COMP2022: Formal Languages and Logic

2017, Semester 1, Week 8

Joseph Godbehere

Adapted from slides by A/Prof Kalina Yacef

May 2, 2017

THE UNIVERSITY OF SYDNEY

Feedback

Detailed feedback next week (when there's more time to spare)

- 94% thought the lectures were helpful
- 92% thought the tutorials were helpful

Comments

- Quizzes are too short $\rightarrow$ quizzes will be longer
- Lecture typos $\rightarrow$ I'll try!
- Desire for chocolate $\rightarrow$ mmm
- Voice is too quiet $\rightarrow$ ask me to turn up the mic!

Outline

- Tautologies and Contradictions
- Equivalence laws
  - Transformational proofs
- Inference rules
  - Deductive proofs
- Formal proofs using the Natural Deduction System

Revision

A *well-formed formula (wff)* is either:

- an atomic proposition
  - A statement which might be evaluated as either *true* or *false*
- a complex proposition constructed from other wff and connectives:
  - $\neg A$ negation
  - $(A \wedge B)$ conjunction
  - $(A \vee B)$ disjunction
  - $(A \rightarrow B)$ implication ($\Rightarrow$ has a similar meaning)
  - $(A \leftrightarrow B)$ equivalence ($\Leftrightarrow$ has a similar meaning)

Tautologies

A wff is a *tautology* if all the truth table values for the wff are true (1)

e.g. $(\neg P \vee P)$ and $(P \rightarrow P)$ are tautologies:

P	$\neg P$	$(\neg P \vee P)$	$(P \rightarrow P)$
1	0	1	1
0	1	1	1

Contradictions

A wff is a *contradiction* if all the truth table values for the wff are false (0)

e.g. $(P \wedge \neg P)$ and $(P \leftrightarrow \neg P)$ are both contradictions:

P	$\neg P$	$(P \wedge \neg P)$	$(P \leftrightarrow \neg P)$
1	0	0	0
0	1	0	0

Contingencies

A wff is a *contingency* if it is neither a tautology nor a contradiction. i.e. the truth table for the wff contains both truth and false values.

e.g. $(P \wedge Q)$ is a contingency, because its truth value is *contingent* on the assignment of truth values to P and Q

P	Q	$(P \wedge Q)$
1	1	1
1	0	0
0	1	0
0	0	0

Logical equivalence

Two propositions A and B are *logically equivalent* if and only if their equivalence is a tautology. We write $A \equiv B$

Example 1: $(P \wedge Q) \equiv (Q \wedge P)$ because $((P \wedge Q) \leftrightarrow (Q \wedge P))$ is a tautology.

P	Q	$(P \wedge Q)$	$(Q \wedge P)$	$((P \wedge Q) \leftrightarrow (Q \wedge P))$
1	1	1	1	1
1	0	0	0	1
0	1	0	0	1
0	0	0	0	1

Logical equivalence

Two propositions A and B are *logically equivalent* if and only if their equivalence is a tautology. We write $A \equiv B$

Example 2: $(\neg P \vee Q) \equiv (P \to Q)$ because $((\neg P \vee Q) \leftrightarrow (P \to Q))$ is a tautology.

P	Q	$\neg P$	$(\neg P \vee Q)$	$(P \to Q)$	$((\neg P \vee Q) \leftrightarrow (P \to Q))$
1	1	0	1	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

Logical vs Material equivalence

The truth of a *material equivalence* is contingent on the truth values assigned to its atomic propositions, whereas a logical equivalence is by definition always true (i.e. it is a tautology)

- $(A \leftrightarrow B)$ is a material equivalence
- $(A \leftrightarrow \neg\neg A)$ is a logical equivalence

Are these logical or material equivalences?

- The school is closed if and only if it is not the case that the school is open.
- Bill won the game if and only if Jim lost it.
- Eclipses occur if and only if the Moon comes in between the Sun and the Earth.

Logical vs Material equivalence

The truth of a *material equivalence* is contingent on the truth values assigned to its atomic propositions, whereas a logical equivalence is by definition always true (i.e. it is a tautology)

- $(A \leftrightarrow B)$ is a material equivalence
- $(A \leftrightarrow \neg\neg A)$ is a logical equivalence

Are these logical or material equivalences?

- The school is closed if and only if it is not the case that the school is open. $(C \leftrightarrow \neg\neg C)$
- Bill won the game if and only if Jim lost it.
- Eclipses occur if and only if the Moon comes in between the Sun and the Earth.

Logical vs Material equivalence

The truth of a *material equivalence* is contingent on the truth values assigned to its atomic propositions, whereas a logical equivalence is by definition always true (i.e. it is a tautology)

- $(A \leftrightarrow B)$ is a material equivalence
- $(A \leftrightarrow \neg\neg A)$ is a logical equivalence

Are these logical or material equivalences?

- The school is closed if and only if it is not the case that the school is open. $(C \leftrightarrow \neg\neg C)$ logical
- Bill won the game if and only if Jim lost it.
- Eclipses occur if and only if the Moon comes in between the Sun and the Earth.

Logical vs Material equivalence

The truth of a *material equivalence* is contingent on the truth values assigned to its atomic propositions, whereas a logical equivalence is by definition always true (i.e. it is a tautology)

- $(A \leftrightarrow B)$ is a material equivalence
- $(A \leftrightarrow \neg\neg A)$ is a logical equivalence

Are these logical or material equivalences?

