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On the Comparison of the Spectra in AdS/CFT Correspondence,

Construction of the Symmetry from Flat Space Limit

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Contents

0.1 Abstract

In recent years, there has been a considerable interest in the study of $\mathcal{N} = 4$ SYM states. For instance, in [20], it was developed an efficient numerical algorithm for the computation of the spectrum of anomalous dimensions at finite coupling; in addition, in [15] , it were studied operators at strong coupling, with general spin and $SO(6)$ symmetric traceless representations, in order to compute their structure constants with two arbitrary chiral primary operators. In these analysis, the most important theoretical concept is the Maldacena correspondence, which relates $\mathcal{N} = 4$ SYM theory with Superstring theory on $AdS_5 \times S^5$ space.

In this context, we have developed an algorithm to compare the spectrum of long single trace operators in planar $\mathcal{N} = 4$ SYM, with their dual, the IIB Superstring states on $AdS_5 \times S^5$; in particular, we want to understand how the representations of Superstring states organize in the $PSU(2, 2|4)$ group, starting from the flat space, where they have Super-Poincaré $10D$ symmetry. In this analysis, we have mainly followed [16] and [18].

As a starting point, we have analyzed the massive states for open Superstring theory in flat space. These states were obtained in two different ways: by direct application of the superstring ladder operators on the ground state, we have evaluated the firsts massive levels; on the other hand, by evaluation of the unrefined and refined string partition function, we have evaluated highers string levels; then, by the tensor product of states at the same string level, we have obtained the IIB massive Superstring spectrum on flat space, which can be recast in the famous form:

$$T_l^{IIB} = T_1^2 \otimes (\mathrm{vac}_l \otimes \mathrm{vac}_l) \tag{1}$$

were T_1 is the first massive open string state; all these states are naturally expressed in $SO(9)$ representations. We have obtained explicit evaluations up to string level $l = 8$.

To perform the analysis in the context of AdS/CFT correspondence, we should evaluate the string spectrum in curve space; for this reason, we have decomposed the $SO(9)$ representations in their maximal subalgebra $SO(4) \times SO(5)$; we have obtained this decomposition using the characters from Lie algebra theory, from which we have derived the specifics branching rules.

A substantial part of this thesis job has regarded the full understanding of the Kaluza Klein theory applied to the $SO(5)$ subgroup. Indeed, the component $SO(4) \subset SO(4, 2)$, describes the AdS_5 group, while the $SO(5)$ component needs to be expanded in representations of $SO(6)$ as Kaluza Klein towers, in order to obtain the states on the curved space $AdS_5 \times S^5$.

In this analysis, we have recognized as wrong a formula written on several papers, which describes the Kaluza Klein expansion for a state in $SO(5)$ representation (for instance, formula 3.4 of [16], but also present in [20] and [15]). We have demonstrated that the right formula is:

$$KK_{[m,n]} = \sum_{k=0}^{m} \sum_{s=0}^{n} \sum_{q=m-k}^{\infty} [a, n + k - s, k + s] \tag{2}$$

Where, on the left, the $SO(5)$ representation is identified through the Dynkin labels $[m, n]$; on the right, the formula is written in terms of the three $SO(6)$ Dynkin labels; we are sure about the correctness of our result, because we have demonstrated it in two different ways.

Initially, we have found the branching rule for the decomposition $SO(6) \to SO(5)$, which we have shown to be:

$$[p, j, l] \xrightarrow[SO(6)\to SO(5)]{} \sum_{m=0}^{p} \sum_{k=0}^{\min(j,l)} [m + k, j - 2k + l] \tag{3}$$

Applying this formula on a general $SO(6)$ state and requiring the consistency conditions from Kaluza Klein theory, we have demonstrated the equation (2).

On the other side, with an independent analysis, we have shown that the original Kaluza Klein

formula written on papers is inconsistent with formula 3.6 of [3]:

$$KK_{\mathcal{R}_{SO(5)}} = \sum_{n=0}^{\infty} [n,0,0] \times \hat{\mathcal{R}}_{SO(5)} \tag{4}$$

where $\hat{\mathcal{R}}_{SO(5)}$ is a representation of $SO(6)$ which can be obtained by a direct lifting of the reducible representation of $SO(5)$, $\mathcal{R}_{SO(5)}$. On the other side, our Kaluza Klein formula is perfectly consistent with (4).

The paper [15], which contains the wrong Kaluza Klein formula, considers low Dynkin labels representations, for which the wrong and the right formula are equivalent; hence, the validity of this paper is still confirmed; on the other hand, further analysis are necessary for the paper [20], in order to understand how this formula were used. Formula (4) is our starting point for the analysis of the correspondence between string states and $\mathcal{N}=4$ SYM states; in any case, our formula (2) is more general than (4) and could be an alternative starting point for future analysis.

