CSC/MAT-220: Discrete Structures

Solution 2

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Book Problems

Problem 8.12

a. There are 10 choices for each of the 9 digits; so,10^9.

b. There are 10 choices for the first 8 digits, then 4 choices for the last digit; so, $10^8 \cdot 4$.

c. There are 4 choices for each of the 9 digits; so, 4^9.

d. There are 10 choices for the first 5 digits; so, 10^5.

e. There are 9 choices for each of the 9 digits; so, 9^9.

f. This is the complement of none of their digits equal to 8; so, $10^9 - 9^9$.

g. There are 9 choices for the remaining 8 digits, move around the 8 to all 9 locations gives: $9^8 \cdot 9$

Problem 9.7

n	$n!$	Stirling's Formula	Relative Error
10	3.629E+06	3.599E+06	8.296E-03
20	2.433E+18	2.423E+18	4.158E-03
30	2.653E+32	2.645E+32	2.774E-03
40	8.159E+47	8.142E+47	2.081E-03
50	3.041E+64	3.036E+64	1.665E-03

Problem 9.18

Let n be a natural number. Then using integration by parts we have

$$\begin{aligned}\int_0^\infty x^n e^{-x} dx &= -x^n e^{-x}|_0^\infty + n\int_0^\infty x^{n-1}e^{-x}dx \\ &= n\int_0^\infty x^{n-1}e^{-x}dx.\end{aligned}$$

We can repeat this process until the power on the x in the integrand is 0, thereby giving us

$$\begin{aligned}\int_0^\infty x^n e^{-x} dx &= n(n-1)\cdots(1)\int_0^\infty e^{-x}dx \\ &= n!\end{aligned}$$

Next, we approach the more difficult integral $\int_0^\infty x^{1/2}e^{-x}dx$ by making the substitution $x = u^2$, which gives us

$$\begin{aligned}2\int_0^\infty u^2 e^{-u^2}du &= 2\int_0^\infty u(ue^{-u^2})du\\ &= 2\left(-\frac{ue^{-u^2}}{2}|_0^\infty + \frac{1}{2}\int_0^\infty e^{-u^2}du\right)\\ &= \int_0^\infty e^{-u^2}du.\end{aligned}$$

Now, the above integral is somewhat famous, it is computed using the following trick.

$$\begin{aligned}\left(\int_0^\infty e^{-u^2}du\right)^2 &= \int_0^\infty e^{-u^2}du\int_0^\infty e^{-v^2}dv\\ &= \int_0^\infty e^{-(u^2+v^2)}dudv\end{aligned}$$

By making the substitutions $u = r\cos(\theta)$ and $v = r\sin(\theta)$ we have

$$\begin{aligned}\left(\int_0^\infty e^{-u^2}du\right)^2 &= \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}rdrd\theta\\ &= \int_0^{\frac{\pi}{2}}\frac{1}{2}d\theta\\ &= \frac{\pi}{4}.\end{aligned}$$

It follows that $\int_0^\infty x^{1/2}e^{-x}dx = \frac{\sqrt{\pi}}{2}$.

Problem 10.13

Let c and d be integers.

Proposition. *The sets* $C = \{x \in \mathbb{Z} : x|c\}$ *and* $D = \{x \in \mathbb{Z} : x|d\}$ *satisfy* $C \subseteq D$ *if and only if* $c|d$.

Proof. Suppose that $c|d$, then there exists an integer n such that $cn = d$. Further, let $x \in C$, then there exists an integer m such that $xm = c$. Therefore, $x(mn) = d$ and it follows that $x \in D$. This shows that $c|d$ is a sufficient condition for $C \subseteq D$.

Now, suppose that C is not a subset of D. Then, there exists an $x \in C$ such that $x \notin D$. Therefore, there exists an integer m such that $xm = c$, and for all integers n, $xn \neq d$. It follows that $cn \neq md$ for all integers n. This implies that c does not divide md, and therefore cannot divide d. This shows that $c|d$ is a necessary condition for $C \subseteq D$. □

Problem 12.21

We provide proof for the following statements that are true, and a counterexample for those that are false.

a. False. The set difference operator is not commutative. For example, let $A = \{1, 2, 3\},\ B = \{3, 4\},\ C = \{3, 4\}$. Then $A - (B - C) = \{1, 2, 3\}$ and $(A - B) - C = \{1, 2\}$.

b. True.

