MAT – 450: Advanced Linear Algebra

Solution 2

Thomas R. Cameron

2/9/2018

Other Problems

Problem 1. Let V be a vector space of dimension n, and let $T\colon V \to V$ be linear. Suppose that W is a subspace of V with ordered basis $\gamma = \{x_1, \ldots, x_k\}$.

Theorem 1. *If W is T-invariant, then the ordered basis*

$$\beta = \{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$$

for V satisfies $[T]_\beta = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$, where $B_{11} = [T_W]_\gamma$.

Proof. Note that $[T]_\beta = [a_{ij}]$ is a $n \times n$ matrix such that

$$T(x_j) = \sum_{i=1}^{n} a_{ij} x_i, \qquad j = 1, \ldots, n.$$

Since W is T-invariant, it follows that $a_{ij} = 0$ for all $i > k$ and $j = 1, \ldots, k$. Thus,

$$[T]_\beta = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

where B_{11} is a $k \times k$ matrix. For $j = 1, \ldots, k$ let

$$T_W(x_j) = \sum_{i=1}^{k} \hat{a}_{ij} x_i.$$

Then, since $T_W(x) = T(x)$ for all $x \in W$, we have

$$\sum_{i=1}^{k} \hat{a}_{ij} x_i = \sum_{i=1}^{k} a_{ij} x_i.$$

Furthermore, since the the x_i's are linearly independent, it follows that $\hat{a}_{ij} = a_{ij}$ for $i, j = 1, \ldots, k$. Therefore, $[T_W]_\gamma = B_{11}$. $\square$

Theorem 2. *If the ordered basis γ satisfies*

$$span(x_1, \ldots, x_j)$$

being T-invariant for $j = 1, \ldots, k$, then $[T_W]_\gamma$ is a $k \times k$ upper triangular matrix.

Proof. Suppose that γ satisfies $span(x_1, \ldots, x_j)$ being T-invariant for $j = 1, \ldots, k$. Then, it is clear that $W = span(x_1, \ldots, x_k)$ is T-invariant and it follows that T_W is linear. Therefore, $[T_W]_\gamma = [a_{ij}]$ is a $k \times k$ matrix, where

$$T(x_j) = \sum_{i=1}^{k} a_{ij} x_i, \qquad j = 1, \ldots, k.$$

Since $T(x_j) \in span(x_1, \ldots, x_j)$, it follows that $a_{ij} = 0$ for all $i > j$. Therefore, $[T_W]_\gamma$ is upper-triangular. □

Problem 2. Let l^2 denote the sequence space of all real or complex value sequences $x = (x_1, x_2, \ldots)$ such that

$$\left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} < \infty.$$

Further define $T\colon\ l^2 \to\ l^2$ by $T(x) = (0, x_1, x_2, \ldots)$ and $U\colon\ l^2 \to\ l^2$ by $U(x) = (x_2, x_3, \ldots)$.

Theorem 3. *T is linear, one-to-one, but not onto.*

Proof. Let $x, y \in l^2$ and $\alpha \in \mathbb{F}$, where $\mathbb{F}$ is the real or complex numbers. Then

$$\begin{aligned} T(x + \alpha y) &= (0, x_1 + \alpha y_1, x_2 + \alpha y_2, \ldots) \\ &= (0, x_1, x_2, \ldots) + \alpha(0, y_1, y_2, \ldots) \\ &= T(x) + \alpha T(y). \end{aligned}$$

It is clear from the above equations that T is linear. Furthermore, if $T(x) = T(y)$, then it is clear that $x = y$. Thus, T is one-to-one. However, T cannot map to any sequences whose first component is not zero, so T is not onto. □

Theorem 4. *U is linear, onto, but not one-to-one.*

Proof. Let $x, y \in l^2$ and $\alpha \in \mathbb{F}$, where $\mathbb{F}$ is the real or complex numbers. Then

$$\begin{aligned} U(x + \alpha y) &= (x_2 + \alpha y_2, x_3 + \alpha y_3, \ldots) \\ &= (x_2, x_3, \ldots) + \alpha(y_2, y_3, \ldots) \\ &= U(x) + \alpha U(y). \end{aligned}$$

It is clear from the above equations that U is linear. Furthermore, for any $y = (y_1, y_2, \ldots)$ define $x = (0, y_1, y_2, \ldots)$. Then, $U(x) = y$ and it follows that U is onto. However, since we could replace the first entry of x with any nonzero element, it follows that U is not one-to-one. □

Theorem 5. *T is isometric, but U is not.*

Proof. To show that T is isometric, note that

$$\begin{aligned} d(T(x), T(y)) &= \left(\sum_{i=1}^{\infty} |T(x)_i - T(y)_i|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=2}^{\infty} |x_{i-1} - y_{i-1}|^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{\frac{1}{2}} \\ &= d(x, y). \end{aligned}$$

Since T preserves distances it is isometric.

Conversely, let $x = (1, 0, \ldots)$ and $y = (2, 0, \ldots)$. Then $d(x, y) = 1$ and $d(U(x), U(y)) = 0$. Therefore, U is not isometric. □

Problem 3. Let $P_n(\mathbb{F})$ denote the set of all polynomials over $\mathbb{F}$ of degree n or less, and let $\mathbb{F}^{n+1}$ denote the set of all $(n+1)$-tuples made up of elements from $\mathbb{F}$.

Theorem 6. *$P_n(\mathbb{F}$ is isomorphic to $\mathbb{F}^{n+1}$ for all $n \in \mathbb{N}$.*

Proof. Define $\phi\colon\ P_n(\mathbb{F}) \to \mathbb{F}^{n+1}$ by

$$\phi(p) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix},$$

where $p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$. It is easy to show that ϕ is linear and bijective and therefore an isomorphism. □

References

[1] S.H. Friedberg, A.H. Insel, and L.E. Spence. *Linear Algebra*. Pearson Education, Upper Saddle River, NJ, 4th edition, 2003.