MAT-150: Linear Algebra
EFY 5

September 25, 2017

Part 1: Volume and Orientation of Parallelepipeds

Please do each of the following.

i. State the definition of a parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$.

$$\mathcal{P} = \{c_1 a_1 + \cdots + c_n a_n \colon\ 0 \leq c_i \leq 1,\ i = 1, \ldots, n\}$$

- What is the dimension of $\mathcal{P}$?
  The number of linearly independent vectors in the set $\{a_1, \ldots, a_n\}$.
- What is the ambient space of $\mathcal{P}$?
  $\mathbb{R}^n$

ii. Let $a \in \mathbb{R}$

- Give a geometric description of the parallelepiped $\mathcal{P}$ determined by a.
  The interval on the real line described by $[0, a]$*, if* $a \geq 0$*, or* $[a, 0]$*, if* $a < 0$.
- What is $V(\mathcal{P})$?
  $|a|$
- What is $O(\mathcal{P})$?
  $sgn(a)$

iii. State the definition of a proper parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$.
The parallelepiped is proper if $span(a_1, \ldots, a_i) \subset span(e_1, \ldots, e_i)$ *for all* $i = 1, \ldots, n$.

- Which of the following Parallelepipeds are proper? Why?

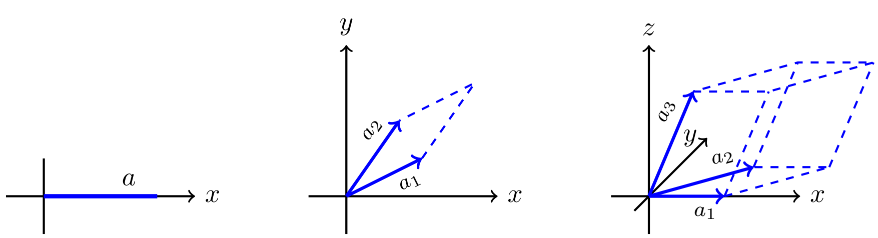

Figure 1: Parallelepipeds in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

Note that every parallelepiped in $\mathbb{R}$ *is proper, and the parallelepiped in* $\mathbb{R}^3$ *is also proper since* a_1 *lies on the x-axis and* a_1 *and* a_2 *lie in the xy-plane. The parallelepiped in* $\mathbb{R}^2$ *is not proper, since* a_1 *does not lie on the x-axis.*

- State the definition of the base of a proper parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$
  The base of a proper parallelepiped is the first (n-1) entries of the vectors $a_1, \ldots, a_{n-1}$.
- State the definition of the height of a proper parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$
  The height of a proper parallelepiped is $|a_n(n)|$.

iv. State the definition of the image of the parallelepiped $\mathcal{P}$ under the linear transformation T.
The image parallelepiped under T is defined by

$$T(\mathcal{P}) = \{c_1 T(a_1) + \cdots + c_n T(a_n) \colon 0 \leq c_i \leq 1,\ i = 1, \ldots, n\}$$

v. Let

$$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- Denote by $\mathcal{P}$ the parallelepiped determined by a_1 and a_2. Explain why $\mathcal{P}$ is not proper.
  The parallelepiped $\mathcal{P}$ is not proper, since a_1 does not lie on the x-axis.
- Find the matrix representation of the linear transformation T such that the image $T(\mathcal{P})$ will be proper.

$$\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

- Compute the vectors $T(a_1)$ and $T(a_2)$ that determine the proper parallelepiped $T(\mathcal{P})$

$$T(a_1) = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \text{ and } T(a_2) = \begin{bmatrix} 3\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

- Compute $V(\mathcal{P})$ and $O(\mathcal{P})$.

$$V(\mathcal{P}) = 1 \text{ and } O(\mathcal{P}) = 1.$$

vi. Let

$$a_1 = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} b \\ d \end{bmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

