COMP2022: Formal Languages and Logic

2017, Semester 1, Week 11

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Adapted from slides by A/Prof Kalina Yacef

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Announcements

Assignment 3:

- Due on Thursday of week **12**
- May be completed individually or in pairs
- Proofs on the Logic Tutor
- 4 marks for correctness
- 1 mark based on proof length

Outline

Predicate Logic (continued)

- Substitution of variables
- Semantics in predicate logic
- Validity, Satisfiability
- "Free to replace"
- Equivalence Laws involving quantifiers
- Inference Rules involving quantifiers
- Tautologies

Substitution for Free variables

Suppose W is a wff, x is a free variable in W, and t is a term

$W[t/x]$ is the wff obtained by substituting t for all *free* occurrences of x

i.e. replacing all *free* occurrences of x with t

Examples:

- If $W = \forall x P(x, y)$ then
  - $W[t/x] = \forall x P(x, y) = W$ (x was already bound)
  - $W[t/y] = \forall x P(x, t) \neq W$
- If $Z = (P(x, y) \vee \exists y Q(x, y))$ then
  - $Z[t/x] = (P(t, y) \vee \exists y Q(t, y)) \neq Z$
  - $Z[t/y] = (P(x, t) \vee \exists y Q(x, y)) \neq Z$

Meaning (semantics) of a wff

The meaning of a wff with respect to an interpretation with domain D, is the truth value obtained by applying the following rules:

- If the wff has no quantifier, then the truth value of the proposition is obtained by applying the interpretation to the wff
- $\forall x W$ is true if $W[d/x]$ is true for every $d \in D$. Otherwise it is false.
- $\exists x W$ is true if $W[d/x]$ is true for some $d \in D$. Otherwise it is false.

Example

Existential quantification on the domain of interpretation
$D = \{a_1, a_2..., a_n\}$
$\exists x P(x) \equiv P(a_1) \vee P(a_2) \vee ... \vee P(a_n)$
e.g. suppose

- $D = \{a, b, c, d, e\}$
- $P(a), P(b), P(c), Q(a), Q(b), R(c), R(d)$ are true, all others false.

Then is $\exists x (P(x) \wedge (\neg Q(x) \to R(x)))$ true or false?

Example

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Then is $\exists x(P(x) \wedge (\neg Q(x) \rightarrow R(x)))$ true or false?

- Let $W = (P(x) \wedge (\neg Q(x) \rightarrow R(x)))$
- Notice that $W[c/x] = (P(c) \wedge (\neg Q(c) \rightarrow R(c)))$ is true
- Therefore $\exists x W$ is true

Example

Universal quantification on the domain of interpretation
$D = \{a_1, a_2 ..., a_n\}$
$\forall x P(x) \equiv P(a_1) \land P(a_2) \land ... \land P(a_n)$
e.g. suppose

- $D = \{a, b, c, d, e\}$
- $P(a), P(b), P(c), Q(a), Q(b), R(c), R(d)$ are true, all others false.

Then is $\forall x (P(x) \rightarrow (Q(x) \lor R(x)))$ true or false?

Example

Universal quantification on the domain of interpretation

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$\forall x P(x) \equiv P(a_1) \wedge P(a_2) \wedge ... \wedge P(a_n)$

e.g. suppose

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Then is $\forall x(P(x) \rightarrow (Q(x) \vee R(x)))$ true or false?

- Let $W = (P(x) \rightarrow (Q(x) \vee R(x)))$
- $W[a/x] = (P(a) \rightarrow (Q(a) \vee R(a))) = T \rightarrow (T \vee F) = T$
- $W[b/x] = (P(b) \rightarrow (Q(b) \vee R(b))) = T \rightarrow (T \vee F) = T$
- $W[c/x] = (P(c) \rightarrow (Q(c) \vee R(c))) = T \rightarrow (F \vee T) = T$
- $W[d/x] = (P(d) \rightarrow (Q(d) \vee R(d))) = F \rightarrow (F \vee T) = T$
- $W[e/x] = (P(e) \rightarrow (Q(e) \vee R(e))) = F \rightarrow (F \vee F) = T$
- so $\forall x W \equiv W[a/x] \wedge W[b/x] \wedge W[c/x] \wedge W[d/x] \wedge W[e/x]$ is true

Examples

$P(a, c)$ "a is a parent of c", Domain = humans

Suppose Jim is a parent of Andy, but Fred is not a parent of anyone

P(Jim, Andy)	true
P(Fred, Andy)	false
$\exists a P(a,$ Andy$)$	
$\forall a P(a,$ Andy$)$	
$\forall a \exists c P(a, c)$	
$\forall c \exists a P(a, c)$	
$\exists a \forall c P(a, c)$	
$\exists c \forall a P(a, c)$	
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$P(\text{Jim, Andy})$	true
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EXAMPLES

