CSC/MAT-220: Discrete Structures

Solution 5

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November 10, 2017

Book Problems

Problem 23.2

a. By solving the associated linear equation we know that the general solution has the form $a_n = c\left(\frac{2}{3}\right)^n$, where $a_0 = c = 4$. Therefore,

$$a_n = 4\left(\frac{2}{3}\right)^n,$$

and $a_9 = 4\left(\frac{2}{3}\right)^9 = 0.104049$.

f. By Proposition 23.1, the general solution has the form $a_n = c_1(-2)^n + c_2$, where

$$\begin{aligned} c_1 + c_2 &= 0 \\ -2c_1 + c_2 &= 4 \end{aligned}$$

Therefore, $c_2 = 4/3$ and $c_1 = -4/3$. It follows that

$$a_n = \frac{4}{3} - \frac{4}{3}(-2)^n,$$

and $a_9 = 4/3 + 4/3(2)^9 = 684$.

i. By Theorem 23.9, the recurrence relation has the general form $a_n = c_1 3^n + c_2 5^n$, where

$$\begin{aligned} c_1 + c_2 &= 1 \\ 3c_1 + 5c_2 &= 4 \end{aligned}$$

Therefore, $c_2 = 1/2$ and $c_1 = 1/2$. It follows that

$$a_n = \frac{1}{2}3^n + \frac{1}{2}5^n,$$

and $a_9 = 1/2(3^9 + 5^9) = 986404$.

Problem 24.19 Let n be a positive integer and define

$$A_n = \{d \in \mathbb{N}\colon\ d|n,\ d < \sqrt{n}\} \text{ and } B_n = \{d \in \mathbb{N}\colon\ d|n,\ d > \sqrt{n}\}.$$

a. Define $f\colon\ A_n \to B_n$ by $(a, b) \in f$ provided that ab=n, where $a \in A_n$ and $b \in B_n$. Note that for $b \in B_n$ there is a unique $a \in A_n$ such that $ab = n$. It follows that f is both onto and one-to-one. Therefore, $|A_n| = |B_n|$.

b. From part a. it follows that the number of divisors of n is equal to

$$\begin{cases} 2|A_n| \text{ if } n \text{ is not a perfect square,} \\ 2|A_n| + 1 \text{ if } n \text{ is a perfect square.} \end{cases}$$

Problem 25.17 Let $f\colon \mathbb{E} \to \mathbb{Z}$ be defined by $f(x) = \frac{1}{2}x$. Since f is a linear function, it is clearly a bijection. It follows that $\mathbb{E}$ and $\mathbb{Z}$ have the same cardinality.

Other Problems

Problem 1

i. Define $f\colon\ S \to T$ by $f(x) = 2x$. Since f is linear it is clearly bijective. Therefore, $|S| = |T|$.

ii. Define $f\colon\ S \to T$ by $((x, y), z) \in f$ provided that $x = \cos(\theta - \pi/2)$, $y = \sin(\theta - \pi/2)$, and $z = \tan(\theta)$, where $\theta \in (-\pi/2, \pi/2)$. It follows that for each θ we have uniquely paired a point $(x, y) \in S$ with a point $z \in \mathbb{R}$. Therefore, $|S| = |T|$.

Problem 2

i. Let $n \in \mathbb{N}$ and denote by $\mathbb{P}_n(\mathbb{Z})$ the set of nth degree polynomials with integer coefficients. When $n = 0$, it is clear that $\mathbb{N} = \mathbb{P}_n(\mathbb{Z})$. When $n = 1$, we simplify split the integers in half, then one half maps to the coefficient a_0 and the other half maps to a_1. In general, we split z into $n + 1$ classes, each class maps to a different coefficient of the polynomial. It follows that $\mathbb{P}_n(\mathbb{Z})$ is countable has the same cardinality as $\mathbb{Z}$ and is therefore countable.

ii. By the fundamental theorem of algebra, every polynomial of degree n has exactly n roots. It follows that, since $\mathbb{P}_n(\mathbb{Z})$ is countable, so to is the set of algebraic numbers.

iii. Since the algebraic numbers are countable and the transcendental numbers are everything else, it follows that the transcendental numbers must be uncountable. Therefore, there are more transcendental numbers than algebraic numbers.