- The school is closed if and only if it is not the case that the school is open. $(C \leftrightarrow \neg\neg C)$ logical
- Bill won the game if and only if Jim lost it. $(B \leftrightarrow J)$
- Eclipses occur if and only if the Moon comes in between the Sun and the Earth.

Logical vs Material equivalence

The truth of a *material equivalence* is contingent on the truth values assigned to its atomic propositions, whereas a logical equivalence is by definition always true (i.e. it is a tautology)

- $(A \leftrightarrow B)$ is a material equivalence
- $(A \leftrightarrow \neg\neg A)$ is a logical equivalence

Are these logical or material equivalences?

- The school is closed if and only if it is not the case that the school is open. $(C \leftrightarrow \neg\neg C)$ logical
- Bill won the game if and only if Jim lost it. $(B \leftrightarrow J)$ material
- Eclipses occur if and only if the Moon comes in between the Sun and the Earth.

Logical vs Material equivalence

The truth of a *material equivalence* is contingent on the truth values assigned to its atomic propositions, whereas a logical equivalence is by definition always true (i.e. it is a tautology)

- $(A \leftrightarrow B)$ is a material equivalence
- $(A \leftrightarrow \neg\neg A)$ is a logical equivalence

Are these logical or material equivalences?

- The school is closed if and only if it is not the case that the school is open. $(C \leftrightarrow \neg\neg C)$ logical
- Bill won the game if and only if Jim lost it. $(B \leftrightarrow J)$ material
- Eclipses occur if and only if the Moon comes in between the Sun and the Earth. $(E \leftrightarrow M)$

Logical vs Material equivalence

The truth of a *material equivalence* is contingent on the truth values assigned to its atomic propositions, whereas a logical equivalence is by definition always true (i.e. it is a tautology)

- $(A \leftrightarrow B)$ is a material equivalence
- $(A \leftrightarrow \neg\neg A)$ is a logical equivalence

Are these logical or material equivalences?

- The school is closed if and only if it is not the case that the school is open. $(C \leftrightarrow \neg\neg C)$ logical
- Bill won the game if and only if Jim lost it. $(B \leftrightarrow J)$ material
- Eclipses occur if and only if the Moon comes in between the Sun and the Earth. $(E \leftrightarrow M)$ material

Laws of Equivalence

If $(E \leftrightarrow F)$ is a tautology, then E and F are logically equivalent, denoted $E \equiv F$

$(E \leftrightarrow F)$	E	F
1	1	?
1	0	?

This means that in any complex wff R, we can replace any occurrence of E with F, or vice versa, without changing the truth value of R.

Laws of Equivalence

If $(E \leftrightarrow F)$ is a tautology, then E and F are logically equivalent, denoted $E \equiv F$

$(E \leftrightarrow F)$	E	F
1	1	? 1
1	0	?

This means that in any complex wff R, we can replace any occurrence of E with F, or vice versa, without changing the truth value of R.

Laws of Equivalence

If $(E \leftrightarrow F)$ is a tautology, then E and F are logically equivalent, denoted $E \equiv F$

$(E \leftrightarrow F)$	E	F
1	1	? 1
1	0	? 0

This means that in any complex wff R, we can replace any occurrence of E with F, or vice versa, without changing the truth value of R.

Example: commutation

The law of *commutation* is defined as $(P \wedge Q) \equiv (Q \wedge P)$

We can clearly see from the truth table that the two formulas are equivalent:

P	Q	$(P \wedge Q)$	$(Q \wedge P)$	$((P \wedge Q) \rightarrow (Q \wedge P))$
1	1	1	1	1
1	0	0	0	1
0	1	0	0	1
0	0	0	0	1

Laws of Equivalence: Introduction

Once proven to be tautologies, logical equivalences can be transformed into *Laws of Equivalence*

We may then apply these laws without needing to prove them again.

Commutation of conjunction is a Law of Equivalence, so whenever you need to replace $(P \wedge Q)$ with $(Q \wedge P)$, you can justify it by stating that you are using the commutation law without needing to prove that the two formulas are equivalent.

Laws of Equivalence

De Morgan's Laws (DeM)	$\neg(F \wedge G) \equiv (\neg F \vee \neg G)$ $\neg(F \vee G) \equiv (\neg F \wedge \neg G)$
Commutation (Comm)	$(F \vee G) \equiv (G \vee F)$ $(F \wedge G) \equiv (G \wedge F)$
Association (Assoc)	$(F \vee (G \vee H)) \equiv ((F \vee G) \vee H)$ $(F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H)$
Distribution (Dist)	$(F \wedge (G \vee H)) \equiv ((F \wedge G) \vee (F \wedge H))$ $(F \vee (G \wedge H)) \equiv ((F \vee G) \wedge (F \vee H))$
Double Negation (DN)	$F \equiv \neg\neg F$

Laws of Equivalence

Transposition (Trans) (also known as contrapositive)	$(F \to G) \equiv (\neg G \to \neg F)$
Def. of Implication	$(F \to G) \equiv (\neg F \lor G)$
Material Equivalence (Equiv)	$(F \leftrightarrow G) \equiv ((F \to G) \land (G \to F))$
Def. of Equivalence	$(F \leftrightarrow G) \equiv ((F \land G) \lor (\neg F \land \neg G))$
Exportation (Exp)	$((F \land G) \to H) \equiv (F \to (G \to H))$
Idempotence (Idem)	$F \equiv (F \lor F)$ $F \equiv (F \land F)$

Substitution Principles

Laws of Equivalence give general schemes.

Consider DeMorgan's Law: $\neg(F \land G) \equiv (\neg F \lor \neg G)$

F and G don't need to be atomic propositions, they can be *any* wffs.

