Section 5.2

2. Relative maximum value of 1 at $x = 4$

3. Relative maximum value of 3 at $x = -2$

6. Relative minimum value of -6 at $x = 1$ and a relative maximum value of 2 at $x = 5$.

8. Relative minimum value of 0 at $x = \pm 3$ and a relative maximum value of 4 at $x = 0$.

15. Since $f'(x) = 3x^2 + 12x + 9 = 3(x+1)(x+3)$, f has critical numbers $x = -1, -3$. These yield the intervals

$(-\infty, -3)$: $f'(x) > 0$ so f is increasing on this interval.

$(-3, -1)$: $f'(x) < 0$ so f is decreasing on this interval.

$(-1, +\infty)$: $f'(x) > 0$ so f is increasing on this interval.

Hence $f(-3) = -8$ is a relative maximum value and $f(-1) = -12$ is a relative minimum value.

20. Since $f'(x) = 4x^3 - 16x = 4x(x-2)(x+2)$, f has critical numbers $x = -2, 0, 2$. These yield the intervals

$(-\infty, -2)$: $f'(x) < 0$ so f is decreasing on this interval.

$(-2, 0)$: $f'(x) > 0$ so f is increasing on this interval.

$(0, 2)$: $f'(x) < 0$ so f is decreasing on this interval.

$(2, +\infty)$: $f'(x) > 0$ so f is increasing on this interval.

Hence $f(-2) = f(2) = -7$ is a relative minimum value and $f(0) = 9$ is a relative maximum value.

23. Since $f'(x) = 2 + 2x^{-1/3} = \dfrac{2\sqrt[3]{x} + 2}{\sqrt[3]{x}}$, f has critical numbers $x = -1, 0$. These yield the intervals

$(-\infty, -1)$: $f'(x) > 0$ so f is increasing on this interval.

$(-1, 0)$: $f'(x) < 0$ so f is decreasing on this interval.

$(0,+\infty)$: $f'(x) > 0$ so f is increasing on this interval.

Hence $f(-1) = 1$ is a relative maximum value and $f(0) = 0$ is a relative minimum value.

27. Since $f'(x) = \dfrac{(2x-2)(x-3) - (x^2-2x+1)(1)}{(x-3)^2} = \dfrac{x^2-6x+5}{(x-3)^2} = \dfrac{(x-1)(x-5)}{(x-3)^2}$, f has critical numbers $x = 1, 5$. These divide the domain $(-\infty, 3) \cup (3, +\infty)$ of f into the intervals

$(-\infty, 1)$: $f'(x) > 0$ so f is increasing on this interval.

$(1, 3)$: $f'(x) < 0$ so f is decreasing on this interval.

$(3, 5)$: $f'(x) < 0$ so f is decreasing on this interval.

$(5, +\infty)$: $f'(x) > 0$ so f is increasing on this interval.

Hence $f(1) = 0$ is a relative maximum value and $f(5) = 8$ is a relative minimum value.

30. Since $f'(x) = 3xe^x + 3e^x = 3e^x(x+1)$, f has a single critical number $x = -1$. This critical number creates two intervals

$(-\infty, -1)$: $f'(x) < 0$ so f is decreasing on this interval.

$(-1, +\infty)$: $f'(x) > 0$ so f is increasing on this interval.

Hence $f(-1) = 2 - 3e^{-1} \approx 0.896$ is a relative minimum value.

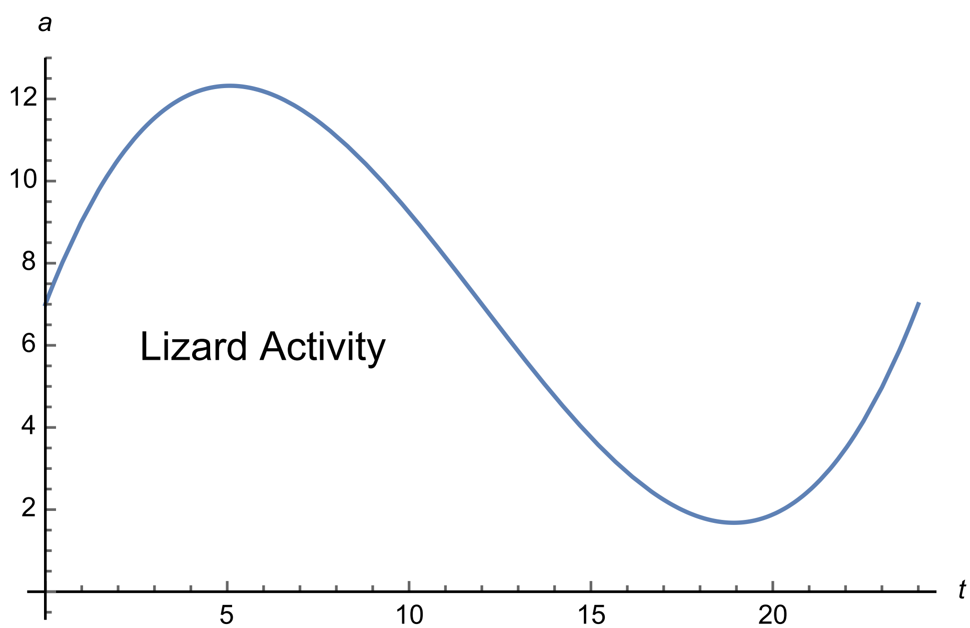
32. The domain of f is $(0,1) \cup (1, +\infty)$. Since $f'(x) = \dfrac{2x\ln x - x^2 \cdot (1/x)}{(\ln x)^2} = \dfrac{x(2\ln x - 1)}{(\ln x)^2}$ is defined everywhere on the domain of f the only critical numbers will be values of x such that $f'(x) = 0$. Now, if $f'(x) = 0$ then $x(2\ln x - 1) = 0$ which implies that either $x = 0$ or $2\ln x - 1 = 0$. We discard $x = 0$ since this number does not belong to the domain of f. If $2\ln x - 1 = 0$ then $\ln x = 1/2$ which implies $x = e^{1/2} = \sqrt{e} \approx 1.65$. This critical number creates the two intervals

