^n \alpha_j v_j = \sum_{j=1}^n \alpha_j (v + u_j) = v + \sum_{j=1}^n \alpha_j u_j.$$

Since U is a subspace of V, $u := \sum_{j=1}^n \alpha_j u_j \in U$. Therefore, $\sum_{j=1}^n \alpha_j v_j = v + u \in S$, and it follows that $(i) \implies (iii)$.

If we let $n = 2$, then it is clear that $(iii) \implies (ii)$. Now, suppose that (ii) holds, fix $v \in S$ and define $U = \{x - v\colon\ x \in S\}$. Then $S = \{v\} + U$, and to show that (i) holds it is enough to show that U is a subspace of V. To this end, let $u, u' \in U$ and $\alpha \in F$. Then, $u = x - v$ for some $x \in S$. Since $x, v \in S$, we have $\alpha x + (1-\alpha)v \in S$. Thus

$$\alpha u = \alpha(x - v) = \alpha x + (1-\alpha)v - v \in U.$$

Similarly, $u' = x' - v$ for some $x' \in S$. Thus

$$\begin{aligned} u + u' &= x + x' - 2v \\ &= 2\left(\frac{x + x'}{2} - v\right) \in U. \end{aligned}$$

It follows that S is an affine space associated with the subspace U in V. ∎

References

[1] S.H. Friedberg, A.H. Insel, and L.E. Spence. *Linear Algebra.* Pearson Education, Upper Saddle River, NJ, 4th edition, 2003.