For example, we can deduce:

$$\neg((P \land Q) \land R) \equiv (\neg(P \land Q) \lor \neg R)$$

by assigning $F = (P \land Q)$ and $G = R$

Substitution Principles

Substitutions using equivalence laws can be in sub-formula

Consider Commutation: $(F \vee G) \equiv (G \vee F)$

We can deduce:

$$(P \rightarrow (P \vee (Q \rightarrow R))) \equiv (P \rightarrow ((Q \rightarrow R) \vee P))$$

by assigning $F = P$ and $G = (Q \rightarrow R)$

Transformational proofs

A *transformational proof* is an argument made using (only) the Laws of Equivalence.

It proves that two wffs are equivalent, by a series of consecutive steps which each substitute some part of the previous wff with another logically equivalent wff.

- Each substitution is *justified* by the law used

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$((A \vee (B \vee C)) \wedge (C \vee A))$$

$$\equiv$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
&((A \vee (B \vee C)) \wedge (C \vee A)) && \\
\equiv&((A \vee B) \vee C)) \wedge (C \vee A)) && \text{by Association} \\
\equiv& &&
\end{aligned}
$$

Example

Proof of $((A \lor (B \lor C)) \land (C \lor A)) \equiv ((A \land B) \lor (C \lor A))$:

$$
\begin{aligned}
&((A \lor (B \lor C)) \land (C \lor A)) && \\
\equiv&((A \lor B) \lor C)) \land (C \lor A)) && \text{by Association} \\
\equiv&(C \lor (A \lor B))) \land (C \lor A)) && \text{by Commutation} \\
\equiv& &&
\end{aligned}
$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
&((A \vee (B \vee C)) \wedge (C \vee A)) && \\
\equiv&(((A \vee B) \vee C)) \wedge (C \vee A)) && \text{by Association} \\
\equiv&(C \vee (A \vee B))) \wedge (C \vee A)) && \text{by Commutation} \\
\equiv&(C \vee ((A \vee B) \wedge A)) && \text{by Distribution} \\
\equiv&
\end{aligned}
$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
& ((A \vee (B \vee C)) \wedge (C \vee A)) & \\
\equiv & ((A \vee B) \vee C)) \wedge (C \vee A)) & \text{by Association} \\
\equiv & (C \vee (A \vee B))) \wedge (C \vee A)) & \text{by Commutation} \\
\equiv & (C \vee ((A \vee B) \wedge A)) & \text{by Distribution} \\
\equiv & (C \vee (A \wedge (A \vee B))) & \text{by Commutation} \\
\equiv &
\end{aligned}
$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
& ((A \vee (B \vee C)) \wedge (C \vee A)) & \\
\equiv & ((A \vee B) \vee C)) \wedge (C \vee A)) & \text{by Association} \\
\equiv & (C \vee (A \vee B))) \wedge (C \vee A)) & \text{by Commutation} \\
\equiv & (C \vee ((A \vee B) \wedge A)) & \text{by Distribution} \\
\equiv & (C \vee (A \wedge (A \vee B))) & \text{by Commutation} \\
\equiv & (C \vee ((A \wedge A) \vee (A \wedge B))) & \text{by Distribution} \\
\equiv & &
\end{aligned}
$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
& ((A \vee (B \vee C)) \wedge (C \vee A)) & \\
\equiv & ((A \vee B) \vee C)) \wedge (C \vee A)) & \text{by Association} \\
\equiv & (C \vee (A \vee B))) \wedge (C \vee A)) & \text{by Commutation} \\
\equiv & (C \vee ((A \vee B) \wedge A)) & \text{by Distribution} \\
\equiv & (C \vee (A \wedge (A \vee B))) & \text{by Commutation} \\
\equiv & (C \vee ((A \wedge A) \vee (A \wedge B))) & \text{by Distribution} \\
\equiv & (C \vee (A \vee (A \wedge B))) & \text{by Idempotence} \\
\equiv &
\end{aligned}
$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
& ((A \vee (B \vee C)) \wedge (C \vee A)) & \\
\equiv & (((A \vee B) \vee C)) \wedge (C \vee A)) & \text{by Association} \\
\equiv & (C \vee (A \vee B))) \wedge (C \vee A)) & \text{by Commutation} \\
\equiv & (C \vee ((A \vee B) \wedge A)) & \text{by Distribution} \\
\equiv & (C \vee (A \wedge (A \vee B))) & \text{by Commutation} \\
\equiv & (C \vee ((A \wedge A) \vee (A \wedge B))) & \text{by Distribution} \\
\equiv & (C \vee (A \vee (A \wedge B))) & \text{by Idempotence} \\
\equiv & ((C \vee A) \vee (A \wedge B)) & \text{by Association} \\
\equiv &
\end{aligned}
$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
&((A \vee (B \vee C)) \wedge (C \vee A)) && \\
\equiv&((A \vee B) \vee C)) \wedge (C \vee A)) && \text{by Association} \\
\equiv&(C \vee (A \vee B))) \wedge (C \vee A)) && \text{by Commutation} \\
\equiv&(C \vee ((A \vee B) \wedge A)) && \text{by Distribution} \\
\equiv&(C \vee (A \wedge (A \vee B))) && \text{by Commutation} \\
\equiv&(C \vee ((A \wedge A) \vee (A \wedge B))) && \text{by Distribution} \\
\equiv&(C \vee (A \vee (A \wedge B))) && \text{by Idempotence} \\
\equiv&((C \vee A) \vee (A \wedge B)) && \text{by Association} \\
\equiv&((A \wedge B) \vee (C \vee A)) && \text{by Commutation}
\end{aligned}
$$