$(1, \sqrt{e})$: Since $f'(1.5) \approx -1.73$, $f'(x) < 0$ on this interval and f is decreasing.

$(\sqrt{e}, +\infty)$: Since $f'(2) \approx 1.61$, $f'(x) > 0$ on this interval and f is increasing.

Hence $f(\sqrt{e}) = \dfrac{(\sqrt{e})^2}{\ln(e^{1/2})} = \dfrac{e}{1/2} = 2e \approx 5.44$ is a relative minimum value.

45. The domain of $a(t)$ is the interval $0 \leq t \leq 24$ since t is the time of day expressed as the number of hours after 12 noon. A graph of the activity level is below.



To determine when the activity level is the highest (lowest) we will find all the relative maximum (minimum) values of $a(t)$ and pick the largest (smallest) of these. Since $a'(t) = 0.024t^2 - 0.576t + 2.304$ is defined for all values of $0 \leq t \leq 24$, the critical numbers of $a(t)$ are the solutions to the equation $0 = 0.024t^2 - 0.576t + 2.304$. Solving this equation with either the quadratic formula of the "solve" feature of the TI89 we get $t \approx 5.07$ (hrs) and $t \approx 18.93$ (hrs). These critical numbers divided the interval $0 < t < 24$ as follows:

$(0, 5.07)$: $a'(t) > 0$ so $a(t)$ is increasing on this interval.

$(5.07, 18.93)$: $a'(t) < 0$ so $a(t)$ is decreasing on this interval.

$(18.93, 24)$: $a'(t) > 0$ so $a(t)$ is increasing on this interval.

Hence, $a(5.07) \approx 12.32$ and $a(24) = 7$ are the relative maximum values, and $a(0) = 7$ and $a(18.93) \approx 1.68$ are the relative minimum values. Therefore, the maximum activity occurs for $t = 5.07$ or 5:04 PM and the minimum activity occurs at $t = 18.93$ of 6:56 AM.

54. The domain of $D(x)$ is $x \geq 0$ and

$$D'(x) = -4x^3 + 24x^2 + 160x = -4x(x - 10)(x + 4)$$

Note that $D'(x)$ is defined everywhere and $D'(x) = 0$ at $x = -4, 0, 10$. We discard $x = -4$ since it's not in the domain of $D(x)$. Since $x = 0$ is an endpoint, we simply use $x = 10$ to divide the interval $(0, +\infty)$ into:

$(0, 10) : D'(1) = 180 > 0$ so D is increasing on the interval $(0, 10)$.

$(10, +\infty) : D'(11) = -660 < 0$ so D is decreasing on $(10, +\infty)$. Since there is a relative max at $x = 10$ (actually it's also the absolute maximum of D) the speaker should shoot for a discrepancy of $x = 10$.

Section 5.3

1. $f''(x) = 30x - 14$ so $f''(0) = -14$ and $f''(2) = 46$.

23. $f'''(x) = 18(x+2)^{-4}, \qquad f^{(4)}(x) = -72(x+2)^{-5}$

28. (a) $f'(x) = x^{-1} \qquad f''(x) = -x^{-2} \qquad f'''(x) = 2x^{-3} \qquad f^{(4)}(x) = -6x^{-4} \qquad f^{(5)}(x) = 24x^{-5}$

(b) $f^{(n)}(x) = (-1)^{(n-1)}(n-1)!x^{-n}$ where $(n-1)! = 1 \cdot 2 \cdot \ldots \cdot (n-2) \cdot (n-1)$.

