MAT-150: Linear Algebra
Solution 1

Thomas R. Cameron

September 1, 2017

Other Problems

Problem 1

In this problem we will denote the $m \times n$ matrix A by its entries $a_{i,j}$, where $1 \leq i \leq m$ and $1 \leq j \leq n$. We will denote the vector b in $\mathbb{R}^m$ by its entries b_i. In addition, we denote r_i as the ith row of A. Since we stop the algorithm in the presence of a free variable, we may assume that $n \leq m$.

Algorithm 1 Solving the matrix equation $Ax = b$

```
for j = 1 to n do
    amax ← max{|a_{j,j}|, |a_{j+1,j}|, ..., |a_{m,j}|}
    if amax = 0 then
        Stop Algorithm, Free Variable
    end if
    k ← smallest index ≥ j such that |a_{k,j}| = amax
    Swap r_k and r_j
    Swap b_k and b_j
    for i = j + 1 to m do
        r_i ← r_i − (a_{i,j}/a_{j,j}) r_j
        b_i ← b_i − (a_{i,j}/a_{j,j}) b_j
    end for
    if max{|b_{n+1}|, ..., |b_m|} > 0 then
        Stop Algorithm, Inconsistent System
    end if
    x_k ← 0 for k = n + 1, ..., m
    for i = n to 1 do
        x_i ← b_i
        for j = i + 1 to n do
            x_i ← x_i − a_{i,j} x_i
        end for
        x_i ← x_i / a_{i,i}
    end for
end for
```

Problem 2

Theorem 1. *A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. If the solution set is consistent, then the solution set contains either a unique solution, or infinitely many solutions.*

Proof. Let $x_1, \ldots, x_n$ denote the unknown variables in a system of linear equations. If the system is consistent, then the last column of the augmented matrix cannot be a pivot column, since otherwise we would have a solution to $0x_1 + \cdots + 0x_n = b$, where b is a nonzero number. Furthermore, if the last column of the augmented matrix is a pivot column, then no solution can exist. It follows that the system is consistent if and only if the last column of the augmented matrix is not a pivot column.

Now, if the system is consistent, then either each column of the coefficient matrix is a pivot column or not. It follows that the solution is either unique or there are infinitely many solutions. $\square$

Problem 3

Theorem 2. *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent.*

(a) For each b in $\mathbb{R}^m$, the equation $Ax = b$ has a solution.

(b) Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Proof. We first show that $(a) \to (b) \to (c)$. Since the matrix-vector product Ax is a linear combination of the column vectors of A, it follows that $(a) \to (b)$, i.e. (a) implies (b). Since the span is the set of all linear combinations, it follows that $(b) \to (c)$. These arguments can easily be reversed to show that $(c) \to (b) \to (a)$.

Now, we show that $(a) \to (d)$. If $Ax = b$ is consistent for every b, then there must be a pivot position in every row; otherwise, the last column of the augmented matrix would be a pivot column, which would violate Theorem 1. Again, this argument can easily be reversed to show that $(d) \to (a)$. $\square$

Problem 4

Theorem 3. *Suppose $Ax = b$ is consistent for some b, and let p denote the particular solution. Then the solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogeneous equation.*

Proof. Suppose $w = p + v_h$, then $Aw = A(p + v_h) = Ap + Av_h = b$. Therefore, w is a solution of the matrix equation $Ax = b$.

Now, suppose that w is a solution to the matrix equation $Ax = b$. Define $v_h = w - p$, then it follows that $Av_h = Aw - Ap = 0$. Therefore, v_h is a solution to the homogeneous equation, and $w = p + v_h$ as desired. $\square$