   Proof. Suppose that $x \in (A - B) - C$, then

$$\begin{aligned} &x \in (A - B) \wedge x \notin C \\ \rightarrow &(x \in A \wedge x \notin B) \wedge x \notin C. \end{aligned}$$

   Using the associativity and commutativity of the conjunction operator, it follows that $x \in (A - C) - B$. Next, suppose that $x \in (A - C) - B$, then

$$\begin{aligned} &x \in (A - C) \wedge x \notin B \\ \rightarrow &(x \in A \wedge x \notin C) \wedge x \notin B. \end{aligned}$$

   Again, using the associativity and commutativity of the conjunction operator, it follows that $x \in (A - B) - C$. □

c. False. Let $A = \{1, 2, 3, 4\},\ B = \{3, 4, 5, 6\},\ C = \{2, 3, 5, 7\}$. Then $(A \cup B) - C = \{1, 4, 6\}$ and $(A - C) \cap (B - C) = \{4\}$.

d. False. If the set C has any element not in B, then this implication does not hold.

e. False. If $A \cap C \neq \emptyset$, then this implication does not hold.

f. False. If B has more elements then A, then $|A| - |B|$ is negative, and therefore this equality cannot hold.

g. False. If B contains a element not in A, then this equality cannot hold.

h. False. If $A \cap B \neq \emptyset$, then this equality cannot hold.

Other Problems

I. We write each definition and its negation using quantifiers and logical symbolism.

   a. A function $f\colon D \to$ is *continuous* at $c \in D \ \leftrightarrow\ \forall \epsilon > 0,\ \exists \delta > 0\ \ni\ (x \in D \wedge |x - c| < \delta) \rightarrow |f(x) - f(c)| < \epsilon$.

      A function $f\colon D \to$ is *not continuous* at $c \in D \leftrightarrow\ \exists \epsilon > 0\ \ni\ \forall \delta > 0,\ \exists x \in D\ \ni\ |x - c| < \delta \wedge |f(x) - f(c)| \geq \epsilon$.

b. A function f is *uniformly continuous* on a set $S \leftrightarrow \forall \epsilon > 0, \exists \delta > 0 \ni (x, y \in S \wedge |x - y| < \delta) \rightarrow |f(x) - f(y)| < \epsilon$.

A function f is *not uniformly continuous* on a set $S \leftrightarrow \exists \epsilon > 0 \ni \forall \delta > 0, \exists x, y \in S \ni |x - y| < \delta \wedge |f(x) - f(y)| \geq \epsilon$.

Proposition. *Let A be a subset of U, then $A \cup (U - A) = U$.*

Proof. Suppose that $x \in A \cup (U - A)$, then $x \in A$ or $x \in (U - A)$. If $x \in A$, then $x \in U$, since A is a subset of U. If $x \in (U - A)$, then $x \in U$ by definition of set-minus. $\square$

II. Let f_n denote the number of ways to tile a board of n squares (n-board), using squares and dominoes (two squares joined together). We define $f_0 = 1$, since there is 1 way to do nothing.

i.

Proposition. *For $n \geq 0$, $f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$.*

Proof. We build this proof around the following question: How many tilings of an $(n + 2)$-board use at least one domino?

By definition, there are f_{n+2} tilings of a $(n + 2)$-board; excluding the "all square" tiling gives $f_{n+2} - 1$ tilings with at least one domino.

Furthermore, there are f_k tilings where the last domino covers cells $k + 1$ and $k + 2$. Indeed, cell 1 through k can be tiled in f_k ways, cells $k + 1$ and $k + 2$ must be covered by squares. Hence the total number of tilings with at least one domino is $f_0 + f_1 + f_2 + \cdots + f_n$.

Therefore, both $f_{n+2} - 1$ and $f_0 + f_1 + f_2 + \cdots + f_n$ denote the number of tilings of an $(n + 2)$-board that use at least one domino, and the result follows. $\square$

ii.

Proposition. *For $n \geq 0$, $f_0 + f_2 + f_4 + \cdots + f_{2n} = f_{2n+1}$.*

Proof. We build this proof around the following question: How many tilings of a $(2n + 1)$-board exist?

By definition, there are f_{2n+1} tilings of a $(2n + 1)$ board.

Furthermore, since the board has odd length there must be at least one square and the last square occupies an odd-numbered cell. There are f_{2k} tilings where the last square occupies cell $(2k + 1)$, and hence the total number of tilings is $f_0 + f_2 + f_4 + \cdots + f_{2n}$.

Therefore, both f_{2n+1} and $f_0 + f_2 + f_4 + \cdots + f_{2n}$ denote the number of tilings of a $(2n + 1)$-board, and the result follows. $\square$

iii.

Proposition. *For $n \geq 1$, $f_1 + f_3 + f_5 + \cdots + f_{2n-1} = f_{2n} - 1$.*

Proof. We build this proof around the following question: How many tilings of a $2n$-board use at least one square?

By definition, there f_{2n} tilings of a $2n$-board; excluding the "all domino" tiling gives $f_{2n} - 1$ tilings with at least one square.

Note that the last square cannot occupy an odd number cell, since the remaining odd number of cells cannot be occupied by only dominoes. Furthermore, there are f_{2k-1} ways to tile the $(2n)$-board when the last square is in cell $2k$, and hence the total number of tilings is $f_1 + f_3 + f_5 + \cdots + f_{2n-1}$

Therefore, both $f_{2n} - 1$ and $f_1 + f_3 + f_5 + \cdots + f_{2n-1}$ denote the number of tilings of a $(2n)$-board that use at least one square, and the result follows. $\square$