CSC/MAT-220 **Name:** ____________________

Fall 2017

Midterm Solution

Handout: 10/6, Due: 10/16 **Pledge:** ____________________

Each question topic and point value is recorded in the tables below. You may review these topics from any resource at your leisure. Once you decide to start an exam problem, you are on the clock and you must work without any external resources. Each problem can be done one at a time, but must be finished in a single sitting. Answer each question in the space provided, if you run out of room, then you may continue on the back of the page. It is your responsibility to plan out your time to ensure that you can finish all problems within the 7.0 hours allotted. All SML problems should be put in a single file titled *name_midterm.sml*. Use comments so that I may easily see where each problem begins and ends, and drop this file in your dropbox folder by the due date. By writing your name and signing the pledge you are stating that your work adheres to these terms and the Davidson honor code.

Scoring Table

Question	Points	Score
1	10	
2	15	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	15	
Total:	100	

Topics Table

Question	Topic
1	Logical Connectives and Quantifiers
2	Sets
3	Relations
4	Binomial Coefficients
5	Proof by Contradiction
6	Proof by Induction
7	Logic Puzzle
8	SML Concepts: Bindings, Aggregate Data Types, and Functions.
9	SML Programming Clausal Function Expressions, Let Expressions, Binary and Ascii.

Start Time: **End Time:**

1. Let x and y be boolean variables.

   (a) (6 points) Create a truth tables for the following expressions.

      i. $x \implies y$

      Solution:

x	y	$x \implies y$
1	1	1
1	0	0
0	1	1
0	0	1

      ii. $\neg x \vee y$

      Solution:

x	y	$\neg x \vee y$
1	1	1
1	0	0
0	1	1
0	0	1

      iii. $x \wedge \neg y$

      Solution:

x	y	$x \wedge \neg y$
1	1	0
1	0	1
0	1	0
0	0	0

   (b) (4 points) A function $f\colon A \to B$ is *strictly decreasing* if and only if for every $x, y \in A$, if $x < y$, then $f(x) > f(y)$.

      i. Write the above statement using logical connectives and quantifiers.

      Solution: A function $f\colon A \to B$ is *strictly decreasing* $\Leftrightarrow$ $\forall x, y \in A,\ x < y \implies f(x) > f(y)$.

      ii. Write the negation of the above statement using logical connectives and quantifiers.

      Solution: A function $f\colon A \to B$ is *not strictly decreasing* $\Leftrightarrow$ $\exists x, y \in A,\ x < y\ \wedge\ (f) \leq f(y)$.

Start Time: **End Time:**

2. Let A and B be sets.

   (a) (10 points) State the definition of each of the following.

      i. $A \subseteq B$.
      Solution: For all $x \in A$, $x \in B$.

      ii. $A = B$.
      Solution: $A \subseteq B$ and $B \subseteq A$.

      iii. $A \cup B$.
      Solution: $\{x : x \in A \vee x \in B\}$.

      iv. $A \cap B$.
      Solution: $\{x : x \in A \wedge x \in B\}$.

      v. $A - B$.
      Solution: $\{x : x \in A \wedge x \notin B\}$.

   (b) (5 points) Prove: $A - (B \cup C) = (A - B) \cap (A - C)$

   Proof. Let $x \in A - (B \cup C)$, then $x \in A$ and $x \notin (B \cup C)$. Therefore, $x \in A$, $x \notin B$, and $x \notin C$. It follows that $x \in (A - B)$ and $x \in (A - C)$, so $x \in (A - B) \cap (A - C)$.
   Conversely, let $x \in (A - B) \cap (A - C)$. Then, $x \in A$ and $x \notin B$, and $x \in A$ and $x \notin C$. Therefore, $x \in A$ and $x \notin B \cup C$, so $x \in A - (B \cup C)$. $\square$

Start Time: **End Time:**

3. Let A and B be sets.

   (a) (4 points) State the definition of each of the following.

   i. A relation R between A and B.
   Solution: $R \subseteq A \times B$.

   ii. An equivalence relation R on A.
   Solution: $R \subseteq A \times A$ and satisfies the following
   - Reflexive: for all $x \in A$, xRx.
   - Symmetric: if xRy, then yRx.
   - Transitive: if xRy and yRz, then xRz.

