MAT-150: Linear Algebra
EFY 9

Due: November 10, 2017

Problem 1.

a. Let W be a subspace of $\mathbb{R}^n$. State the definition of the orthogonal complement of W.

b. Let
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$
Show, by example, that Theorem 3 of Section 6.1 holds for the matrix A.

Solution.

a. The orthogonal complement of W, denoted $W^{\perp}$, is the set of all vectors that are orthogonal to every vector in W.

b. Well your instructor really messed up on this problem. Since the matrix A is invertible, both Null A and Null A^T contain only the zero vector. Furthermore, the zero vector is orthogonal to every vector, so
$$(\text{Row } A)^{\perp} = \text{Nul } A \quad \text{and} \quad (\text{Col } A)^{\perp} = \text{Nul } A^T$$
follows easily.

Problem 2. The vectors $\{u_1, u_2, \ldots, u_n\}$ in $\mathbb{R}^n$ are said to be orthonormal if
$$\langle u_i, u_j \rangle = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} .$$
Let $U = [u_1 \ u_2 \ \ldots \ u_n]$ be a matrix with orthonormal columns. Prove that $U^T U = I$, where I is the $n \times n$ identity. Then justify the following statements.

a. U^T is the inverse of U.

b. The fact that the matrix Q in the QR Algorithm (and Francis's Algorithm) is orthonormal makes the algorithm far more efficient.

Solution.

a. We can write the matrix U^T in row oriented form, and it follows that
$$\begin{aligned} U^T U &= \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \\ &= \begin{bmatrix} u_1^T u_1 & u_1^T u_2 & \cdots & u_1^T u_n \\ u_2^T u_1 & u_2^T u_2 & \cdots & u_2^T u_n \\ \vdots & & \ddots & \vdots \\ u_n^T u_1 & u_n^T u_2 & \cdots & u_n^T u_n \end{bmatrix}. \end{aligned}$$
Since $u_i^T u_j = \langle u_i, u_j \rangle$, the result follows.

b. As noted in Lab 5, the QR algorithm is essentially subspace iteration on $\{e_1, e_2, \ldots, e_n\}$ with a change of coordinate system after each iteration. This shows up in each iteration as follows

- The QR factorization of A is computed
- The matrix A is updated via a similarity transformation

$$A \leftarrow QAQ^T$$

  Essentially the same process is performed in Francis's algorithm (with some shifts and other nuances to improve convergence and efficiency). For us, it is sufficient to note that the similarity transformation, in general, requires the computation of an inverse matrix, which is expensive. However, since Q has orthonormal columns, its inverse is its transpose which is much cheaper to find.