COMP2022: Formal Languages and Logic

2017, Semester 1, Week 9

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Outline

Propositional logic (continued)

- Indirect and Conditional Proofs
- Proving invalidity
- Proving tautologies
  - in the Natural Deduction System (NDS)
  - Quine's method

Rules of Inference

Rules involving the manipulation of premises:

- Premises are shown to the left of the symbol $\vdash$
- Conclusion is shown to the right of the symbol $\vdash$
- $premise_1, premise_2, ...premise_n \vdash conclusion$
- S denotes a set of premises, other letters denote single premises not in S

Conditional Proof (CP)	$\dfrac{S,\ A \vdash B}{S \vdash (A \to B)}$
Indirect Proof (IP)	$\dfrac{S,\ \neg C \vdash (A \wedge \neg A)}{S \vdash C}$

Conditional Proof

Key idea:

- Have the normal premises of an argument
- Assume some formula A is true (new premise)
- Derive, using this *assumption*, some formula B
- Then we can conclude $(A \to B)$ without the assumption A

Premises	Line	Formula	Justification	Refs.
$1 \ldots n$	$1 \ldots n$	premises 1 to n	Premise	
$n+1$	m	assumption A	Premise	
...	...	derivation ...	...	...
{premises including $n+1$}	j	some formula B	...	...
{premises of line j *excluding* $n+1$}	$j+1$	$(A \to B)$	CP	m, j

Conditional Proof

Ideal for proving implications!

- To prove $A \to B$, assume A and try to derive B

Also good for disjuctions:

- Recall that $(\neg A \vee B) \equiv (A \to B)$ (Definition of Implication)

CONDITIONAL PROOF EXAMPLE

$$(\neg A \vee C), (\neg B \vee A) \vdash (B \to C)$$

Premises	Line	Formula	Justification	References
1	1	$(\neg A \vee C)$	Premise	
2	2	$(\neg B \vee A)$	Premise	
3	3	B	Premise	
3	4	$\neg\neg B$	DN	3
2,3	5	A	DS	2,4
2,3	6	$\neg\neg A$	DN	5
1,2,3	7	C	DS	1,6
1,2	8	$(B \to C)$	CP	3,7

Conditional Proof example

May be used several times in a proof

$$(A \to (B \to C)), (B \to (C \to D)) \vdash (A \to (B \to D))$$

Premises	Line	Formula	Justification	References
1	1	$(A \to (B \to C))$	Premise	
2	2	$(B \to (C \to D))$	Premise	
3	3	A	Premise	
1,3	4	$(B \to C)$	MP	1,3
4	5	B	Premise	
1,3,4	6	C	MP	4,5
2,4	7	$(C \to D)$	MP	2,5
1,2,3,4	8	D	MP	6,7
1,2,3	9	$(B \to D)$	CP	5,8
1,2	10	$(A \to (B \to D))$	CP	3,9

Indirect Proof

Key idea:

- Have the normal premises of an argument
- Assume that the conclusion C is false (new premise $\neg C$)
- Derive, using this *assumption*, a contradiction of the form $(A \wedge \neg A)$ – A can be any wff.
- Then we can conclude C without the assumption $\neg C$

Premises	Line	Formula	Justification	Refs.
$1 ... n$	$1 ... n$	premises 1 to n	Premise	
$n+1$	m	assumption $\neg C$	Premise	
...	...	derivation ...	...	...
{premises including $n+1$}	j	contradiction $(A \wedge \neg A)$	...	...
{premises of line j *excluding* $n+1$}	$j+1$	C	IP	m, j

Indirect Proof example

$$(A \to B), (\neg B \wedge C) \vdash (\neg A \vee D)$$

Premises	Line	Formula	Justification	References
1	1	$(A \to B)$	Premise	
2	2	$(\neg B \wedge C)$	Premise	
3	3	$\neg(\neg A \vee D)$	Premise	
3	4	$(\neg\neg A \wedge \neg D)$	DeMorgan's	3
3	5	$(A \wedge \neg D)$	DN	4
3	6	A	Simpl.	5
1,3	7	B	MP	1,6
2	8	$\neg B$	Simpl.	2
1,2,3	9	$(B \wedge \neg B)$	Conj.	7
1,2	10	$(\neg A \vee D)$	IP	3,8