   (b) (6 points) Determine which of the following are equivalence relations.

   i. Let A be the set of all lines in a plane and let R be the relation "is perpendicular to."
   Solution: A line cannot be perpendicular to itself, so this relation is not reflexive and therefore not an equivalence relation.

   ii. Let A be the set of real numbers and let R be the relation "$>$".
   Solution: A number cannot be greater than itself, so this relation is not reflexive and therefore not an equivalence relation.

   iii. Let A be the set of all triangles in a plane and let R be the relation "is similar to."
   Solution: Two triangles are similar if and only if corresponding angles have the same measure. Since equality of real numbers is an equivalence relation, it is clear that R is also an equivalence relation.

Start Time: **End Time:**

4. Let $n, k \in \mathbb{N}$.

   (a) (2 points) State the definition of the binomial coefficient $\binom{n}{k}$.

   Solution: The binomial coefficient $\binom{n}{k}$ is the number of size k subsets can be formed from a set of size n.

   (b) (8 points) Give a combinatorial proof for each of the following identities.

   i. $\binom{n}{2} = 1 + 2 + \cdots + (n-1) = \sum_{k=1}^{n-1} k$.

   Proof. We build this proof around the question: How many subsets of size 2 can be created from a set of size n? By definition it is clear that one answer to this question is $\binom{n}{2}$. For the second answer, suppose the set of size n contains elements $x_1, x_2, \ldots, x_n$. Then of all subsets of size 2 that can be created from these elements, there are $(n-1)$ of the form $\{x_1, \_\}$, there are $(n-2)$ of the form $\{x_2, \_\}$, and so on. In general there are $(n-k)$ subsets of the form $\{x_k, \_\}$, for $k = 1, \ldots, n-1$. Therefore, there are $\sum_{k=1}^{n-1} k$ subsets of size 2 that can be created from a set of size n. $\square$

   ii. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

   Proof. We build this proof around the question: How many subsets of any size can be created from a set of size n. It is clear that we can make subsets of size k, where $k = 0, 1, \ldots, n$. Furthermore, by definition, for each k there are $\binom{n}{k}$ subsets of size k that can be created. Therefore, there are $\sum_{k=0}^{n} \binom{n}{k}$ total subsets that can be created from a set of size n. For the second answer, we decide element by element whether or not to put that element in the subset. Since there are two possibilities for each element (in or out), there are 2^n possible subsets. $\square$

5. Prove the following statements using proof by contradiction.

   (a) (5 points) $\log_2(7)$ is irrational.

   Proof. Suppose that $\log_2(7)$ is rational. Then, there exists integers m and n such that $\log_2(7) = \frac{m}{n}$, and $n \neq 0$. This implies that $7^n = 2^m$. However, since the product of two odd numbers is always odd and the product of two even numbers is always even, this can only be true if $m = n = 0$, which contradicts $n \neq 0$. Therefore, $\log_2(7)$ is irrational. □

   (b) (5 points) $7^n - 4^n$ is divisible by 3 for all $n \in \mathbb{N}$.

   Proof. Define the set of counterexamples

$$X = \{n \in \mathbb{N} \colon 7^n - 4^n \text{ is not divisible by } 3\},$$

   and suppose that X is non-empty. Then, by the well-ordering principle there exists a smallest element $m \in X$. Note that $m > 1$, since the result clearly holds for $n = 0$ and $n = 1$. Therefore, $m - 1 \geq 1$ and $7^{m-1} - 4^{m-1}$ is divisible by 3. Note that

$$\begin{aligned} 7^m - 4^m &= 7 \cdot 7^{m-1} - 7 \cdot 4^{m-1} + 7 \cdot 4^{m-1} - 4^m \\ &= 7(7^{m-1} - 4^{m-1}) + 4^{m-1}(7 - 4). \end{aligned}$$

□

   Since both factors on the right hand side of the above expression are divisible by 3, it follows that $m \notin X$. Therefore, X is in fact empty and the result holds.

Start Time: **End Time:**

6. Prove the following statements using induction.

   (a) (5 points) For every natural number $n \geq 1$

$$1 + 2 + \cdots + n = \frac{1}{2}n(n+1).$$

Proof. Note that the result clearly holds for $n = 1$. Then, suppose that for some integer $k \geq 1$,

$$1 + 2 + \cdots + k = \frac{1}{2}k(k+1).$$

Adding $(k+1)$ to both sides gives

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

By the principle of mathematical induction, the result holds for all integers $n \geq 1$. □

   (b) (5 points) For every natural number $n \geq 1$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} = \frac{n}{n+1}.$$

Proof. Note that the result clearly holds for $n = 1$. Then, suppose that for some integer $k \geq 1$,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$

Adding $\frac{1}{(k+1)(k+2)}$ to both sides gives

$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ &= \frac{k(k+2) + 1}{(k+1)(k+2)} \\ &= \frac{(k+1)}{(k+1)+1}. \end{aligned}$$

By the principle of mathematical induction, the result holds for all integers $n \geq 1$. □

7. This logic puzzle is derived from a famous story entitled "The Lady or the Tiger?". Each of the following signs are placed on a door, behind each door is either a lady *or* a tiger. Suppose you want to choose the room with the lady, use logical reasoning to determine which door you should open.

