# COMP2022: Formal Languages and Logic

## 2017, Semester 1, Week 10

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Adapted from slides by A/Prof Kalina Yacef

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THE UNIVERSITY OF SYDNEY

# Announcements

Assignments:

- Assignment 2 is due on Thursday
- Assignment 3 will be released on Thursday, due in week **12**

Advanced Seminar

- Where: here
- When: Friday 5pm
- Topic: Greg will demonstrate how to prove the *completeness* of the version of the Natural Deduction System that we used for predicate logic.
- Non-assessable, optional, but interesting!

# Outline

Proof by Resolution

Introduction to Predicate Logic

# Normal Forms and Resolution

When premises are put into Conjunctive Normal Form (CNF), then it is possible to perform proofs which use only one rule called *resolution*, together with *indirect proof* to deduce the conclusion.

This is the way the programming language Prolog works.

# Conjunctive Normal Form (CNF)

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- $(A \vee B), (A \vee \neg B), (A \vee (B \vee C)), ...$
- Note: $A$ is also a disjunction

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- $(A \vee B), (A \vee \neg B), (A \vee (B \vee C)), ...$
- Note: $A$ is also a disjunction

A wff is is CNF if it is built using a conjunction of disjunctions:

- $((A \vee B) \wedge (C \vee D))$
- $((A \vee B) \wedge C)$
- $B$
- $(A \wedge C)$

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Using only *equivalence laws*, since it must be an equivalent wff

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1. If the wff contains the operators $\rightarrow$ or $\leftrightarrow$, rewrite the formula using $\neg, \wedge, \vee$ only:
   - Definition of Implication: $(A \rightarrow B) \equiv (\neg A \vee B)$
   - Material Equivalence: $(F \leftrightarrow G) \equiv ((F \rightarrow G) \wedge (G \rightarrow F))$
     $+$ Definition of Implication: $\equiv ((\neg F \vee G) \wedge (\neg G \vee F))$

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2. Push negations inwards and eliminate double negations using DeMorgan's Laws and Double Negation:
   - DeMorgan's Laws (DeM):
     - $\neg(F \wedge G) \equiv (\neg F \vee \neg G)$
     - $\neg(F \vee G) \equiv (\neg F \wedge \neg G)$
   - Double Negation (DN): $F \equiv \neg\neg F$

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     - $\neg(F \vee G) \equiv (\neg F \wedge \neg G)$
   - Double Negation (DN): $F \equiv \neg\neg F$
3. Distribute $\vee$ over $\wedge$
   - Distribution (Dist): $(F \vee (G \wedge H)) \equiv ((F \vee G) \wedge (F \vee H))$

# Omitting parentheses in complex conjunctions/disjunctions

By the laws of association for $\vee$, the location of the parentheses does not change the truth value of a complex wff containing only disjunctions. Therefore we allow omitting the parentheses:

$$(F \vee (G \vee H)) \equiv ((F \vee G) \vee H) \equiv (F \vee G \vee H)$$

Similarly for $\wedge$

$$(F \wedge (G \wedge H)) \equiv ((F \wedge G) \wedge H) \equiv (F \wedge G \wedge H)$$

This is similar to dropping parentheses around additions, e.g.

$$(5 + (4 + 8)) = ((5 + 4) + 8) = (5 + 4 + 8)$$

# Example

$$
\begin{aligned}
(P \rightarrow (Q \rightarrow \neg R)) &\equiv (P \rightarrow (\neg Q \vee \neg R)) && \text{(Def. Implication)} \\
&\equiv (\neg P \vee (\neg Q \vee \neg R)) && \text{(Def. Implication)} \\
&\equiv (\neg P \vee \neg Q \vee \neg R) && \text{(Associativity)}
\end{aligned}
$$

This is now in CNF (it is a conjunction of a single disjunction)

# Example 2

$$((A \wedge B) \rightarrow (C \wedge (A \vee D)))$$

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$$
\begin{aligned}
&((A \wedge B) \rightarrow (C \wedge (A \vee D))) \\
\equiv&(\neg(A \wedge B) \vee (C \wedge (A \vee D))) && \text{(Def. Implication)}
\end{aligned}
$$

# Example 2

$$
\begin{aligned}
&((A \wedge B) \rightarrow (C \wedge (A \vee D))) && \\
\equiv&(\neg(A \wedge B) \vee (C \wedge (A \vee D))) && \text{(Def. Implication)} \\
\equiv&((\neg A \vee \neg B) \vee (C \wedge (A \vee D))) && \text{(De Morgan's)}
\end{aligned}
$$

