Spectral Theorem Preliminaries

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Let V be an inner product space over the field $\mathbb{F}$. We will make use of the following results.

Theorem 1. *Let W be a finite-dimensional subspace of V and let $y \in V$. Then there exists a unique $u \in W$ and $z \in W^{\perp}$ such that $y = u + z$. Furthermore, if $\{v_1, \ldots, v_k\}$ is a basis for W, then*

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

Proof. See [1, Theorem 6.6]. □

This in fact holds for any closed subspace W. Hence, we can write $V = W \oplus W^{\perp}$ for any closed subspace W of V.

Theorem 2. *If V is finite-dimensional and $\mathbb{F} = \mathbb{C}$. Then $T \in \mathcal{L}(V)$ is normal if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.*

Proof. See [1, Theorem 6.16]. □

Theorem 3. *If V is finite-dimensional and $\mathbb{F} = \mathbb{R}$. Then $T \in \mathcal{L}(V)$ is self-adjoint if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.*

Proof. See [1, Theorem 6.17]. □

References

[1] S.H. Friedberg, A.H. Insel, and L.E. Spence. *Linear Algebra.* Pearson Education, Upper Saddle River, NJ, 4th edition, 2003.