Indirect Proof example

$$(A \to (B \wedge C)), ((B \vee D) \to E), (D \vee A) \vdash E$$

Premises	Line	Formula	Justification	References
1	1	$(A \to (B \wedge C)$	Premise	
2	2	$((B \vee D) \to E)$	Premise	
3	3	$(D \vee A)$	Premise	
4	4	$\neg E$	Premise	
2,4	5	$\neg(B \vee D)$	MT	2,4
2,4	6	$(\neg B \wedge \neg D)$	DeM	5
2,4	7	$(\neg D \wedge \neg B)$	Comm	6
2,4	8	$\neg D$	Simp	7
2,3,4	9	A	DS	3,8
1,2,3,4	10	$(B \wedge C)$	MP	1,9
1,2,3,4	11	B	Simp	10
2,4	12	$\neg B$	Simp	6
1,2,3,4	13	$B \wedge \neg B$	Conj	11,12
1,2,3	14	E	IP	4,13

CP and IP together

$$(A \vee C), (A \to D), (C \to E) \vdash (\neg E \to D)$$

Premises	Line	Formula	Justification	References
1	1	$(A \vee C)$	Premise	
2	2	$(A \to D)$	Premise	
3	3	$(C \to E)$	Premise	
4	4	$\neg E$	Premise	
5	5	$\neg D$	Premise	
3,4	6	$\neg C$	MT	3,4
2,5	7	$\neg A$	MT	2,5
1,2,5	8	C	DS	1,7
1,2,3,4,5	9	$(C \wedge \neg C)$	Conj	6,8
1,2,3,4	10	D	IP	5,9
1,2,3	11	$(\neg E \to D)$	CP	4,10

Direct proof

Sometimes the direct proof is easier (or harder)

$$(A \vee C), (A \to D), (C \to E) \vdash (\neg E \to D)$$

Premises	Line	Formula	Justification	Refs
1	1	$(A \vee C)$	Premise	
2	2	$(A \to D)$	Premise	
3	3	$(C \to E)$	Premise	
2,3	4	$((A \to D) \wedge (C \to E))$	Conj.	2,3
1,2,3	5	$(D \vee E)$	CD	1,4
1,2,3	6	$(E \vee D)$	Comm	5
1,2,3	7	$(\neg\neg E \vee D)$	DN	6
1,2,3	8	$(\neg E \to D)$	DefI	7

One example, four proofs (1)

$$((P \wedge Q) \vee R) \vdash (\neg P \rightarrow R)$$

Direct proof:

Premises	Line	Formula	Justification	Refs
1	1	$((P \wedge Q) \vee R)$	Premise	
1	2	$(R \vee (P \wedge Q))$	Comm	1
1	3	$((R \vee P) \wedge (R \vee Q))$	Dist	2
1	4	$(R \vee P)$	Simp	3
1	5	$(P \vee R)$	Comm	4
1	6	$(\neg\neg P \vee R)$	DN	5
1	7	$(\neg P \rightarrow R)$	Def Impl	6

One example, four proofs (2)

$$((P \wedge Q) \vee R) \vdash (\neg P \rightarrow R)$$

Using Conditional proof:

Premises	Line	Formula	Justification	Refs
1	1	$((P \wedge Q) \vee R)$	Premise	
2	2	$\neg P$	Premise	1
2	3	$(\neg P \vee \neg Q)$	Add	2
2	4	$\neg(P \wedge Q)$	DeM	3
1,2	5	R	DS	1,4
1	6	$(\neg P \rightarrow R)$	CP	1,5

One example, four proofs (3)

$$((P \wedge Q) \vee R) \vdash (\neg P \rightarrow R)$$

Using Indirect proof:

