MAT-150 **Name:** ____________________

Fall 2017

Final

Handout: 12/6, Due: 12/14 **Pledge:** ____________________

Each question topic and point value is recorded in the tables below. You may review these topics from any resource at your leisure. Once you decide to start an exam problem, you are on the clock and you must work without any external resources. Each problem can be done one at a time, but must be finished in a single sitting. Answer each question in the space provided, if you run out of room, then you may continue on the back of the page. It is your responsibility to plan out your time to ensure that you can finish all problems within the 4.0 hours allotted. By writing your name and signing the pledge you are stating that your work adheres to these terms and the Davidson honor code.

Scoring Table

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	15	
6	15	
7	15	
8	15	
Total:	100	

Topics Table

Question	Topic
1	Eigenvalues and Eigenvectors
2	Invariant Subspaces
3	Orthogonal Decomposition
4	Gram-Schmidt Process
5	Least-Squares Problem
6	Spectral Theorem
7	Quadratic Forms
8	Singular Value Decomposition

Time Table

Question	1	2	3	4	5	6	7	8
Time								

1. Let $A \in \mathbb{R}^{n\times n}$.

   (a) (2 points) State the definition of an eigenvalue-eigenvector pair (λ, v) of the matrix A.

   (b) (2 points) Show that the eigenvalues of A are the roots of $\det(\lambda I - A)$ (characteristic polynomial).

   (c) (2 points) Suppose that A is an upper-triangular matrix. Show that the eigenvalues of A are its main diagonal entries.

   (d) (4 points) Let v be an eigenvector of A. Use v to find the corresponding eigenvalue of A.

2. (a) (2 points) Let $A \in \mathbb{R}^{n\times n}$. State the definition of an invariant subspace under A.

   (b) (8 points) Define $A = \begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Show that $\text{span}\{v\}$ is an invariant subspace under A. Then, use v to deflate the problem of finding the eigenvalues of A to a smaller 2×2 problem. What are the eigenvalues of A?

3. (a) (5 points) Let W be a subspace of $\mathbb{R}^n$. State the Orthogonal Decomposition Theorem.

   (b) (5 points) Let $W = \text{span}\{u_1, u_2\}$, where $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Compute the orthogonal projection for each of the following vectors

$$y_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{and} \quad y_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

   Which is closer to the subspace W?

4. Let $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

   (a) (5 points) Draw a graph that includes x_1 and x_2, along with any other vectors relevant in the construction, via the Gram-Schmidt process, of an orthogonal basis for $\mathbb{R}^2$.

   (b) (5 points) Use the Gram-Schmidt process to compute an orthonormal basis for $\mathbb{R}^2$ from the vectors x_1 and x_2.

5. Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Suppose further that the system $Ax = b$ does not have a solution.
   (a) (5 points) What can you say about b and the columns of A?

   (b) (5 points) Suppose the columns of A are orthogonal. Given an itemized outline of how to find the best approximation to the solution of $Ax = b$, and provide justification of each step.

   (c) (5 points) Suppose the columns of A are not orthogonal. Given an itemized outline of how to find the best approximation to the solution of $Ax = b$, and provide justification of each step.

6. Let $A \in \mathbb{C}^{n\times n}$. By Schur's theorem, there exists an $n \times n$ unitary matrix Q such that $Q^*AQ = T$, where T is an upper-triangular matrix.

   (a) (5 points) Show that if A is Hermitian, then T is diagonal and its main diagonal entries are real.

   (b) (5 points) As a corollary to (1), show that if A is Hermitian, then its eigenvalues are all real and the corresponding eigenvectors are orthogonal.

   (c) (5 points) Let A be a 5×5 matrix with eigenvalues $\{1, 2, 3, 3, 4\}$. Suppose further that the corresponding eigenspaces satisfy

   $$\dim\ E_1 = 1,\ \dim\ E_2 = 1,\ \dim\ E_3 = 1,\ \text{and}\ \dim\ E_4 = 1.$$

   Explain why A cannot be Hermitian.

7. (a) (5 points) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. State the Principal Axes Theorem.

   (b) (10 points) Let
$$A = \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix}.$$
   Find the orthogonal change of variable that results in a quadratic form with no cross-product term. Finally, compute this quadratic form and give its classification.

8. Let $A \in \mathbb{R}^{m \times n}$ have rank $r > 0$.

   (a) (5 points) State the Singular Value Decomposition (SVD) Theorem.

   (b) (5 points) Use the SVD to justify that $A = \hat{U}\hat{\sum}\hat{V}^T$, where $\hat{\sum}$ is an $r \times r$ diagonal matrix whose diagonal entries are all nonzero. What are the sizes of $\hat{U}$ and $\hat{V}$?

   (c) (5 points) The pseudo-inverse of A is defined by

   $$A^\dagger = \hat{V}\hat{\sum}^{-1}\hat{U}^T$$

   Justify why the inverse of $\hat{\sum}$ must exist. Then, show that $A^\dagger A = I_n$ (the $n \times n$ identity) and $AA^\dagger = I_m$ (the $m \times m$ identity).