MAT-150: Linear Algebra
EFY 5

September 25, 2017

Part 1: Volume and Orientation of Parallelepipeds

Please do each of the following.

i. State the definition of a parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$.

- What is the dimension of $\mathcal{P}$?
- What is the ambient space of $\mathcal{P}$?

ii. Let $a \in \mathbb{R}$

- Give a geometric description of the parallelepiped $\mathcal{P}$ determined by a.
- What is $V(\mathcal{P})$?
- What is $O(\mathcal{P})$?

iii. State the definition of a proper parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$.

- Which of the following Parallelepipeds are proper? Why?

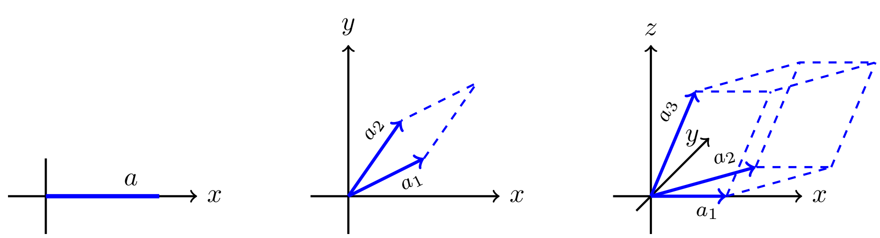

Figure 1: Parallelepipeds in $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

- State the definition of the base of a proper parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$
- State the definition of the height of a proper parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$

iv. State the definition of the image of the parallelepiped $\mathcal{P}$ under the linear transformation T.

v. Let
$$a_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

- Denote by $\mathcal{P}$ the parallelepiped determined by a_1 and a_2. Explain why $\mathcal{P}$ is not proper.
- Find the matrix representation of the linear transformation T such that the image $T(\mathcal{P})$ will be proper.
- Compute the vectors $T(a_1)$ and $T(a_2)$ that determine the proper parallelepiped $T(\mathcal{P})$
- Compute $V(\mathcal{P})$ and $O(\mathcal{P})$.

vi. Let
$$a_1 = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad a_2 = \begin{bmatrix} b \\ d \end{bmatrix},$$
where $a, b, c, d \in \mathbb{R}$.

- Define
$$\cos(\theta) = \frac{a}{\sqrt{a^2 + c^2}} \quad \text{and} \quad \sin(\theta) = \frac{c}{\sqrt{a^2 + c^2}},$$
and
$$Q = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$
- Show that Qa_1 and Qa_2 determine a proper parallelepiped.
- Compute the volume and orientation of this parallelepiped.

Part 2: The Determinant and its Properties

We note that every parallelepiped may be transformed via rotations to a proper one with the same n-volume and orientation. Furthermore, if $\mathcal{P}$ is a proper parallelepiped in $\mathbb{R}^n$, then
$$V(P) = \left| \prod_{i=1}^{n} a_i(i) \right| \quad \text{and} \; O(P) = \operatorname{sgn}\left(\prod_{i=1}^{n} a_i(i) \right).$$

Also, we may denote the volume and orientation of a parallelepiped in $\mathbb{R}^n$ by $V(a_1, \ldots, a_n)$ and $O(a_1, \ldots, a_n)$, respectively. Please do each of the following.

i. Show that for any parallelepiped $\mathcal{P}$ in $\mathbb{R}^n$, determined by the vectors $a_1, \ldots, a_n$, the following holds for all $\alpha \in \mathbb{R}$
$$V(a_1, \ldots, \alpha a_k, \ldots, a_n) = |\alpha| V(a_1, \ldots, a_k, \ldots, a_n),$$
$$O(a_1, \ldots, \alpha a_k, \ldots, a_n) = \operatorname{sgn}(\alpha) O(a_1, \ldots, a_k, \ldots, a_n).$$

Hint: You may assume that $\mathcal{P}$ is proper.

ii. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. State the definition of volume magnification and orientation change under T.

iii. Show that $V(\mathcal{P}) = 0$ and $O(\mathcal{P}) = 0$ if and only if the vectors $a_1, \ldots, a_n$ are linearly dependent. *Hint: You may assume that $\mathcal{P}$ is proper.*

iv. What conversation did we have in class that shows that our definitions of volume magnification and orientation change is well defined? Now, state the definition of the determinant of the linear transformation T.

v. Use the result from (iii) in your homework: Other Problems - Problem 2.

vi. Draw a picture that relates to your homework: Other Problems - Problems 2 - 4.

Part 3: Computing the Determinant

We note that so far we have only discussed the determinant of a linear transformation. In this section, we define the determinant of a matrix and discuss the most efficient method for computing the determinant. Lastly, we motivate the Laplace expansion (cofactor expansion). Please do each of the following.

i. State the definition of the determinant of a matrix.

ii. Give an argument for why the determinant of an upper triangular matrix is equal to the product of its diagonal entries. *Hint: relate it back to a proper parallelepiped.*

iii. Justify how the following row operations affect the determinant: a constant multiple of one row and swapping two rows. Use this to outline an algorithm for computing the determinant of any matrix.

iv. Let $a_1, \ldots, a_k, \ldots, a_n$ and b_k denote column vectors and $\alpha \in \mathbb{R}$. Then justify the following properties.

- $\det(I) = 1$, where I is the $n \times n$ identity matrix.
- $$\begin{aligned} \det(a_1, \ldots, a_k + b_k, \ldots, a_n) &= \det(a_1, \ldots, a_k, \ldots, a_n) \\ &\quad + \det(a_1, \ldots, b_k, \ldots, a_n), \\ \det(a_1, \ldots, \alpha a_k, \ldots, a_n) &= \alpha \det(a_1, \ldots, a_k, \ldots, a_n). \end{aligned}$$
- If $a_i = a_j$ $(i \neq j)$ then $\det(a_1, \ldots, a_n) = 0$.

The result from (iv) can be used to show that the determinant is a unique alternating multilinear function of the column vectors of A. There is only one function that satisfies this property and it turns out this function is described algebraically by the Laplace expansion (cofactor expansion), which can be found on p. 167 of your text-book. This is why your book is justified in using cofactor expansion to define the determinant.