Example

Proof of $((A \vee (B \vee C)) \wedge (C \vee A)) \equiv ((A \wedge B) \vee (C \vee A))$:

$$
\begin{aligned}
&((A \vee (B \vee C)) \wedge (C \vee A)) && \\
\equiv&((A \vee B) \vee C)) \wedge (C \vee A)) && \text{by Association}\\
\equiv&(C \vee (A \vee B))) \wedge (C \vee A)) && \text{by Commutation}\\
\equiv&(C \vee ((A \vee B) \wedge A)) && \text{by Distribution}\\
\equiv&(C \vee (A \wedge (A \vee B))) && \text{by Commutation}\\
\equiv&(C \vee ((A \wedge A) \vee (A \wedge B))) && \text{by Distribution}\\
\equiv&(C \vee (A \vee (A \wedge B))) && \text{by Idempotence}\\
\equiv&((C \vee A) \vee (A \wedge B)) && \text{by Association}\\
\equiv&((A \wedge B) \vee (C \vee A)) && \text{by Commutation}
\end{aligned}
$$

We could also have started from the last wff to reach the first one, as all of these wffs are *logically equivalent*.

Logical implication

A proposition A *logically implies* B if and only if the implication $(A \to B)$ is a tautology. We write $A \vdash B$

Example 1: $(P \wedge Q) \vdash P$ because $((P \wedge Q) \to P)$ is a tautology.

P	Q	$(P \wedge Q)$	$((P \wedge Q) \to P)$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	1

Logical implication

A proposition A *logically implies* B if and only if the implication $(A \to B)$ is a tautology. We write $A \vdash B$

Example 2: $((P \to Q) \land P) \vdash Q$ because $(((P \to Q) \land P) \to Q)$ is a tautology.

P	Q	$(P \to Q)$	$((P \to Q) \land P)$	$(((P \to Q) \land P) \to Q)$
1	1	1	1	1
1	0	0	0	1
0	1	1	0	1
0	0	1	0	1

Rules of Inference

If a wff is known to be true, we can use it to *deduce* other wffs

For example, if we know that $(E \rightarrow F)$ is true, and that E is true, then we can deduce that F must also be true:

$(E \rightarrow F)$	E	F
1	1	?

We say that F is a *logical consequence* of $(E \rightarrow F)$ and E. We denote this as:

$$(E \rightarrow F), E \vdash F$$

Beware! This does ***not*** mean we can substitute E with F. Rules of Inference cannot be used in the same way as Laws of Equivalence are.

Rules of Inference

If a wff is known to be true, we can use it to *deduce* other wffs

For example, if we know that $(E \rightarrow F)$ is true, and that E is true, then we can deduce that F must also be true:

$(E \rightarrow F)$	E	F
1	1	? 1

We say that F is a *logical consequence* of $(E \rightarrow F)$ and E. We denote this as:

$$(E \rightarrow F), E \vdash F$$

Beware! This does ***not*** mean we can substitute E with F. Rules of Inference cannot be used in the same way as Laws of Equivalence are.

Example: Modus Ponens (MP)

From the two sentences

- "If it is raining, I take an umbrella", and
- "It is raining"

We can deduce "I take an umbrella"

Concentrating on only the form of the argument, we get the rule

$$\frac{(A \to B),\ A}{B}$$

i.e. $(((A \to B) \wedge A) \to B)$ is a tautology

Because B is a *logical consequence* of $(A \to B)$ and A, any evaluation which makes both $(A \to B)$ and A *true* will also make B *true*:

$(A \to B)$	A	B
1	1	?

Example: Modus Ponens (MP)

From the two sentences

- "If it is raining, I take an umbrella", and
- "It is raining"

We can deduce "I take an umbrella"

Concentrating on only the form of the argument, we get the rule

$$\frac{(A \to B),\ A}{B}$$

i.e. $(((A \to B) \wedge A) \to B)$ is a tautology

Because B is a *logical consequence* of $(A \to B)$ and A, any evaluation which makes both $(A \to B)$ and A *true* will also make B *true*:

$(A \to B)$	A	B
1	1	? 1

Example: Modus Tollens (MT)

From the two sentences

- "If it is raining, I take an umbrella", and
- "I do not take an umbrella"

We deduce "It is not raining"

Concentrating on only the form of the argument, we get the rule

$$\frac{(A \to B),\ \neg B}{\neg A}$$

i.e. $(((A \to B) \wedge \neg B) \to \neg A)$ is a tautology

Because $\neg A$ is a *logical consequence* of $(A \to B)$ and $\neg B$, any evaluation which makes both $(A \to B)$ and $\neg B$ *true* will also make $\neg A$ *true*:

$(A \to B)$	$\neg B$	$\neg A$
1	1	?

Example: Modus Tollens (MT)

From the two sentences

- "If it is raining, I take an umbrella", and
- "I do not take an umbrella"

We deduce "It is not raining"

Concentrating on only the form of the argument, we get the rule

$$\frac{(A \to B),\ \neg B}{\neg A}$$

i.e. $(((A \to B) \wedge \neg B) \to \neg A)$ is a tautology

Because $\neg A$ is a *logical consequence* of $(A \to B)$ and $\neg B$, any evaluation which makes both $(A \to B)$ and $\neg B$ *true* will also make $\neg A$ *true*:

$(A \to B)$	$\neg B$	$\neg A$
1	1	? 1

Rules of Inference: introduction

Once proven to be tautologies, logical implications can be transformed into Rules of Inference to deduce a new formula

Which we may use without always proving that they are tautologies

Modus Ponens is a Rule of Inference. Whenever you need to deduce B from $(A \to B)$ and A, you can justify it using the Modus Ponens Rule of Inference, without needing to prove it again.