34. Concave up on $(-\infty, 3)$, concave down on $(3, +\infty)$, inflection point $(3, 7)$.

38. Concave up on $(-\infty, 0)$, concave down on $(0, +\infty)$, no inflection points.

41. $f'(x) = -6x^2 + 18x + 168$ has domain $(-\infty, +\infty)$. Since $f''(x) = -12x + 18$ is defined everywhere and since $0 = f''(x) = -12x + 18 \Rightarrow x = 3/2$, f' has the single critical number $x = 3/2$. This critical number divides the domain of f' into the intervals

$(-\infty, 3/2)$: $f''(x) > 0$ so f' is increasing and the graph of f is concave up over this interval.

$(3/2, +\infty)$: $f''(x) < 0$ so f' is decreasing and the graph of f is concave down over this interval. The point $(3/2, 525/2)$ is an inflection point.

45. $f(x) = x^3 + 10x^2 + 25x \Rightarrow f'(x) = 3x^2 + 20x + 25$ has domain $(-\infty, +\infty)$. Since $f''(x) = 6x + 20$ is defined everywhere and since $0 = f''(x) = 6x + 20 \Rightarrow x = -10/3$, f' has the single critical number $x = -10/3$. This critical number divides the domain of f' into the intervals

$(-\infty, -10/3)$: $f''(x) < 0$ so f' is decreasing and the graph of f is concave down on this interval.

$(-10/3, +\infty)$: $f''(x) > 0$ so f' is increasing and the graph of f is concave up on this interval. The point $(-10/3, -250/27)$ is an inflection point.

47. $f'(x) = 18 + 18e^{-x}$ has domain $(-\infty, +\infty)$ and $f''(x) = -18e^{-x}$ is defined and negative for all x. Hence f' is always decreasing and the graph of f is concave down on $(-\infty, +\infty)$.

49. The domain of f is $(-\infty, +\infty)$ and the same is true for $f'(x) = \frac{8}{3}x^{5/3} - \frac{20}{3}x^{2/3}$. We have $f''(x) = \frac{40}{9}x^{2/3} - \frac{40}{9}x^{-1/3} = \frac{40}{9}\left(\frac{x-1}{\sqrt[3]{x}}\right)$ which is undefined for $x = 0$ and is equal to 0 for $x = 1$. Hence, f' has critical numbers $x = 0, 1$. These divide the domain of f' into the intervals

$(-\infty, 0)$: $f''(x) > 0$ so f' is increasing and the graph of f is concave up over this interval.

$(0, 1)$: $f''(x) < 0$ so f' is decreasing and the graph of f is concave down over this interval.

$(1, +\infty)$: $f''(x) > 0$ so f' is increasing and the graph of f is concave up over this interval.

There are inflection points at $(0, 0)$ and $(1, -3)$.

77. Consider the function $f(x) = -e^{-x}$. We have $f'(x) = e^{-x} > 0$ for all x so f is increasing on the entire number line $(-\infty, +\infty)$. We also have $f''(x) = -e^{-x} < 0$ for all x so the graph of f is concave down over the entire number line $(-\infty, +\infty)$.

78. (a) The initial population (b) The population at the point the population is growing most rapidly (c) An upper bound for the population

97. $v(t) = s'(t) = 256 - 32t$ and $a(t) = -32$. The maximum height is achieved when the velocity is 0, that is when $0 = 256 - 32t$ or at the time $t = 8$. Therefore, the maximum height is $s(8) = 1024$ feet. The cannonball hits the ground when $0 = s(t) = t(256 - 16t)$ which yields $t = 0, 16$. Since we want the time at which the cannonball has returned to the ground, $t = 16$ seconds.

Section 5.3

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$(-\infty, 0)$: $f''(x) > 0$ so f' is increasing and the graph of f is concave up over this interval.

$(0, 1)$: $f''(x) < 0$ so f' is decreasing and the graph of f is concave down over this interval.

$(1, +\infty)$: $f''(x) > 0$ so f' is increasing and the graph of f is concave up over this interval.

There are inflection points at $(0, 0)$ and $(1, -3)$.

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Section 5.3

66. $f'(x) = 2x - 12$ so f has the single critical number 6. Since $f''(x) = 2 > 0$ for all x (and in particular for $x = 6$) there is a relative minimum value of f at $x = 6$.

67. $f'(x) = 9x^2 - 6x = 3x(3x - 2)$ so f has critical numbers $x = 0, 2/3$. We have $f''(x) = 18x - 6$ with $f''(0) = -6 < 0$ and $f''(2/3) = 6 > 0$ so there is a relative maximum value of f at $x = 0$ and a relative minimum value of f at $x = 2/3$.

68. $f'(x) = 6x^2 - 8x = 2x(3x - 4)$ so f has critical numbers $x = 0, 4/3$. We have $f''(x) = 12x - 8$ with $f''(0) = -8 < 0$ and $f''(4/3) = 8 > 0$ so there is a relative maximum value of f at $x = 0$ and a relative minimum value of f at $x = 4/3$.

