MAT – 450: Advanced Linear Algebra

Solution 1

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Other Problems

Problem 1. Let V be a vector space over a field F.

Theorem 1. *The zero vector and additive inverse are unique.*

Proof. Suppose there exists two zero vectors 0 and $0'$. Then, by [1, VS 3], for all $x \in V$ we have

$$x + 0 = x + 0'.$$

It follows from [1, VS 1] that $0' = 0' + 0 = 0 + 0' = 0$. Therefore, $0' = 0$ and it follows that the zero vector is unique. Throughout this course we will denote the zero vector by 0.

Similarly, let $x \in V$ and suppose there exists two additive inverses y and y'. Then, by [1, VS 4], we have

$$x + y = 0 \quad \text{and} \quad x + y' = 0.$$

It follows from [1, VS 1 and VS 2] that

$$y = y + 0 = y + (x + y') = (y + x) + y' = 0 + y' = y' + 0 = y'.$$

Therefore, $y' = y$ and it follows that the additive inverse is unique. Throughout this course we will denote the additive inverse by $-x$. □

Theorem 2. *For $0 \in F$, we have $0x = 0$ (zero vector) for all $x \in V$.*

Proof. Let $x \in V$. It follows from [1, VS 8] that $x + 0x = (1 + 0)x = x$. So, by [1, VS 3] and Theorem 1, it follows that $0x = 0$ (zero vector). □

Theorem 3. *For $(-1) \in F$, we have $(-1)x = -x$ (additive inverse) for all $x \in V$.*

Proof. Let $x \in V$. It follows from [1, VS 8] that $x + (-1)x = (1 - 1)x = 0x$. Therefore, by [1, VS 4] and Theorem 2, it follows that $(-1)x = -x$ (additive inverse). □

Problem 2. Let $P_n(F)$ denote the vector space of polynomials of degree n over the field F. Let S denote the set of polynomials that are zero at $t_1, \ldots, t_j$, where $j \leq n$.

Theorem 4. *The set S is a subspace of $P_n(F)$.*

Proof. Let $x, y \in P_n(F)$ and $\alpha \in F$. Then, $x(\lambda) = \hat{x}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j)$ and $y(\lambda) = \hat{y}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j)$, where $\hat{x}, \hat{y} \in P_{(n-j)}(F)$. Note that

$$x(\lambda) + y(\lambda) = (\hat{x}(\lambda) + \hat{y}(\lambda))(\lambda - t_1)\cdots(\lambda - t_j) \in S$$

and

$$\alpha x(\lambda) = \alpha\hat{x}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j) \in S.$$

It follows that S is closed under addition of vectors and scalar multiplication, and is therefore a subspace of $P_n(F)$. $\square$

Since each element of S can be represented by $\hat{x}(\lambda)(\lambda - t_1)\cdots(\lambda - t_j)$, where $\hat{x} \in P_{(n-j)}(F)$, it follows that $\dim S = \dim P_{(n-j)}(F) = (n - j) + 1$. Furthermore, as a corollary of [1, §1.6: Problem 35], we know that $\dim P_n(F)/S = \dim P_n(F) - \dim S = j$.

References

[1] S.H. Friedberg, A.H. Insel, and L.E. Spence. *Linear Algebra*. Pearson Education, Upper Saddle River, NJ, 4th edition, 2003.