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Different interpretations lead to different semantics

$W = \exists x \forall y (P(y) \to Q(x,y))$
$Q(x,y)$: "x is equal to y"

- Let $D = \{1\}$ and $P(1) =$ true. Then W is true.
- Let $D = \{1\}$ and $P(1) =$ false. Then W is true.
- Let $D = \{1,2\}$ and $P(1) = P(2) =$ true. Then W is false.
  - Let $x = 1$, then
    - Let $y = 1$, then $(P(1) \to Q(1,1)) =$ true
    - Let $y = 2$, then $(P(1) \to Q(1,2)) =$ false
    - $\mathit{true} \wedge \mathit{false} = \mathit{false}$, so $\forall y (P(y) \to Q(1,y))$ is false
  - Let $x = 2$, then
    - Let $y = 1$, then $(P(2) \to Q(2,1)) =$ false
    - Let $y = 2$, then $(P(2) \to Q(2,2)) =$ true
    - $\mathit{false} \wedge \mathit{true} = \mathit{false}$, so $\forall y (P(y) \to Q(2,y))$ is false
  - $\mathit{false} \vee \mathit{false} = \mathit{false}$, so $\exists x \forall y (P(y) \to Q(x,y))$ is false

Validity and satisfiability

A wff is *valid* if it is *true for all* possible interpretations, otherwise it is *invalid*

A wff is *unsatisfiable* if it is *false for all* possible interpretations, otherwise it is *satisfiable*

- Valid (and satisfiable) $\rightarrow$ tautology
- (Invalid and) unsatisfiable $\rightarrow$ contradiction
- Invalid but satisfiable $\rightarrow$ contingency

Validity and satisfiability

- $\exists x \forall y (P(y) \rightarrow Q(x, y))$ is satisfiable and invalid
  - Satisfiable because there is some interpretation for which it is not false
  - Invalid because it is not true for all interpretations
- $\forall x (P(x) \vee \neg P(x))$ is a tautology
- $\forall x (P(x) \wedge \neg P(x))$ is a contradiction

Logical equivalences/implications

Propositions are predicates of arity 0, so all the tautologies for propositional logic also hold in predicate logic

- $(A \lor \neg A)$
- $(A(x) \lor \neg A(x))$
- $(\forall A(x) \lor \neg \forall A(x))$
- ...

- $A \to B, A \vdash B$
- $A(x) \to B(y), A(x) \vdash B(y)$
- $\forall x A(x) \to \forall y B(y), \forall x A(x) \vdash \forall y B(y)$
- ...

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- $\forall x A(x) \to \forall y B(y), \forall x A(x) \vdash \forall y B(y)$
- ...

Therefore all the laws of equivalence and rules of inferences seen in propositional logic can also be for predicate logic, when each proposition if replaced by a wff for predicate logic.

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- $(A(x) \vee \neg A(x))$
- $(\forall A(x) \vee \neg \forall A(x))$
- ...

- $A \rightarrow B, A \vdash B$
- $A(x) \rightarrow B(y), A(x) \vdash B(y)$
- $\forall x A(x) \rightarrow \forall y B(y), \forall x A(x) \vdash \forall y B(y)$
- ...

Therefore all the laws of equivalence and rules of inferences seen in propositional logic can also be for predicate logic, when each proposition if replaced by a wff for predicate logic.

We also need specific laws and rules involving quantifiers:

- Laws of equivalence involving quantifiers
- Rules of inference involving quantifiers (adding and removing quantifiers)

Natural Deduction in Predicate Logic

Simple example which does not involve reasoning with quantifiers:

Premises as below $\vdash (P(f(x)) \to \forall y P(y))$

Premises	Line	Formula	Just.	Refs.
1	1	$(\forall x P(x) \to Q(f(x), x))$	P	
2	2	$\neg Q(f(x), x)$	P	
3	3	$(\neg Q(f(x), x) \to (\exists y Q(x, y) \to \forall y P(y)))$	P	
4	4	$(\forall x P(x) \vee (P(f(x)) \to \exists y Q(x, y)))$	P	
2,3	5	$(\exists y Q(x, y) \to \forall y P(y))$	MP	2,3
1,2	6	$\neg \forall x P(x)$	MT	1,2
1,2,4	7	$(P(f(x)) \to \exists y Q(x, y))$	DS	4,6
1,2,3,4	8	$(P(f(x)) \to \forall y P(y))$	HS	5,7

Natural Deduction in Predicate Logic

Simple example which does not involve reasoning with quantifiers:

$$\text{Premises as below} \vdash (P(f(x)) \to \forall y P(y))$$

Let $A = P(f(x)), B = \forall y P(y), C = \exists y Q(x, y), D = Q(f(x), x), E = \forall x P(x)$

