MAT-150: Linear Algebra
EFY 1

Thomas R. Cameron

August 25, 2017

Perform an analysis for each of the following linear systems and give detailed information on whether they have no solution, infinitely many solutions, or a unique solution.Your analysis should include interpretation given from the viewpoint of row operations, geometry, and vector equations.

Problem 1.

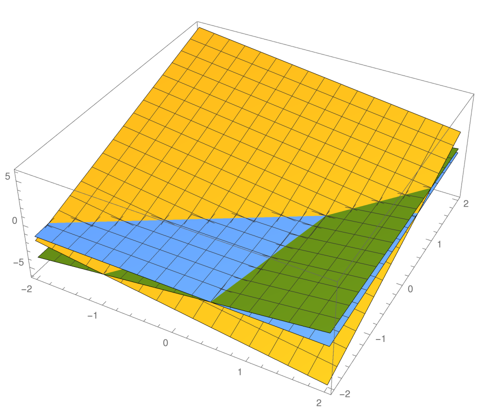
Consider the following augmented matrix and subsequent row operations used to reduce to echelon form.

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} - 5r_1 + r_3 \to r_3 \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$-5r_2 + r_3 \to r_3 \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

At this point, we can see that this system has a unique solution. Therefore, the vector $\begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$ is in the span of the vectors $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -8 \\ -5 \end{bmatrix}$. Furthermore, this means that the planes represented by each equation in the system intersect at exactly one point. This can be seen in the figure below, which was created using the Mathematica command:

Plot3D[{2y-x,(2y-8)/8,(5x-10)/5},{x,-2,2},{y,-2,2}].



Problem 2.

Consider the following augmented matrix and subsequent row operations used to reduce to echelon form.

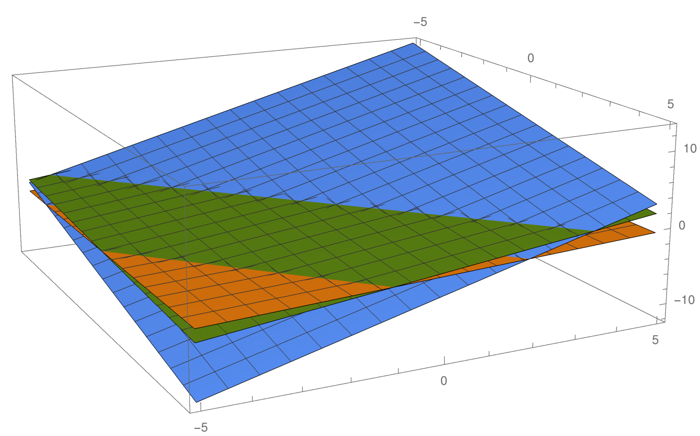
$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix} \text{swap } r_1 \text{ and } r_2 \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}$$

$$-2r_1 + r_2 \to r_2 \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}$$

$$2r_2 + r_3 \to r_3 \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

At this point we can see that this system is inconsistent, since the last column is a pivot column. Therefore, the vector $\begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$ is not in the span of the vectors $\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -3 \\ -8 \end{bmatrix}$, $\begin{bmatrix} -4 \\ 2 \\ 12 \end{bmatrix}$. Furthermore, this means that the planes represented by each equation in the system do not all intersect at a single point. This can be seen in the figure below, which was created using the Mathematica command:

Plot3D[{(y-8)/4,(1-2x+3y)/2,(1-4x+8y)/12}, {x,-5,5}, {y,-5,5}]



Problem 3.

Consider the following augmented matrix and subsequent row operations used to reduce to echelon form.

$$\begin{bmatrix} 1 & 5 & 0 & 21 \\ 0 & 1 & 1 & 4 \\ 1 & 7 & 2 & 29 \end{bmatrix} - 1r_1 + r_3 \to r_3 \begin{bmatrix} 1 & 5 & 0 & 21 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & 8 \end{bmatrix}$$

$$-2r_2 + r_3 \to r_3 \begin{bmatrix} 1 & 5 & 0 & 21 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At this point we can see that this system is consistent, but the solution is not unique, since the third column is a pivot column. Therefore, the vector $\begin{bmatrix} 21 \\ 4 \\ 29 \end{bmatrix}$ is in the span of the first two vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 1 \\ 7 \end{bmatrix}$. Furthermore, it can be seen that the third equation in the system is the sum of the first two equations. Since the system is consistent, it follows that the planes represented by these systems will intersect on a line. This can bee seen in the figure below, which was created using the Mathematica command:

Plot3D[{(21-x)/5, 4-z, (29-x-2z)/7}, {x,-5,5}, {z,-5,5}]