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$$
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\equiv&((\neg A \vee \neg B) \vee (C \wedge (A \vee D))) && \text{(De Morgan's)} \\
\equiv&(((\neg A \vee \neg B) \vee C) \wedge ((\neg A \vee \neg B) \vee (A \vee D))) && \text{(Distribution)}
\end{aligned}
$$

# Example 2

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\equiv&((\neg A \vee \neg B) \vee (C \wedge (A \vee D))) && \text{(De Morgan's)} \\
\equiv&(((\neg A \vee \neg B) \vee C) \wedge ((\neg A \vee \neg B) \vee (A \vee D))) && \text{(Distribution)} \\
\equiv&((\neg A \vee \neg B \vee C) \wedge (\neg A \vee \neg B \vee A \vee D))) && \text{(Associativity)}
\end{aligned}
$$

This is now in CNF (it is a conjunction of a two disjunctions)

# Resolution Rule

$(((A \vee B) \wedge (\neg A \vee C)) \rightarrow (B \vee C))$ is a tautology. How can we prove it?

- Natural Deduction System
- Quine's method
- truth table

New way: Proof by Resolution, a deductive proof which uses one generalised rule

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- $(A \vee B), (\neg A \vee C) \vdash (B \vee C)$

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- $(A \vee B), (\neg A \vee 0) \vdash (B \vee 0)$

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- $(A \vee B), (\neg A \vee 0) \vdash (B \vee 0)$

We also acknowledge the commutativity and associativity of $\vee$

- we ignore parenthese in disjunctions
- we ignore the order of the terms (unlike proofs in the NDS!)

# Proving with Resolution

Use Indirect Proof + Resolution Rule

1. Assume the presmises and the *negation* of the conclusion (as per the strategy of indirect proof)

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4. Use the resolution rule (only!) until reaching a contradiction, denoted $\Box$
5. Hence we can conclude that the original conclusion is true

# Example

$$(R \to U)$$
$$(U \to \neg W)$$
$$(\neg R \to \neg W)$$
$$\vdash \neg W$$

First we put the premises and negated conclusion into CNF:

# Example

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\begin{aligned}
&(R \to U)\\
&(U \to \neg W)\\
&(\neg R \to \neg W)\\
\vdash &\neg W
\end{aligned}
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First we put the premises and negated conclusion into CNF:

$$(R \to U) \equiv (\neg R \vee U)$$

# Example

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First we put the premises and negated conclusion into CNF:

$$
\begin{aligned}
(R \to U) &\equiv (\neg R \vee U) \\
(U \to \neg W) &\equiv (\neg U \vee \neg W)
\end{aligned}
$$

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First we put the premises and negated conclusion into CNF:

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\begin{aligned}
(R \to U) &\equiv (\neg R \vee U) \\
(U \to \neg W) &\equiv (\neg U \vee \neg W) \\
(\neg R \to \neg W) &\equiv (\neg\neg R \vee \neg W) \equiv (R \vee \neg W)
\end{aligned}
$$

# Example

$$
\begin{aligned}
&(R \to U) \\
&(U \to \neg W) \\
&(\neg R \to \neg W) \\
\vdash &\neg W
\end{aligned}
$$

First we put the premises and negated conclusion into CNF:

$$
\begin{aligned}
(R \to U) &\equiv (\neg R \lor U) \\
(U \to \neg W) &\equiv (\neg U \lor \neg W) \\
(\neg R \to \neg W) &\equiv (\neg\neg R \lor \neg W) \equiv (R \lor \neg W) \\
\neg\neg W &\equiv W
\end{aligned}
$$

# Example

Conducting the proof using resolution:

| Line | Formula | References |
|---|---|---|
| 1 | $(\neg R \vee U)$ | |
| 2 | $(\neg U \vee \neg W)$ | |
| 3 | $(R \vee \neg W)$ | |
| 4 | $W$ | |

Note because we only use the Resolution Rule (and IP on the very last line), we only need a column for lines referenced by each formula.