Premises	Line	Formula	Just.	Refs.
1	1	$(E \to D)$	P	
2	2	$\neg D$	P	
3	3	$(\neg D \to (C \to B))$	P	
4	4	$(E \vee (A \to C))$	P	
2,3	5	$(C \to B)$	MP	2,3
1,2	6	$\neg E$	MT	1,2
1,2,4	7	$(A \to C)$	DS	4,6
1,2,3,4	8	$(A \to B)$	HS	5,7

Laws of Equivalence involving quantifiers

Let F and G be arbitrary wffs

Quantifiers and Negation (QNeg)

$$\neg\forall xF \equiv \exists x\neg F$$

$$\neg\exists xF \equiv \forall x\neg F$$

Interchanging of quantifiers of same type (QInter)

Laws of Equivalence involving quantifiers

Let F and G be arbitrary wffs

Quantifiers and Negation (QNeg)

$$\neg\forall xF \equiv \exists x\neg F$$
$$\neg\exists xF \equiv \forall x\neg F$$

Distribution of quantifiers (QDistr)

$$(\forall xF \wedge \forall xG) \equiv \forall x(F \wedge G)$$
$$(\exists xF \vee \exists xG) \equiv \exists x(F \vee G)$$

Interchanging of quantifiers of same type (QInter)

Laws of Equivalence involving quantifiers

Let F and G be arbitrary wffs

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$$\neg\forall xF \equiv \exists x\neg F$$

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Interchanging of quantifiers of same type (QInter)

$$\forall x\forall yF \equiv \forall y\forall xF$$

$$\exists x\exists yF \equiv \exists y\exists xF$$

LAWS OF EQUIVALENCE INVOLVING QUANTIFIERS

Let F and G be arbitrary wffs

Extraction of quantifiers (QExtr)

If x does not occur in G:

$$(\forall x F \wedge G) \equiv \forall x(F \wedge G)$$
$$(\forall x F \vee G) \equiv \forall x(F \vee G)$$
$$(\exists x F \wedge G) \equiv \exists x(F \wedge G)$$
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Laws of Equivalence involving quantifiers

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$$(\exists x F \wedge G) \equiv \exists x(F \wedge G)$$
$$(\exists x F \vee G) \equiv \exists x(F \vee G)$$

similarly:

$$(G \wedge \forall x F) \equiv \forall x(G \wedge F)$$
$$(G \vee \forall x F) \equiv \forall x(G \vee F)$$
$$(G \wedge \exists x F) \equiv \exists x(G \wedge F)$$
$$(G \vee \exists x F) \equiv \exists x(G \vee F)$$

EXAMPLE

Suppose we want to derive the following equivalence:

$$\neg(\exists x P(x,y) \lor \forall z \neg R(z)) \equiv \forall x \exists z (\neg P(x,y) \land R(z))$$

$$\neg(\exists x P(x,y) \lor \forall z \neg R(z))$$

Example

Suppose we want to derive the following equivalence:

$$\neg(\exists x P(x,y) \vee \forall z \neg R(z)) \equiv \forall x \exists z (\neg P(x,y) \wedge R(z))$$

$$\begin{aligned} & \neg(\exists x P(x,y) \vee \forall z \neg R(z)) & \\ \equiv\ & (\neg\exists x P(x,y) \wedge \neg\forall z \neg R(z)) & \qquad \text{DeMorgan's Laws} \end{aligned}$$

Example

Suppose we want to derive the following equivalence:

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$$\begin{aligned} & \neg(\exists x P(x,y) \vee \forall z \neg R(z)) & \\ \equiv\ & (\neg\exists x P(x,y) \wedge \neg\forall z \neg R(z)) & \text{DeMorgan's Laws} \\ \equiv\ & (\forall x \neg P(x,y) \wedge \exists z \neg\neg R(z)) & \text{Quantifier Negation} \end{aligned}$$

Example

Suppose we want to derive the following equivalence:

$$\neg(\exists x P(x,y) \vee \forall z \neg R(z)) \equiv \forall x \exists z(\neg P(x,y) \wedge R(z))$$

$$\begin{aligned}
&\neg(\exists x P(x,y) \vee \forall z \neg R(z)) && \\
\equiv\ &(\neg\exists x P(x,y) \wedge \neg\forall z \neg R(z)) && \text{DeMorgan's Laws} \\
\equiv\ &(\forall x \neg P(x,y) \wedge \exists z \neg\neg R(z)) && \text{Quantifier Negation} \\
\equiv\ &(\forall x \neg P(x,y) \wedge \exists z R(z)) && \text{Double Negation}
\end{aligned}$$

Example

Suppose we want to derive the following equivalence:

$$\neg(\exists x P(x,y) \lor \forall z \neg R(z)) \equiv \forall x \exists z(\neg P(x,y) \land R(z))$$