Rules of Inference

- Above the line: list of wffs which must be assumed to be true before we can apply the inference rule.
- Below the line: the new wff which can be deduced from the ones above the line, by applying the inference rule.
- A, B, C, D denote any wff.

Modus Ponens (MP)	$\dfrac{(A \to B),\ A}{B}$
Modus Tollens (MT)	$\dfrac{(A \to B),\ \neg B}{\neg A}$
Hypothetical Syllogism (HS)	$\dfrac{(A \to B),\ (B \to C)}{(A \to C)}$

Rules of Inference

Disjunctive Syllogism (DS)	$\dfrac{(A \vee B),\ \neg A}{B}$
Constructive Dilemma (CD)	$\dfrac{((A \rightarrow B) \wedge (C \rightarrow D)),\ (A \vee C)}{(B \vee D)}$
Destructive Dilemma (DD)	$\dfrac{((A \rightarrow B) \wedge (C \rightarrow D)),\ (\neg B \vee \neg D)}{(\neg A \vee \neg C)}$
Simplification (Simp)	$\dfrac{(A \wedge B)}{A}$
Conjunction (Conj)	$\dfrac{A,\ B}{(A \wedge B)}$
Addition (Add)	$\dfrac{A}{(A \vee B)}$

Rules of Inference

Rules involving the manipulation of premises

- Premises are shown to the left of the symbol $\vdash$
- Conclusion is shown to the right of the symbol $\vdash$
- $premise_1, premise_2, ...premise_n \vdash conclusion$
- S denotes a set of premises, other letters denote single premises not in S

Conditional Proof (CP)	$\dfrac{S,\ A \vdash B}{S \vdash (A \to B)}$
Indirect Proof (IP)	$\dfrac{S,\ \neg C \vdash (A \wedge \neg A)}{S \vdash C}$

We will discuss these more complex rules next week

Checking the validity of an argument

If F is a *consequence* of some wffs (called premises), then no assignment of truth values to the atomic propositions can make the premises *true* but F *false*

Example: $(A \rightarrow C)$ is a consequence of $(A \rightarrow B)$ and $(B \rightarrow C)$

We could prove this by showing that the truth table for $(((A \rightarrow B) \land (B \rightarrow C)) \rightarrow (A \rightarrow C))$ is a tautology.

Therefore the following constitutes a *valid argument*:

$$\frac{(A \rightarrow B),\ (B \rightarrow C)}{A \rightarrow C}$$

Checking the validity of an argument

However, truth tables are (very, very) tedious!

If you have n atomic propositions, then you need 2^n rows!

We will look a more practical approach: formal deductive proofs

Formal Proofs

Formal way to demonstrate that a logical argument is valid

- Proofs are a highly disciplined way of reasoning
- Each step in the argument is a deduction based on earlier steps
  - Each deduction is formally justified using an equivalence law or an inference rule
  - Performed correctly, this method eliminates the errors in reasoning which can easily arise in less formal formal proofs.

Formal proofs using the Natural Deduction system

- Write the premises (the wffs that are given as true)
- Generate the necessary and justified steps to derive the *conclusion*.
- Each step must be:
  - *line numbered* so that we can refer to it later in the proof
  - *justified* by an equivalence law or an inference rule
  - annotated with the lines *referenced* by the justification
  - annotated with the *premise* numbers which the formula depends on

Premises	Line	Formula	Justification	References
1	1	$(P \leftrightarrow Q)$	Premise	
	2	...		

Formal proofs using the Natural Deduction system

- Each line of the proof states that
  - Given the *premises*, the *formula* is true, because it can be *justified* by applying the law or rule to the *referenced* lines above it.
  - If this holds for every line of the proof, then the argument is valid.

Beware: formality is *crucial*. Unless every step is valid and justified, the argument is not valid!

Premises	Line	Formula	Justification	References
1	1	$(P \leftrightarrow Q)$	Premise	
	2	...		

Natural Deduction system at work

Natural Deduction System = equivalence laws + inference rules

- We are using exactly the set of rules and laws given on the previous slides
- A one page summary of these is on Ed, and will be provided at the exam.

We will look at increasingly complex proofs:

- Simple proofs using laws of equivalence only
- Proofs using laws of inference *except* IP and CP
- Proofs using Conditional Proof and Indirect Proof
- Proofs with no premises (tautologies)

Example transformational proof

Simple example of a transformational proof using laws of equivalence only:

$$(P \leftrightarrow Q) \vdash (Q \leftrightarrow P)$$

Premises	Line	Formula	Justification	References
1	1	$(P \leftrightarrow Q)$	Premise	

Example transformational proof

Simple example of a transformational proof using laws of equivalence only:

$$(P \leftrightarrow Q) \vdash (Q \leftrightarrow P)$$

Premises	Line	Formula	Justification	References
1	1	$(P \leftrightarrow Q)$	Premise	
1	2	$((P \rightarrow Q) \wedge (Q \rightarrow P))$	Equiv.	1

