

# MAT – 450: Advanced Linear Algebra

## Solution 1

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### Other Problems

**Problem 1.** Let  $V$  be a vector space over a field  $F$ .

**Theorem 1.** *The zero vector and additive inverse are unique.*

*Proof.* Suppose there exists two zero vectors  $0$  and  $0'$ . Then, by [1, VS 3], for all  $x \in V$  we have

$$x + 0 = x + 0'.$$

It follows from [1, VS 1] that  $0' = 0' + 0 = 0 + 0' = 0$ . Therefore,  $0' = 0$  and it follows that the zero vector is unique. Throughout this course we will denote the zero vector by  $0$ .

Similarly, let  $x \in V$  and suppose there exists two additive inverses  $y$  and  $y'$ . Then, by [1, VS 4], we have

$$x + y = 0 \quad \text{and} \quad x + y' = 0.$$

It follows from [1, VS 1 and VS 2] that

$$y = y + 0 = y + (x + y') = (y + x) + y' = 0 + y' = y' + 0 = y'.$$

Therefore,  $y' = y$  and it follows that the additive inverse is unique. Throughout this course we will denote the additive inverse by  $-x$ .  $\square$

**Theorem 2.** *For  $0 \in F$ , we have  $0x = 0$  (zero vector) for all  $x \in V$ .*

*Proof.* Let  $x \in V$ . It follows from [1, VS 8] that  $x + 0x = (1 + 0)x = x$ . So, by [1, VS 3] and Theorem 1, it follows that  $0x = 0$  (zero vector).  $\square$

**Theorem 3.** *For  $(-1) \in F$ , we have  $(-1)x = -x$  (additive inverse) for all  $x \in V$ .*

*Proof.* Let  $x \in V$ . It follows from [1, VS 8] that  $x + (-1)x = (1 - 1)x = 0x$ . Therefore, by [1, VS 4] and Theorem 2, it follows that  $(-1)x = -x$  (additive inverse).  $\square$

**Problem 2.** Let  $P_n(\mathbb{R})$  denote the vector space of polynomials of degree  $n$  over the field  $\mathbb{R}$ . Let  $S$  denote the set of polynomials that are zero at  $t_1, \dots, t_j \in \mathbb{R}$ , where  $j \leq n$ .

**Theorem 4.** *The set  $S$  is a subspace of  $P_n(\mathbb{R})$ .*

*Proof.* Let  $x, y \in P_n(\mathbb{R})$  and  $\alpha \in \mathbb{R}$ . Then,  $x(\lambda) = \hat{x}(\lambda)(\lambda - t_1) \cdots (\lambda - t_j)$  and  $y(\lambda) = \hat{y}(\lambda)(\lambda - t_1) \cdots (\lambda - t_j)$ , where  $\hat{x}, \hat{y} \in P_{(n-j)}(\mathbb{R})$  and  $\lambda$  is a variable. Note that

$$x(\lambda) + y(\lambda) = (\hat{x}(\lambda) + \hat{y}(\lambda))(\lambda - t_1) \cdots (\lambda - t_j) \in S$$

and

$$\alpha x(\lambda) = \alpha \hat{x}(\lambda)(\lambda - t_1) \cdots (\lambda - t_j) \in S.$$

It follows that  $S$  is closed under addition of vectors and scalar multiplication, and is therefore a subspace of  $P_n(\mathbb{R})$ .  $\square$

Since each element of  $S$  can be represented by  $\hat{x}(\lambda)(\lambda - t_1) \cdots (\lambda - t_j)$ , where  $\hat{x} \in P_{(n-j)}(\mathbb{R})$ , it follows that  $\dim S = \dim P_{(n-j)}(\mathbb{R}) = (n - j) + 1$ . Furthermore, as a corollary of [1, §1.6: Problem 35], we know that  $\dim P_n(\mathbb{R})/S = \dim P_n(\mathbb{R}) - \dim S = j$ .

**Problem 3.** Let  $X$  be a finite-dimensional vector space, and let  $U$  and  $V$  be two subspaces of  $X$  such that  $X = U + V$ .

