MAT – 450: Advanced Linear Algebra

Solution 2

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2/9/2018

Other Problems

Problem 1. Let V be a vector space of dimension n, and let $T\colon V \to V$ be linear. Suppose that W is a subspace of V with ordered basis $\gamma = \{x_1, \ldots, x_k\}$.

Theorem 1. *If W is T-invariant, then the ordered basis $\beta = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ for V satisfies $[T]_\beta = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$, where $B_{11} = [T_W]_\gamma$.*

Proof. □

Theorem 2. *The ordered basis γ satisfies*

$$span(x_1, \ldots, x_j)$$

being T-invariant for $j = 1, \ldots, k$ if and only if $[T_W]_\gamma$ is a $k \times k$ upper triangular matrix.

Proof. Suppose that γ satisfies $span(x_1, \ldots, x_j)$ being T-invariant for $j = 1, \ldots, k$. Then, it is clear that $W = span(x_1, \ldots, x_k)$ is T-invariant and it follows that T_W is linear. Therefore, $[T_W]_\gamma = [a_{ij}]$ is a $k \times k$ matrix, where

$$T(x_j) = \sum_{i=1}^{k} a_{ij} x_i, \qquad j = 1, \ldots, k.$$

Since $T(x_j) \in span(x_1, \ldots, x_j)$, it follows that $a_{ij} = 0$ for all $i > j$. Therefore, $[T_W]_\gamma$ is upper-triangular.

□

References

[1] S.H. Friedberg, A.H. Insel, and L.E. Spence. *Linear Algebra*. Pearson Education, Upper Saddle River, NJ, 4th edition, 2003.