Section 2.3

There are three basic types of applied problems in this section: (i) Express y as a function of t in the form $y = y_0 e^{kt}$ where y_0 and k are *known* numbers.

(ii) Given *three* of the parameters y, y_0, k, and t, find the fourth.

(iii) Given *two* of k, t and the ratio y/y_0 (the last often expressed as a percentage), find the third.

4. The easiest approach is to solve our half–life equation, $T = -(\ln 2)/k$, for k. Another approach, working from scratch, is as follows. If $y = y_0 e^{kt}$ then $y_0/2 = y_0 e^{kT}$ which implies that $1/2 = e^{kT}$. Hence, $\ln(1/2) = -\ln 2 = kT$ or that $k = -(\ln 2)/T$.

5. Since a radioactive substance decays exponentially we have $y = y_0 e^{kt}$ for some $k < 0$. From Exercise 4 we know that

$$k = -(\ln 2)/T = \frac{(-1) \cdot \ln 2}{T} = \frac{\ln 2^{-1}}{T} = \frac{\ln(1/2)}{T}$$

Using properties of exponents and substituting this expression for k gives

$$y = y_0 e^{kt} = y_0 (e^k)^t = y_0 (e^{\ln(1/2)/T})^t = y_0 (e^{\ln(1/2) \cdot (1/T)})^t = y_0 (e^{\ln(1/2)})^{t/T} = y_0 \left(\frac{1}{2}\right)^{t/T}$$

11. Let y_0 denote the number of women at the beginning of the study. Since the exercise gives us the survival rate as the proportion y/y_0 (expressed as a percentage) we write the equation for exponential decay in the form $y/y_0 = e^{kt}$. The 37% 5–year survival rate means that if we set $t = 5$ then $0.37 = y/y_0 = e^{5k}$. Therefore,

$$0.37 = e^{5k} \Rightarrow \ln 0.37 = 5k \Rightarrow k = \frac{\ln 0.37}{5} \approx -0.1989$$

in agreement with the mortality rate given in the statement of the exercise. (Note that we have a minus sign in addition to the numerical value.)

13. Let y_0 denote the initial amount of C–14 in the shrub. We showed in class that the half–life of C–14 is 5600 years. Using the formula in Exercise 4 gives $k = -\frac{\ln 2}{T} = -\frac{\ln 2}{5600} \approx -0.0001238$. Hence, $y = y_0 e^{-0.0001238t}$. Since the exercise asks us about a proportion ("percent of the original carbon–14") we write this equation in the form $y/y_0 = e^{-0.0001238t}$. Setting $t = 43{,}000$ yields

$$y/y_0 = e^{-(0.0001238)\cdot(43{,}000)} = e^{-5.3234} \approx 0.00488$$

So about 0.488% of the original C–14 was present in the charcoal.

20. We use the formula from Exercise 5.

(a) $y = 4 \cdot \left(\frac{1}{2}\right)^{t/13}$. When $t = 100$ this yields $y = 4 \cdot \left(\frac{1}{2}\right)^{100/13} \approx 0.0193$ g.

(b) We need to solve the equation $0.1 = 4 \cdot \left(\frac{1}{2}\right)^{t/13}$ for t. This equation is equivalent to $0.025 = \left(\frac{1}{2}\right)^{t/13}$. Taking the natural logarithm of both sides of this equation give

$$\ln(0.025) = \frac{t}{13}\ln(1/2) \Rightarrow t = \frac{13\ln(0.025)}{\ln(1/2)} \approx 69.19 \text{ years}$$

24. (a) $y = 40e^{-0.004\cdot 180} \approx 19.47$ watts.

(b) Substituting $k = -0.004$ into our formula $T = -(\ln 2)/k$ for the half–life T yields $T \approx 173.29$ days. (You can also solve $\frac{1}{2} = 0.5 = e^{-0.004T}$ for $\ln(0.5) = -0.004T$ or $T \approx 173.29$ days.)

(c) According to this model the power will never be completely gone since no matter what the value of t the value of y will always be positive (but will be incredibly small when t is very large).

25. In this exercise t denotes temperature instead of time and $y_0 = 10$ is the amount that dissolves at temperature $t = 0$. Hence our formula has the form $y = 10e^{kt}$.

(a) We are given that $11 = 10e^{10k}$. But then, $1.1 = e^{10k}$ or $\ln(1.1) = 10k$ from which it follows that $k = [\ln(1.1)]/10 \approx 0.009531$. Hence, $y = 10e^{0.009531t}$.

(b) Solve $15 = 10e^{0.009531t}$ for $t = \ln(1.5)/0.009531 \approx 42.5417$ degrees Celsius.

28. We have $f(t) = 18 - 14.6e^{-0.6t}$ and we need to solve the equation $10 = 18 - 14.6e^{-0.6t}$ for t. This equation is equivalent to

$$\frac{8}{14.6} = e^{-0.6t} \Rightarrow \ln\left(\frac{8}{14.6}\right) = -0.6t \Rightarrow t = -\frac{1}{0.6}\ln\left(\frac{8}{14.6}\right) \approx 1.00263 \text{ hours}$$

That is, it takes about 1 hour.

CSI problem. Assume that we start measuring time t from the moment $t = 0$ at which death occurred. At the moment of death the body temperature was 98.6 degrees. Since the thermostat was set at 68 we have $T_0 = 68$. According to Newton's Law of Cooling the temperature t hours after death will be $T = 68 + Ce^{-kt}$ for some constants C and k. Since at time $t = 0$ we have $98.6 = T = 68 + C$, we find that $C = 30.6$ and thus $T = 68 + 30.6e^{-kt}$. Now, let t denote the specific number of hours since death at 10:30 am. We are told that $80 = 68 + 30.6e^{-kt}$ and that $78.5 = 68 + 30.6e^{-k(t+1)}$. We can solve these two equations simultaneously either "by hand" or by using the "solve" feature of the TI89. In either case we get $k = 0.1335314$ and $t = 7.0103$. So 10:30 am is about 7 hours since death occurred, and thus the moment of death was around 3:30 am.

Here's the "by hand" solution. Since $80 = 68 + 30.6e^{-kt}$ and $78.5 = 68 + 30.6e^{-k(t+1)}$ we have $12 = 30.6e^{-kt}$ and $10.5 = 30.6e^{-k(t+1)}$ and thus

$$\frac{12}{10.5} = \frac{30.6e^{-kt}}{30.6e^{-k(t+1)}} = \frac{e^{-kt}}{e^{-kt-k}} = e^{k} \Rightarrow k = \ln\left(\frac{12}{10.5}\right) \approx 0.1335314$$

Therefore,

$$12 = 30.6e^{-kt} \Rightarrow \frac{12}{30.6} = e^{-kt} \Rightarrow \ln\left(\frac{12}{30.6}\right) = -kt \Rightarrow t = \frac{\ln\left(\frac{12}{30.6}\right)}{-k} = \frac{\ln\left(\frac{12}{30.6}\right)}{-0.1335314} \approx 7.0103$$