Premises	Line	Formula	Justification	References
1	1	$((P \wedge Q) \vee R)$	Premise	
2	2	$\neg(\neg P \rightarrow R)$	Premise	
2	3	$\neg(\neg\neg P \vee R)$	DefI	2
2	4	$\neg(P \vee R)$	DN	3
2	5	$(\neg P \wedge \neg R)$	DeM	4
2	6	$(\neg R \wedge \neg P)$	Comm	5
2	7	$\neg R$	Simp	6
1	8	$(R \vee (P \wedge Q))$	Comm	1
1,2	9	$(P \wedge Q)$	DS	7,8
1,2	10	P	Simp	9
2	11	$\neg P$	Simp	5
1,2	12	$(P \wedge \neg P)$	Conj	10,11
1	13	$(\neg P \rightarrow R)$	IP	2,12

One example, four proofs (4)

$$((P \wedge Q) \vee R) \vdash (\neg P \to R)$$

Using both indirect proof and conditional proof:

Premises	Line	Formula	Justification	References
1	1	$((P \wedge Q) \vee R)$	Premise	
2	2	$\neg P$	Premise	
3	3	$\neg R$	Premise	
1	4	$R \vee (P \wedge Q)$	Commutation	1
1,3	5	$(P \wedge Q)$	DS	1,3
1,3	6	P	Simp.	5
1,2,3	7	$(P \wedge \neg P)$	Conj.	2,6
1,2	8	R	IP	3,7
1	9	$(\neg P \to R)$	CP	2,8

After a few unsuccessfull attempts at proving an argument is valid, we might suppose that the argument is NOT valid

You might then want to try to establish that the argument is *invalid* (i.e. that the premises do *not* logically imply the conclusion)

BEWARE: we cannot conclude that an argument is invalid just because we could not find a proof. We need to prove it!

HOW? By giving a counter-example – an allocation of truth values to the atomic propositions for which the argument does not hold. i.e. such that

- the premises are TRUE, but
- the conclusion is FALSE

Establishing Invalidity

Is the argument below valid?

$$(A \to B),\ (C \to D),\ (B \vee C)\ \vdash\ (A \vee D)$$

Method:

- Start with false conclusion, here $(A \vee D)$ is false, therefore both A and D must be false.
- Then try to allocate truth values to the other atomic propositions such that the premises are still true
- If you can find a combination for which the premises are truth, but the conclusion is false, then the argument is *invalid*

Establishing Invalidity

Is the argument below valid?

$$(A \to B),\ (C \to D),\ (B \lor C)\ \vdash\ (A \lor D)$$

1. $(A \lor D)$ is false, therefore A and D are both false

Establishing Invalidity

Is the argument below valid?

$$(A \to B),\ (C \to D),\ (B \vee C)\ \vdash\ (A \vee D)$$

1. $(A \vee D)$ is false, therefore A and D are both false
2. $(C \to D)$ is true, but D is false, therefore C is false

Establishing Invalidity

Is the argument below valid?

$$(A \to B),\ (C \to D),\ (B \vee C)\ \vdash\ (A \vee D)$$

1. $(A \vee D)$ is false, therefore A and D are both false
2. $(C \to D)$ is true, but D is false, therefore C is false
3. $(B \vee C)$ is true and C is false, therefore B is true

Establishing Invalidity

Is the argument below valid?

$$(A \to B),\ (C \to D),\ (B \vee C)\ \vdash\ (A \vee D)$$

1. $(A \vee D)$ is false, therefore A and D are both false
2. $(C \to D)$ is true, but D is false, therefore C is false
3. $(B \vee C)$ is true and C is false, therefore B is true
4. $(A \to B)$ holds true

Establishing Invalidity

Is the argument below valid?

$$(A \to B),\ (C \to D),\ (B \lor C)\ \vdash\ (A \lor D)$$

1. $(A \lor D)$ is false, therefore A and D are both false
2. $(C \to D)$ is true, but D is false, therefore C is false
3. $(B \lor C)$ is true and C is false, therefore B is true
4. $(A \to B)$ holds true

We've found an assignment of truth values which satisfy all the premises, but for which the conclusion is false. Therefore the argument is invalid.

