MAT – 112: Calculus I and Modeling

Logarithm and Exponent Derivatives

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Instructions

Read the section on the natural exponent derivative. Use the writing style I use in the proof as a guideline for the next two sections, where you are asked to prove derivative rules for exponent and logarithm functions.

Natural Exponent Derivative. The following holds true

$$\frac{d}{dx}e^x = e^x.$$

Proof. Using the limit definition of the derivative we have

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x\left(e^h-1\right)}{h}\\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h}.\end{aligned}$$

All that remains is to show that $\lim_{h\to 0}\frac{e^h-1}{h} = 1$. To this end, we introduce the change of variables

$$u = e^h - 1,$$

which implies that $\ln(u+1) = h$. Since $u \to 0$ as $h \to 0$ we have

$$\lim_{h\to 0}\frac{e^h-1}{h} = \lim_{u\to 0}\frac{u}{\ln(u+1)}.$$

Next, we multiply the numerator and denominator by $(1/u)$ to give

$$\lim_{u\to 0}\frac{1}{\frac{1}{u}\ln(u+1)} = \lim_{u\to 0}\frac{1}{\ln(1+u)^{1/u}}.$$

Let $n = 1/u$ and note that $n \to \infty$ as $u \to 0$. Therefore, we have

$$\lim_{n\to\infty}\frac{1}{\ln(1+\frac{1}{n})^n} = \frac{1}{\ln(e)} = 1.$$

$\square$

Exponent Derivative. Use the previous result to show that

$$\frac{d}{dx}a^x = (\ln a)a^x.$$

Logarithm Derivative. Use the previous result to show that

$$\frac{d}{dx}\log_a(x) = \frac{1}{(\ln a)x}.$$

Then deduce formula for $\frac{d}{dx}\ln(x)$.

Applications.

1. The altitude (feet) of a plane t (min) after takeoff is given by

$$h(t) = 2000\ln(t+1), \qquad 0 \le t \le 5.$$

2. The sound pressure P (dB) for a given sound can be modeled by

$$P = 20\log_{10}\frac{W}{W_0},$$

where W is the size of the variable energy source and W_0 is a constant. Compute $\frac{dP}{dt}$ at $t = 3$, if $W = 7.2$ and $\frac{dW}{dt} = 0.5$ at $t = 3$.

3. The charge of a capacitor in a circuit containing a capacitance C, resistance R, and source voltage E is given by

$$q = CE\left(1 - e^{-t/RC}\right).$$

Show that the follow equation holds true

$$R\frac{dq}{dt} + \frac{q}{C} = E$$