   (a) (3 points) One of the signs is true, but the other one is false.
   Sign 1: "In this room there is a lady, and in the other room there is a tiger."
   Sign 2: "In one of these rooms there is a lady, and in one of these rooms there is a tiger."

   Solution: If Sign 1 is true, then Sign 2 is also true. Since one of the signs must be false, this cannot be. Therefore, Sign 1 is false and Sign 2 is true. It follows that the lady is in room 2.

   (b) (3 points) Either both signs are true or both are false.
   Sign 1: "At least one of these rooms contains a lady."
   Sign 2: "A tiger is in the other room."

   Solution: If both signs are false, then no room contains a lady and room 1 does not contain a tiger, which leads to a contradiction. Therefore, both signs must be true, and it follows that the lady is in room 2.

   (c) (4 points) This time there are three rooms and three signs. One of the three rooms contains a lady, another a tiger, and the third room is empty. The sign on the door of the room containing the lady is true, the sign on the door of the tiger is false, and the sign on the door of the empty room can be either true or false.
   Sign 1: "Room 3 is empty."
   Sign 2: "The tiger is in room 1."
   Sign 3: "This room is empty."
   Determine which room is empty, which room contains the lady, and which room contains the tiger.
   Solution: Suppose room 3 is empty, then both Sign 1 and Sign 3 are true. Therefore, the lady must be in room 1, which means that the tiger is in room 2 and the sign on the door is false (as it should be). Note that the assumption of any other room being empty leads to a contradiction.

8. Answer the following SML concept questions.

   (a) (2 points) Describe the following expression in terms of bindings.

   val logb = fn (x:real,b:real) => Math.ln(x)/Math.ln(b)

   Re-write this expression using the *fun* syntax.

   Solution: The function value *fn (x:real,b:real) => Math.ln(x)/Math.ln(b)* is being bound to the variable *logb*. This variable is of the type $real * real \rightarrow real$. We can re-write this using fun syntax as follows:

   fun logb (x:real,b:real) = Math.ln(x)/Math.ln(b)

   (b) (4 points) Consider the value bindings below.

   *val x:real*real = (1.0,2.0) and y:int*int = (1,2)*

   i. What is wrong with this expression?

   if x=y then "true" else "false"

   Solution: Since x and y are not of the same type they cannot be compared, SML does not do implicit type conversions.

   ii. What does this expression return?

   let val (y1:int,y2:int) = y in y = (y2,y1) end

   Solution: false.

   (c) (4 points) Write pseudocode for creating a record type called *address* that has labels for *state*, *city*, *zipcode*, *street_name*, and *street_number*.

   Solution:

$$
\begin{aligned}
&\mathit{val\ address}\ = \\
&\quad \{\ \mathit{state}\ :\ \mathit{string}, \\
&\quad \mathit{city}\ :\ \mathit{string}, \\
&\quad \mathit{zipcode}\ :\ \mathit{int}, \\
&\quad \mathit{street\_name}\ :\ \mathit{string}, \\
&\quad \mathit{street\_number}\ :\ \mathit{int}\ \}
\end{aligned}
$$

Start Time: **End Time:**

9. In the space provided, include your scratch work to highlight your thought process when solving each problem. Then write the code in your *sml file* (this is what you will be graded on).

   (a) (3 points) Use a clausal function expression to define a function *impl* that returns the logical implication $x \implies y$, where x and y are the two boolean arguments.

   (b) (6 points) Write a function *bin_to_char* that returns the ascii character associated with the binary number argument. Use the following hints and guidelines.

   - Store a binary number as an 8-tuple, where each element has type int.
   - Make use of a clausal function expression and a let expression.
   - Make use of the member function *chr* of the SML Char Signature.
   - Test this on the binary number 01100001, you should get the character a.

   (c) (6 points) Write a function *char_to_bin* that returns the binary number associated with the ascii character argument. Use the following hints and guidelines.

   - Store a binary number as an 8-tuple, where each element has type int.
   - Make use of a let expression.
   - Make use of the integer operations mod and int.
   - Make use of the member function *ord* of the SML Char Signature.
   - Test this on the character a, you should get the binary number 01100001.