MAT-150: Linear Algebra

Solution 2

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Other Problems

Problem 1

Theorem. *Let T be any linear transformation, $v_1, \ldots, v_p$ be elements in the domain of T, and $c_1, \ldots, c_p$ be scalars. Then,*

$$T(c_1v_1 + \cdots + c_pv_p) = c_1T(v_1) + \cdots + c_pT(v_p).$$

Proof. From the definition of a linear transformation, we know that T preserves the operation of vector addition and scalar multiplication. Therefore,

$$\begin{aligned} T(c_1v_1 + \cdots + c_pv_p) &= c_1T(v_1) + T(c_2v_2 + \cdots + c_pv_p) \\ &\vdots \\ &= c_1T(v_1) + \cdots + c_pT(v_p). \end{aligned}$$

□

Corollary. *Let T be any linear transformation, then*

$$T(0) = 0.$$

Proof. Let $c_1 = \cdots = c_p = 0$, then the result follows directly from the previous theorem. □

Problem 2

Theorem. *Let A be an. $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then,*

I. $A(B + C) = AB + AC$

II. $(B + C)A = BA + CA$

Proof. I. Let B and C be $n \times p$ matrices. Then, from the definition of matrix multiplication, it follows that

$$\begin{aligned} A(B + C) &= A\left[(b_1 + c_1)\ (b_2 + c_2)\ \cdots\ (b_p + c_p)\right] \\ &= \left[A(b_1 + c_1)\ A(b_2 + c_2)\ \cdots\ A(b_p + c_p)\right] \\ &= \left[Ab_1\ Ab_2\ \cdots\ Ab_p\right] + \left[Ac_1\ Ac_2\ \cdots\ Ac_p\right] \\ &= AB + AC. \end{aligned}$$

II. Let B and C be $p \times m$ matrices. Then

$$\begin{aligned}(B+C)A &= (B+C)\,[a_1\ a_2\ \cdots\ a_n] \\ &= [(B+C)a_1\ (B+C)a_2\ \cdots\ (B+C)a_p] \\ &= [Ba_1\ Ba_2\ \cdots\ Ba_p] + [Ca_1\ Ca_2\ \cdots\ Ca_p] \\ &= BA + BC.\end{aligned}$$

□

Problem 3

Proof of Theorem 8 on p. 114 of our book.

Proof. First, we show that $(a) \to (j)$ and $(k) \to (d)$ and (f) and (g) and $(i) \to (e)$ and $(h) \to (c)$ and (b).

Suppose (a), then, by definition of an invertible matrix, there exists a matrix C such that $AC = I$ and $CA = I$, and (j) and (k) follow. From (j), it follows that $Ax = 0$ has only the trivial solution (multiply both sides of the homogenous equation by C on the left), and from (k), it follows that $Ax = b$ has a solution for every $b \in \mathbb{R}^n$ (let $x = Db$). Therefore, both (d) and (f) follow from (j), and both (g) and (i) follow from (k). Then, (d) implies (e), by the definition of linear independence and (g) implies (h), from the definition of span. Furthermore, it follows that A has n pivots and therefore can be row reduced to the $n \times n$ identity matrix; that is, both (c) and (b) follow.

Next, we show that $(b) \to (j)$ and $(k) \to (a)$.

Suppose (b), then there exists a matrix C (product of elementary matrices) such that $CA = I$. Furthermore, $Ax = b$ has a unique solution for every b, and it follows that there exists a matrix D such that $AD = I$ (solve column by column). Therefore, both (j) and (k) hold. Furthermore, by the associative property of matrix multiplication, it follows that $C = D$ (consider the product CAD), and it follows that the matrix A is invertible. Therefore, (a) holds.

Finally, we show that $(a) \leftrightarrow (l)$

Note that (a) holds if and only if there exists a matrix C such that $AC = I$ and $CA = I$. Taking the transpose of both equations gives $C^T A^T = I$ and $A^T C^T = I$. Therefore, by definition of invertibility, A^T is also invertible. □