MAT – 112: Calculus I and Modeling

Improper Integral Extra Credit

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Due: April 16, 2018

Instructions

Below is a problem that upon accurate completion with full justification is worth up to 10 extra credit points. The points you receive will be distributed uniformly to your three review scores. This work must be completed on your own with absolutely no external resources outside of your class notes and addressing questions to your instructor.

Problem. Find a formula for the following improper integral

$$\int_0^\infty x^n e^{-x} dx,$$

where n is any natural number.
Hint: Try the integral with $n = 0$, $n = 1$, and $n = 2$ and note that no matter what n is, $x^n e^{-x}$ approaches zero as $x \to \infty$.Do you notice a pattern? If so, use that pattern to justify a general formula.
Solution: When $n = 0$, we have

$$\begin{aligned}\int_0^\infty e^{-x} dx &= \lim_{b\to\infty} \int_0^b e^{-x} dx \\ &= \lim_{b\to\infty} -e^{-x}|_0^b \\ &= \lim_{b\to\infty} \left(e^0 - e^{-b}\right) = 1.\end{aligned}$$

When $n = 1$, we have

$$\begin{aligned}\int_0^\infty xe^{-x} dx &= \lim_{b\to\infty} \int_0^b xe^{-x} dx \\ &= \lim_{b\to\infty} \left(-xe^{-x}|_0^b + \int_0^b e^{-x} dx\right) \\ &= \lim_{b\to\infty} \int_0^b e^{-x} dx = 1.\end{aligned}$$

When $n = 2$, we have

$$\begin{aligned}\int_0^\infty x^2 e^{-x} dx &= \lim_{b\to\infty} \int_0^b x^2 e^{-x} dx \\ &= \lim_{b\to\infty} \left(-x^2 e^{-x} + 2 \int_0^b x e^{-x} dx \right) \\ &= 2 \lim_{b\to\infty} \int_0^b x e^{-x} dx = 2(1) = 2.\end{aligned}$$

In general, we see that

$$\int_0^\infty x^n e^{-x} dx = n \int_0^\infty x^{n-1} e^{-x} dx,$$

when $n \geq 1$. It follows that

$$\int_0^\infty x^n e^{-x} dx = n!$$