

# MAT – 450: Advanced Linear Algebra

## Solution 2

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### Other Problems

**Problem 1.** Let  $V$  be a vector space of dimension  $n$ , and let  $T: V \rightarrow V$  be linear. Suppose that  $W$  is a subspace of  $V$  with ordered basis  $\gamma = \{x_1, \dots, x_k\}$ .

**Theorem 1.** *If  $W$  is  $T$ -invariant, then the ordered basis*

$$\beta = \{x_1, \dots, x_k, x_{k+1}, \dots, x_n\}$$

*for  $V$  satisfies  $[T]_\beta = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$ , where  $B_{11} = [T_W]_\gamma$ .*

*Proof.* Note that  $[T]_\beta = [a_{ij}]$  is a  $n \times n$  matrix such that

$$T(x_j) = \sum_{i=1}^n a_{ij} x_i, \quad j = 1, \dots, n.$$

Since  $W$  is  $T$ -invariant, it follows that  $a_{ij} = 0$  for all  $i > k$  and  $j = 1, \dots, k$ . Thus,

$$[T]_\beta = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

where  $B_{11}$  is a  $k \times k$  matrix. For  $j = 1, \dots, k$  let

$$T_W(x_j) = \sum_{i=1}^k \hat{a}_{ij} x_i.$$

Then, since  $T_W(x) = T(x)$  for all  $x \in W$ , we have

$$\sum_{i=1}^k \hat{a}_{ij} x_i = \sum_{i=1}^k a_{ij} x_i.$$

Furthermore, since the  $x_i$ 's are linearly independent, it follows that  $\hat{a}_{ij} = a_{ij}$  for  $i, j = 1, \dots, k$ . Therefore,  $[T_W]_\gamma = B_{11}$ .  $\square$

**Theorem 2.** *If the ordered basis  $\gamma$  satisfies*

$$\text{span}(x_1, \dots, x_j)$$

*being  $T$ -invariant for  $j = 1, \dots, k$ , then  $[T_W]_\gamma$  is a  $k \times k$  upper triangular matrix.*

*Proof.* Suppose that  $\gamma$  satisfies  $\text{span}(x_1, \dots, x_j)$  being  $T$ -invariant for  $j = 1, \dots, k$ . Then, it is clear that  $W = \text{span}(x_1, \dots, x_k)$  is  $T$ -invariant and it follows that  $T_W$  is linear. Therefore,  $[T_W]_\gamma = [a_{ij}]$  is a  $k \times k$  matrix, where

$$T(x_j) = \sum_{i=1}^k a_{ij} x_i, \quad j = 1, \dots, k.$$

Since  $T(x_j) \in \text{span}(x_1, \dots, x_j)$ , it follows that  $a_{ij} = 0$  for all  $i > j$ . Therefore,  $[T_W]_\gamma$  is upper-triangular.  $\square$

**Problem 2.** Let  $l^2$  denote the sequence space of all real or complex value sequences  $x = (x_1, x_2, \dots)$  such that

$$\left( \sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}} < \infty.$$

Further define  $T: l^2 \rightarrow l^2$  by  $T(x) = (0, x_1, x_2, \dots)$  and  $U: l^2 \rightarrow l^2$  by  $U(x) = (x_2, x_3, \dots)$ .

**Theorem 3.**  *$T$  is linear, one-to-one, but not onto.*

*Proof.* Let  $x, y \in l^2$  and  $\alpha \in \mathbb{F}$ , where  $\mathbb{F}$  is the real or complex numbers. Then

$$\begin{aligned} T(x + \alpha y) &= (0, x_1 + \alpha y_1, x_2 + \alpha y_2, \dots) \\ &= (0, x_1, x_2, \dots) + \alpha(0, y_1, y_2, \dots) \\ &= T(x) + \alpha T(y). \end{aligned}$$

It is clear from the above equations that  $T$  is linear. Furthermore, if  $T(x) = T(y)$ , then it is clear that  $x = y$ . Thus,  $T$  is one-to-one. However,  $T$  cannot map to any sequences whose first component is not zero, so  $T$  is not onto.  $\square$

**Theorem 4.**  *$U$  is linear, onto, but not one-to-one.*

*Proof.* Let  $x, y \in l^2$  and  $\alpha \in \mathbb{F}$ , where  $\mathbb{F}$  is the real or complex numbers. Then

$$\begin{aligned} U(x + \alpha y) &= (x_2 + \alpha y_2, x_3 + \alpha y_3, \dots) \\ &= (x_2, x_3, \dots) + \alpha(y_2, y_3, \dots) \\ &= U(x) + \alpha U(y). \end{aligned}$$

It is clear from the above equations that  $U$  is linear. Furthermore, for any  $y = (y_1, y_2, \dots)$  define  $x = (0, y_1, y_2, \dots)$ . Then,  $U(x) = y$  and it follows that  $U$  is onto. However, since we could replace the first entry of  $x$  with any nonzero element, it follows that  $U$  is not one-to-one.  $\square$

**Theorem 5.** *T is isometric, but U is not.*

*Proof.* To show that  $T$  is isometric, note that

$$\begin{aligned} d(T(x), T(y)) &= \left( \sum_{i=1}^{\infty} |T(x)_i - T(y)_i|^2 \right)^{\frac{1}{2}} \\ &= \left( \sum_{i=2}^{\infty} |x_{i-1} - y_{i-1}|^2 \right)^{\frac{1}{2}} \\ &= \left( \sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{\frac{1}{2}} \\ &= d(x, y). \end{aligned}$$

Since  $T$  preserves distances it is isometric.

Conversely, let  $x = (1, 0, \dots)$  and  $y = (2, 0, \dots)$ . Then  $d(x, y) = 1$  and  $d(U(x), U(y)) = 0$ . Therefore,  $U$  is not isometric.  $\square$

**Problem 3.**

## References

- [1] S.H. Friedberg, A.H. Insel, and L.E. Spence. *Linear Algebra*. Pearson Education, Upper Saddle River, NJ, 4th edition, 2003.