MAT – 112: Calculus I and Modeling

Solution 9

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Other Problems

Problem 1. Consider the differential equation

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

We are interested in approximating this differential equation with a step size of h. Therefore, the next point (t_1, y_1) satisfies $t_1 = t_0 + h$ and

$$\begin{aligned} y_1 - y_0 &= \int_{y_0}^{y_1} dy \text{ (by the F.T.C.)} \\ &= \int_{t_0}^{t_1} f(t, y)dt \text{ (since } dy = f(t, y)dt). \end{aligned}$$

It follows that we can approximate the next y-value (y_1) by using numerical integration techniques. If we employ the left-hand rule, then we have

$$y_1 \approx y_0 + hf(t_0, y_0).$$

This is Euler's method. If we use the trapezoidal method to approximate the integral, then we have

$$y_1 \approx y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_1, y_1)\right).$$

The problem here is that we have y_1 on both sides of the equation. For this reason, we employ Euler's method to approximate y_1 on the right hand side of the equation. Thus, we have

$$y_1 \approx y_0 + \frac{h}{2}\left(f(t_0, y_0) + f(t_0 + h, y_0 + hf(t_0, y_0))\right).$$

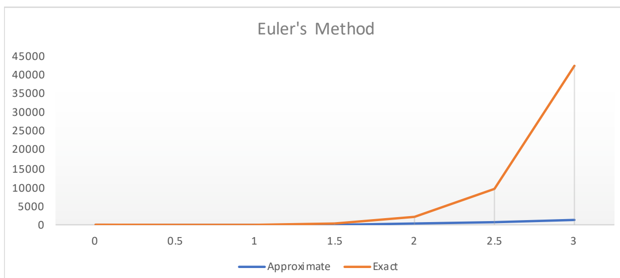
This is what we called the *Trapezoidal method*. In practice, this is referred to as one of the many multi-step methods. Perhaps the most famous multi-step method is the Runge-Kutta method, which can be derived from Simpson's rule.

Problem 2. See the figures below.

Differential Equation	Initial Condition	Step Size
dy/dx=(6y+e^x)/2	y(0)=5	h=1/2

Euler's Method

x	y	f(x,y)	delta y	Exact	Relative Error
0	5	15.5	7.75	5	0
0.5	12.75	39.0743606	19.5371803	23.1166873	0.448450384
1	32.2871803	98.2206819	49.1103409	104.769498	0.691826526
1.5	81.3975213	246.433408	123.216704	471.469517	0.827353586
2	204.614225	617.537204	308.768602	2116.1539	0.903308438
2.5	513.382828	1546.23973	773.119865	9489.17705	0.945898066
3	1286.50269	3869.55085	1934.77542	42536.1692	0.969755088



Differential Equation	Initial Condition	Step Size
dy/dx=(6y+e^x)/2	y(0)=5	h=1/2

Trapezoidal Method

x	y	f1	f2	delta y	Exact	Relative Error
0	5	15.5	39.0743606	13.6435902	5	0
0.5	18.6435902	56.7551311	142.422608	49.7944348	23.1166873	0.193500785
1	68.438025	206.673216	517.564743	181.05949	104.769498	0.346775292
1.5	249.497515	750.733388	1878.28715	657.255136	471.469517	0.47080881
2	906.75265	2723.95248	6812.27792	2384.0576	2116.1539	0.571509119
2.5	3290.81025	9878.522	24700.2565	8644.69463	9489.17705	0.653203831
3	11935.5049	35816.5574	89547.9085	31341.1165	42536.1692	0.71940339