Proof: angles for rotating disc and moving grabber

(coordinate system: origin at top-left, x rightwards, y downwards)

Problem setup. We have a circular disc of radius R with centre at the point $\mathbf{D} = (D_x, D_y)$ (Cartesian coordinates, origin at top-left, x increasing to the right, y increasing downwards). A point on the disc is $\mathbf{A} = (A_x, A_y)$. Independently there is a circular "grabber" of fixed radius r whose centre is at $\mathbf{G} = (G_x, G_y)$. The grabber centre lies due south of the disc centre and the northern-most point of the grabber circle coincides with the disc centre; hence

$$\mathbf{G} = \mathbf{D} + (0, r) \qquad \text{(i.e. } G_x = D_x,\ G_y = D_y + r\text{)}.$$

The grabber can move along its circular arc; we measure the grabber angular position θ as the clockwise angle from the NS line (the vertical line through $\mathbf{G}$ and $\mathbf{D}$), so that $\theta = 0$ corresponds to the northernmost point of the grabber circle (which is the disc centre). The disc may be rotated by an angle α (clockwise positive) about its centre $\mathbf{D}$.

We seek expressions for (i) the rotation angle α that places the given point on the disc onto the grabber circle, and (ii) the corresponding grabber angle θ at which the grabber meets the (now rotated) point.

Notation. Let

$$\begin{aligned}
\Delta x &= A_x - D_x, \\
\Delta y &= A_y - D_y, \\
\mathbf{u} &= \mathbf{A} - \mathbf{D} = (\Delta x, \Delta y), \\
L &= |\mathbf{u}| = \sqrt{\Delta x^2 + \Delta y^2} \quad \text{(distance from } \mathbf{D} \text{ to } \mathbf{A}\text{)}.
\end{aligned}$$

Define the angle β of the vector $\mathbf{u}$ measured clockwise from the positive x-axis (this choice is consistent with the screen coordinates where y increases downward):

$$\beta = \operatorname{atan2}(\Delta y, \Delta x).$$

Finally set

$$S := \frac{L}{2r}. \qquad \text{(note: a necessary condition for a solution is } L \le 2r\text{, i.e. } S \le 1.\text{)}$$

Step 1: equation for the rotated point lying on the grabber circle. Translate coordinates so the disc centre is at the origin: working with vectors relative to $\mathbf{D}$ is algebraically convenient. The vector from $\mathbf{D}$ to $\mathbf{A}$ is $\mathbf{u} = L(\cos\beta, \sin\beta)$ (where cosine and sine follow the same orientation convention as β). After rotating the disc clockwise by α, the image of $\mathbf{A}$ relative to the disc centre is

$$\mathbf{u}' = L\big(\cos(\beta + \alpha),\ \sin(\beta + \alpha)\big).$$

The position of the rotated point relative to the grabber centre $\mathbf{G}$ is

$$\mathbf{u}' - (0, r), \qquad \text{because } \mathbf{G} = \mathbf{D} + (0, r).$$

Requiring that the rotated point lies on the grabber circle of radius r centred at $\mathbf{G}$ yields

$$\left\|\mathbf{u}' - (0, r)\right\| = r.$$

Square both sides and expand:

$$\begin{aligned}\left|\mathbf{u}' - (0, r)\right|^2 &= \left|\mathbf{u}'\right|^2 + r^2 - 2r\big(\text{the } y\text{-component of } \mathbf{u}'\big)\\ &= L^2 + r^2 - 2r\big(L\sin(\beta+\alpha)\big) = r^2.\end{aligned}$$

Cancel r^2 and solve for $\sin(\beta+\alpha)$:

$$L^2 - 2rL\sin(\beta+\alpha) = 0 \quad\Longrightarrow\quad \sin(\beta+\alpha) = \frac{L}{2r} = S.$$

Step 2: solutions for α. The identity $\sin(\beta+\alpha) = S$ gives the two (principal) solutions for $\beta+\alpha$ modulo 2π:

$$\beta + \alpha = \arcsin(S) \quad\text{or}\quad \beta+\alpha = \pi - \arcsin(S) \pmod{2\pi}.$$

Therefore the corresponding solutions for the disc rotation angle α (clockwise positive) are

$$\boxed{\alpha = \arcsin(S) - \beta \quad\text{or}\quad \alpha = \pi - \arcsin(S) - \beta} \tag{1}$$

(Each value is understood modulo 2π and chosen according to which intersection on the grabber circle is required.)