On the CFT side, we have evaluated the spectrum of single trace operators in $\mathcal{N}=4$ SYM; these states were obtained using the enumeration of states in Polya theory, from which we have obtained the Super Yang Mills generating function; in order to compare these states with the massive string states, we only need long superconformal states; for this reason, we have explicitly decomposed the SYM states in long states and in $\frac{1}{2}$ BPS states, holding only the long states; in addition, we have removed all the descendants, dividing by the superconformal multiplet, which is generated by the unconstrained action of all supercharges and superderivatives on the superconformal singlet, obtaining:

$$\mathcal{Z}_{\text{sconf}} = \frac{\mathcal{Z}_{SYM} - \mathcal{Z}_{BPS}}{\mathcal{T}_{sconf}} \tag{5}$$

where $\mathcal{Z}_{\text{sconf}}, \mathcal{Z}_{SYM}, \mathcal{Z}_{BPS}$ are the generating functions for the superconformal, general Super Yang Mills and $\frac{1}{2}$BPS multiplets, respectively. Using (6.9) and character theory, we have explicitly derived the superconformal states up to conformal dimension $\Delta = 6$.

As a final analysis, the spectra of AdS and CFT side were compared; we started with the superconformal multiplet and the $l=1$ closed string level on $SO(5)\times SO(4)$; we have evaluated the character of the superconformal multiplet, removing the information about the conformal dimension and breaking its representations from $SO(6)\times SO(4)$ to $SO(5)\times SO(4)$, finding that it is equivalent to the character of the first superstring massive state. In this way, using formula (4), we were able to write the level $l=1$ KK tower as

$$\mathcal{H}_1 = \sum_{n=0}^{\infty} \mathcal{T}_{sconf} \times [n,0,0]_{[0,0]} \tag{6}$$

Then, we have lifted the representations vac_l of higher string levels to $SO(6)\times SO(4)$, in order to obtain their KK towers. From them, we have obtained the SYM spectrum; in particular, we have shown that the first 5 string levels KK towers reproduce all the states of SYM up to conformal dimension $\Delta = 6$.

In the lifting algorithm for vac_l, the conformal dimensions were assigned quite arbitrarily, usually saturating their unitary bound; however, we have demonstrated that this assignment is confirmed when we analyze the work of Gromov [20], which relates, with an independent analysis, the conformal dimension from weak to strong coupling for SYM states; knowing that at strong coupling the conformal dimension depends only on the string level: $\Delta \simeq 2\sqrt{l}\lambda^{\frac{1}{4}}$, we were able to confirm the assignment of conformal dimensions in the weak limit in our analysis.

0.2 Introduction

In recent years there has been a considerable interest in the study of states of $\mathcal{N} = 4$ SYM. For instance, in [20], it was developed an efficient numerical algorithm for computing the spectrum of anomalous dimension in $\mathcal{N} = 4$ SYM at finite coupling; on the other side, in [15] , it were studied short operators in planar $\mathcal{N} = 4$ SYM at strong coupling, for general spin and $SO(6)$ symmetric traceless representations, in order to compute their structure constants with two arbitrary chiral primary operators.

On the other hand, the study of these states is very important also for evaluation of correlation functions; for instance, in the very recent article [12] , it was studied the CFT data of the massive short strings exchanged in the operator product expansion of the four point function dual to the Virasoro-Shapiro amplitude. In addition, in [4] it were studied the correlators of four half-BPS operators in stroungly coupled SYM, using new bootstrap methods based on the integrability of SYM; moreover, in the same article, it was discovered an unexpected ten-dimensional conformal symmetry, enjoyed by tree-level supergravity,

In this context, my thesis job wants to be a systematic study on the reorganization of states from IIB superstrings flat space to $AdS_5 \times S^5$ space, where they organizes in multiplets of $SO(4) \times SO(6)$ group; in this way, this thesis wants to fully explore the duality between these states and the $\mathcal{N} = 4$ SYM single trace states, finding precise relations between them; in this context, our work will mostly follow [6] for the derivation of superstring states using a refined generating function, and [16],[18] for the derivation of SYM states and the comparison to superstring states.

The mathematic tool that we will mostly use, is the character theory from Lie algebra theory; for this reason, the first chapter of this thesis is a review of Lie groups and Lie algebra, with particular focus on the character theory, where we give some explicit characters formulae for the most analyzed groups. chapters 2,3,4 and 5 are still very theorical and give a brief review of Superstring theory, Super yang Mills and the Maldacena conjecture; in chapter 6 we derived the IIB supersting spectrum in flat space and we construct its implementation on $AdS_5 \times S^5$. Chapter 7 is fully devoted to Kaluza Klein theory and, in particular, to the Kaluza Klein expansion of spherical harmonics from $SO(5)$ to $SO(6)$ representations. In chapter 8 we derive the SYM spectrum using Polya theory and the refined generating function, while in chapter 9 we explicitly verify Maldacena duality for SYM and superstring states.