69. $f'(x) = 4(x + 3)^3$ so f has critical number $x = -3$. We have $f''(x) = 12(x + 3)^2$ with $f''(-3) = 0$ so the Second Derivative Test gives no information. However $f'(x) < 0$ on the interval $(-\infty, -3)$ and $f'(x) > 0$ on the interval $(-3, +\infty)$ so there is a relative minimum at $x = -3$.

Section 5.4

3. Domain: $(-\infty, +\infty)$; The y–intercept is at -10 and the x–intercepts are (approximately) at$x = -10, .1, 5.37$.

$f'(x) = -6x^2 - 18x + 108 = -6(x-3)(x+6)$ so f has critical numbers $x = -6, 3$. Checking the resulting intervals gives:

$(-\infty, -6)$; $f'(x) < 0$ so f is decreasing

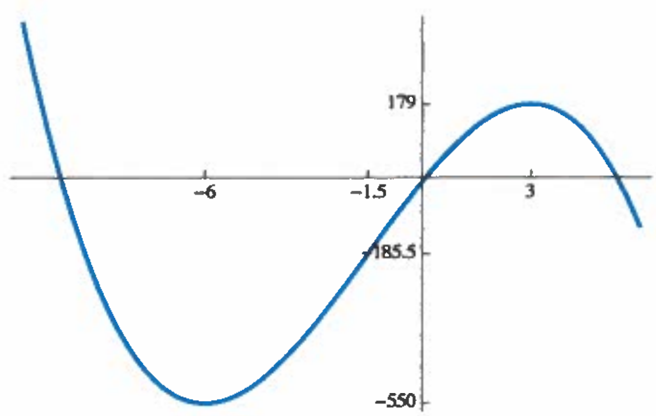
$(-6, 3)$; $f'(x) > 0$ so f is increasing

$(3, +\infty < 0)$; $f'(x)$ so f is decreasing

$f(-6) = -550$ is a relative minimum value of f and $f(3) = 179$ is a relative maximum value. $f''(x) = -12x - 18 = -6(2x+3)$ so we use $x = -3/2 = -1.5$ as a break point. Checking the resulting intervals gives:
$(-\infty, -3/2)$; $f''(x) > 0$ so the graph of f is concave up;

$(-3/2, +\infty)$; $f''(x) < 0$ so the graph of f is concave down;

There is an inflection point at $(-1.5, -185.5)$.



Graph of $f(x) = -2x^3 - 9x^2 + 108x - 10$

6. Domain: $(-\infty, +\infty)$; The y-intercept is -11 and the x intercept is (approximately) $x = 3.44$.

$f'(x) = 3x^2 - 12x + 12 = 3(x-2)^2$ so f has critical number $x = 2$. Checking the resulting intervals gives:

$(-\infty, 2)$; $f'(x) > 0$ so f is increasing

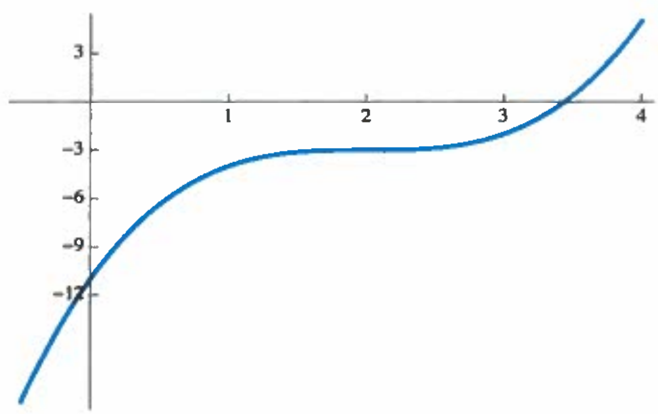
$(2, +\infty)$; $f'(x) > 0$ so f is increasing. We conclude that f is increasing on $(-\infty, +\infty)$.

$f''(x) = 6x - 12 = 6(x-2)$ and we have already used $x = 2$ as a break point. Checking the resulting intervals gives:

$(-\infty, 2)$; $f''(x) < 0$ so the graph of f is concave down

$(2, +\infty)$; $f''(x) > 0$ so the graph of f is concave up.

There is an inflection point at $(2, -3)$.



Graph of $f(x) = x^3 - 6x^2 + 12x - 11$

9. Domain: $(-\infty; +\infty)$; The y-intercept is 0 and the x-intercepts are 0 and 4.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ so f has critical numbers $x = 0, 3$. Checking the resulting intervals gives:

$(-\infty, 0)$; $f'(x) < 0$ so f is decreasing

$(0, 3)$; $f'(x) < 0$ so f is decreasing

$(3, +\infty)$; $f'(x) > 0$ so f is increasing.

$f(3) = -27$ is a relative minimum value.

