MAT-150 **Name:** ______________________

Fall 2017

Midterm Solution

Handout: 10/6, Due: 10/16 **Pledge:** ______________________

Each question topic and point value is recorded in the tables below. You may review these topics from any resource at your leisure. Once you decide to start an exam problem, you are on the clock and you must work without any external resources. Each problem can be done one at a time, but must be finished in a single sitting. Answer each question in the space provided, if you run out of room, then you may continue on the back of the page. It is your responsibility to plan out your time to ensure that you can finish all problems within the 3.5 hours allotted. By writing your name and signing the pledge you are stating that your work adheres to these terms and the Davidson honor code.

Scoring Table

Question	Points	Score
1	12	
2	10	
3	8	
4	10	
5	10	
6	10	
7	16	
8	14	
Total:	90	

Topics Table

Question	Topic
1	Matrix Equations and Solution Sets
2	Linear Independence
3	Linear Transformations
4	Matrix Representation of Linear Transformations
5	The Determinant's Properties
6	Computing the Determinant
7	Subspaces, Basis, and Dimension
8	Change of Basis

Start Time: **End Time:**

1. Let A be an $m \times n$ matrix and consider the matrix equation $Ax = b$.

   (a) (2 points) If the matrix equation has a solution, what can you say about the vector b in terms of the column vectors of A?
   Solution: The vector b is a linear combination of the column vectors of A; that is, $b \in \mathrm{Col}(A)$.

   (b) (2 points) If $m < n$, then what can you say about the solution set to the matrix equation? Is the system consistent, are the solutions unique: always, sometimes, never?
   Solution: We know that there is at most m pivots. Since $m < n$, not every column of A has a pivot. Therefore, if the system is consistent, then the solution is not unique.

   (c) (2 points) If $m > n$, then what can you say about the solution set to the matrix equation? Is the system consistent, are the solutions unique: always, sometimes, never?
   Solution: We know that there is at most n pivots. Since $n < m$, not every row of A has a pivot. Therefore, the system is not always consistent.

   (d) (2 points) If $m = n$, then what is a necessary and sufficient condition for the matrix equation to always be consistent. If consistent, is the solution always unique?
   Solution: The matrix equation is always consistent if and only if the number of pivots is equal to m. Since $m = n$, it follows that the solution is always unique.

   (e) (4 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & 6 \\ -5 & 6 & 15 \end{bmatrix}$$

   Find a vector b for which $Ax = b$ has a solution, and compute the general solution vector.
   Solution: Consider the following augmented matrix and row operations

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ -3 & 2 & 6 & b_2 \\ -5 & 6 & 15 & b_3 \end{bmatrix} \xrightarrow[5r_1+r_3\to r_3]{3r_1+r_2\to r_2} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 8 & 15 & b_2+3b_1 \\ 0 & 16 & 30 & b_3+5b_1 \end{bmatrix}$$

$$\xrightarrow{-2r_2+r_3\to r_3} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 8 & 15 & b_2+3b_1 \\ 0 & 0 & 0 & b_3-2b_2-b_1 \end{bmatrix}$$

   At this point, it is clear that the system has a solution if and only if $b_3 - 2b_2 - b_1$. For instance, let $b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$. Then the solution can be formed as follows

$$\begin{aligned} x_3 &= t \quad \text{(free variable)} \\ x_2 &= \frac{1}{2} - \frac{15}{8}t \\ x_3 &= 1 - 3t - 2x_2 = \frac{3}{4}t \end{aligned}$$

Start Time: **End Time:**

2. Let $S = \{v_1, \ldots, v_p\}$ be a set of vectors in $\mathbb{R}^n$.

(a) (2 points) State the definition of S being linearly independent and dependent.
Solution: The set S is linearly independent if the vector equation

$$x_1 v_1 + \cdots + x_p v_p$$

has only the trivial solution $x_1 = \cdots = x_p = 0$. Otherwise, the set S is linearly dependent.

(b) (2 points) If $p > n$ what can you say about the linear independence of the set S?
Solution: If $p > n$, then the matrix $A = [v_1 \cdots v_p]$ has more columns than rows. Therefore, not every column of A has a pivot and it follows that the vectors are linearly dependent.

(c) (2 points) If $v_i = 0$ for any $i \in \{1, \ldots, p\}$ what can you say about the linear independence of the set S?
Solution: Suppose $v_i = 0$. Then,

$$0v_1 + \cdots + 0v_{i-1} + v_i + 0v_{i+1} + \cdots + 0v_n = 0,$$

and it follows that the vectors are linearly dependent.

