

CSC/MAT-220: Discrete Structures

Solution 4

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Book Problems

Problem 17.18

Consider an $m \times n$ chessboard, which has $m + 1$ horizontal lines and $n + 1$ vertical lines. In order to make a rectangle we need to select 2 horizontal lines and 2 vertical lines. There are $\binom{m+1}{2}$ ways to select 2 horizontal lines and $\binom{n+1}{2}$ ways to select 2 vertical lines, and it follows that there are $\binom{m+1}{2}\binom{n+1}{2}$ different rectangles that can be formed from an $m \times n$ chessboard.

Problem 17.25

For all natural numbers $n \geq 1$,

$$1 + 2 + \cdots + (n - 1) + n + (n - 1) + \cdots + 2 + 1 = n^2.$$

Algebraic Proof. Let $n \geq 1$ be a natural number and define

$$\begin{aligned} s(n) &= 1 + 2 + \cdots + (n - 1) \\ &= (n - 1) + (n - 2) + \cdots + 1. \end{aligned}$$

It is clear from the above equation that $2s(n) = n(n - 1)$. Finally, it is also clear that $n + 2s(n) = n + n(n - 1) = n^2$. \square

Geometric Proof. Consider an $n \times n$ chess board, where each square is considered to have area 1 unit². Then, the chess board clearly has area n^2 unit². To prove the identity, consider deconstructing the $n \times n$ chessboard into $2(n - 1) + 1$ pieces as follows:

- Remove the first row of area n unit², leaving a $(n - 1) \times n$ chessboard.
- Remove the first column of area $(n - 1)$ unit², leaving a $(n - 1) \times (n - 1)$ chessboard.
- Repeat until you are left with a 2×2 chessboard with area equal to $2 + 1 + 1$ unit².

Then, the sum of the area of the deconstructed pieces is $n + 2(n - 1) + \cdots + 2(2) + 2(1)$ unit². Since this sum of areas must be equal to the area of the whole, the result follows. \square

Problem 20.7

If the sum of two primes is prime, then one of the primes must be 2.

Proof. The following is a proof by contradiction. Suppose that p and q are prime numbers whose sum is prime, but neither is 2. By definition of a prime number, only even prime number is 2. Therefore, both p and q are odd prime numbers. Since the sum of two odd numbers is even this implies that $p+q$ must be even. But this is absurd, since $p+q$ must be prime, and $p+q > 2$. Therefore, either p or q must be 2. \square

Problem 21.8

For $n \in \mathbb{N}$, let F_n denote the n th Fibonacci number and let $S_n = F_0 + F_1 + \cdots + F_n$ denote the sum of the first n Fibonacci numbers. Then,

$$S_n = F_{n+2} - 1$$

for all $n \geq 0$.

Proof. Let us first establish that this result holds for $n = 0$:

$$S_0 = F_0 = 1 \quad \text{and} \quad F_2 - 1 = 2 - 1 = 1.$$

What follows is a proof by contradiction (lack of counterexample).

Suppose there exists a $k \in \mathbb{N}$ such that $S_k \neq F_{k+2} - 1$ and define the set of all counterexamples by

$$X = \{n \in \mathbb{N} : S_n \neq F_{n+2} - 1\}.$$

Since $k \in X$, we know that X is non-empty, and by the Well Ordering Principle it follows that there exists a least element of X , which we denote by l . Note that $l > 0$, since $S_0 = F_2 - 1$. Therefore, $l - 1 \geq 0$ and satisfies

$$S_{l-1} = F_{l+1} - 1.$$

\square

Adding F_l to both sides of the above equation, and noting that $S_l = S_{l-1} + F_l$ and $F_{l+2} = F_l + F_{l+1}$, gives

$$S_l = F_{l+2} - 1,$$

which contradict $l \in X$. Therefore, X is empty, and the result therefore holds.

Problem 22.13

- a. The given proof shows that for every $n \in \mathbb{N}$, any subset of \mathbb{N} of size n has a least element. But the Well Ordering Principle is a statement about all subsets of \mathbb{N} , regardless of size.
- b. Below is a correct proof by induction of the Well Ordering Principle.

Proof. Let $T \subseteq \mathbb{N}$ and suppose that T has no least element. We will show by induction that the set

$$S = \{n \in \mathbb{N} : n < t, \forall t \in T\}$$

is equal to \mathbb{N} . To this end, note that $0 \in S$, since otherwise $0 \in T$ which would contradict T having no least element. Next, suppose that $k \in S$, for some $k \in \mathbb{N}$, and suppose that $(k+1) \notin S$. Then, there exists a $t_1 \in T$ such that $t_1 \leq (k+1)$. Furthermore, since T has no least element, there exists a $t_2 \in T$ such that $t_2 < t_1$ and therefore satisfies $t_2 \leq k$. This clearly contradicts the fact that $k \in S$, and it follows that $k \in S \implies (k+1) \in S$. Therefore, by the principle of mathematical induction $S = \mathbb{N}$, which implies that $T = \emptyset$. Thus, the only set with no least element is the empty set, which proves the Well Ordering Principle. \square

Problem 22.14

Let A_1, A_2, \dots, A_n be sets, where $n \geq 2$. If for any two sets A_i and A_j either $A_i \subseteq A_j$ or $A_j \subseteq A_i$, then one of these n sets is a subset of all of them.