Logical Tautologies

Recall that a tautology is a formula which is true all the time (i.e. it does not rely on any premise.)

We can prove that a wff A is a tautology by proving that it can be derived with an empty premise set.

Conditional and Indirect proofs are the only two rules which allow us to "remove premises". We can use them to derive tautologies.

Example using CP

Prove that $(\neg A \rightarrow (A \rightarrow B))$ is a tautology. i.e.

$$\vdash (\neg A \rightarrow (A \rightarrow B))$$

Premises	Line	Formula	Justification	References
1	1	$\neg A$	Premise	
1	2	$(\neg A \vee B)$	Add	1
1	3	$(A \rightarrow B)$	DefI	2
	4	$(\neg A \rightarrow (A \rightarrow B))$	CP	1,3

Example using CP

$$\vdash (P \rightarrow (Q \rightarrow P))$$

Premises	Line	Formula	Justification	References
1	1	P	Premise	
1	2	$(P \vee \neg Q)$	Add	1
1	3	$(\neg Q \vee P)$	Comm	2
1	4	$(Q \rightarrow P)$	DefI	3
	5	$(P \rightarrow (Q \rightarrow P))$	CP	1,4

Example using IP

$$\vdash ((P \rightarrow Q) \vee (P \rightarrow \neg Q))$$

Prem.	Line	Formula	Justification	Refs
1	1	$\neg((P \rightarrow Q) \vee (P \rightarrow \neg Q))$	Premise	
1	2	$(\neg(P \rightarrow Q) \wedge \neg(P \rightarrow \neg Q))$	DeM	1
1	3	$(\neg(\neg P \vee Q) \wedge \neg(P \rightarrow \neg Q))$	DefI	2
1	4	$(\neg(\neg P \vee Q) \wedge \neg(\neg P \vee \neg Q))$	DefI	3
1	5	$((\neg\neg P \wedge \neg Q) \wedge \neg(\neg P \vee \neg Q))$	DeM	4
1	6	$((\neg\neg P \wedge \neg Q) \wedge (\neg\neg P \wedge \neg\neg Q))$	DeM	5
1	7	$((\neg\neg P \wedge \neg Q) \wedge (\neg\neg Q \wedge \neg\neg P))$	Comm	6
1	8	$(((\neg\neg P \wedge \neg Q) \wedge \neg\neg Q) \wedge \neg\neg P))$	Assoc	7
1	9	$((\neg\neg P \wedge \neg Q) \wedge \neg\neg Q)$	Simp	8
1	10	$(\neg\neg P \wedge (\neg Q \wedge \neg\neg Q))$	Assoc	9
1	11	$((\neg Q \wedge \neg\neg Q) \wedge \neg\neg P)$	Comm	10
1	12	$(\neg Q \wedge \neg\neg Q)$	Simp	11
	13	$((P \rightarrow Q) \vee (P \rightarrow \neg Q))$	IP	1,12

- Conditional proofs are often better suited when you need to prove an implication (or a disjunction)

- You can use several IP and CP in the same proof, as long as you are careful with the assumption and removal of premises

- With the Natural Deduction System we use here,
  - A proof exists for every argument that is valid (the NDS is *complete*)
  - Any formula derived from an empty set of premises is a tautology (the NDS is *sound*)

Quine's Method

Another way to prove tautologies.

- Build a binary tree, substituting T(rue) and F(alse) for each atomic proposition.
- Simplify the wffs by replacing:
  - $\neg T$ with F, and $\neg F$ with T
  - $(P \to T)$ with T
  - $(F \to P)$ with T
  - $(T \to P)$ with P
  - $(T \lor P)$ with T
  - $(F \lor P)$ with P
  - $(T \land P)$ with P
  - $(F \land P)$ with F
- If all the leaves of the tree are True, then it must be a tautology
  - (similarly, we can deduce contradiction or contingency.)

