MAT – 112: Calculus I and Modeling

Solution 8

Thomas R. Cameron

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Other Problems

Problem 1. Let $f(x)$ be continuous on the interval $[a, b]$ and let $F(x)$ be any antiderivative of $f(x)$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Recall that the indefinite integral

$$\int f(x)dx = F(x) + C$$

denotes the entire family of antiderivatives of $f(x)$. The Fundamental Theorem of Calculus (part 1) states that you can take any function from that family of antiderivatives and use it to compute the definite integral.

Problem 2. Let $y = f(x)$ and let a and b be any two real numbers such that $f(x)$ is continuous on the interval $[a, b]$. Furthermore, define $y_1 = f(a)$ and $y_2 = f(b)$. Then

$$\begin{aligned}
\int_a^b f^{'}(x)dx &= f(b) - f(a) \text{ (by the F.T.C.)} \\
&= y_2 - y_1 \\
&= \int_{y_1}^{y_2} dy \text{ (by the F.T.C.)}
\end{aligned}$$

It follows that we have further justification for the curious relationship among differentials; that is,

$$dy = f^{'}(x)dx.$$

Problem 3. Let $f(x)$ be a continuous function on an open interval $[a, b]$ and define

$$F(x) = \int_a^x f(t)dt,$$

then $F^{'}(x) = f(x)$. Recall that we call $F(x)$ an accumulator function and it measures the growth or decay in the area bounded by f, the x-axis, and the interval $[a, x]$. The Fundamental Theorem of Calculus (part 2) states that the rate of change of the accumulator function is equal to the integrand evaluated at x, i.e., $f(x)$. Another interpretation lies in noting that

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x).$$

Thus, the derivative and integral are pseudo-inverses.

Problem 4. Let $f(x)$ be a continuous function and let c be any real number. If u and v are differential functions of x, then

$$\begin{aligned}\frac{d}{dx}\left(\int_u^v f(t)dt\right) &= \frac{d}{dx}\left(\int_u^c f(t)dt + \int_c^v f(t)dt\right) \text{ (by property 4 on p. 400)}\\ &= \frac{d}{dx}\left(-\int_c^u f(t)dt + \int_c^v f(t)dt\right) \text{ (by property 5 on p. 400).}\end{aligned}$$

Define $G(x) = \int_c^x f(t)dt$, then $G^{'}(x) = f(x)$ by the F.T.C. Furthermore, by the chain rule we have

$$\frac{d}{dx}\int_c^u f(t)dt = G^{'}(u)\frac{du}{dx}$$

and

$$\frac{d}{dx}\int_c^v f(t)dt = G^{'}(v)\frac{dv}{dx}.$$

Therefore,

$$\frac{d}{dx}\left(\int_u^v f(t)dt\right) = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}.$$