$\neg(\exists x P(x,y) \lor \forall z \neg R(z))$	
$\equiv (\neg\exists x P(x,y) \land \neg\forall z \neg R(z))$	DeMorgan's Laws
$\equiv (\forall x \neg P(x,y) \land \exists z \neg\neg R(z))$	Quantifier Negation
$\equiv (\forall x \neg P(x,y) \land \exists z R(z))$	Double Negation
$\equiv \forall x(\neg P(x,y) \land \exists z R(z))$	Quantifier Extraction

Example

Suppose we want to derive the following equivalence:

$$\neg(\exists x P(x,y) \vee \forall z \neg R(z)) \equiv \forall x \exists z(\neg P(x,y) \wedge R(z))$$

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$\equiv (\forall x \neg P(x,y) \wedge \exists z \neg\neg R(z))$	Quantifier Negation
$\equiv (\forall x \neg P(x,y) \wedge \exists z R(z))$	Double Negation
$\equiv \forall x(\neg P(x,y) \wedge \exists z R(z))$	Quantifier Extraction
$\equiv \forall x \exists z(\neg P(x,y) \wedge R(z))$	Quantifier Extraction

Reasoning with quantifiers

Simple example: if a property holds for everything then it holds for a particular thing.

- So, it seems reasonable to deduce $W(a)$ from $\forall x W(x)$
  - Similarly ($W(b)$, $W(x)$, $W(f(x)), \ldots$ etc.)

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Simple example: if a property holds for everything then it holds for a particular thing.

- So, it seems reasonable to deduce $W(a)$ from $\forall x W(x)$
  - Similarly $(W(b), W(x), W(f(x)), \ldots$ etc.)
- *But not always!*

Consider $\forall x \exists y P(x, y)$

- Can we deduce $\exists y P(a, y)$?
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Reasoning with quantifiers

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Consider $\forall x \exists y P(x, y)$

- Can we deduce $\exists y P(a, y)$? YES
- Can we deduce $\exists y P(x, y)$?
- Can we deduce $\exists y P(y, y)$?

Reasoning with quantifiers

Simple example: if a property holds for everything then it holds for a particular thing.

- So, it seems reasonable to deduce $W(a)$ from $\forall x W(x)$
  - Similarly $(W(b), W(x), W(f(x)), \dots$ etc.)
- *But not always!*

Consider $\forall x \exists y P(x, y)$

- Can we deduce $\exists y P(a, y)$? YES
- Can we deduce $\exists y P(x, y)$? YES
- Can we deduce $\exists y P(y, y)$?

REASONING WITH QUANTIFIERS

Simple example: if a property holds for everything then it holds for a particular thing.

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  - Similarly ($W(b)$, $W(x)$, $W(f(x))$, ... etc.)
- *But not always!*

Consider $\forall x \exists y P(x, y)$

- Can we deduce $\exists y P(a, y)$? YES
- Can we deduce $\exists y P(x, y)$? YES
- Can we deduce $\exists y P(y, y)$? NO

Reasoning with quantifiers

e.g. If P(x,y) means x is the child of y then:

- $\forall x \exists y P(x, y)$ means
- $\exists y P(x, y)$ means
- $\exists y P(y, y)$ means

REASONING WITH QUANTIFIERS

e.g. If P(x,y) means x is the child of y then:

- $\forall x \exists y P(x, y)$ means everyone is the child of someone
- $\exists y P(x, y)$ means
- $\exists y P(y, y)$ means

Reasoning with quantifiers

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Reasoning with quantifiers

e.g. If P(x,y) means x is the child of y then:

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- $\exists y P(y, y)$ means someone is their own child (?!)

(Beware the time travel paradox quokka!)

Trouble arises when we try to infer $W(t)$ from $\forall x W(x)$ where t:

1. Contains an occurrence of a quantified variable, and
2. x occurs free within the scope of that quantifier

Definition: Free to replace

Suppose t is a term and x is a free variable in F

t is *free to replace* x in F if either:

1. no variable in t occurs bound to a quantifier in F
2. or, x does not occur free within the scope of a quantifier in F

i.e. both F and $F[t/x]$ have the *same* bound occurrences of variables.

(Recall that $F[t/x]$ is the wff in obtained from F by substituting t for all free occurrences of x in F)

Examples: Free to replace

t is *free to replace* x in F if either:

1. no variable in t occurs bound to a quantifier in F
2. or, x does not occur free within the scope of a quantifier in F

Example: $\forall y(P(y) \rightarrow G(y, x))$

- Is $f(y)$ free to replace x?
- Is $f(x, y)$ free to replace x?
- Is $f(x)$ free to replace x?

EXAMPLES: FREE TO REPLACE

t is *free to replace* x in F if either:

1. no variable in t occurs bound to a quantifier in F
2. or, x does not occur free within the scope of a quantifier in F

Example: $\forall y(P(y) \rightarrow G(y, x))$

- Is $f(y)$ free to replace x? NO, because a variable of $f(y)$ occurs bound and x occurs free within the scope of a quantifier.