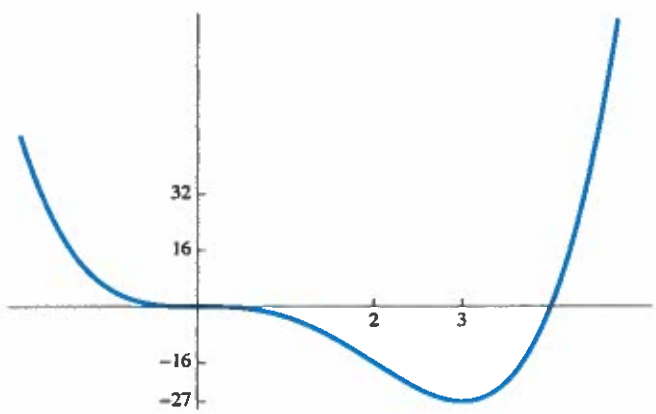
$f''(x) = 12x^2 - 24x = 12x(x-2)$ so we use $x = 0, 2$ as break points. Checking the resulting intervals gives:

$(-\infty, 0)$; $f''(x) > 0$ so the graph of f is concave up;

$(0, 2)$; $f''(x) < 0$ so the graph of f is concave down;

$(2, +\infty)$; $f''(x) > 0$ so the graph of f is concave up.

There are inflection points at $(0, 0)$ and $(2, -16)$.



Graph of $f(x) = x^4 - 4x^3$

14. Domain: $(-\infty, 2) \cup (2, +\infty)$; Both the x-intercept and the y-intercept is the origin.

$f'(x) = \dfrac{-6}{(x-2)^2} < 0$ so f has no critical numbers and is decreasing on both $(-\infty, 2)$ and $(2, +\infty)$.

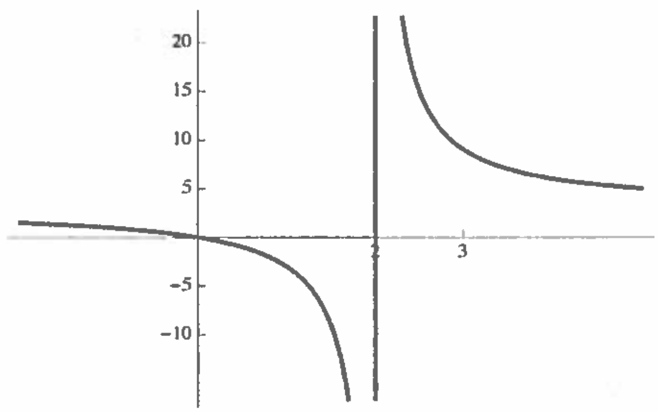
$f''(x) = \dfrac{12}{(x-2)^3}$ and the only break point is $x = 2$. Checking the resulting intervals gives:

$(-\infty, 2)$; $f''(x) < 0$ so the graph of f is concave down

$(2, +\infty)$; $f''(x) > 0$ so the graph of f is concave up.

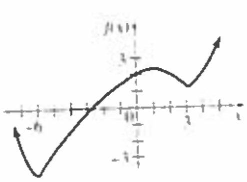
Checking limits gives

$$\lim_{x\to-\infty} \frac{3x}{x-2} = 3 = \lim_{x\to+\infty} \frac{3x}{x-2}, \quad \lim_{x\to 2^-} \frac{3x}{x-2} = -\infty \quad \text{and} \lim_{x\to 2^+} \frac{3x}{x-2} = +\infty$$

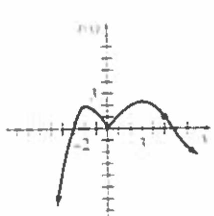


Graph of $f(x) = \dfrac{3x}{x-2}$

38.



40.



44. We are given that the domain is the closed interval $[0, 1]$ and from the factored form the ν–intercepts are $\nu = 0, 0.25, 1$. We solve $f'(\nu) = -3\nu^2 + 2.5\nu - .25 = 0$ to find critical numbers at $\nu \approx 0.12, 0.72$. Using these critical numbers as break points, we check the resulting intervals.

$(0, 0.12)$: $f'(\nu) < 0$ so f is decreasing

$(0.12, 0.72)$: $f'(\nu) > 0$ so f is increasing

$(0.72, 1)$: $f'(\nu) < 0$ so f is decreasing

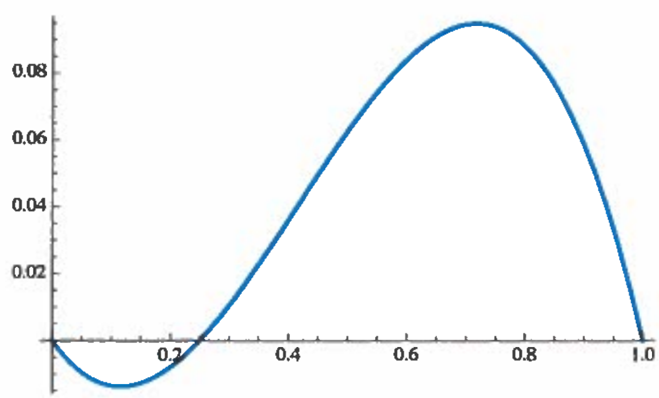
Therefore there is a relative minimum value $f(0.12) \approx -.02$ and a relative maximum value of $f(0.72) \approx 0.09$.