# Example

Conducting the proof using resolution:

| Line | Formula | References |
|---|---|---|
| 1 | $(\neg R \vee U)$ | |
| 2 | $(\neg U \vee \neg W)$ | |
| 3 | $(R \vee \neg W)$ | |
| 4 | $W$ | |
| 5 | $R$ | 3,4 |

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| 2 | $(\neg U \vee \neg W)$ | |
| 3 | $(R \vee \neg W)$ | |
| 4 | $W$ | |
| 5 | $R$ | 3,4 |
| 6 | $U$ | 1,5 |

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| Line | Formula | References |
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| 3 | $(R \vee \neg W)$ | |
| 4 | $W$ | |
| 5 | $R$ | 3,4 |
| 6 | $U$ | 1,5 |
| 7 | $\neg U$ | 2,4 |

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| 4 | $W$ | |
| 5 | $R$ | 3,4 |
| 6 | $U$ | 1,5 |
| 7 | $\neg U$ | 2,4 |
| 8 | $\square$ | 6,7 |

Note because we only use the Resolution Rule (and IP on the very last line), we only need a column for lines referenced by each formula.

# Example 2

$$\begin{aligned} &(A \to (B \to C)) \\ &(B \to (C \to D)) \\ \vdash &(A \to (B \to D)) \end{aligned}$$

First we transform the premises and *negated* conclusion into CNF:

$$\begin{aligned} (A \to (B \to C)) &\equiv (\neg A \lor \neg B \lor C) \\ (B \to (C \to D)) &\equiv (\neg B \lor \neg C \lor D) \\ \neg(A \to (B \to D)) & \end{aligned}$$

# Example 2

$$
\begin{aligned}
&(A \rightarrow (B \rightarrow C)) \\
&(B \rightarrow (C \rightarrow D)) \\
\vdash &(A \rightarrow (B \rightarrow D))
\end{aligned}
$$

First we transform the premises and *negated* conclusion into CNF:

$$
\begin{aligned}
(A \rightarrow (B \rightarrow C)) &\equiv (\neg A \vee \neg B \vee C) \\
(B \rightarrow (C \rightarrow D)) &\equiv (\neg B \vee \neg C \vee D) \\
\neg(A \rightarrow (B \rightarrow D)) &\equiv \neg(\neg A \vee (\neg B \vee D))
\end{aligned}
$$

# Example 2

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&(A \to (B \to C)) \\
&(B \to (C \to D)) \\
\vdash &(A \to (B \to D))
\end{aligned}
$$

First we transform the premises and *negated* conclusion into CNF:

$$
\begin{aligned}
(A \to (B \to C)) &\equiv (\neg A \lor \neg B \lor C) \\
(B \to (C \to D)) &\equiv (\neg B \lor \neg C \lor D) \\
\neg(A \to (B \to D)) &\equiv \neg(\neg A \lor (\neg B \lor D)) \\
&\equiv (\neg\neg A \land \neg(\neg B \lor D))
\end{aligned}
$$

# Example 2

$$
\begin{aligned}
&(A \to (B \to C)) \\
&(B \to (C \to D)) \\
\vdash &(A \to (B \to D))
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First we transform the premises and *negated* conclusion into CNF:

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\begin{aligned}
(A \to (B \to C)) &\equiv (\neg A \vee \neg B \vee C) \\
(B \to (C \to D)) &\equiv (\neg B \vee \neg C \vee D) \\
\neg(A \to (B \to D)) &\equiv \neg(\neg A \vee (\neg B \vee D)) \\
&\equiv (\neg\neg A \wedge \neg(\neg B \vee D)) \\
&\equiv (\neg\neg A \wedge (\neg\neg B \wedge \neg D))
\end{aligned}
$$

# Example 2

$$\begin{aligned} &(A \to (B \to C)) \\ &(B \to (C \to D)) \\ \vdash &(A \to (B \to D)) \end{aligned}$$

First we transform the premises and *negated* conclusion into CNF:

$$\begin{aligned} (A \to (B \to C)) &\equiv (\neg A \lor \neg B \lor C) \\ (B \to (C \to D)) &\equiv (\neg B \lor \neg C \lor D) \\ \neg(A \to (B \to D)) &\equiv \neg(\neg A \lor (\neg B \lor D)) \\ &\equiv (\neg\neg A \land \neg(\neg B \lor D)) \\ &\equiv (\neg\neg A \land (\neg\neg B \land \neg D)) \\ &\equiv (A \land B \land \neg D) \end{aligned}$$