Proof. The base case when $n = 2$ is clear: if either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$, then either A_1 or A_2 is a subset of both of them. Suppose that the result holds for k subsets, where $k \geq 2$, and consider the $(k+1)$ subsets: A_1, \dots, A_k, A_{k+1} . By the induction hypothesis, there exists an integer p such that $1 \leq p \leq k$ and A_p is a subset of all the sets A_1, \dots, A_k . Furthermore, either $A_p \subseteq A_{k+1}$ or $A_{k+1} \subseteq A_p$. In the former case it follows that A_p is a subset of all $(k+1)$ sets, and in the later case it follows that A_{k+1} is a subset of all $(k+1)$ sets. In either case, it follows that the result holds for $(k+1)$ sets, if it holds for k sets. Therefore, by the principle of mathematical induction the result holds for all $n \geq 2$. \square

Problem 22.15

If the grid has $a + 1$ vertical lines and $b + 1$ horizontal lines, there are $\binom{a+b}{a}$ lattice paths from the lower left to the upper right corner.

Proof. The following is a proof by strong induction on all grids that have a path of length n , where $n \in \mathbb{N}$. The base case is for a path of the length zero. The only grid construction that is possible is a single point, where both $a = 0$ and $b = 0$. In this case, the result clearly holds since $\binom{0}{0} = 1$, which denotes the 1 path of length zero.

Now, consider a path of length $n \in \mathbb{N}$ and suppose the result holds for all $a, b \in \mathbb{N}$ such that $a + b = n$ (i.e. for all grids that have a lattice path from the lower left to upper right corner of length n). To get a path of length $n + 1$, we can either add 1 to a (add a vertical line), or we can add 1 to b (add a horizontal line). Consider the former, then we have a grid with $(a + 1) + 1$ vertical lines and $b + 1$ horizontal lines. To get to the point $(a + 1, b)$, the path must pass through the point (a, b) or $(a + 1, b - 1)$, both of which are the right corner of a grid (grid 1 and grid 2, respectively) with a lattice path of length n . Therefore, by the induction hypothesis, there are $\binom{a+b}{a}$ lattice paths in grid 1, and $\binom{a+b}{a+1}$ lattice paths in grid 2. Adding these combinations together and using Pascals identity gives

$$\binom{a+b}{a} + \binom{a+b}{a+1} = \binom{a+1+b}{a+1},$$

and it follows that there are $\binom{a+1+b}{a+1}$ lattice paths on a grid with $(a + 1) + 1$ vertical lines and $b + 1$ horizontal lines. In a similar fashion, we can show that there are $\binom{a+b+1}{a}$ lattice paths on a grid with $a + 1$ vertical lines $(b + 1) + 1$ horizontal lines. Therefore, the result follows. \square

Other Problems

Problem 1

Theorem. *Let $m \in \mathbb{N}$ and let $P(n)$ be a statement that is either true or false for each $n \geq m$. Then $P(n)$ is true for all $n \geq m$, provided that*

- i. $P(m)$ is true, and*
- ii. for each $k \geq m$, if $P(k)$ is true, then $P(k + 1)$ is true.*

Proof. The following is a proof by contradiction. Suppose that both (i) and (ii) hold true, but there exists a natural number $k \geq m$ such that $P(k)$ is false. Define the set of all counterexamples by

$$X = \{n \in \mathbb{N} : P(n) \text{ is false}\}.$$

Since $P(k)$ is false, we know that X is non-empty. Furthermore, by the Well Ordering Principle, it follows that there exists a least element of X , which we denote by l . Note that $l > m$, since $P(m)$ is true due to (i), and therefore, $l - 1 \geq m$. It follows from (ii) that since $P(l - 1)$ must be true, so to must $P(l)$ be true. But this contradicts l being an element of X . Therefore, X is in fact empty, and the result follows. \square

Problem 2

Theorem. *Let A be a subset of the natural numbers. If*

- i. $0 \in A$, and*

ii. for all $k \in \mathbb{N}$, if $0, 1, \dots, k \in A$, then $k + 1 \in A$.

then $A = \mathbb{N}$

Proof. The following is a proof by contradiction. Suppose that both (i) and (ii) hold true, but there exists a natural number $k \in \mathbb{N}$ such that $k \notin A$. Define the set of all elements not in A by

$$X = \{n \in \mathbb{N}: n \notin A\}.$$

Since $k \notin A$, we know that X is non-empty. Furthermore, by the Well Ordering Principle, it follows that there exists a least element of X , which we denote by l . Note that $l > 0$, since $0 \in A$ due to (i). Therefore $l - 1 \geq 0$, and $0, \dots, l - 1 \in A$. It follows from (ii) that $l \in A$ which contradicts l being an element of X . Therefore, X is in fact empty, and the result follows. \square