Step 3: relationship giving the grabber angle θ. Let θ denote the clockwise angle along the grabber circle measured from the NS line (vertical through $\mathbf{G}$ and $\mathbf{D}$), with $\theta = 0$ at the northernmost point (which equals $\mathbf{D}$). The parametric coordinates of a point on the grabber circle with angle θ are (in the same screen-coordinate convention)

$$\mathbf{P}(\theta) = \mathbf{G} + r\big(\sin\theta,\, -\cos\theta\big).$$

The rotated point we found also equals $\mathbf{P}(\theta)$; comparing the y-components of the equality $\mathbf{u}' - (0, r) = r(\sin\theta, -\cos\theta)$ gives

$$L\sin(\beta+\alpha) - r = -r\cos\theta \quad\Longrightarrow\quad L\sin(\beta+\alpha) = r(1-\cos\theta).$$

Using $\sin(\beta+\alpha) = S$, we get

$$L\cdot S = r(1-\cos\theta) \quad\Longrightarrow\quad 1-\cos\theta = \frac{L^2}{2r^2}.$$

Hence

$$\cos\theta = 1 - \frac{L^2}{2r^2} = 1 - 2S^2.$$

Write this in half-angle form: recall $1-\cos\theta = 2\sin^2(\theta/2)$. Therefore

$$2\sin^2\left(\frac{\theta}{2}\right) = \frac{L^2}{2r^2} = 4S^2\cdot\frac{1}{4} = 2S^2 \quad\Longrightarrow\quad \sin^2\left(\frac{\theta}{2}\right) = S^2.$$

Thus

$$\frac{\theta}{2} = \arcsin(S) \quad \text{or} \quad \frac{\theta}{2} = \pi - \arcsin(S) \pmod{2\pi}.$$

Consequently the grabber angular solutions are

$$\boxed{\theta = 2\arcsin(S) \quad \text{or} \quad \theta = 2\pi - 2\arcsin(S)} \tag{2}$$

(Again θ is understood modulo 2π; take the branch in $[0, 2\pi)$ appropriate to the chosen intersection.)

Remarks and conditions.

- The quantity $S = \dfrac{L}{2r}$ must satisfy $0 \le S \le 1$ for real solutions; therefore a necessary and sufficient condition for an intersection is $L \le 2r$. Geometrically this says the point on the disc (after rotation) must be at most diameter distance apart from the grabber centre so that the grabber circle can reach it.
- The disc radius R does not appear in the algebraic expressions for α and θ except insofar as the given point $\mathbf{A}$ must lie on the disc (so that $L \le R$). The grabber centre $\mathbf{G}$ is used only via the relation $\mathbf{G} = \mathbf{D} + (0, r)$ (the northernmost point of the grabber circle is at $\mathbf{D}$) and the grabber radius r appears explicitly.
- The two algebraic branches for α correspond to the two different intersection points on the grabber circle (the one on the left-hand side of the vertical through the centres and the one on the right-hand side). Similarly the two values of θ denote the clockwise and counter-clockwise arc positions that are symmetric with respect to the NS line.

Final boxed formulas.

$$\boxed{\alpha = \arcsin\left(\frac{L}{2r}\right) - \beta \quad \text{or} \quad \alpha = \pi - \arcsin\left(\frac{L}{2r}\right) - \beta}$$

$$\boxed{\theta = 2\arcsin\left(\frac{L}{2r}\right) \quad \text{or} \quad \theta = 2\pi - 2\arcsin\left(\frac{L}{2r}\right)}$$

where $L = |\mathbf{A} - \mathbf{D}|$ and $\beta = \operatorname{atan2}(\Delta y, \Delta x)$ (clockwise from the positive x-axis). □

(Optional) short coordinate check. If one places the disc centre at the origin and chooses a concrete $\mathbf{u}$, the algebra above reduces to the elementary trigonometric equalities used in the derivation; the two branches correspond to the two solutions of the sine and cosine equations.