- Define

$$\cos(\theta) = \frac{a}{\sqrt{a^2 + c^2}} \quad \text{and} \quad \sin(\theta) = \frac{c}{\sqrt{a^2 + c^2}},$$

  and

$$Q = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

- Show that Qa_1 and Qa_2 determine a proper parallelepiped.

$$Qa_1 = \begin{bmatrix} \sqrt{a^2+c^2} \\ 0 \end{bmatrix} \text{ and } Qa_2 = \begin{bmatrix} (ab+cd)/\sqrt{a^2+c^2} \\ (ad-bc)/\sqrt{a^2+c^2} \end{bmatrix}$$

- Compute the volume and orientation of this parallelepiped.

$$V(\mathcal{P}) = |ad-bc| \text{ and } O(\mathcal{P}) = \operatorname{sgn}(ad-bc)$$

Part 2: The Determinant and its Properties

We note that every parallelepiped may be transformed via rotations to a proper one with the same n-volume and orientation. Furthermore, if $\mathcal{P}$ is a proper parallelepiped in $\mathbb{R}^n$, then

$$V(P) = \left|\prod_{i=1}^{n} a_i(i)\right| \quad \text{and } O(P) = \operatorname{sgn}\left(\prod_{i=1}^{n} a_i(i)\right).$$

Also, we may denote the volume and orientation of a parallelepiped in $\mathbb{R}^n$ by $V(a_1,\dots,a_n)$ and $O(a_1,\dots,a_n)$, respectively. Please do each of the following.

i. Show that for any parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$, determined by the vectors $a_1,\dots,a_n$, the following holds for all $\alpha \in \mathbb{R}$

$$V(a_1,\dots,\alpha a_k,\dots,a_n) = |\alpha| V(a_1,\dots,a_k,\dots,a_n),$$
$$O(a_1,\dots,\alpha a_k,\dots,a_n) = \operatorname{sgn}(\alpha) O(a_1,\dots,a_k,\dots,a_n).$$

Hint: You may assume that $\mathcal{P}$ is proper.

Since $\mathcal{P}$ is proper, we can apply the formula above for volume and orientation, at which point the result is clear.

ii. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. State the definition of volume magnification and orientation change under T.

Let $a_1,\dots,a_n\}$ be a basis for $\mathbb{R}^n$ and let $\mathcal{P}$ be the parallelepiped determined by these vectors. Then the volume magnification and orientation change under T is defined as follows.

$$V_T = \frac{V(T(\mathcal{P}))}{V(\mathcal{P})} \text{ and } O_T = \frac{O(T(\mathcal{P}))}{O(\mathcal{P})}$$

iii. Show that $V(\mathcal{P}) = 0$ and $O(\mathcal{P}) = 0$ if and only if the vectors $a_1,\dots,a_n$ are linearly dependent. *Hint: You may assume that $\mathcal{P}$ is proper.*

Proof. We may assume, without loss of generality, that $\mathcal{P}$ is a proper parallelepiped. The vectors $\{a_1,\dots,a_n\}$ are linearly dependent if and only if there exists an $i \in \{1,\dots,n\}$ such that $a_i \in \operatorname{span}\{a_1,\dots,a_{i-1}\}$. If $\mathcal{P}$ is proper, then this is true if and only if $a_i(i) = 0$, and the result follows from our equation of volume and orientation of a proper parallelepiped. □

iv. What conversation did we have in class that shows that our definitions of volume magnification and orientation change is well-defined? Now, state the definition of the determinant of the linear transformation T.

First, we are not embarrassed by dividing by zero since the vectors that determine $\mathcal{P}$ are linearly independent (they form a basis for $\mathbb{R}^n$). Second, it can be shown that both V_T and O_T are independent of the parallelepiped $\mathcal{P}$ and are intrinsic properties of the linear transformation.

v. Use the result from (iii) in your homework: Other Problems - Problem 2.

See homework solutions.

vi. Draw a picture that relates to your homework: Other Problems - Problems 2 - 4.

See class notes.