# Example 2

Conducting the proof using resolution:

| Line | Formula | References | |
|---|---|---|---|
| 1 | $(\neg A \vee \neg B \vee C)$ | | (from premise 1) |
| 2 | $(\neg B \vee \neg C \vee D)$ | | (from premise 2) |
| 3 | $A$ | | (from the negated conclusion) |
| 4 | $B$ | | (from the negated conclusion) |
| 5 | $\neg D$ | | (from the negated conclusion) |
| 6 | $(\neg B \vee C)$ | 1,3 | |
| 7 | $(\neg C \vee D)$ | 2,4 | |
| 8 | $\neg C$ | 5,7 | |
| 9 | $\neg B$ | 6,8 | |
| 10 | $\square$ | 4,9 | |

Note: we didn't need to justify the use of simplification (and commutation) to break the conjunctions into disjunctions

# Example 2

Alternatively, we could make different inferences on lines 8 and 9:

| Line | Formula | References | |
|---|---|---|---|
| 1 | $(\neg A \vee \neg B \vee C)$ | | (from premise 1) |
| 2 | $(\neg B \vee \neg C \vee D)$ | | (from premise 2) |
| 3 | $A$ | | (from the negated conclusion) |
| 4 | $B$ | | (from the negated conclusion) |
| 5 | $\neg D$ | | (from the negated conclusion) |
| 6 | $(\neg B \vee C)$ | 1,3 | |
| 7 | $(\neg C \vee D)$ | 2,4 | |
| 8 | $(\neg \mathbf{B} \vee \mathbf{D})$ | **6,7** | |
| 9 | $\neg B$ | **5**,8 | |
| 10 | $\square$ | 4,9 | |

# PROPOSITIONAL LOGIC - SUMMING UP

- Basic building block for formal logic
- Formal reasoning
  - Natural Deduction System N (Laws of Equivalence + Rules of Inference)
  - Resolution (Indirect proof, Laws of Equivalence to put wffs in CNF, then resolution rule)
- Laws of Equivalence can be applied to subparts of a wff
  e.g. $((A \wedge B) \rightarrow C) \equiv ((B \wedge A) \rightarrow C)$ by commutation
- Rules of Inference CANNOT be applied to subparts of a wff
  e.g. $((A \wedge B) \rightarrow C) \not\vdash (A \rightarrow C)$ by simplification. INVALID

# Predicate Logic

Consider the following argument:

- All humans are mortal
- Socrates is a human
- Therefore, Socrates is mortal

With Propositional Logic...?

The meaning of the words *"all"*, *"not"*, *"is mortal"* are important to understand this argument.

# Limits of Propositional Logic

How can we express that one particular individual, Socrates, has several properties, like:

- Socrates is a human
- Socrates is a philosopher
- Socrates is Greek

How can we express that two particular individuals have the same property, like:

- Aristotle is Greek
- Socrates is Greek

How do we express that several individuals are linked by a common property, like:

- Alex and Chris are siblings

We need a more powerful logic – *predicate logic*

# Predicates

- "Socrates is a human" is a true proposition
- "My dog is a human" is a false proposition
- "x is a human" is *not* a proposition, because its truth value depends on x

$H$: "is a human" is a *predicate*

- $H(x)$ means "$x$ is a human"
- $H(\text{Socrates})$ means "Socrates is a human"

$G$: "is greater than" is a *predicate*

- $G(x, y)$ means "$x$ is greater than $y$"

Predicates have some arity $k$ (i.e. they can have $k$ arguments)

# Predicates

Predicates correspond to attributes/properties or relationships
They enable us to assert that certain properties or relationships hold for certain objects

Examples of unary predicates:

| | |
|---|---|
| Socrates is a human being | is_a_human($x$) |
| Socrates is an adult | is_an_adult($x$) |
| Lucy is blond | is_blond($x$) |
| Emma studies at University | studies_at_university($x$) |

Examples of binary predicates:

| | |
|---|---|
| Cathy is Lucy's mother | is_the_mother_of($x, y$) |
| Cathy loves Lucy | loves($x, y$) |
| Lucy is taller than Emma | is_taller_than($x, y$) |
| 7 is greater than 5 | is_greater_than($x, y$) |

Example of an n-ary predicate:

$t$ is the product of $x, y$ and $z$ $\quad P(t, x, y, z)$

# Quantifiers

*Domain of interpretation* $D$ (e.g. humans, integers, numbers, etc.)