- Is $f(x, y)$ free to replace x?

- Is $f(x)$ free to replace x?

Examples: Free to replace

t is *free to replace* x in F if either:

1. no variable in t occurs bound to a quantifier in F
2. or, x does not occur free within the scope of a quantifier in F

Example: $\forall y(P(y) \rightarrow G(y, x))$

- Is $f(y)$ free to replace x? NO, because a variable of $f(y)$ occurs bound and x occurs free within the scope of a quantifier.
- Is $f(x, y)$ free to replace x? NO, because a variable of $f(x, y)$ occurs bound and x occurs free within the scope of a quantifier.
- Is $f(x)$ free to replace x?

Examples: Free to replace

t is *free to replace* x in F if either:

1. no variable in t occurs bound to a quantifier in F
2. or, x does not occur free within the scope of a quantifier in F

Example: $\forall y(P(y) \rightarrow G(y, x))$

- Is $f(y)$ free to replace x? NO, because a variable of $f(y)$ occurs bound and x occurs free within the scope of a quantifier.
- Is $f(x, y)$ free to replace x? NO, because a variable of $f(x, y)$ occurs bound and x occurs free within the scope of a quantifier.
- Is $f(x)$ free to replace x? YES, because no variable of $f(x)$ occurs bound in the resulting formula

Rules of inference involving quantifiers

∃-elimination (Existential instantiation)

$\dfrac{S \vdash \exists x F}{S \vdash F[c/x]}$	where c is a new constant

∀-elimination (Universal Instantiation)

$\dfrac{S \vdash \forall x F}{S \vdash F[t/x]}$	where t is free to replace x in F

∃-introduction (Existential generalisation)

$\dfrac{S \vdash F}{S \vdash \exists x F[x/t]}$	where x is free to replace t in F

∀-introduction (Universal generalisation)

$\dfrac{S \vdash F}{S \vdash \forall x F}$	x is not free in any of the premises used to deduce F, and x is not free in any wff constructed by ∃-elimination

∃-elimination (Existential instantiation)

If a property holds for something, then it holds for a particular thing

$$\frac{S \vdash \exists x F}{S \vdash F[c/x]} \quad \text{where } c \text{ is a new constant}$$

i.e. c has not previously been defined (either earlier in the argument, or in the conclusion.)

$\exists$-elimination (Existential instantiation)

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$$\frac{S \vdash \exists x F}{S \vdash F[c/x]} \quad \text{where } c \text{ is a new constant}$$

i.e. c has not previously been defined (either earlier in the argument, or in the conclusion.)

From $\exists x \forall y P(x, y)$ we can deduce $\forall y P(c, y)$ if c is a new constant.

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i.e. c has not previously been defined (either earlier in the argument, or in the conclusion.)

From $\exists x \forall y P(x, y)$ we can deduce $\forall y P(c, y)$ if c is a new constant.

But these arguments don't work:

1	1	$\exists x P(x)$	P	
2	2	$\exists x Q(x)$	P	
1	3	$P(c)$	$\exists$-elim	1
2	4	$Q(c)$	$\exists$-elim	2

Line 4 is an error, as c was previously defined on line 3.

Attempt to prove $\exists x P(x) \vdash P(c)$

1	1	$\exists x P(x)$	P	
1	2	$P(c)$	$\exists$-elim	1

Line 2 is an error, as c was previously defined in the conclusion.

$\forall$-elimination (Universal Instantiation)

If a property holds for everything, then it holds for any particular thing:

$$\frac{S \vdash \forall x F}{S \vdash F[t/x]} \qquad \text{where } t \text{ is free to replace } x \text{ in } F$$

Suppose we know $F = \forall x \exists y G(x, y, f(x))$. Can we deduce:

- $\exists y G(2, y, f(2))$

$\forall$-ELIMINATION (UNIVERSAL INSTANTIATION)

If a property holds for everything, then it holds for any particular thing:

$$\frac{S \vdash \forall x F}{S \vdash F[t/x]}$$ where t is free to replace x in F

Suppose we know $F = \forall x \exists y G(x, y, f(x))$. Can we deduce:

- $\exists y G(2, y, f(2))$ YES
- $\exists y G(x, y, f(x))$

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If a property holds for everything, then it holds for any particular thing:

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Suppose we know $F = \forall x \exists y G(x, y, f(x))$. Can we deduce:

- $\exists y G(2, y, f(2))$ YES
- $\exists y G(x, y, f(x))$ YES
- $\exists y G(f(x), y, f(f(x)))$

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If a property holds for everything, then it holds for any particular thing:

$$\frac{S \vdash \forall x F}{S \vdash F[t/x]} \quad \text{where } t \text{ is free to replace } x \text{ in } F$$

