CSC/MAT-220: Discrete Structures
Solution 3

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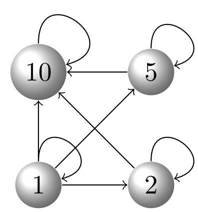
Book Problems

Problem 14.16

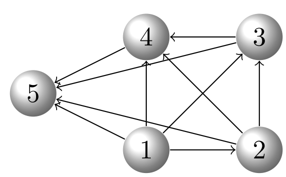
All this proof has shown is that $xRy \implies xRx$; that is, if there is a y such that xRy, then xRx under symmetry and transitivity. However, if x is not related to any y, then the result does not follow. Hence, R is not necessarily reflexive.

Problem 14.17

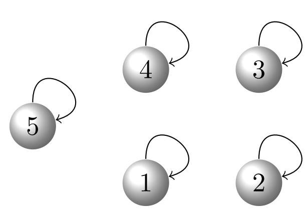
a.



b.

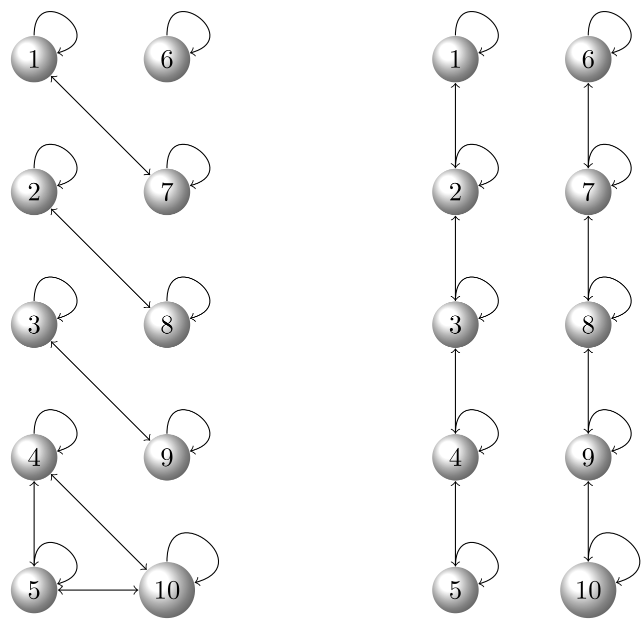


c.



Problem 15.14

a. **Equivalence Relation 1** **Equivalence Relation 2**



b. The equivalence classes for Relation 1 are as follows

$$[1] = \{1, 7\},\ [2] = \{2, 8\},\ [3] = \{3, 9\},\ [4] = \{4, 5, 10\},\ [6] = \{6\}.$$

The equivalence classes for Relation 2 are as follows

$$[1] = \{1, 2, 3, 4, 5\},\ [6] = \{6, 7, 8, 9, 10\}.$$

c. The equivalence relations look like single loops (reflexive) and cycles (transitive) that go both ways (symmetric).

Problem 16.2

In each problem below, A will denote the number of arrangements that can be made from the letters in the given word, where repeated letters are denoted with an index to make them distinct. Then, the relation R on A is defined by xRy if x and y denote the same arrangement when the indices on the repeated letters are removed. Clearly R denotes an equivalence relation, and therefore partitions A into a union of piecewise disjoint equivalence classes. Furthermore, each equivalence class has the same number of elements m: the number of arrangements of the indexed repeated letters. Therefore, the number of distinct equivalence classes is equal to $|A|/m$. Each distinct equivalence class is a unique anagram that can be made by the given word.

a. STAPLE has 6! different anagrams that can be made.

b. DISCRETE has $\frac{8!}{2!}$ different anagrams that can be made.

c. MATHEMATICS has $\frac{11!}{2!2!2!}$ different anagrams that can be made.

d. SUCCESS has $\frac{7!}{3!2!}$ different anagrams that can be made.

e. MISSISSIPPI has $\frac{11!}{4!4!2!}$ different anagrams that can be made.

Problem 16.18

Let A denote the set of all arrangements of two distinct coins on the chess board. Note that $|A| = 64^2$, since there are 64 choices for each coin. Then, the relation R on A is defined by xRy if the coins that make up the arrangement x can be moved diagonally to get the arrangement y. Clearly R denotes an equivalence relation, and therefore partitions A into a union of piecewise disjoint equivalence classes. Furthermore, each equivalence class has the same number of elements $m = 32^2$, since each coin can be moved diagonally to land on 32 different squares. Therefore, the number of distinct equivalence classes is equal to $64^2/32^2 = 4$. It follows that there are only 4 non-equivalent ways to arrange these distinct coins on the chessboard.

Other Problems

I. To show that R is an equivalence relation, we must show that R that is reflexive, symmetric, and transitive. To this end, let $(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}$, and note the following:

- We have $(a, b)R(a, b)$, since $a = a$.
- If $(a, b)R(c, d)$, then $(c, d)R(a, b)$, since $c = a$.
- If $(a, b)R(c, d)$ and $(c, d)R(e, f)$, then $(a, b)R(e, f)$, since $a = c = e$.

The equivalence class $[(a, b)]$ is a vertical line through the point (a, b).

II. Based on the partition $\mathcal{P}$ and the fact that R is an equivalence relation, we arrive at the following

$$R = \{(a, a),\ (b, b),\ (c, c),\ (b, c),\ (c, b),\ (d, d)\}$$

III.

Proposition. *A relation R is an equivalence relation if and only if it is reflexive and circular.*

Proof. Suppose that R is an equivalence relation, then R is reflexive, symmetric, and transitive. Suppose that aRb and bRc, then aRc via the transitive property, and cRa via the symmetric property. Therefore, R is reflexive and circular.

Suppose that R is reflexive and circular. If aRb, then, since bRb, it follows that bRa. Therefore, R is symmetric. Further, if aRb and bRc, then cRa and by the symmetric property aRc. Therefore, R is transitive, and the result follows. $\square$