*Variables* such as $x, y, z$

The *Univeral Quantifier*:

- $\forall$ is the universal quantifier
- $\forall x P(x)$ means that for every $x$ in $D$, $P(x)$ is true

The *Existential Quantifier*:

- $\exists$ is the existential quantifier
- $\exists x P(x)$ means that there exists at least one $x$ in $D$ such that $P(x)$ is true

# Quantifiers

- Universal quantifier: $\forall$ "for every", "for all"
- Existential quantifier: $\exists$ "there exists"

$\forall x$ is_an_adult$(x)$
$\forall x$ is_blond$(x)$
$\exists x$ is_an_adult$(x)$
$\exists x$ is_blond$(x)$
$\forall x \forall y$ loves$(x, y)$
$\exists x \forall y$ loves$(x, y)$
$\forall x \exists y$ loves$(x, y)$
$\exists x \exists y$ loves$(x, y)$
$\forall y \exists x$ loves$(x, y)$

# Quantifiers

- Universal quantifier: $\forall$ "for every", "for all"
- Existential quantifier: $\exists$ "there exists"

$\forall x$ is_an_adult$(x)$ Everybody is an adult
$\forall x$ is_blond$(x)$
$\exists x$ is_an_adult$(x)$
$\exists x$ is_blond$(x)$
$\forall x \forall y$ loves$(x, y)$
$\exists x \forall y$ loves$(x, y)$
$\forall x \exists y$ loves$(x, y)$
$\exists x \exists y$ loves$(x, y)$
$\forall y \exists x$ loves$(x, y)$

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- Universal quantifier: $\forall$ "for every", "for all"
- Existential quantifier: $\exists$ "there exists"

| | |
|---|---|
| $\forall x$ is_an_adult$(x)$ | Everybody is an adult |
| $\forall x$ is_blond$(x)$ | Everybody is blond |
| $\exists x$ is_an_adult$(x)$ | |
| $\exists x$ is_blond$(x)$ | |
| $\forall x \forall y$ loves$(x, y)$ | |
| $\exists x \forall y$ loves$(x, y)$ | |
| $\forall x \exists y$ loves$(x, y)$ | |
| $\exists x \exists y$ loves$(x, y)$ | |
| $\forall y \exists x$ loves$(x, y)$ | |

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- Universal quantifier: $\forall$ "for every", "for all"
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| | |
|---|---|
| $\forall x$ is_an_adult$(x)$ | Everybody is an adult |
| $\forall x$ is_blond$(x)$ | Everybody is blond |
| $\exists x$ is_an_adult$(x)$ | There is at least one adult |
| $\exists x$ is_blond$(x)$ | |
| $\forall x \forall y$ loves$(x, y)$ | |
| $\exists x \forall y$ loves$(x, y)$ | |
| $\forall x \exists y$ loves$(x, y)$ | |
| $\exists x \exists y$ loves$(x, y)$ | |
| $\forall y \exists x$ loves$(x, y)$ | |

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|---|---|
| $\forall x$ is_an_adult$(x)$ | Everybody is an adult |
| $\forall x$ is_blond$(x)$ | Everybody is blond |
| $\exists x$ is_an_adult$(x)$ | There is at least one adult |
| $\exists x$ is_blond$(x)$ | At least someone is blond |
| $\forall x \forall y$ loves$(x, y)$ | |
| $\exists x \forall y$ loves$(x, y)$ | |
| $\forall x \exists y$ loves$(x, y)$ | |
| $\exists x \exists y$ loves$(x, y)$ | |
| $\forall y \exists x$ loves$(x, y)$ | |

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| | |
|---|---|
| $\forall x$ is_an_adult($x$) | Everybody is an adult |
| $\forall x$ is_blond($x$) | Everybody is blond |
| $\exists x$ is_an_adult($x$) | There is at least one adult |
| $\exists x$ is_blond($x$) | At least someone is blond |
| $\forall x \forall y$ loves($x, y$) | Everyone loves everyone |
| $\exists x \forall y$ loves($x, y$) | |
| $\forall x \exists y$ loves($x, y$) | |
| $\exists x \exists y$ loves($x, y$) | |
| $\forall y \exists x$ loves($x, y$) | |

