CSC/MAT-220: Discrete Structures

Solution 2

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Book Problems

Other Problems

I. For each statement below, we indicate which of the strategies (i.) or (ii.) is more appropriate.

   a. Strategy (ii.)
   b. Strategy (i.)
   c. Strategy (i.)

II.

Proposition. *Let A be a subset of U, then $A \cup (U - A) = U$.*

Proof. Suppose that $x \in A \cup (U - A)$, then $x \in A$ or $x \in (U - A)$. If $x \in A$, then $x \in U$, since A is a subset of U. If $x \in (U - A)$, then $x \in U$ by definition of set-minus. $\square$

III. Let f_n denote the number of ways to tile a board of n squares, using squares and dominoes (two squares joined together).

   i.

Proposition. *For $n \geq 0$, $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.*

Proof. We build this proof around the following question: How many tilings of an $(n + 2)$-board use at least one domino?

By definition, there are f_{n+2} tilings of a $(n + 2)$-board; excluding the "all square" tiling gives $f_{n+2} - 1$ tilings with at least one domino.

Furthermore, there are f_k tilings where the last domino covers cells $k + 1$ and $k + 2$. Indeed, cell 1 through k can be tiled in f_k ways, cells $k + 1$ and $k + 2$ must be covered by squares. Hence the total number of tilings with at least one domino is $f_0 + f_1 + f_2 + \cdots + f_n$.

Therefore, both $f_{n+2} - 1$ and $f_0 + f_1 + f_2 + \cdots + f_n$ denote the number of tilings of an $(n + 2)$-board that use at least one domino, and the result follows. $\square$

   ii.

Proposition. *For* $n \geq 0$, $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$.

Proof. We build this proof around the following question: How many tilings of a $(2n+1)$-board exist?

By definition, there are f_{2n+1} tilings of a $(2n+1)$ board.

Furthermore, since the board has odd length there must be at least one square and the last square occupies an odd-numbered cell. There are f_{2k} tilings where the last square occupies cell $(2k+1)$, and hence the total number of tilings is $f_0 + f_2 + f_4 + \cdots + f_{2n}$.

Therefore, both f_{2n+1} and $f_0 + f_2 + f_4 + \cdots + f_{2n}$ denote the number of tilings of a $(2n+1)$-board, and the result follows. □