Example transformational proof

Simple example of a transformational proof using laws of equivalence only:

$$(P \leftrightarrow Q) \vdash (Q \leftrightarrow P)$$

Premises	Line	Formula	Justification	References
1	1	$(P \leftrightarrow Q)$	Premise	
1	2	$((P \rightarrow Q) \wedge (Q \rightarrow P))$	Equiv.	1
1	3	$((Q \rightarrow P) \wedge (P \rightarrow Q))$	Comm.	2

Example transformational proof

Simple example of a transformational proof using laws of equivalence only:

$$(P \leftrightarrow Q) \vdash (Q \leftrightarrow P)$$

Premises	Line	Formula	Justification	References
1	1	$(P \leftrightarrow Q)$	Premise	
1	2	$((P \rightarrow Q) \wedge (Q \rightarrow P))$	Equiv.	1
1	3	$((Q \rightarrow P) \wedge (P \rightarrow Q))$	Comm.	2
1	4	$(Q \leftrightarrow P)$	Equiv.	3

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \vee (B \vee C)) \wedge (C \vee A)) \vdash ((A \wedge B) \vee (C \vee A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \vee (B \vee C)) \wedge (C \vee A))$	Premise	
1	2		Comm.	1
1	3		Assoc.	2
1	4		Idem.	3
1	5		Assoc.	4
1	6		Comm.	5
1	7		Dist.	6
1	8		Comm.	7

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \vee (B \vee C)) \wedge (C \vee A)) \vdash ((A \wedge B) \vee (C \vee A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \vee (B \vee C)) \wedge (C \vee A))$	Premise	
1	2	$(((B \vee C) \vee A) \wedge (C \vee A))$	Comm.	1
1	3		Assoc.	2
1	4		Idem.	3
1	5		Assoc.	4
1	6		Comm.	5
1	7		Dist.	6
1	8		Comm.	7

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \vee (B \vee C)) \wedge (C \vee A)) \vdash ((A \wedge B) \vee (C \vee A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \vee (B \vee C)) \wedge (C \vee A))$	Premise	
1	2	$(((B \vee C) \vee A) \wedge (C \vee A))$	Comm.	1
1	3	$((B \vee (C \vee A)) \wedge (C \vee A))$	Assoc.	2
1	4		Idem.	3
1	5		Assoc.	4
1	6		Comm.	5
1	7		Dist.	6
1	8		Comm.	7

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \vee (B \vee C)) \wedge (C \vee A)) \vdash ((A \wedge B) \vee (C \vee A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \vee (B \vee C)) \wedge (C \vee A))$	Premise	
1	2	$(((B \vee C) \vee A) \wedge (C \vee A))$	Comm.	1
1	3	$((B \vee (C \vee A)) \wedge (C \vee A))$	Assoc.	2
1	4	$((B \vee (C \vee A)) \wedge (C \vee (A \vee A)))$	Idem.	3
1	5		Assoc.	4
1	6		Comm.	5
1	7		Dist.	6
1	8		Comm.	7

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \vee (B \vee C)) \wedge (C \vee A)) \vdash ((A \wedge B) \vee (C \vee A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \vee (B \vee C)) \wedge (C \vee A))$	Premise	
1	2	$(((B \vee C) \vee A) \wedge (C \vee A))$	Comm.	1
1	3	$((B \vee (C \vee A)) \wedge (C \vee A))$	Assoc.	2
1	4	$((B \vee (C \vee A)) \wedge (C \vee (A \vee A)))$	Idem.	3
1	5	$((B \vee (C \vee A)) \wedge ((C \vee A) \vee A))$	Assoc.	4
1	6		Comm.	5
1	7		Dist.	6
1	8		Comm.	7

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \vee (B \vee C)) \wedge (C \vee A)) \vdash ((A \wedge B) \vee (C \vee A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \vee (B \vee C)) \wedge (C \vee A))$	Premise	
1	2	$(((B \vee C) \vee A) \wedge (C \vee A))$	Comm.	1
1	3	$((B \vee (C \vee A)) \wedge (C \vee A))$	Assoc.	2
1	4	$((B \vee (C \vee A)) \wedge (C \vee (A \vee A)))$	Idem.	3
1	5	$((B \vee (C \vee A)) \wedge ((C \vee A) \vee A))$	Assoc.	4
1	6	$((B \vee (C \vee A)) \wedge (A \vee (C \vee A)))$	Comm.	5
1	7		Dist.	6
1	8		Comm.	7

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \lor (B \lor C)) \land (C \lor A)) \vdash ((A \land B) \lor (C \lor A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \lor (B \lor C)) \land (C \lor A))$	Premise	
1	2	$(((B \lor C) \lor A) \land (C \lor A))$	Comm.	1
1	3	$((B \lor (C \lor A)) \land (C \lor A))$	Assoc.	2
1	4	$((B \lor (C \lor A)) \land (C \lor (A \lor A)))$	Idem.	3
1	5	$((B \lor (C \lor A)) \land ((C \lor A) \lor A))$	Assoc.	4
1	6	$((B \lor (C \lor A)) \land (A \lor (C \lor A)))$	Comm.	5
1	7	$((B \land A) \lor (C \lor A))$	Dist.	6
1	8		Comm.	7

Another example using only laws of equivalence (same argument as earlier, but a shorter proof)
$((A \vee (B \vee C)) \wedge (C \vee A)) \vdash ((A \wedge B) \vee (C \vee A))$:

