CSC/MAT-220: Discrete Structures
Infinity and George Cantor

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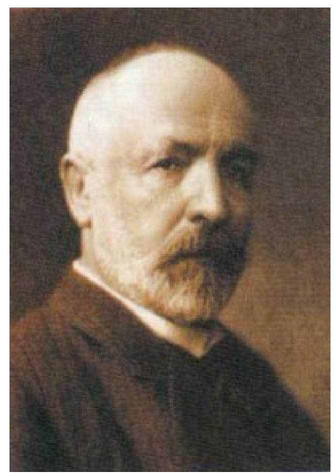
George Cantor was born in St. Petersburg Russia in 1845, where he live until he was 11. Then, his family moved to Germany, where he spent the rest of his life.
Cantor started his math career as a number theorist, and his first major results were in the area of calculus. However, he is most known for his development of Set Theory and a detailed investigation of infinity.
Before Cantor's investigation, infinity was viewed only as a way of describing a process (sum, limit, etc.) that continued forever. Anything deeper was avoided by all mathematicians.
Cantor started a mini revolution within the mathematical community by showing that there are different sizes of infinity, and even more strange: infinities can be added and multiplied!



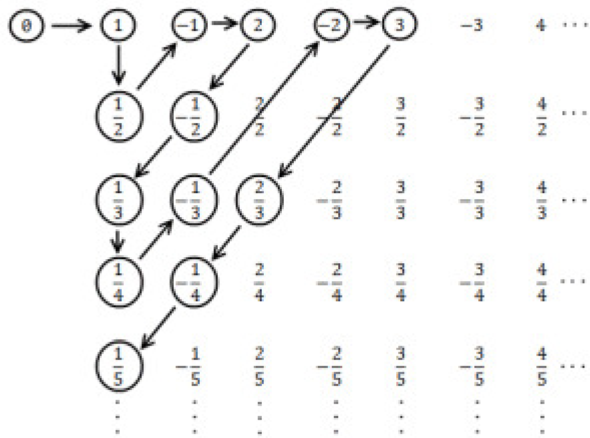
Problem 1. Show that the two sets

$$A = \{1, 2, 3, 4, \ldots\} \quad \text{and} \quad B = \{10, 20, 30, 40, \ldots\}$$

have the same cardinality. Explain why this is surprising.

The rational numbers are countably infinite.

Cantor compared the infinite set of rational numbers with the infinite set of natural numbers by a procedure of listing and enumerating all the rationals...



...and then pairing each rational in this list with the successive natural numbers:

$$
\begin{array}{llllllll}
\mathbb{N}:\{ & 1 & 2 & 3 & 4 & 5 & 6 & \ldots\} \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{Q}:\{ & 0 & 1 & \frac{1}{2} & -1 & 2 & -\frac{1}{2} & \ldots\}
\end{array}
$$

In this way, he showed that the rational numbers are denumerable (or countable), and that the infinity of rational numbers is the same size as the infinity of natural numbers.

Cantor imagined an infinite set of numbers made up of an infinite pattern of just two digits (e.g. 1 and 0, 5 and 4, or m and w in the example below).

He showed how a new number could always be created by making sure that the first digit of the new number is different from the first digit of the first number in the set, the second digit is different from the second digit of the second number, etc, etc.

It is known as the "diagonal argument" because the new number (blue) is different in every place from the diagonal digits (red).

In this way, the new number could never be a duplicate of any number in the infinitely long set, and Cantor proved through this that even an infinite set of numbers cannot contain all possible numbers, and indeed that there are more sets of numbers than there are numbers!

$$
\begin{array}{l}
E_0 = \mathrm{m\ m\ m\ m\ m\ m\ m\ m\ m\ m\ m\ m} \cdots \\
E_1 = \mathrm{w\ w\ w\ w\ w\ w\ w\ w\ w\ w\ w\ w} \cdots \\
E_2 = \mathrm{m\ w\ m\ w\ m\ w\ m\ w\ m\ w\ m\ w} \cdots \\
E_3 = \mathrm{w\ m\ w\ m\ w\ m\ w\ m\ w\ m\ m\ w} \cdots \\
E_4 = \mathrm{w\ m\ m\ w\ w\ m\ m\ w\ m\ w\ m\ w} \cdots \\
E_5 = \mathrm{m\ w\ m\ w\ w\ m\ w\ m\ w\ m\ w\ m} \cdots \\
E_6 = \mathrm{m\ w\ m\ w\ w\ m\ w\ w\ m\ w\ m\ w} \cdots \\
E_7 = \mathrm{w\ m\ m\ w\ m\ w\ m\ w\ m\ w\ m\ w} \cdots \\
E_8 = \mathrm{m\ m\ w\ m\ w\ m\ w\ m\ w\ m\ w\ m} \cdots \\
E_9 = \mathrm{w\ m\ w\ m\ m\ w\ w\ m\ w\ w\ m\ w} \cdots \\
E_{10} = \mathrm{w\ w\ m\ w\ m\ w\ m\ w\ m\ m\ w\ m} \cdots \\
E_{11} = \mathrm{m\ w\ m\ w\ w\ m\ w\ m\ m\ w\ m\ m} \cdots \\
\vdots \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots \\
E_u \neq \mathrm{w\ m\ w\ w\ m\ w\ m\ m\ m\ m\ m\ w} \cdots
\end{array}
$$

Problem 2. Show that the irrational numbers between $(0, 1)$ are uncountably infinite

Paradoxes.

(i.) Let A denote the set of all sets. Use Cantor's theorem to show that A has a proper subset with cardinality larger than the cardinality of A.

(ii.) Image that you have a countably infinite number of balls indexed by $1, 2, 3, \ldots$. Furthermore, you place each ball one at a time, in an infinitely large box, where if the index of the ball is a perfect square then you remove its square root. This process is executed over a 1 minute time interval as follows:

- After 30 seconds the 1st ball is placed in the box (and instantly removed),
- After 45 seconds, balls 1, 2, 3, and 4 are placed in the box, and balls 1 and 2 have been removed.
- After 52.5 seconds, balls $1, 2, \ldots, 9$ are placed in the box, and ball 1, 2, and 3 have been removed.
- This process repeats, where the time to get to the next perfect square is always cut in half.

After 1 minute, how many balls are in the box? Explain what makes this paradoxical.