MAT-150: Linear Algebra
EFY 6

Due: October 20, 2017

Definition of the Eigenvalue and Eigenvector. Let A be an $n \times n$ matrix. Give two equivalent definitions of an eigenvalue and corresponding eigenvector of A.

Solution:

i. The scalar λ is an eigenvalue of A, if there exists a nonzero vector x such that $Ax = \lambda x$.

ii. The scalar λ is an eigenvalue of A, if the matrix $A - \lambda I$ is not invertible. Any nonzero vector $x \in \text{Nul}(A - \lambda I)$ is a corresponding eigenvector.

Upper-Triangular Matrix Let A be an $n \times n$ upper-triangular matrix. Prove that the eigenvalues of A are the diagonal entries of A.

Proof. If A is an upper triangular matrix, then so to is $A - \lambda I$. Therefore, $\det(A - \lambda I)$ is equal to the product of the diagonal entries of $(A - \lambda I)$, which we denote by

$$\det(A - \lambda I) = \prod_{i=1}^{n} (a_{ii} - \lambda),$$

where a_{ii} is the *ith* diagonal entry of A. It follows that $\lambda = a_{ii}$ for any $i = 1, \ldots, n$ will force $\det(A - \lambda I) = 0$, and therefore λ is an eigenvalue of A. $\square$

Linear Independence of Eigenvectors. Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$. Prove that the corresponding eigenvectors $v_1, \ldots, v_p$ are linearly independent.
Hint: See Theorem 2 in Section 5.1, but be sure to put it into your own words.

Proof. Suppose that $v_1, \ldots, v_p$ are linearly dependent, then there exists a smallest index $k > 1$ (since v_1 is nonzero), such that v_k is a linear combination of the vectors $v_1, \ldots, v_{k-1}$. Therefore, there exists scalars $c_1, \ldots, c_{k+1}$ such that

$$c_1 v_1 + \cdots + c_{k-1} v_{k-1} = v_k. \tag{1}$$

Now, multiply both sides of (1) on the left by A and noting that $Av_i = \lambda_i v_i$ results in the following

$$c_1 \lambda_1 v_1 + \cdots + c_{k-1} \lambda_{k-1} v_{k-1} = \lambda_k v_k. \tag{2}$$

By multiplying both sides of (1) by λ_k and subtracting from (2) we arrive at the following

$$c_1(\lambda_1 - \lambda_k) v_1 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1} = 0. \tag{3}$$

Since $v_1, \ldots, v_{k-1}$ are linearly independent (k was smallest index), it follows that the scalars in (3) must all be zero. Since the eigenvalues are distinct, the scalars $c_1, \ldots, c_{k-1}$ must all be zero. But, this implies that $v_k = 0$, which cannot be true since it is an eigenvector. □

Similar Matrices. Let A and B be $n \times n$ matrices that are similar. Prove that A and B have the same characteristic polynomial, and therefore share the same eigenvalues.
Hint: See Theorem 4 in Section 5.2, but be sure to put it into your own words.

Proof. Since A and B are similar, there exists a matrix S such that $A = SBS^{-1}$. Therefore, the characteristic polynomial of A may be written as

$$\begin{aligned}
\det(A - \lambda I) &= \det(SBS^{-1} - \lambda SS^{-1}) \\
&= \det(S(B - \lambda I)S^{-1}) \\
&= \det(S)\det(B - \lambda I)\det(S^{-1}) \\
&= \det(S)\det(S^{-1})\det(B - \lambda I) \\
&= \det(SS^{-1})\det(B - \lambda I) \\
&= \det(I)\det(B - \lambda I) \\
&= \det(B - \lambda I)
\end{aligned}$$

Therefore, the characteristic polynomial of A and B are equal. It follows that the eigenvalues of A and B are the same. □