CSC/MAT-220: Discrete Structures
EFY 1

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August 25, 2017

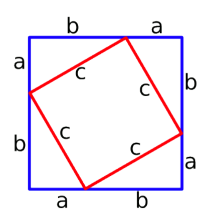
Develop a proof of the Pythagorean theorem and its converse. Be sure to state your starting assumptions, and to guide the reader from those assumptions to your conclusion in a clear and precise manner.

Solution.

Throughout this development, we assume knowledge of simple geometric shapes, area formula for these shapes, and the definition of sin and cos using the angles of a right triangle.

Theorem 1 (Pythagorean Theorem)**.** *Let a, b be the legs, and c the hypotenuse of a right-triangle, then $c^2 = a^2 + b^2$.*

Proof. Take 4 copies of the right triangle and join them together, thereby forming a square with sides $(a + b)$ as shown below.



The area of the above square is the sum of the area of 4 right triangles with sides a, b, c, and a square with sides c. Therefore,

$$(a + b)^2 = c^2 + 4\left(\frac{1}{2}ab\right)$$

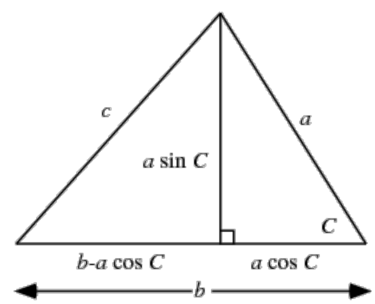
The result follows by expanding and canceling out like terms from the above equation. □

We are now ready to prove the Law of Cosines, as a direct corollary of the Pythagorean theorem. In the following proof, we take advantage of the identity $\sin^2(\theta) + \cos^2(\theta) = 1$ for all real θ, note that this result can be proven using the definition of sin and cos and the Pythagorean theorem.

Proposition 2 (Law of Cosines)**.** *For any triangle with sides a, b, c and opposite angles A, B, C, the following equations hold.*

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$
$$b^2 = a^2 + c^2 - 2ac\cos(B)$$
$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Proof. It suffices to prove the latter of the equations stated in the proposition. To this end, consider the triangle below.



Focusing on the leftmost right triangle in the above figure, we note the following equation follows from the Pythagorean theorem

$$(b - a\cos(C))^2 + (a\sin(C))^2 = c^2.$$

By expanding and canceling out like terms in the above equation, and noting the identity $\cos(\theta)^2 + \sin(\theta)^2 = 1$ for all real θ, we see that

$$c^2 = a^2 + b^2 - 2ab\cos(C).$$

□

Now, we can prove the converse of the Pythagorean theorem as a directly corollary of Theorem 1 and Proposition 2.

Theorem 3. *Any triangle with sides a, b, c that satisfy $c^2 = a^2 + b^2$ must be a right triangle.*

Proof. Suppose a triangle with sides a, b, c satisfies $c^2 = a^2 + b^2$. Then, according to Proposition 2, the angle C, opposite to side c, satisfies $\cos(C) = 0$. Therefore, C must be a 90 degree angle, and the result follows. □