(d) (4 points) Let

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; v_3 = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}.$$

Determine if the given vectors are linearly independent or dependent.
Solution: Consider the following matrix and row operations

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -5 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow[-2r_1 + r_3 \to r_3]{1r_1 + r_2 \to r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

At this point, it is clear that not every column has a pivot. Therefore, the vectors are linearly dependent.

3. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a transformation.

   (a) (2 points) State the definition of T being a linear transformation.
   Solution: The transformation T is linear provided that the following holds for all $u, v \in \mathbb{R}^n$ and for all $c \in \mathbb{R}$

   - $T(u + v) = T(u) + T(v)$
   - $T(cu) = cT(u)$

   (b) (6 points) Prove the following statement.

   The linear transformation T is one-to-one if and only if $T(x) = 0$ has only the trivial solution.

   Solution: Suppose that T is one-to-one. Then, the equation $T(x) = 0$ has at most one solution, and it follows that the only solution is the trivial solution. Conversely, suppose that $T(x) = 0$ has only the trivial solution and for some $b \in \mathbb{R}^n$, there exists two vectors x and $\hat{x}$ such that $T(x) = b$ and $T(\hat{x}) = b$. Since T is a linear, it follows that $T(x - \hat{x}) = 0$. But, since $T(x) = 0$ has only the trivial solution, this implies that $x = \hat{x}$. Therefore, at most one solution exists to the equation $T(x) = b$, and it follows that T is one-to-one.

Start Time: **End Time:**

4. For each part below, the action of a linear transformation will be described. Use that description to find the matrix representation of the linear transformation, and answer the given question.
   (a) (5 points) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by a projection on the x_2-axis followed by a reflection about the x_1-axis. Is this transformation one-to-one, is it onto?

   (b) (5 points) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by a counter-clockwise rotation about the x_3-axis followed by a reflection about the x_1x_3-plane. Is this transformation invertible?

Start Time: **End Time:**

5. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Denote by $\{a_1, \ldots, a_n\}$ a basis for $\mathbb{R}^n$, and by $\mathcal{P}$ the parallelepiped determined by these vectors.

   (a) (4 points) State the definition of $\det(T)$ in terms of volume magnification and orientation change.

   (b) (6 points) Provide a sketch and brief justification for each of the following properties of the determinant.

      i. $\det(T) = 0$ if and only if T is not invertible.

      ii. If T is invertible, then $\det(T^{-1}) = \frac{1}{\det(T)}$

      iii. If $U\colon \mathbb{R}^n \to \mathbb{R}^n$ is also a linear transformation, then the composition $TU(x) = T(U(x))$ satisfies $\det(TU) = \det(T)\det(U)$.

Start Time: **End Time:**

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6. Let

$$A = \begin{bmatrix} 1 & 3 & -3 \\ 3 & 5 & -2 \\ -4 & 4 & -6 \end{bmatrix}.$$

(a) (6 points) Compute $\det(A)$.

(b) (4 points) Answer each of the following questions regarding the matrix A.

i. Is A invertible?

ii. If the 3×3 matrix B satisfies $\det(B) = 0.5$, then what is the value of $\det(AB)$?

iii. Let $T(x) = Ax$ and $\mathcal{P}$ be a parallelepiped in $\mathbb{R}^3$ with volume 3 units^3 and orientation $+1$. What is the volume and orientation of the image parallelepiped $T(\mathcal{P})$?

Start Time: **End Time:**

7. Let S be a set of vectors in $\mathbb{R}^n$.

   (a) (4 points) State the definition of S being a subspace.

   (b) (4 points) State the definition of a basis and the dimension of the subspace S.

   (c) (8 points) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

   Compute a basis for the Null Space and Column Space of A and note the dimension of both spaces.

Start Time: **End Time:**

8. Let $\mathbb{P}_3$ denote the set of polynomials of degree 3 or less.

   (a) (4 points) Find the change-of-coordinates matrix from the basis $\beta = \{1, x, 2x^2 - 1, 4x^3 - 3x\}$ to the standard basis $\mathcal{E} = \{1, x, x^2, x^3\}$. Denote this matrix by $P_{\mathcal{E} \leftarrow \beta}$.

   (b) (4 points) Let $[p]_\beta = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Find $[p]_{\mathcal{E}}$.

   (c) (6 points) Find $(P_{\mathcal{E} \leftarrow \beta})^{-1}$ and use it to show that you can get $[p]_\beta$ from $[p]_{\mathcal{E}}$, as defined in part (b).