Prem.	Ln	Formula	Justification	Refs.
1	1	$((A \vee (B \vee C)) \wedge (C \vee A))$	Premise	
1	2	$(((B \vee C) \vee A) \wedge (C \vee A))$	Comm.	1
1	3	$((B \vee (C \vee A)) \wedge (C \vee A))$	Assoc.	2
1	4	$((B \vee (C \vee A)) \wedge (C \vee (A \vee A)))$	Idem.	3
1	5	$((B \vee (C \vee A)) \wedge ((C \vee A) \vee A))$	Assoc.	4
1	6	$((B \vee (C \vee A)) \wedge (A \vee (C \vee A)))$	Comm.	5
1	7	$((B \wedge A) \vee (C \vee A))$	Dist.	6
1	8	$((A \wedge B) \vee (C \vee A))$	Comm.	7

Example deductive proof (using simple rules of inference)

$$(A \to B), (C \to D), (\neg B \vee \neg D), \neg\neg A, ((E \wedge F) \to C) \vdash \neg(E \wedge F)$$

Premises	Line	Formula	Justification	References
1	1	$(A \to B)$	Premise	
2	2	$(C \to D)$	Premise	
3	3	$(\neg B \vee \neg D)$	Premise	
4	4	$\neg\neg A$	Premise	
5	5	$((E \wedge F) \to C)$	Premise	
	6	$((A \to B) \wedge (C \to D))$		
	7	$(\neg A \vee \neg C)$		
	8	$\neg C$		
	9	$\neg(E \wedge F)$		

Example deductive proof (using simple rules of inference)

$$(A \to B), (C \to D), (\neg B \vee \neg D), \neg\neg A, ((E \wedge F) \to C) \vdash \neg(E \wedge F)$$

Premises	Line	Formula	Justification	References
1	1	$(A \to B)$	Premise	
2	2	$(C \to D)$	Premise	
3	3	$(\neg B \vee \neg D)$	Premise	
4	4	$\neg\neg A$	Premise	
5	5	$((E \wedge F) \to C)$	Premise	
1,2	6	$((A \to B) \wedge (C \to D))$	Conj.	1,2
	7	$(\neg A \vee \neg C)$		
	8	$\neg C$		
	9	$\neg(E \wedge F)$		

Example deductive proof (using simple rules of inference)

$$(A \to B), (C \to D), (\neg B \vee \neg D), \neg\neg A, ((E \wedge F) \to C) \vdash \neg(E \wedge F)$$

Premises	Line	Formula	Justification	References
1	1	$(A \to B)$	Premise	
2	2	$(C \to D)$	Premise	
3	3	$(\neg B \vee \neg D)$	Premise	
4	4	$\neg\neg A$	Premise	
5	5	$((E \wedge F) \to C)$	Premise	
1,2	6	$((A \to B) \wedge (C \to D))$	Conj.	1,2
1,2,3	7	$(\neg A \vee \neg C)$	DD	3,6
	8	$\neg C$		
	9	$\neg(E \wedge F)$		

Example deductive proof (using simple rules of inference)

$$(A \to B), (C \to D), (\neg B \vee \neg D), \neg\neg A, ((E \wedge F) \to C) \vdash \neg(E \wedge F)$$

Premises	Line	Formula	Justification	References
1	1	$(A \to B)$	Premise	
2	2	$(C \to D)$	Premise	
3	3	$(\neg B \vee \neg D)$	Premise	
4	4	$\neg\neg A$	Premise	
5	5	$((E \wedge F) \to C)$	Premise	
1,2	6	$((A \to B) \wedge (C \to D))$	Conj.	1,2
1,2,3	7	$(\neg A \vee \neg C)$	DD	3,6
1,2,3,4	8	$\neg C$	DS	4,7
	9	$\neg(E \wedge F)$		

Example deductive proof (using simple rules of inference)

$$(A \rightarrow B), (C \rightarrow D), (\neg B \vee \neg D), \neg\neg A, ((E \wedge F) \rightarrow C) \vdash \neg(E \wedge F)$$

Premises	Line	Formula	Justification	References
1	1	$(A \rightarrow B)$	Premise	
2	2	$(C \rightarrow D)$	Premise	
3	3	$(\neg B \vee \neg D)$	Premise	
4	4	$\neg\neg A$	Premise	
5	5	$((E \wedge F) \rightarrow C)$	Premise	
1,2	6	$((A \rightarrow B) \wedge (C \rightarrow D))$	Conj.	1,2
1,2,3	7	$(\neg A \vee \neg C)$	DD	3,6
1,2,3,4	8	$\neg C$	DS	4,7
1,2,3,4,5	9	$\neg(E \wedge F)$	MT	5,8

Full example: english to formal proof

Prove formally the argument below:

Either the Attorney General has imposed a strict censorship *or*
If Black mailed the letter he wrote
Then Davis received a warning

If our lines of communication have not broken down completely
Then if Davis received a warning
Then Emory was informed about the matter

If the Attorney General has imposed a strict censorship
Then our lines of communication have broken down completely

Our lines of communication have not broken down completely
Therefore if Black mailed the letter he wrote then Emory was informed about the matter

Method

1. Find the atomic propositions
2. Write the premises and conclusion
3. Conduct the formal proof

Formalising the argument

Find the atomic propositions:

- A = Attorney General has imposed a strict censorship
- B = Black mailed the latter he wrote
- C = our lines of Communication have broken down completely
- D = Davis recieved a warning
- E = Emory was informed about the matter

Write the premises and conclusion:

$(A \vee (B \to D))$	(premise 1)
$(\neg C \to (D \to E))$	(premise 2)
$(A \to C)$	(premise 3)
$\neg C$	(premise 4)
$\vdash (B \to E)$	(conclusion)

Conduct the formal proof

Conclusion: $(B \to E)$

Premises	Line	Formula	Justification	References
1	1	$(A \vee (B \to D))$	Premise	
2	2	$(\neg C \to (D \to E))$	Premise	
3	3	$(A \to C)$	Premise	
4	4	$\neg C$	Premise	
...	...	...	...	