Since $f''(\nu) = 2.5 - 6\nu = 0$ gives $\nu \approx 0.417$ we use this number as a break point:

$(0, 0.417)$: $f''(\nu) > 0$ so the graph of f is concave up

$(0.417, 1)$: $f''(\nu) < 0$ so the graph of f is concave down

Therefore, there is a inflection point at $(0.417, 0.04)$.



Graph of $f(x) = \nu(0.25 - \nu)(\nu - 1)$

Section 6.1

1. There is an absolute maximum at $x = x_3$ and no absolute minimum.

7. There is an absolute maximum at $x = x_1$ and an absolute minimum at $x = x_2$.

12. $f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2)$ so f has critical numbers $x = -2, 4$.

x	-3	-2	4	6
$f(x)$	23	33	-75	-31

There is an absolute maximum at $x = -2$ and an absolute minimum at $x = 4$.

15. $f'(x) = 4x^3 - 36x = 4x(x-3)(x+3)$ so f has critical numbers $x = -3, 0, 3$.

x	-4	-3	0	3	4
$f(x)$	-31	-80	1	-80	-31

There is an absolute maximum at $x = 0$ and an absolute minimum at $x = \pm 3$.

20. $f'(x) = \dfrac{2 - x^2}{(x^2+2)^2}$ so f has critical numbers $x = \pm\sqrt{2}$. However, only $x = \sqrt{2}$ is in the interval $[0, 4]$.

x	0	$\sqrt{2}$	4
$f(x)$	0	$\sqrt{2}/4 \approx 0.35$	$2/9 \approx 0.22$

There is an absolute maximum at $x = \sqrt{2}$ and an absolute minimum at $x = 0$.

24. $f'(x) = \dfrac{2}{x^{1/3}} + 1 = \dfrac{x^{1/3} + 2}{x^{1/2}}$ so f has critical numbers $x = -8, 0$.

x	-10	-8	0	1
$f(x)$	$3x^{2/3} - 10 \approx 3.92$	4	0	4

There is an absolute maximum at $x = -8, 1$ and an absolute minimum at $x = 0$.

34. $f'(x) = \dfrac{(3+x)(3-x)}{x^2}$ so f has the single critical number $x = 3$ in the interval $(0, +\infty)$. Since this interval doesn't contain endpoints, we compute the value of $f(3)$ and use limits for the endpoints:

$$f(3) = 6 \qquad \lim_{x\to 0^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to +\infty} f(x) = -\infty$$

We conclude that f has an absolute maximum value of 6 at $x = 3$ and no absolute minimum value.

There is another way of obtaining this same conclusion that avoids limits. Checking the sign of $f'(x)$ on the intervals $(0, 3)$ and $(3, +\infty)$ we find that $f'(x) > 0$ on $(0, 3)$ and $f'(x) < 0$ on $(3 + \infty)$. Hence, f is increasing on the interval $(0, 3)$ and decreasing on the interval $(3, +\infty)$. We conclude that f has an absolute maximum value at $x = 3$. If f had an absolute minimum value it would have to be at a second critical number of f (since there are no endpoints to be checked). Since f has only the single critical number $x = 3$ we conclude that f has no absolute minimum value on the interval $(0, +\infty)$.

37. The denominator of f is never 0 and thus the domain of f is $(-\infty, +\infty)$. Since

$$f'(x) = \frac{(4-x)(x+2)}{(x^2+2x+6)^2}$$

f has two critical numbers $x = -2, 4$. We have $f(-2) = -1/2$ and $f(4) = 1/10$. In addition,

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x-1}{x^2+2x+6} = \lim_{x\to -\infty} \frac{1/x - 1/x^2}{1 + 2/x + 6/x^2} = 0$$

Likewise $\lim_{x\to +\infty} f(x) = 0$.

We conclude that there is an absolute maximum at of $1/10$ at $x = 4$ and an absolute minimum of $-1/2$ at $x = -2$.

50. The domain of the function S is the interval $(0, +\infty)$. Since $S = a \ln kx - bx = a(\ln k + \ln x) - bx = a \ln k + a \ln x - bx$ and

$$\frac{dS}{dx} = \frac{a}{x} - b = 0 \Rightarrow x = a/b$$

the function S has the single critical number $x = a/b$. Checking the sign of d^2S/dx^2 we find that $d^2S/dx^2 < 0$ at the critical number $x = a/b$ and we conclude that there is a relative maximum value of $a\ln(ka/b) - a$ for S at $x = a/b$. Were S to have a larger value at some other point then the graph of S would have to "bottom out" somewhere and then climb above the value $a\ln(ka/b) - a$ of S. But, the "bottom out" position would imply the existence of another critical number, a contradiction. Hence, the maximum satisfaction occurs for $x = a/b$.

