MAT-150: Linear Algebra
EFY 3

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September 4, 2017

Problem Statement: Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, and define $T(x) = Ax$.

i. Describe the linear transformation T in terms of its action on a vector x.

ii. Is T one-to-one? Is T onto? Why?

iii. Is $Ax = b$ consistent? Always, sometimes, never? Explain.

Solution: The action of T on a vector x in $\mathbb{R}^2$ can be broken up as follows.

$$T(x)\colon \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} \to \begin{bmatrix} x_2 \\ x_1 \\ x_1 + x_2 \end{bmatrix}$$

Thus, you can see that T first takes the vector x in $\mathbb{R}^2$ and *reflects* it about the line $x_2 = x_1$. Then, a third component, equal to $x_1 + x_2$, is *appended* to the vector.

Since T is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$, it is immediately clear that T *cannot be onto*. Now, consider a vector x in $\mathbb{R}^2$, such that $T(x) = 0$. Then, based on the action of T it is clear that $x = 0$. Now, let b be a vector in $\mathbb{R}^3$ and suppose there are two vectors x_1 and x_2, such that $T(x_1) = b$ and $T(x_2) = b$. Then, it follows that $T(x_1 - x_2) = 0$, and base on our previous observation $x_1 - x_2 = 0 \to x_1 = x_2$. Therefore, T *is one-to-one*.

The domain of T is $\mathbb{R}^2$ and the range of T is the plane in $\mathbb{R}^3$ determined by the equation

$$x_1 + x_2 - x_3 = 0. \tag{1}$$

It follows that the matrix equation $Ax = b$ is *consistent* if and only if the vector b lies in the plane determined by (1).