Using rules of inference

- Rules of inference are *very different* to Laws of equivalence
- Laws of equivalence can be applied to sub-parts of wffs
- Rules of inference can only apply to the *whole* wff
- Example: suppose we have $((E \wedge F) \rightarrow C)$
  - We can justify writing $((F \wedge E) \rightarrow C)$ using commutation because it is a law of equivalence, $(E \wedge F)$ is equivalent to $(F \wedge E)$.
  - We *cannot* justify writing $(E \rightarrow C)$ using simplification on the sub-formula $(E \wedge F)$. Test this yourself using truth tables!

Natural Deduction at work

Another example, mixing laws of equivalence and simple rules of inference.

$$(A \vee C), (A \rightarrow D), (C \rightarrow \neg E), E \vdash D$$

Premises	Line	Formula	Justification	References
1	1	$(A \vee C)$	Premise	
2	2	$(A \rightarrow D)$	Premise	
3	3	$(C \rightarrow \neg E)$	Premise	
4	4	E	Premise	

Natural Deduction at work

Another example, mixing laws of equivalence and simple rules of inference.

$$(A \vee C), (A \to D), (C \to \neg E), E \vdash D$$

Premises	Line	Formula	Justification	References
1	1	$(A \vee C)$	Premise	
2	2	$(A \to D)$	Premise	
3	3	$(C \to \neg E)$	Premise	
4	4	E	Premise	
4	5	$\neg\neg E$	Double Negation	4

Natural Deduction at work

Another example, mixing laws of equivalence and simple rules of inference.

$$(A \vee C), (A \to D), (C \to \neg E), E \vdash D$$

Premises	Line	Formula	Justification	References
1	1	$(A \vee C)$	Premise	
2	2	$(A \to D)$	Premise	
3	3	$(C \to \neg E)$	Premise	
4	4	E	Premise	
4	5	$\neg\neg E$	Double Negation	4
3,4	6	$\neg C$	Modus Tollens	3,5

Natural Deduction at work

Another example, mixing laws of equivalence and simple rules of inference.

$$(A \vee C), (A \rightarrow D), (C \rightarrow \neg E), E \vdash D$$

Premises	Line	Formula	Justification	References
1	1	$(A \vee C)$	Premise	
2	2	$(A \rightarrow D)$	Premise	
3	3	$(C \rightarrow \neg E)$	Premise	
4	4	E	Premise	
4	5	$\neg\neg E$	Double Negation	4
3,4	6	$\neg C$	Modus Tollens	3,5
1	7	$C \vee A$	Commutation	1

Natural Deduction at work

Another example, mixing laws of equivalence and simple rules of inference.

$$(A \vee C), (A \to D), (C \to \neg E), E \vdash D$$

Premises	Line	Formula	Justification	References
1	1	$(A \vee C)$	Premise	
2	2	$(A \to D)$	Premise	
3	3	$(C \to \neg E)$	Premise	
4	4	E	Premise	
4	5	$\neg\neg E$	Double Negation	4
3,4	6	$\neg C$	Modus Tollens	3,5
1	7	$C \vee A$	Commutation	1
1,3,4	8	A	Disjunctive Syllogism	6,7

Natural Deduction at work

Another example, mixing laws of equivalence and simple rules of inference.

$$(A \vee C), (A \to D), (C \to \neg E), E \vdash D$$

Premises	Line	Formula	Justification	References
1	1	$(A \vee C)$	Premise	
2	2	$(A \to D)$	Premise	
3	3	$(C \to \neg E)$	Premise	
4	4	E	Premise	
4	5	$\neg\neg E$	Double Negation	4
3,4	6	$\neg C$	Modus Tollens	3,5
1	7	$C \vee A$	Commutation	1
1,3,4	8	A	Disjunctive Syllogism	6,7
1,2,3,4	8	D	Modus Ponens	6,7

Typical mistakes

Using the Natural Deduction System, you must use the laws and rules *exactly* as they are given.
People often make the mistake of skipping steps

- This opens the door to errors in reasoning
- Formal proofs must not be error prone

What is wrong in this proof? (1)

Premises	Line	Formula	Justification	References
1	1	$(\neg P \to S)$	Premise	
2	2	$(S \to \neg(B \land D))$	Premise	
3	3	$(\neg B \to T)$	Premise	
4	4	$\neg T$	Premise	
5	5	D	Premise	
3,4	6	B	Modus Tollens	3,4
3,4,5	7	$(B \land D)$	Conjunction	5,6
2,3,4,5	8	$\neg S$	Modus Tollens	7,2
1,2,3,4,5	9	P	Modus Tollens	1,8

What is wrong in this proof? (2)

Premises	Line	Formula	Justification	References
1	1	$(\neg P \rightarrow S)$	Premise	
2	2	$(S \rightarrow \neg(B \wedge D))$	Premise	
3	3	$(\neg B \rightarrow T)$	Premise	
4	4	$\neg P$	Premise	
1,4	5	S	Modus Ponens	1,4
1,2,4	6	$\neg(B \vee D)$	Modus Ponens	2,5
1,2,4	7	$(\neg B \wedge \neg D)$	DeMorgan's	4,6
1,2,4	8	$\neg D$	Simplification	7