MAT-150: Linear Algebra

Solution 3

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September 29, 2017

Other Problems

Problem 1

Let A be an $m \times n$ matrix. Then, the column-space of A is a subspace of $\mathbb{R}^m$ and the null-space of A is a subspace of $\mathbb{R}^n$.

Proof. The column-space of A is denoted by $Col(A)$ and is defined by the span of the column vectors of A, which we denote by $a_1, \ldots, a_n$. Since each column vector is in $\mathbb{R}^m$, it is clear that any linear combination of these vectors is also in $\mathbb{R}^m$. Suppose $x, y \in Col(A)$, then

$$x = c_1 a_1 + \cdots + c_n a_n \quad \text{and} \quad y = \hat{c}_1 a_1 + \cdots + \hat{c}_n a_n,$$

where $c_1, \hat{c}_1, \ldots, c_n, \hat{c}_n$ are scalars. It follows that

$$x + y = (c_1 + \hat{c}_1)a_1 + \cdots + (c_n + \hat{c}_n)a_n,$$

which is clearly and linear combination of the column vectors of A and therefore in $Col(A)$. In a similar fashion we can show that $cx \in Col(A)$ for any scalar c. Therefore, $Col(A)$ is a subspace of $\mathbb{R}^m$.

The null-space of A is denoted by $Nul(A)$ and is defined as the set of all solutions to the homogeneous equation $Ax = 0$. Note that solution vectors x must be in $\mathbb{R}^n$. Suppose $x, y \in Nul(A)$, then

$$A(x + y) = Ax + Ay = 0,$$

since matrix vector multiplication is a linear transformation. In a similar fashion we can show that $cx \in Nul(A)$ for any scalar c. Therefore, $Nul(A)$ is a subspace of $\mathbb{R}^n$. $\square$

Problem 2

Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then T is invertible if and only if $\det(T) \neq 0$.

Proof. T is invertible if and only if T is one-to-one, which is true if and only if the vectors $\{T(a_1), \ldots, T(a_n)\}$ are linearly independent, where the set $\{a_1, \ldots, a_n\}$ forms a basis for $\mathbb{R}^n$. Therefore, T is invertible if and only if the volume of the parallelepiped determined by the vectors $\{T(a_1), \ldots, T(a_n)\}$ is nonzero, and the result follows. $\square$

Problem 3

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If T is invertible, then

$$\det(T^{-1}) = \frac{1}{\det(T)}.$$

Proof. Suppose the linear transformation T maps a parallelepiped $\mathcal{P}_1$ in $\mathbb{R}^n$ of dimension n to a parallelepiped $\mathcal{P}_2$ in $\mathbb{R}^n$ of dimension n. Then $\det(T) = \frac{O(\mathcal{P}_2)}{O(\mathcal{P}_1)} \cdot \frac{V(\mathcal{P}_2)}{V(\mathcal{P}_1)}$, and the inverse transformation maps $\mathcal{P}_2$ back to $\mathcal{P}_1$. Therefore,

$$\det(T^{-1}) = \frac{O(\mathcal{P}_1)}{O(\mathcal{P}_2)} \cdot \frac{V(\mathcal{P}_1)}{V(\mathcal{P}_2)} = \frac{1}{\det(T)}.$$

□

Problem 4

Let $T, U : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations. Then the composition, $TU(x) = T(U(x))$ for all $x \in \mathbb{R}^n$, satisfies

$$\det(TU) = \det(T)\det(U).$$

Proof. Let $\{a_1, \ldots, a_n\}$ be any basis for $\mathbb{R}^n$, and denote by $\mathcal{P}_1$ the parallelepiped of dimension n that it determines. Then $\{U(a_1), \ldots, U(a_n)\}$ is also a basis for $\mathbb{R}^n$ if and only if U is an invertible transformation. If U is not invertible, then TU is not invertible and the result follows from Theorem 7.

If U is invertible, then the vectors $\{U(a_1), \ldots, U(a_n)\}$ determine a parallelepiped of dimension n, which we denote by $\mathcal{P}_2$. By definition, we have

$$\det(TU) = \frac{O(TU(\mathcal{P}_1))}{O(\mathcal{P}_1)} \cdot \frac{V(TU(\mathcal{P}_1))}{V(\mathcal{P}_1)}.$$

Furthermore, it is easy to see that

$$\frac{O(TU(\mathcal{P}_1))}{O(\mathcal{P}_1)} = \frac{O(T(\mathcal{P}_2))}{O(\mathcal{P}_2)} \cdot \frac{O(U(\mathcal{P}_1))}{O(\mathcal{P}_1)},$$
$$\frac{V(TU(\mathcal{P}_1))}{V(\mathcal{P}_1)} = \frac{V(T(\mathcal{P}_2))}{V(\mathcal{P}_2)} \cdot \frac{V(U(\mathcal{P}_1))}{V(\mathcal{P}_1)},$$

and the result follows. □