Alternatively, we can check the limits

$$\lim_{x\to 0^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to +\infty} f(x) = -\infty$$

and reach the same conclusion. (However, the computation of the second limit requires us to know the fact that $\ln x$ grows much more slowly than bx, as x increases without bound.)

51. The total area enclosed is

$$A = A(x) = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{12-x}{4}\right)^2$$

which has derivative

$$A'(x) = \frac{x}{2\pi} - \frac{12-x}{8} = \frac{4x - \pi(12-x)}{8\pi} = \frac{(4+\pi)x - 12\pi}{8\pi}$$

and the single critical number $x = \dfrac{12\pi}{4+\pi} \approx 5.28$. The domain of $A(x)$ is $[0, 12]$ (where $x = 0$ corresponds to not cutting the wire and using all the wire to make the square and $x = 12$ corresponds to not cutting the wire and using all the wire to make the circle). Checking the values of $A(x)$ at the critical number and two endpoints of the domain yields the table

x	0	$\frac{12\pi}{4+\pi}$	12
$A(x)$	9	$\frac{36}{4+\pi} \approx 5.04$	$\frac{36}{\pi} \approx 11.46$

Hence the cut should be made at $x = \dfrac{12\pi}{4+\pi} \approx 5.28$ in order to minimize the total area enclosed.

Section 6.2

4. (a) Since $x + y = 105$ and x, y are nonnegative we can solve for $y = 105 - x$ for (c) $0 \le x \le 105$. Then (b) $xy^2 = x(105 - x)^2 = f(x)$ and we want to find the maximum value of f on the closed interval $0 \le x \le 105$. (d) We find that $f'(x) = 3(x - 105)(x - 35)$ and f has critical numbers $x = 35, 105$. (The critical number $x = 105$ is also an endpoint of the interval.) (e) Checking the values of f at the endpoints of the interval $0 \le x \le 105$ and at the critical number $x = 35$ yields

$$f(0) = 0, \qquad f(35) = 171{,}500, \qquad f(105) = 0$$

(f) Hence the maximum value of the expression xy^2 (subject to the constraints) is 171,500 which is realized for $x = 35$ and $y = 70$.

13. Assume we choose units of energy such that the pigeon expends 1 unit of energy flying 1 mile over land. Then, the pigeon will expend 4/3 units of energy flying 1 mile over water. Refer to the diagram in the book and let x denote the distance from point A to point P. Then $2 - x$ will be the distance between P and L and by the Pythagorean Theorem the distance from B to P will be $\sqrt{1 + x^2}$. The total energy E expended by the bird's flight will be

$$E = E(x) = \frac{4\sqrt{1 + x^2}}{3} + 2 - x \qquad (0 \le x \le 2)$$

We find that

$$E'(x) = \frac{4x}{3\sqrt{1 + x^2}} - 1$$

Using the solve feature of the calculator we have

$$0 = E'(x) = \frac{4x}{3\sqrt{1 + x^2}} - 1 \Rightarrow x = \frac{3\sqrt{7}}{7} \approx 1.13$$

so that E has a single critical number in the interior of the interval $[0, 2]$. We now have two ways of finishing the problem.

Approach #1. We compute

$$E''(x) = \frac{4}{3(1 + x^2)^{3/2}}$$

which is positive for all values of x, and in particular, is positive at the single critical number of E. Hence, E has a relative minimum value at this critical number by the Second Derivative Test, and it follows that E has an absolute minimum value at $x = \frac{3\sqrt{7}}{7}$ by the only critical number in town test. It follows that the pigeon should fly to the point P that is about 1.13 miles "east" of point A.

Approach #2. We check the values of $E(x)$ at the critical number $x = 3\sqrt{7}/7$ and at the endpoints $x = 0, 2$ of the interval.

$$E(0) = 10/3 = 3.33333..., \qquad E(3\sqrt{7}/7) = \sqrt{7}/3 + 2 \approx 2.8819, \qquad E(2) = 4\sqrt{5}/3 \approx 2.981$$

Again, we see that $x = 3\sqrt{7}/7 \approx 1.13$ yields the minimum energy and the pigeon should fly to the point P that is about 1.13 miles "east" of point A.

20. (a) $l = 1400 - 2x \quad (0 < x < 700)$ (b) $A = A(x) = x(1400 - 2x) \quad (0 < x < 700)$

(c) $A'(x) = 1400 - 4x$ which implies A has a single critical number $x = 1400/4 = 350$ in its domain. Since $A''(x) = -4 < 0$ for all x, it follows that A has a relative maximum value at $x = 350$ by the Second Derivative Test. Since $x = 350$ is the only critical number in the domain of A, the absolute maximum value of A occurs for $x = 350$ by the only critical number in town test.

