MAT – 112: Calculus I and Modeling

Solution 1

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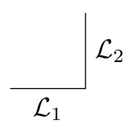
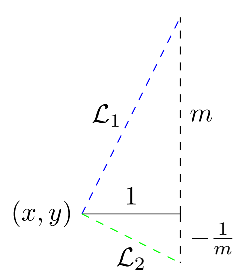
1/26/2018

Other Problems

Problem 1. By definition, two lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are perpendicular if $\mathcal{L}_1$ is horizontal and $\mathcal{L}_2$ is vertical, or the slope of $\mathcal{L}_1$ is $m \neq 0$, and the slope of $\mathcal{L}_2$ is $-\frac{1}{m}$. The horizontal and verticle case is clear, since the intersection of the two lines creates a right angle (side of a square) as seen below.



The case where $\mathcal{L}_1$ has a slope of $m \neq 0$ and $\mathcal{L}_2$ has a slope of $-\frac{1}{m}$ is shown below. Starting at the point (x, y), the line $\mathcal{L}_1$ denotes going over 1 and up m, whereas the line $\mathcal{L}_2$ denotes going over 1 and down $\frac{1}{m}$.



In order to show that the angle of intersection is a 90 degree angle, we can show that the triangle outlined by the blue, green, and black dashed lines is a right triangle. Note that the both smaller triangles must be right triangles, since one of their angles is the intersection of a horizontal and vertical line. Therefore, the length of the blue dashed line is $\sqrt{1+m^2}$ and the length of the green dashed line is $\sqrt{1+1/m^2}$. Furthermore, note that

$$\sqrt{(1+m^2)+(1+1/m^2)} = \sqrt{(m+1/m)^2} = m + 1/m$$

which is the length of the black dashed line. It follows that the triangle is outlined by the blue, green, and black dashed lines is a right triangle.

Problem 2. To show that $h = g \circ f$ is a function from A to C, we must show that h takes each element of A to exactly one element of C. To this end, let α be an element of A. Then, $f(\alpha)$ is a unique element in B, which is the domain of the function g. It follows that g takes $f(\alpha)$ to exactly one element of C. Therefore, $h(\alpha) = g(f(\alpha))$ is a unique element in C.

Problem 3. Using the method of completing the square, we can transfer the quadratic $ax^2 + bx + c$, where $a \neq 0$, into vertex form as follows

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \end{aligned}$$

Once in vertex form, we can identify the vertex, axis of symmetry, x-intercept, and y-intercept.
Vertex: $(-\frac{b}{2a}, \frac{4ac-b^2}{4a})$
Axis of Symmetry: $x = -\frac{b}{2a}$
y-intercept: $(0, c)$
x-intercept: The x-intercepts occur when the quadratic equals zero. Thus, we can solve for x as follows

$$\begin{aligned} a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a} \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\ x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Note that we have derived the quadratic formula.