Suppose we know $F = \forall x \exists y G(x, y, f(x))$. Can we deduce:

- $\exists y G(2, y, f(2))$ YES
- $\exists y G(x, y, f(x))$ YES
- $\exists y G(f(x), y, f(f(x)))$ YES
- $\exists y G(y, y, f(y))$

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If a property holds for everything, then it holds for any particular thing:

$$\frac{S \vdash \forall x F}{S \vdash F[t/x]}$$ where t is free to replace x in F

Suppose we know $F = \forall x \exists y G(x, y, f(x))$. Can we deduce:

- $\exists y G(2, y, f(2))$ YES
- $\exists y G(x, y, f(x))$ YES
- $\exists y G(f(x), y, f(f(x)))$ YES
- $\exists y G(y, y, f(y))$ NO. Why not?

$\forall$-elimination (Universal Instantiation)

If a property holds for everything, then it holds for any particular thing:

$$\frac{S \vdash \forall x F}{S \vdash F[t/x]} \quad \text{where } t \text{ is free to replace } x \text{ in } F$$

Suppose we know $F = \forall x \exists y G(x, y, f(x))$. Can we deduce:

- $\exists y G(2, y, f(2))$ YES
- $\exists y G(x, y, f(x))$ YES
- $\exists y G(f(x), y, f(f(x)))$ YES
- $\exists y G(y, y, f(y))$ NO. Why not?

Suppose we know $\forall x G(x, y, f(x))$. Can we deduce:

- $G(y, y, f(y))$

$\forall$-elimination (Universal Instantiation)

If a property holds for everything, then it holds for any particular thing:

$$\frac{S \vdash \forall x F}{S \vdash F[t/x]} \qquad \text{where } t \text{ is free to replace } x \text{ in } F$$

Suppose we know $F = \forall x \exists y G(x, y, f(x))$. Can we deduce:

- $\exists y G(2, y, f(2))$ YES
- $\exists y G(x, y, f(x))$ YES
- $\exists y G(f(x), y, f(f(x)))$ YES
- $\exists y G(y, y, f(y))$ NO. Why not?

Suppose we know $\forall x G(x, y, f(x))$. Can we deduce:

- $G(y, y, f(y))$ YES. Why?

∃-INTRODUCTION (EXISTENTIAL GENERALISATION)

If a property holds a particular thing, then it holds for something:

$$\frac{S \vdash F}{S \vdash \exists x F[x/t]}$$ where x is free to replace t in F

Examples:

$\exists$-introduction (Existential generalisation)

If a property holds a particular thing, then it holds for something:

$$\frac{S \vdash F}{S \vdash \exists x F[x/t]}$$ where x is free to replace t in F

Examples:

- From $\forall y P(c, y)$ we can deduce $\exists x \forall y P(x, y)$

$\exists$-introduction (Existential generalisation)

If a property holds a particular thing, then it holds for something:

$$\frac{S \vdash F}{S \vdash \exists x F[x/t]}$$ where x is free to replace t in F

Examples:

- From $\forall y P(c, y)$ we can deduce $\exists x \forall y P(x, y)$
- From $\forall y P(2, y)$ we can deduce $\exists x \forall y P(x, y)$

$\exists$-introduction (Existential generalisation)

If a property holds a particular thing, then it holds for something:

$$\frac{S \vdash F}{S \vdash \exists x F[x/t]}$$ where x is free to replace t in F

Examples:

- From $\forall y P(c, y)$ we can deduce $\exists x \forall y P(x, y)$
- From $\forall y P(2, y)$ we can deduce $\exists x \forall y P(x, y)$
- From $\forall y P(f(x, z), y)$ we can deduce any of these:
  - $\exists x \forall y P(x, y)$ (replaced term $f(x, z)$ with bound x)
  - $\exists x \forall y P(f(x, z), y)$
  - $\exists w \forall y P(f(w, z), y)$

$\exists$-INTRODUCTION (EXISTENTIAL GENERALISATION)

If a property holds a particular thing, then it holds for something:

$$\frac{S \vdash F}{S \vdash \exists x F[x/t]} \quad \text{where } x \text{ is free to replace } t \text{ in } F$$

Examples:

- From $\forall y P(c, y)$ we can deduce $\exists x \forall y P(x, y)$
- From $\forall y P(2, y)$ we can deduce $\exists x \forall y P(x, y)$
- From $\forall y P(f(x, z), y)$ we can deduce any of these:
  - $\exists x \forall y P(x, y)$ (replaced term $f(x, z)$ with bound x)
  - $\exists x \forall y P(f(x, z), y)$
  - $\exists w \forall y P(f(w, z), y)$
- But from $\forall y P(f(x, y), y)$ we cannot deduce $\exists x \forall y P(x, y)$
  Why not?

