CSC/MAT-220: Discrete Structures
EFY 7

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October 2, 2017

Find the counterfeit. You are given a pan balance and 12 coins. All 12 coins are identical with the exception that one is of a different weight. Find the minimum number of weighings needed to guarantee that you can find the coin that differs.

Provide a clear and concise explanation of your answer.

Solution: We can find the counterfeit coin, and identify if it is heavier or lighter, in 3 weighings. We start by splitting our 12 coins into 3 sets of 4 as follows:

$$S_1 = \{c_1, \ldots, c_4\} \text{ and } S_2 = \{c_5, \ldots, c_8\} \text{ and } S_3 = \{c_9, \ldots, c_{12}\}.$$

Our *first weighing* consists of weighing S_1 against S_2. We then split the problem into two cases, $|S_1| = |S_2|$ and $|S_1| \neq |S_2|$, where the absolute value denotes the weight of the set of coins.

If $|S_1| = |S_2|$, then the counterfeit is in S_3. The *second weighing* consists of weighing $\{c_9, c_{10}, c_{11}\}$ against $\{c_1, c_2, c_3\}$. If $|\{c_1, c_2, c_3\}| = |\{c_9, c_{10}, c_{11}\}|$, then the counterfeit coin is c_{12}. By weighing c_{12} against c_1 it can be determined if it is heavier or lighter. If $|\{c_1, c_2, c_3\}| < |\{c_9, c_{10}, c_{11}\}|$, then the counterfeit coin is either c_9, c_{10}, or c_{11}, and it is heavier. By weighing c_9 against c_{10} we can easily determine which is the counterfeit. A similar conclusion holds if $|\{c_1, c_2, c_3\}| > |\{c_9, c_{10}, c_{11}\}|$, but the counterfeit coin is lighter. Note that the weighing of c_{12} against c_1, or c_9 against c_{10} constitutes as our *third weighing*.

If $|S_1| \neq |S_2|$, then the counterfeit is in either S_1 or S_2. Without loss of generality suppose that $|S_1| < |S_2|$. Then, our *second weighing* consists of weighing $\{c_1, c_5, c_6\}$ against $\{c_2, c_7, c_8\}$. If $|\{c_1, c_5, c_6\}| = |\{c_2, c_7, c_8\}|$, then the counterfeit is either c_3 or c_4, and we know the counterfeit is lighter. By weighing c_3 against c_4 we can determine which is the counterfeit by noting the lighter of the two. If $|\{c_1, c_5, c_6\}| < |\{c_2, c_7, c_8\}|$, then the counterfeit is either c_7 or c_8 (heavier), or the counterfeit is c_1 (lighter). This can easily be determined by weighing c_7 against c_8. If $|\{c_1, c_5, c_6\}| > |\{c_2, c_7, c_8\}|$, then the counterfeit is either c_5 or c_6 (heavier), or the counterfeit is c_2 (lighter). Again, this can be determined by weighing c_5 against c_6. Note that the weighing of c_3 against c_4, c_7 against c_8, or c_5 against c_6 constitutes our *third weighing*.

Find John and the liar. There are two twins, one of whom name is John and the other name I don't remember. What I do remember is that one of them always lies and the other always tells the truth. Suppose you meet the two brothers on the street one day. Devise a three word question, answerable by yes or no, to determine which one is John. Next, devise a three word question, answerable by yes or no, to determine whether John is the liar or the one who tells the truth.

Use truth tables in your explanation.

Solution: For the first question, we want to determine which one is John. This can be done by using the question q_1: "Does John Lie?". For the second question, we want to determine if John is the liar or the one who tells the truth. This can be done by using the question q_2: "Are you John?". Consider the following truth table

	q_1	q_2
John (liar)	0	0
John (truth)	0	1
Not-John (liar)	1	1
Not-John (truth)	1	0

where 0 denotes the person answering no and 1 denotes the person answering yes. Note that whether or not John lies we are able to determine which one is John with question q_1, since John will answer no in both cases. For the second question, both John and Not-John will answer the same in both scenarios, and we can use their answer to determine whether John is the truth teller or the liar.

Find the problem. The Barber of Seville lived in Seville and shaved all of those and only those inhabitants of Seville who did not shave themselves. Did the Barber of Seville shave himself?

Provide a short logical description of your answer.

Solution: No such barber can exist, since if he shaves himself, then he must not, and if he does not shave himself, then he must.