

# MAT-150: Linear Algebra

## Solution 4

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### Other Problems

**Problem 1.** Let  $A$  be an  $n \times n$  matrix. By definition, if  $\lambda$  is an eigenvalue of  $A$  then there exists an eigenvector  $v$  such that  $Av = \lambda v$ . It follows that  $(A - \lambda I)v = 0$  for a nonzero vector  $v$ . Therefore,  $(A - \lambda I)$  is non-invertible, so  $\det(A - \lambda I) = 0$ .

**Problem 2.** Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ . Then, the eigenvalues are  $\lambda = 1, 2, 2$ .

- When  $\lambda = 1$ ,  $\text{Nul}(A - \lambda I) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$
- When  $\lambda = 2$ ,  $\text{Nul}(A - \lambda I) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Since both eigenspaces are only 1-dimensional, it follows that there is not a basis for  $\mathbb{R}^3$  that consists of only eigenvectors of  $A$ . Therefore,  $A$  is not diagonalizable.

**Problem 3.** Let  $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + \lambda^n$  be a monic polynomial of degree  $n$  and define the  $n \times n$  matrix

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}.$$

Let  $\lambda$  be an eigenvalue of  $A$  with associated eigenvector  $v$ . Then,

$$Av = \begin{bmatrix} -a_{n-1}v_1 - a_{n-2}v_2 - \cdots - a_0v_n \\ v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix} \quad \text{and} \quad \lambda v = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \\ \vdots \\ \lambda v_n \end{bmatrix}. \quad (1)$$

It follows that  $v_{n-1} = \lambda v_n, v_{n-2} = \lambda^2 v_n, \dots, v_1 = \lambda^{n-1} v_n$ . So,

$$v = v_n \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}. \quad (2)$$

Furthermore, by equating the first row of  $Av$  and  $\lambda v$  from (1), we have

$$-a_{n-1}\lambda^{n-1}v_n - a_{n-2}\lambda^{n-2}v_n - \dots - a_0v_n = \lambda^n v_n.$$

Since  $v$  is an eigenvector  $v_n \neq 0$ , and we can cancel  $v_n$  from both sides of the above equation. By moving the left hand side over to the right hand side, we see that  $p(\lambda) = 0$ . Therefore  $\lambda$  is a root of the polynomial  $p(\lambda)$ .

Conversely, suppose that  $p(\lambda) = 0$  and define the vector  $v$  as in (2), where  $v_n = 1$ . Then, it follows from (1) that  $Av = \lambda v$ , so  $\lambda$  is an eigenvalue of  $A$  corresponding to eigenvector  $v$ .