MAT-150 **Name:** ______________________

Fall 2017

Midterm

Handout: 10/6, Due: 10/16 **Pledge:** ______________________

Each question topic and point value is recorded in the tables below. You may review these topics from any resource at your leisure. Once you decide to start an exam problem, you are on the clock and you must work without any external resources. Each problem can be done one at a time, but must be finished in a single sitting. Answer each question in the space provided, if you run out of room, then you may continue on the back of the page. It is your responsibility to plan out your time to ensure that you can finish all problems within the 3.5 hours allotted. By writing your name and signing the pledge you are stating that your work adheres to these terms and the Davidson honor code.

Scoring Table

Question	Points	Score
1	12	
2	10	
3	8	
4	10	
5	10	
6	10	
7	16	
8	14	
Total:	90	

Topics Table

Question	Topic
1	Matrix Equations and Solution Sets
2	Linear Independence
3	Linear Transformations
4	Matrix Representation of Linear Transformations
5	The Determinant's Properties
6	Computing the Determinant
7	Subspaces, Basis, and Dimension
8	Change of Basis

Start Time: **End Time:**

1. Let A be an $m \times n$ matrix and consider the matrix equation $Ax = b$.

   (a) (2 points) If the matrix equation has a solution, what can you say about the vector b in terms of the column vectors of A?

   (b) (2 points) If $m < n$, then what can you say about the solution set to the matrix equation? Is the system consistent, are the solutions unique: always, sometimes, never?

   (c) (2 points) If $m > n$, then what can you say about the solution set to the matrix equation? Is the system consistent, are the solutions unique: always, sometimes, never?

   (d) (2 points) If $m = n$, then what is a necessary and sufficient condition for the matrix equation to always be consistent. If consistent, is the solution always unique?

   (e) (4 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & 6 \\ -5 & 6 & 15 \end{bmatrix}$$

   Find a vector b for which $Ax = b$ has a solution, and compute the general solution vector.

Start Time: **End Time:**

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2. Let $S = \{v_1, \ldots, v_p\}$ be a set of vectors in $\mathbb{R}^n$.

   (a) (2 points) State the definition of S being linearly independent and dependent.

   (b) (2 points) If $p > n$ what can you say about the linear independence of the set S?

   (c) (2 points) If $v_i = 0$ for any $i \in \{1, \ldots, p\}$ what can you say about the linear independence of the set S?

   (d) (4 points) Let

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; v_3 = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}.$$

   Determine if the given vectors are linearly independent or dependent.

Start Time: **End Time:**

3. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a transformation.

   (a) (2 points) State the definition of T being a linear transformation.

   (b) (6 points) Prove the following statement.

   The linear transformation T is one-to-one if and only if $T(x) = 0$ has only the trivial solution.

4. For each part below, the action of a linear transformation will be described. Use that description to find the matrix representation of the linear transformation, and answer the given question.

   (a) (5 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by a projection on the x_2-axis followed by a reflection about the x_1-axis. Is this transformation one-to-one, is it onto?

   (b) (5 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by a counter-clockwise rotation about the x_3-axis followed by a reflection about the x_1x_3-plane. Is this transformation invertible?

Start Time: **End Time:**

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Denote by $\{a_1, \ldots, a_n\}$ a basis for $\mathbb{R}^n$, and by $\mathcal{P}$ the parallelepiped determined by these vectors.

   (a) (4 points) State the definition of $\det(T)$ in terms of volume magnification and orientation change.

   (b) (6 points) Provide a sketch and brief justification for each of the following properties of the determinant.

      i. $\det(T) = 0$ if and only if T is not invertible.

      ii. If T is invertible, then $\det(T^{-1}) = \frac{1}{\det(T)}$

      iii. If $U\colon \mathbb{R}^n \to \mathbb{R}^n$ is also a linear transformation, then the composition $TU(x) = T(U(x))$ satisfies $\det(TU) = \det(T)\det(U)$.

Start Time: **End Time:**

6. Let

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 3 & 5 & -2 \\ -4 & 4 & -6 \end{bmatrix}.$$

(a) (6 points) Compute $\det(A)$.

(b) (4 points) Answer each of the following questions regarding the matrix A.

i. Is A invertible?

ii. If the 3×3 matrix B satisfies $\det(B) = 0.5$, then what is the value of $\det(AB)$?

iii. Let $T(x) = Ax$ and $\mathcal{P}$ be a parallelepiped in $\mathbb{R}^3$ with volume 3 units^3 and orientation $+1$. What is the volume and orientation of the image parallelepiped $T(\mathcal{P})$?

Start Time: **End Time:**

7. Let S be a set of vectors in $\mathbb{R}^n$.

   (a) (4 points) State the definition of S being a subspace.

   (b) (4 points) State the definition of a basis and the dimension of the subspace S.

   (c) (8 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

   Compute a basis for the Null Space and Column Space of A and note the dimension of both spaces.

Start Time: **End Time:**

8. Let $\mathbb{P}_3$ denote the set of polynomials of degree 3 or less.

   (a) (4 points) Find the change-of-coordinates matrix from the basis $\beta = \{1, x, 2x^2 - 1, 4x^3 - 3x\}$ to the standard basis $\mathcal{E} = \{1, x, x^2, x^3\}$. Denote this matrix by $P_{\mathcal{E}\leftarrow\beta}$.

   (b) (4 points) Let $[p]_\beta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find $[p]_{\mathcal{E}}$.

   (c) (6 points) Find $(P_{\mathcal{E}\leftarrow\beta})^{-1}$ and use it to show that you can get $[p]_\beta$ from $[p]_{\mathcal{E}}$, as defined in part (b).