∀-introduction (Universal generalisation)

If a property holds for an arbitrary thing, then it holds for all things:

$\dfrac{S \vdash F}{S \vdash \forall x F}$	x is not free in any of the premises used to deduce F, and x is not free in any wff constructed by ∃-elimination

We let x be an arbitrary but fixed element of the domain D. Next, we construct a proof that F is true for x. Then we can say that since x was arbitrarily chosen, it follows that F is true for any x of the domain.

However, if during our argument we made any assumptions about the value of x (i.e. if we used it in a formula inferred by ∃-elimination), then we can no longer claim x was arbitrarily chosen.)

$\forall$-introduction (Universal generalisation)

Example of incorrect usage

Requirements:

- x is not free in any of the premises used to deduce F
- x is not free in any wff constructed by $\exists$-elimination

Premises	Line	Formula	Justification	References
1	1	$P(x)$	P	
1	2	$\forall x P(x)$	$\forall$-intro	1

x is free in premise 1, so we cannot justify line 2.

Example: Domain of natural numbers, $P(x)$: x is prime

∀-introduction (Universal generalisation)

Example of incorrect usage

Requirements:

- x is not free in any of the premises used to deduce F
- x is not free in any wff constructed by ∃-elimination

Premises	Line	Formula	Justification	References
1	1	$\forall x \exists y P(x, y)$	P	
1	2	$\exists y P(x, y)$	∀-elim	1
1	3	$P(x, c)$	∃-elim	2
1	4	$\forall x P(x, c)$	∀-intro	3

x was free in a wff constructed by ∃-elimination, therefore we cannot justify line 4.

Example: Domain of natural numbers, $P(x, y) : x < y$

Formal proof examples (1)

All humans are mortal. Socrates is human. Hence, Socrates is mortal.

$$\forall x(H(x) \to M(x)),\ H(Soc)\ \vdash\ M(Soc)$$

Premises	Line	Formula	Justification	References
1	1	$\forall x(H(x) \to M(x))$	P	
2	2	$H(Soc)$	P	
1	3	$(H(Soc) \to M(Soc))$	$\forall$-elim	1
1,2	4	$M(Soc)$	MP	2,3

Formal proof examples (2)

Prove $\forall x P(x),\ \exists x Q(x)\ \vdash\ \exists x(P(x) \wedge Q(x))$

P.	L.	Formula	Just.	Refs.
1	1	$\forall x P(x)$	P	
2	2	$\exists x Q(x)$	P	
2	3	$Q(c)$ (c is a new constant)	$\exists$-elim	2
1	4	$P(c)$ (c free to replace x)	$\forall$-elim	1
1,2	5	$(P(c) \wedge Q(c))$	Conj.	3,4
1,2	6	$\exists x(P(x) \wedge Q(x))$ (c free to replace x)	$\exists$-intro	5

The order of lines 3 and 4 is very important here. If we did the $\forall$-elim first, then we could not have used c in the $\exists$-elim, because it would not have been a new constant.

FORMAL PROOF EXAMPLES (3)

Prove

$\forall x(A(x) \to B(x)),\ \forall x(B(x) \to C(x)) \vdash \forall x(A(x) \to C(x))$

P.	L.	Formula	Just.	Refs.
1	1	$\forall x(A(x) \to B(x))$	P	
2	2	$\forall x(B(x) \to C(x))$	P	

FORMAL PROOF EXAMPLES (3)

Prove

$\forall x(A(x) \to B(x)),\ \forall x(B(x) \to C(x)) \vdash \forall x(A(x) \to C(x))$

P.	L.	Formula	Just.	Refs.
1	1	$\forall x(A(x) \to B(x))$	P	
2	2	$\forall x(B(x) \to C(x))$	P	
1	3	$(A(x) \to B(x))$	$\forall$-elim	1

FORMAL PROOF EXAMPLES (3)

Prove

$$\forall x(A(x) \to B(x)),\ \forall x(B(x) \to C(x))\ \vdash \forall x(A(x) \to C(x))$$

P.	L.	Formula	Just.	Refs.
1	1	$\forall x(A(x) \to B(x))$	P	
2	2	$\forall x(B(x) \to C(x))$	P	
1	3	$(A(x) \to B(x))$	$\forall$-elim	1
2	4	$(B(x) \to C(x))$	$\forall$-elim	2

Formal proof examples (3)

Prove
$\forall x(A(x) \to B(x)),\ \forall x(B(x) \to C(x)) \vdash \forall x(A(x) \to C(x))$

P.	L.	Formula	Just.	Refs.
1	1	$\forall x(A(x) \to B(x))$	P	
2	2	$\forall x(B(x) \to C(x))$	P	
1	3	$(A(x) \to B(x))$	$\forall$-elim	1
2	4	$(B(x) \to C(x))$	$\forall$-elim	2
1,2	5	$(A(x) \to C(x))$	HS	3,4

