MAT-150: Linear Algebra
EFY 8

Due: October 30, 2017

Problem Statement. Let $\beta = \{q_1, \ldots, q_n\}$ be a basis for R^n such that

$$S_j = \operatorname{span}\{q_1, \ldots, q_j\}$$

is a $j-$dimensional invariant subspace under A, for $j = 1, \ldots, n$. Show that $[A]_\beta$ is an upper triangular matrix.

Solution. Let $Q = [q_1\ q_2\ \cdots\ q_n]$, then $[A]_\beta = Q^{-1}AQ$, which we write as

$$AQ = Q[A]_\beta. \tag{1}$$

We can re-write the left hand side of (1) as

$$[Aq_1\ Aq_2\ \cdots\ Aq_n], \tag{2}$$

and the right hand side of (1) can be written as

$$[q_1\ q_2\ \ldots\ q_n]\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}. \tag{3}$$

Equating each column in (2) with the corresponding column in (3), we find that

$$\begin{aligned} Aq_1 &= c_{11}q_1 + c_{21}q_2 + \cdots + c_{n1}q_n \\ Aq_2 &= c_{12}q_1 + c_{22}q_2 + \cdots + c_{n2}q_n \\ &\vdots \\ Aq_n &= c_{1n}q_1 + c_{2n}q_2 + \cdots + c_{nn}q_n \end{aligned}$$

Furthermore, since S_j is a $j-$dimensional invariant subspace under A, for each $j = 1, \ldots, n$, it follows that

$$\begin{aligned} Aq_1 &= c_{11}q_1 \\ Aq_2 &= c_{12}q_1 + c_{22}q_2 \\ &\vdots \\ Aq_n &= c_{1n}q_1 + c_{2n}q_2 + \cdots + c_{nn}q_n \end{aligned}$$

Therefore,

$$[A]_\beta = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ 0 & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{nn} \end{bmatrix},$$

which is an upper Triangular matrix.