# MAT-150: Linear Algebra

Solution 5

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## Other Problems

**Problem 1.**

a. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix},$$

where the unknown vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ denotes $x_1$ the slope and $x_2$ the y-intercept of the line of best fit. We know there is no solution, since if there was we would have a line that goes through all of the given points, and this is absurd.

b. An orthogonal basis for Col$A$ can be computed using the Gram-Schmidt process as follows

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$
$$u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2/5 \\ -1/5 \end{bmatrix}.$$

Therefore,

$$\mathcal{O} = \begin{bmatrix} 0 & 1 \\ 1 & 2/5 \\ 2 & -1/5 \end{bmatrix}.$$

c. The elements that form the matrix $R$ can be determined from the Gram-Schmidt process in part b. Specifically,

$$R = \begin{bmatrix} 1 & 3/5 \\ 0 & 1 \end{bmatrix}.$$

d. Now, we can solve the matrix equation $Ax = b$ in two parts.

1. Solve $\mathcal{O}y = b$. Since Col$\mathcal{O}$ = Col$A$, there is no solution (see part a.), thus we are seeking a best approximation. To this end, we project $b$ onto the column space of $\mathcal{O}$:

$$\hat{b} = \frac{8}{5}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 1 \\ 2/5 \\ -1/5 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 5/3 \\ 19/6 \end{bmatrix}$$

Therefore, the best approximation to this system is $\hat{y} = \begin{bmatrix} 8/5 \\ 1/6 \end{bmatrix}$.

2. Solve $R\hat{x} = \hat{y}$, which has a unique solution:

$$\hat{x} = \begin{bmatrix} 3/2 \\ 1/6 \end{bmatrix}$$

The error in our approximation can be found by computing the norm of the error in the original projection $\hat{b}$. That is,

$$\|z\| = \left\|b - \hat{b}\right\| = \left\| \begin{bmatrix} -1/6 \\ 1/3 \\ -1/6 \end{bmatrix} \right\| = \frac{1}{\sqrt{6}}$$

**Problem 2.**

a. A good guess is $x = 7$, since it is the average of 9 and 5.

b. We apply a 45 degree clockwise rotation to both sides of the equation to get

$$\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} x = \begin{bmatrix} 7\sqrt{2} \\ -2\sqrt{2} \end{bmatrix}$$

It is clear that the error is $2\sqrt{2}$ and the best approximation solution is 7.