Formal proof examples (3)

Prove

$\forall x(A(x) \to B(x)),\ \forall x(B(x) \to C(x)) \vdash \forall x(A(x) \to C(x))$

P.	L.	Formula	Just.	Refs.
1	1	$\forall x(A(x) \to B(x))$	P	
2	2	$\forall x(B(x) \to C(x))$	P	
1	3	$(A(x) \to B(x))$	$\forall$-elim	1
2	4	$(B(x) \to C(x))$	$\forall$-elim	2
1,2	5	$(A(x) \to C(x))$	HS	3,4
1,2	6	$\forall x(A(x) \to C(x))$	$\forall$-intro	5

- x was always free to replace x
- x was not free in any of the premises used to deduce $(A(x) \to C(x))$, nor was free in a wff constructed by $\exists$-elim.

Formal proof examples (4)

Database, $S \vdash A(Brown, 66)$

P.	L.	Formula	Just.	Refs.
1	1	$CSG(2013, 1377660, 70)$	P	
2	2	$CSG(2013, 1377540, 66)$	P	
3	3	$CSG(2013, 1381660, 55)$	P	
4	4	$SN(1377540, Brown)$	P	
5	5	$SN(1381660, Liu)$	P	
6	6	$\forall s \forall g \forall n((CSG(2013, s, g) \wedge SN(s, n)) \rightarrow A(n, g))$	P	
6	7	$\forall g \forall n((CSG(2013, 1377540, g) \wedge SN(1377540, n)) \rightarrow A(n, g))$	$\forall$-elim	6
6	8	$\forall n((CSG(2013, 1377540, 66) \wedge SN(1377540, n)) \rightarrow A(n, 66))$	$\forall$-elim	6
6	9	$((CSG(2013, 1377540, 66) \wedge SN(1377540, Brown)) \rightarrow A(Brown, 66))$	$\forall$-elim	6
2,4	10	$(CSG(2013, 1377540, 66) \wedge SN(1377540, Brown))$	Conj	2,4
2,4,6	11	$A(Brown, 66)$	MP	9,10

Formal proof examples (5)

Lewis Carroll's logic:

Babies are illogical. Nobody is despised who can manage a crocodile. Illogical persons are despised. Therefore, babies cannot manage crocodiles.

P.	L.	Formula	Just.	Refs.
1	1	$\forall x(B(x) \to I(x))$	P	
2	2	$\forall x(C(x) \to \neg D(x))$	P	
3	3	$\forall x(I(x) \to D(x))$	P	

Tautologies

Tautologies of propositional logic are also tautologies in predicate logic, when the atomic formula is replaced with a wff of predicate logic. e.g.

- $(A \vee \neg A)$ is a tautology, so is $(\forall x P(x) \vee \neg \forall x P(x))$ (replacing A with $\forall x P(x)$)
- $(\neg P \vee (Q \to P))$ is a tautology, so is $(\neg \forall x P(x) \vee (\exists y Q(y) \to \forall x P(x)))$ (replacing P with $\forall x P(x)$ and Q with $\exists y Q(y)$)

Tautologies

Tautologies of propositional logic are also tautologies in predicate logic, when the atomic formula is replaced with a wff of predicate logic. e.g.

- $(A \vee \neg A)$ is a tautology, so is $(\forall x P(x) \vee \neg \forall x P(x))$ (replacing A with $\forall x P(x)$)
- $(\neg P \vee (Q \to P))$ is a tautology, so is $(\neg \forall x P(x) \vee (\exists y Q(y) \to \forall x P(x)))$ (replacing P with $\forall x P(x)$ and Q with $\exists y Q(y)$)

Other tautologies involve manipulating quantifiers, e.g.

- $(\forall x P(x) \to \exists x P(x))$
- $(\forall x P(x) \to \neg \exists x \neg P(x))$
- $\forall x (P(x) \vee \neg P(x))$
- $(P(x, y) \to (\exists x P(x, y) \wedge \exists y P(x, y)))$

Example

Prove $(\forall x P(x) \to \exists x P(x))$

Premises	Line	Formula	Justification	References
1	1		P	
	2			
	3			
	4			
	5			
	6			

Example

Prove $(\forall x P(x) \to \neg\exists x \neg P(x))$

Premises	Line	Formula	Justification	References
1	1		P	
	2			
	3			
	4			
	5			
	6			

EXAMPLE

Prove $\forall x(P(x) \vee \neg P(x))$

Premises	Line	Formula	Justification	References
1	1		P	
	2			
	3			
	4			
	5			
	6			

Example

Prove $(P(x, y) \to (\exists x P(x, y) \wedge \exists y P(x, y)))$

Premises	Line	Formula	Justification	References
1	1		P	
	2			
	3			
	4			
	5			
	6			