(d) $A(350) = 245{,}000 \text{ m}^2$

30. Let x denote the side of the square base and let y denote the height of the box. We are given that $x^2y = 16{,}000$. We wish to minimize the total cost of materials used to construct the box. The area of the top and of the bottom is x^2 and the cost of the top and bottom is

$$\text{cost of top} + \text{cost of bottom} = 3x^2 + 3x^2 = 6x^2$$

Each of the 4 sides of the box has area xy for a total cost of $4 \times (1.5xy) = 6xy$. Hence, the total cost C of the box is $C = 6x^2 + 6xy$. We solve the equation $x^2y = 16{,}000$ for $y = 16{,}000/x^2$ and substitute into the cost formula to get

$$C = C(x) = 6x^2 + 6x\left(\frac{16{,}000}{x^2}\right) = 6x^2 + \frac{96{,}000}{x} \qquad (0 < x)$$

We compute

$$C'(x) = 12x - \frac{96{,}000}{x^2} = 0 \Rightarrow x = 20$$

so $x = 20$ is the only critical number of C. (Note that 0 does not belong to the domain of $C(x)$ and is thus not a critical number.) Since

$$C''(x) = \frac{192{,}000}{x^3} + 12$$

is always positive on the domain $0 < x$ it follows that C has a relative minimum value for $x = 20$ by the Second Derivative Test. Since $x = 20$ is the only critical number in the domain of C we conclude that $x = 20$ yields the absolute minimum value of C by the only critical number in town test. Substituting $x = 20$ into our cost function yields \$7200 as the minimum cost and solving for $y = 40$ gives dimensions $20 \times 20 \times 40$.

40. This problem is similar to the example we did in class. Let x denote the base of the right triangle. Then we want to minimize the travel time

$$T = T(x) = \frac{\sqrt{9 + x^2}}{2} + \frac{8 - x}{5} \qquad (0 \leq x \leq 8)$$

We compute

$$T'(x) = \frac{x}{2\sqrt{x^2 + 9}} - \frac{1}{5}$$

and use the solve feature of the calculator to find

$$0 = T'(x) = \frac{x}{2\sqrt{x^2 + 9}} - \frac{1}{5} \Rightarrow x = \frac{2\sqrt{21}}{7} \approx 1.31$$

Hence, T has a single critical number in the interior of the interval $[0, 8]$. There are now two approaches for finishing the problem.

Approach #1. Since

$$T''(x) = \frac{9}{2(x^2 + 9)^{3/2}}$$

is always positive, T has a relative minimum value at $x = \dfrac{2\sqrt{21}}{7}$ by the Second Derivative Test. Since, T has this single critical number in the interior of its domain, it follows from the only critical number in town test that the absolute minimum value of T occurs when the base of the right triangle is $x = \dfrac{2\sqrt{21}}{7}$. This means that the hunter should travel upriver $8 - \dfrac{2\sqrt{21}}{7} \approx 6.69$ miles before angling off towards his cabin.

Approach #2. We check the values of T at the critical number and at the endpoints.

$$T(0) = 31/10 = 3.1, \qquad T\left(\frac{2\sqrt{21}}{7}\right) = \frac{3\sqrt{21}}{10} + \frac{8}{5} \approx 2.975, \qquad T(8) = \frac{\sqrt{73}}{2} \approx 4.272$$

42. Let l and w denote the length and width of the package respectively. We wish to maximize the volume $V = lw^2$ subject to the constraint that $l + 4w = 108$. We solve for

$l = 108 - 4w$. There are now two approaches, depending upon whether or not we insist that $w > 0$ and $l > 0$.

Approach #1. (Both $w > 0$ and $l > 0$.) Since we want $l = 108 - 4w$ to be positive, the width w must satisfy $0 < w < 27$. In this case we wish to maximize V on the open interval $0 < w < 27$. Making the substitution $l = 108 - 4w$ in V, we want to maximize $V = V(w) = (108 - 4w)w^2 \quad (0 < w < 27)$. We compute $V'(w) = 12w(18 - w)$ which is always defined and is 0 at $w = 0, 18$. Hence V has a single critical number $w = 18$ that belongs to the interior of the interval $(0, 27)$. We find that $V''(w) = 216 - 24w$ and that $V''(18) = -216 < 0$. It follows that V has a relative maximum value when $w = 18$ by the Second Derivative Test and since this is the only critical number in the interval $(0, 27)$ the absolute maximum occurs for $w = 18$ and $l = 36$ by the only critical number in town test.

Approach #2. (We allow $w = 0$ and $l = 0$.) In this case we wish to maximize V on the closed interval $0 \leq w \leq 27$. Checking the value of V at the endpoints and at $w = 18$ yields

$$V(0) = 0, \qquad V(18) = 11664, \qquad V(27) = 0$$

and we conclude that the dimensions $w = 18$ and $l = 36$ maximize the volume.