**Theorem 5.** *Denote by  $W$  the intersection of  $U$  and  $V$ . Then*

$$\dim X = \dim U + \dim V - \dim W.$$

*Proof.* Let  $\{w_1, \dots, w_k\}$  denote a basis for the subspace  $W$ . Then the intersection of  $U$  and  $V$  forms a  $k$ -dimensional subspace of  $X$ . Furthermore, we can extend this basis to form a basis  $\{w_1, \dots, w_k, u_{k+1}, \dots, u_m\}$  for  $U$  and a basis  $\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$  for  $V$ .

By definition of the sum,  $X = \{u + v : u \in U \text{ and } v \in V\}$ , it follows that the set

$$S = \{w_1, \dots, w_k, u_{k+1}, \dots, u_m, w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$$

spans  $X$ . Furthermore, if we remove the repeated vectors  $w_1, \dots, w_k$  we can form a linearly independent set

$$S' = \{w_1, \dots, w_k, u_{k+1}, \dots, u_m, v_{k+1}, \dots, v_n\}.$$

Thus,  $S'$  forms a basis for  $X$  and it follows that

$$\begin{aligned} \dim X &= m + n - k \\ &= \dim U + \dim V - \dim W. \end{aligned}$$

$\square$

**Problem 4.** Any subset of a vector space  $V$  which is equal to  $\{v\} + U$  for some vector  $v \in V$  and some subspace  $U$  of  $V$  is called an *affine space* associated with the subspace  $U$  in  $V$ .

**Theorem 6.** Let  $S$  be a nonempty subset of a vector space  $V$  over a field  $F$  of not characteristic 2. Then, the following are equivalent:

- (i)  $S$  is an affine space in  $V$ .
- (ii) If  $x, y \in S$ , then  $\alpha x + (1 - \alpha)y \in S$  for all  $\alpha \in F$ .
- (iii) For any  $n \in \mathbb{N}$ , if  $v_1, \dots, v_n \in S$  and  $\alpha_1, \dots, \alpha_n \in F$  with  $\sum_{j=1}^n \alpha_j = 1$ , then  $\sum_{j=1}^n \alpha_j v_j \in S$ .

*Proof.* Assume that  $S$  is an affine space in  $V$ . Then  $S = \{v\} + U$  for some  $v \in V$  and for some subspace  $U$  of  $V$ . Let  $v_1, \dots, v_n \in S$  and  $\alpha_1, \dots, \alpha_n \in F$  with  $\sum_{j=1}^n \alpha_j = 1$ . Then  $v_1 = v + u_1, \dots, v_n = v + u_n$  for some  $u_1, \dots, u_n \in U$ . Hence

$$\sum_{j=1}^n \alpha_j v_j = \sum_{j=1}^n \alpha_j (v + u_j) = v + \sum_{j=1}^n \alpha_j u_j.$$

Since  $U$  is a subspace of  $V$ ,  $u := \sum_{j=1}^n \alpha_j u_j \in U$ . Therefore,  $\sum_{j=1}^n \alpha_j v_j = v + u \in S$ , and it follows that (i)  $\implies$  (iii).

If we let  $n = 2$ , then it is clear that (iii)  $\implies$  (ii). Now, suppose that (ii) holds, fix  $v \in S$  and define  $U = \{x - v : x \in S\}$ . Then  $S = \{v\} + U$ , and to show that (i) holds it is enough to show that  $U$  is a subspace of  $V$ . To this end, let  $u, u' \in U$  and  $\alpha \in F$ . Then,  $u = x - v$  for some  $x \in S$ . Since  $x, v \in S$ , we have  $\alpha x + (1 - \alpha)v \in S$ . Thus

$$\alpha u = \alpha(x - v) = \alpha x + (1 - \alpha)v - v \in U.$$

Similarly,  $u' = x' - v$  for some  $x' \in S$ . Thus

$$\begin{aligned} u + u' &= x + x' - 2v \\ &= 2\left(\frac{x + x'}{2} - v\right) \in U. \end{aligned}$$

It follows that  $S$  is an affine space associated with the subspace  $U$  in  $V$ . □

## References

- [1] S.H. Friedberg, A.H. Insel, and L.E. Spence. *Linear Algebra*. Pearson Education, Upper Saddle River, NJ, 4th edition, 2003.