Example (P first)

$$(P \to (Q \to P))$$

$P = T$

$$(T \to (Q \to T))$$
$$= (T \to T)$$
$$= T$$

$P = F$

$$(F \to (Q \to F))$$
$$= T$$

All the leaves are *true*, so $(P \to (Q \to P))$ is a tautology.

Example (Q first)

$$(P \to (Q \to P))$$

$Q = T$

$$(P \to (T \to P))$$
$$= (P \to P)$$

$Q = F$

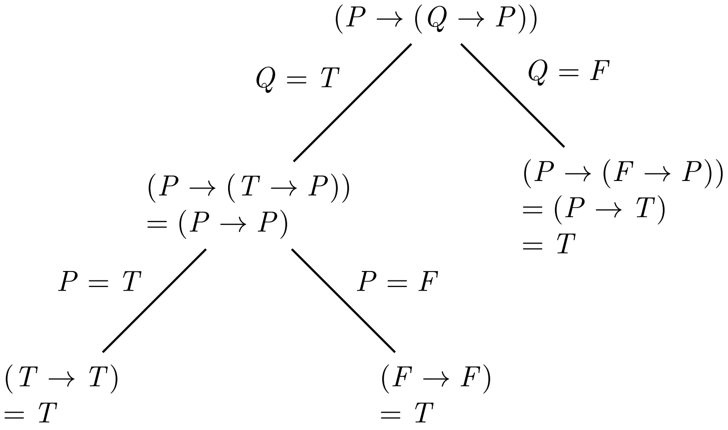
$$(P \to (F \to P))$$
$$= (P \to T)$$
$$= T$$

Example (Q first)

$(P \to (Q \to P))$

- $Q = T$: $(P \to (T \to P))$ $= (P \to P)$
  - $P = T$: $(T \to T)$ $= T$
  - $P = F$: $(F \to F)$ $= T$
- $Q = F$: $(P \to (F \to P))$ $= (P \to T)$ $= T$

Example (Q first)



All the leaves are *true*, so $(P \to (Q \to P))$ is a tautology.

Example

$$(((A \wedge B) \to C) \wedge (A \to B))$$

$A = T$

$A = F$

$$\begin{aligned}&(((T \wedge B) \to C) \wedge (T \to B))\\ &= ((B \to C) \wedge B)\end{aligned}$$

$$\begin{aligned}&(((F \wedge B) \to C) \wedge (F \to B))\\ &= ((F \to C) \wedge T)\\ &= (T \wedge T)\\ &= T\end{aligned}$$

Example

$$(((A \wedge B) \to C) \wedge (A \to B))$$

$A = T$

$$(((T \wedge B) \to C) \wedge (T \to B))$$
$$= ((B \to C) \wedge B)$$

$B = T$

$$((T \to C) \wedge T)$$
$$= C$$

$B = F$

$$((F \to C) \wedge F)$$
$$= F$$

$A = F$

$$(((F \wedge B) \to C) \wedge (F \to B))$$
$$= ((F \to C) \wedge T)$$
$$= (T \wedge T)$$
$$= T$$

Example

$$(((A \wedge B) \to C) \wedge (A \to B))$$

$A = T$

$$(((T \wedge B) \to C) \wedge (T \to B))$$
$$= ((B \to C) \wedge B)$$

$B = T$

$$((T \to C) \wedge T)$$
$$= C$$

$B = F$

$$((F \to C) \wedge F)$$
$$= F$$

$A = F$

$$(((F \wedge B) \to C) \wedge (F \to B))$$
$$= ((F \to C) \wedge T)$$
$$= (T \wedge T)$$
$$= T$$

Some of the leaves are *true* and some are *false*, so $(((A \wedge B) \to C) \wedge (A \to B))$ is a contingency.