MAT-150: Linear Algebra

Solution 4

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November 3, 2017

Other Problems

Problem 1. Let A be an $n \times n$ matrix. By definition, if λ is an eigenvalue of A if and only if there exists a nonzero vector v such that $Av = \lambda v$. This is true if and only if $(A - \lambda I)v = 0$ for a nonzero vector v, which is true if and only if $(A - \lambda I)$ is non-invertible, which is true if and only if $\det(A - \lambda I) = 0$.

Problem 2. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Then, the eigenvalues are $\lambda = 1, 2, 2$.

- When $\lambda = 1$, $\mathrm{Nul}(A - \lambda I) = \mathrm{span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$
- When $\lambda = 2$, $\mathrm{Nul}(A - \lambda I) = \mathrm{span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \}$

Since both eigenspaces are only 1-dimensional, it follows that there is not a basis for $\mathbb{R}^3$ that consists of only eigenvectors of A. Therefore, A is not diagonalizable.

Problem 3. Let $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + \lambda^n$ be a monic polynomial of degree n and define the $n \times n$ matrix

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & -a_{n-3} & \cdots & -a_0 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{bmatrix}.$$

Let λ be an eigenvalue of A with associated eigenvector v. Then,

$$Av = \begin{bmatrix} -a_{n-1}v_1 - a_{n-2}v_2 - \cdots - a_0 v_n \\ v_1 \\ v_2 \\ \vdots \\ v_{n-1} \end{bmatrix} \quad \text{and} \quad \lambda v = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \\ \vdots \\ \lambda v_n \end{bmatrix}. \qquad (1)$$

It follows that $v_{n-1} = \lambda v_n, v_{n-2} = \lambda^2 v_n, \cdots, v_1 = \lambda^{n-1} v_n$. So,

$$v = v_n \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ \lambda^2 \\ \lambda \\ 1 \end{bmatrix}. \tag{2}$$

Furthermore, by equating the first row of Av and λv from (1), we have

$$-a_{n-1}\lambda^{n-1} v_n - a_{n-2}\lambda^{n-2} v_n - \cdots - a_0 v_n = \lambda^n v_n.$$

Since v is an eigenvector $v_n \neq 0$, and we can cancel v_n from both sides of the above equation. By moving the left hand side over to the right hand side, we see that $p(\lambda) = 0$. Therefore λ is a root of the polynomial $p(\lambda)$.

Conversely, suppose that $p(\lambda) = 0$ and define the vector v as in (2), where $v_n = 1$. Then, it follows from (1) that $Av = \lambda v$, so λ is an eigenvalue of A corresponding to eigenvector v.