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|---|---|
| $\forall x$ is_an_adult$(x)$ | Everybody is an adult |
| $\forall x$ is_blond$(x)$ | Everybody is blond |
| $\exists x$ is_an_adult$(x)$ | There is at least one adult |
| $\exists x$ is_blond$(x)$ | At least someone is blond |
| $\forall x \forall y$ loves$(x, y)$ | Everyone loves everyone |
| $\exists x \forall y$ loves$(x, y)$ | There is someone who loves everyone |
| $\forall x \exists y$ loves$(x, y)$ | |
| $\exists x \exists y$ loves$(x, y)$ | |
| $\forall y \exists x$ loves$(x, y)$ | |

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- Universal quantifier: $\forall$ "for every", "for all"
- Existental quantifier: $\exists$ "there exists"

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| $\exists x \exists y$ loves$(x, y)$ | There is someone who loves someone |
| $\forall y \exists x$ loves$(x, y)$ | Everyone is loved by someone |

# Syntax – symbols

- Variables $x, y, z$
- Constants $a, b, c$
- Functions $f, g, h$ (arity $k$)
- Predicates $P, Q, R$ (arity $k$)
- Connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ as in Prop. Logic
- Quantifiers $\exists, \forall$

# Syntax – wffs

Well-formed formulas (wff) for predicate logic:

- A *term* is a variable, constant or function
- If $P$ is a predicate symbol with arity $k$ and $t_1, ..., t_k$ are terms, then $P(t_1, ..., t_k)$ is an atomic formula
- Truth symbols and atomic formulas are wff
- If $P$ and $Q$ are wffs and $x$ is a variable, then: $(P), \neg P, (P \wedge Q), (P \vee Q), (P \rightarrow Q), (P \leftrightarrow Q), \exists xP, \forall xP$ are all wffs

Note: We can rename wffs, such as
$Z = \forall x((P(x) \rightarrow Q(x)) \vee R(y))$

# English and Predicate Logic

- Everybody is tall but there are children

- All second-year students are clever

- One cannot love without respecting

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$$\begin{aligned} &\forall x \exists y (T(x) \land C(y)) \\ \equiv &(\forall x T(x) \land \exists y C(y)) \end{aligned}$$

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# English and Predicate Logic

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(everyone is a clever student)

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$$\begin{aligned} &\forall x (L(x) \to R(x)) \\ \equiv &\neg \exists x (L(x) \wedge \neg R(x)) \end{aligned}$$

- $\forall x(\mathsf{adult}(x) \vee \mathsf{child}(x))$

- $(\forall x \ \mathsf{adult}(x) \vee \exists x \ \mathsf{child}(x))$

- $((\neg \exists x \ \mathsf{short}(x)) \rightarrow \forall x(\mathsf{clever}(x) \wedge \mathsf{child}(x)))$

- $\forall x((\mathsf{clever}(x) \wedge \mathsf{child}(x)) \rightarrow \exists y \ \mathsf{loves}(x, y))$

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  Everyone is an adult or a child
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- $(\forall x\ \text{adult}(x) \vee \exists x\ \text{child}(x))$
  Everyone is an adult, or someone is a child

- $((\neg\exists x\ \text{short}(x)) \to \forall x(\text{clever}(x) \wedge \text{child}(x)))$

- $\forall x((\text{clever}(x) \wedge \text{child}(x)) \to \exists y\ \text{loves}(x, y))$

- $\forall x(\mathsf{adult}(x) \lor \mathsf{child}(x))$
  Everyone is an adult or a child

- $(\forall x\ \mathsf{adult}(x) \lor \exists x\ \mathsf{child}(x))$
  Everyone is an adult, or someone is a child

- $((\neg\exists x\ \mathsf{short}(x)) \rightarrow \forall x(\mathsf{clever}(x) \land \mathsf{child}(x)))$
  If nobody is short, then everyone is a clever child

- $\forall x((\mathsf{clever}(x) \land \mathsf{child}(x)) \rightarrow \exists y\ \mathsf{loves}(x, y))$

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  If nobody is short, then everyone is a clever child

- $\forall x((\text{clever}(x) \land \text{child}(x)) \to \exists y\ \text{loves}(x, y))$
  Every clever child has someone that they love

# Scope of a quantifier

- In the wff $\exists x F$, $F$ is the scope of the quantifier $\exists x$
- In the wff $\forall x F$, $F$ is the scope of the quantifier $\forall x$

The quantifier applies to the formula immediately following it.

