MAT − 450
Spring 2018
Review 2
Dates: 3/15 − 3/20

Name: _____

Pledge: _____

Each question topic and the point value is recorded in the tables below. You may review these topics from any resource at your leisure. Once you decide to start a review problem, you are on the clock and you must work without any external resources. Each problem can be done one at a time but must be finished in a single sitting. Answer each question in the space provided, if you run out of space, then you may continue on the back of the page. It is your responsibility to plan out your time to ensure that you can finish all problems within the 4.0 hours allotted. By writing your name and signing the pledge you are stating that your work adheres to these terms and the Davidson honor code.

Scoring Table

Question	Points	Score
1	10	
2	10	
3	10	
4	15	
Total:	45	

Topics Table

Question	Topic
1	Invariant Subspaces
2	Eigenspaces
3	The Determinant
4	Cayley-Hamilton Theorem

Time Table

Question	1	2	3	4
Time				

1. (a) (2 points) Let $T \colon V \to W$ be linear, where V and W are vector spaces over field $\mathbb{F}$. Define T-invariant subspaces of V.

    (b) (2 points) In your own words, state the invariant subspace problem and describe why it is important.

    (c) (6 points) Show by example that there are linear operators with no non-trivial invariant subspace. Then, describe a non-trivial invariant subspace of the differential operator, denoted by T, on the space of all differentiable functions over $\mathbb{R}$, denoted by $C^1(\mathbb{R})$.

2. Let $T\colon V \to V$ be linear with V a finite dimensional vector space over $\mathbb{C}$.

   (a) (4 points) State the definition of an eigenvalue and eigenvector pair of T, and state the definition of an eigenspace of T.

   (b) (6 points) Prove, without using determinants, that T has an eigenvalue.

3. Let $T\colon V \to V$ be linear with V a finite dimensional vector space over $\mathbb{R}$.

    (a) (2 points) Define the determinant of T in terms of its action, volume magnification, and orientation change.

    (b) (4 points) Use the definition in (a) to prove that if T is invertible then $\det(T^{-1}) = 1/\det(T)$.

    (c) (4 points) Describe the properties that the determinant as defined in (1) possesses, which allows us to uniquely characterize the determinant as an alternating multi-linear function of the columns of a matrix. Provide an explicit formula for this function.

4. Let $T\colon V \to V$ be linear with V a finite dimensional vector space over field $\mathbb{F}$.

   (a) (3 points) State the Cayley-Hamilton theorem.

   (b) (6 points) The Cayley-Hamilton theorem relies on two results

      1. For any T-invariant subspace W of V, the characteristic polynomial of T_W divides the characteristic polynomial of T.

      2. Let $x \in V$ be nonzero and define $W = \mathrm{span}(x, Tx, T^2x, \ldots)$. If $\dim(W) = k$ and

$$a_0 x + a_1 T x + \cdots + a_{k-1} T^{k-1} x + a_k T^k x = 0,$$

     then the characteristic polynomial of T_W is $f(t) = a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} + t^k$.

    Use these two results to provide a proof of the Cayley-Hamilton theorem.

(c) (6 points) Let

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Find the characteristic polynomial of A and show that it satisfies the Cayley-Hamilton theorem. Then find a smaller degree polynomial that also satisfies the Cayley-Hamilton theorem. *Hint: try a polynomial with the same roots, but different multiplicities.* Also, you may use Mathematica (or any other software) to help with the arithmetic.