Part 3: Computing the Determinant

We note that so far we have only discussed the determinant of a linear transformation. In this section, we define the determinant of a matrix and discuss the most efficient method for computing the determinant. Lastly, we motivate the Laplace expansion (cofactor expansion). Please do each of the following.

i. State the definition of the determinant of a matrix.

Let A be an $n \times n$ matrix. Then $T(x) = Ax$ defines a linear transformation, we define the determinant of A as $\det(A) = \det(T)$*, noting that we have already defined the determinant of a linear transformation as* $\det(T) = V_T \cdot O_T$*. Note further that this is equivalent to defining* $\det(A)$ *as the signed volume of the parallelepiped determined by the column vectors of A.*

ii. Give an argument for why the determinant of an upper triangular matrix is equal to the product of its diagonal entries. *Hint: relate it back to a proper parallelepiped.*

The column vectors of an upper triangular matrix determine a proper parallelepiped. Using our formula for the volume and orientation of a proper parallelepiped, we see that $\det(A)$ *is equal to the product of the diagonal entries of A.*

iii. Justify how the following row operations affect the determinant: a constant multiple of one row added to another row, and swapping two rows. Use this to outline an algorithm for computing the determinant of any matrix.

A constant multiple of one row added to another row is a shear transformation. We saw in class that these transformations do not change the volume or orientation of a parallelepiped. Let E denote the matrix representation of a shear transformation, then $\det(E) = 1$*, and it follows that* $\det(EA) = \det(E)\det(A) = \det(A)$*. Therefore, a constant multiple of one row added to another does not change the determinant. Swapping two rows* *is a reflection transformation. We saw in class that a reflection does not change the volume of a parallelepiped, but it does change the sign. Let E denote the matrix representation of a reflection, then* $\det(E) = -1$*, and it follows that* $\det(EA) = -\det(A)$*. Therefore, swapping two rows changes the sign of the determinant.*

iv. Let $a_1, \dots, a_k, \dots, a_n$ and b_k denote column vectors and $\alpha \in \mathbb{R}$. Then justify the following properties.

- $\det(I) = 1$, where I is the $n \times n$ identity matrix.

  The columns of the identity matrix determine the standard unit hypercube, which has volume and orientation equal to 1.

- $$\begin{aligned} \det(a_1, \dots, a_k + b_k, \dots, a_n) &= \det(a_1, \dots, a_k, \dots, a_n) \\ &\quad + \det(a_1, \dots, b_k, \dots, a_n), \\ \det(a_1, \dots, \alpha a_k, \dots, a_n) &= \alpha \det(a_1, \dots, a_k, \dots, a_n). \end{aligned}$$

  The second equation follows from Part 2 (i). For the first equation, note that there exists a product of rotation matrices Q such that the vectors $\{Qa_1, \dots, Q(a_k + b_k), \dots, Qa_n\}$ determine a proper parallelepiped. Therefore,

  $$\begin{aligned} \det(a_1, \dots, a_k + b_k, \dots, a_n) &= \det(Qa_1, \dots, Q(a_k + b_k), \dots, Qa_n) \\ &= \prod_{i=1}^{k-1} Qa_i(i) \cdot (Qa_k(k) + Qb_k(k)) \cdot \prod_{i=k+1}^{n} Qa_i(i). \end{aligned}$$

  The result follows immediately by expanding the above product and recalling that rotations preserve the determinant.

- If $a_i = a_j$ $(i \neq j)$ then $\det(a_1, \dots, a_n) = 0$.

  If $a_i = a_j$ $(i \neq j)$, then the column vectors of A are linearly dependent, and the result follows from Part 2 (iii).

The result from (iv) can be used to show that the determinant is a unique alternating multilinear function of the column vectors of A. There is only one function that satisfies this property and it turns out this function is described algebraically by the Laplace expansion (cofactor expansion), which can be found on p. 167 of your text-book. This is why your book is justified in using cofactor expansion to define the determinant.