- If there are no brackets, the quantifier applies just to the atomic formula.
  e.g. $(\forall x \underline{P(x, y)} \rightarrow Q(x))$
- If there are brackets, the quantifier is applies the whole (sub)formula.
  e.g. $(\forall x \underline{(P(x) \wedge Q(x))} \rightarrow R(x))$
  e.g. $\forall x (\underline{(P(x) \wedge Q(x)) \rightarrow R(x))}$

# Bound and Free variables

An occurrence of a variable in a formula is *bound* iff:

- It lies *within the scope* of a quantifier introducing it
- Or, if it is the quantifier variable itself

Otherwise, the occurrence is *free*

Example: $(\exists x P(x, y) \to Q(x))$

- The first two occurrences of $x$ are bound (to the quantifier $\exists x$)
- The third occurrence of $x$ is free (because it is outside the scope of the quantifier)
- The only occurrence of $y$ is free (because it is not introduced by a quantifier)

# Comparison with variable scope in programming

```
public static void quokka() {

    // neither 'd' nor 'i' are in scope here

    double d = 0.0;
    // 'd' is in scope, but 'i' is not

    for(int i=0; i < NB_DAYS; ++i) {
        // both 'd' and 'i' are in scope
    }

    // 'd' is in scope, but 'i' is not
}
```

# When is a wff true or false?

In propositional logic, $P$ is either *true* or *false*

But in predicate logic, it might be that $P(x)$ is *true*, while $P(y)$ is *false*

Interpretations are needed.

# Interpretation

An iterpretation for a wff is a pair $(D, I)$ where $D$ is a non-empty set called the *domain of interpretation* and $I$ is a mapping which assigns the symbols of the wff to values in $D$ as follows:

1. Each predicate letter is assigned to a relation over $D$. A predicate with no argument (arity 0) is a proposition and must be assigned a truth value.
2. Each function letter is assigned to a function over $D$
3. Each free variable is assigned to a value in $D$. All free occurrences of a variable $x$ are assigned to the same value in $D$.
4. Each constant is assigned to a value in $D$. All occurrences of a given constant are assigned to the same value in $D$.

# Examples of Interpretations

Examples of suitable interpretations of $P(x)$ : $(D, I)$

1. $(D, I)$ :
    - $D$ is the set of humans
    - $I$ assigns to $x$ Socrates and to $P$ the subset of Greek humans
    - Then $P(x)$ will be *true*
2. $(D, I)$ :
    - $D$ is the set of integers
    - $I$ assigns to $x$ the integer 3, and to $P$ the subset of even integers
    - Then $P(x)$ will be *false*

# Examples of Interpretations

Suitable iterpretations of a term $t$

- If $t$ is some variable $x$, then $I(t) = I(x)$
- If $t$ has the form $f(t_1, ..., t_k)$ then $I(f(t_1, ..., t_k)) = I(f)(I(t_1), ..., I(t_k))$

Example:

- Let $D$ be the set of integers
- Let $I(x) = 2, I(y) = 5$
- Let $I(g)$ be the function sum, and $I(f)$ the function square
- Then $I(f(g(x, y))) = I(f)(I(g)(I(x), I(y))) = I(f)(I(g)(2, 5)) = I(f)(2 + 5) = 7^2 = 49$
- If $I$ assigns to $P$ the set of even integers, then $P(f(g(x, y)))$ is *false*

# Renaming free and bound variables

- *Bound* variables can be renamed without altering the meaning of the formula. You rename the quantified variable and *every* variable bound to that quantifier.
- *Free* variables denote a specific object. Therefore we cannot rename them without also changing the domain of interpretation.

Example: $\forall x\big((\text{clever}(x) \wedge \text{child}(x)) \rightarrow \text{plays}(x, y)\big)$

- is equivalent to: $\forall \mathbf{z}\big((\text{clever}(\mathbf{z}) \wedge \text{child}(\mathbf{z})) \rightarrow \text{plays}(\mathbf{z}, y)\big)$
- but it is not equivalent to: $\forall x\big((\text{clever}(x) \wedge \text{child}(x)) \rightarrow \text{plays}(x, \mathbf{t})\big)$

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This is just like how in programming, all the occurences of a scoped variable can be renamed without changing the meaning of the code:

- `for(int i=0; i < NB_DAYS; ++i)`
- is equivalent to: `for(int k=0; k < NB_DAYS; ++k)`
- but NOT to: `for(int i=0; i < NB_YEARS; ++i)`

# Next week

... Conducting formal proofs using predicate logic