Chapter 1

Lie Groups and Lie Algebra

In this chapter, we will give a brief review of Lie group theory, following mostly [9],[14]; In section 1.4, we will explain how we have evaluated the character functions for different groups, using the symbolic software [26].

1.1 The Structure of Simple Lie algebra

1.1.1 Basic Concepts

A group G is a set of elements $g_1, g_2, \ldots$ that has

- An associative multiplication law, under which $g_1 g_2 = g_3$ for each $g_{1,2} \in G$, with $g_3 \in G$ and $(g_1 g_2) g_3 = g_1 (g_2 g_3)$.
- An identity element $I \in G$ with $Ig = gI = g$ for all $g \in G$.
- A unique inverse element g^{-1} for each $g \in G$, such that $gg^{-1} = g^{-1}g = I$

An abelian (commutative) group is a special case, defined by $g_1 g_2 = g_2 g_1$ for all $g_{1,2} \in G$. Otherwise, G is non-abelian.

A Lie group G is a continuous group and, at the same time, a smooth manifold, for which the multiplication law involves differentiable functions of the parameters that label the group elements. Most of the Lie groups of interest in particle physics are compact, which means that the parameters form a compact manifold.

In this way, a Lie group can be defined, at least for infinitesimal transformations, in terms of its associated Lie algebra, which consists of N generators, $T^i, i = 1, 2, \ldots N$ and their commutation rules

$$[T^i, T^j] = ic_{ijk} T^k \tag{1.1}$$

where c_{ijk} are the structure constants of G and N is the dimension of the algebra. In addition, the Lie algebra must satisfy the Jacobi identity:

$$[[T_i, T_j], T_k] + [[T_j, T_k] T_i] + [[T_k, T_i] T_j] = 0 \tag{1.2}$$

A representation of a Lie group refers to the association of every element of g to a linear operator acting on some vector space V, which respects the commutation relation of the algebra. In this way, each element of g can be represented in terms of a square matrix and the basis vectors are represented by column matrices (the commutator corresponds to the usual matrix commutation). A representation is said to be irreducible if the matrices representing the elements of g cannot all be brought in a block-diagonal form by a change of basis.

Two goups G_1 and G_2 commute if $[g_i, \hat{g}_j] = 0 \; \forall g_i \in G_1, \hat{g}_j \in G_2$. Then one can define the direct product group $G = G_1 \times G_2$ with elements $g_i\hat{g}_j$. We define an invariant subgroup H of G, a subgroup of G such that $ghg^{-1} \in H, \; \forall g \in G, \; h \in H$. A simple group is a group that does not contain any invariant subgroups (or less rigorously, it is a non abelian group that is not a direct product), while a semi-simple group is the direct product of two or more simple groups.

1.1.2 Cartan-Weyl basis

In the standard Cartan-Weyl basis, we take the maximal set of commuting Hermitian generators H^i, $i = 1, \ldots, r$ (r is the rank of the algebra):

$$\left[H^i, H^j\right] = 0 \tag{1.3}$$

which form the Cartan subalgebra h; these generators can all be diagonalized simultaneously. The remaining generators are chosen to be those particular combinations of the T^i's that satisfy the following equation:

$$\left[H^i, E^\alpha\right] = \alpha^i E^\alpha \tag{1.4}$$

The vector $\alpha = \left(\alpha^1, \ldots, \alpha^r\right)$ is called a root and E^α is the corresponding ladder operator. The root α naturally maps an element $H^i \in h$ to the number α^i, by $\alpha(H^i) = \alpha^i$. Hence, the roots are elements of the dual of the Cartan subalgebra, $\alpha \in h^*$.

From (1.4) and its Hermitian conjugate, we see that $-\alpha$ is necessarily a root if α is a root.

Root components can be regarded as the nonzero eigenvalues of the H^i in the adjoint representation, for which the Lie algebra itself serves as the vector space on which the generators act.In this representation, we have an identification between the generators and the states of the representation:

$$E^\alpha \to |E^\alpha\rangle = |\alpha\rangle \tag{1.5}$$

$$H^i \to \left|H^i\right\rangle \tag{1.6}$$

Hence, follow from (1.4) that in the adjoint representation, the action of a generator X is represented by $ad(X)$, defined as:

$$ad(X)Y = [X, Y] \tag{1.7}$$

so that

$$ad(H^i)E^\alpha = \alpha^i E^\alpha \quad \to \quad H^i \left|\alpha\right\rangle = \alpha^i \left|\alpha\right\rangle \tag{1.8}$$

By construction, the dimension of the adjoint is equal to the dimension of the algebra, itself equal to the total number of roots plus r.

The Jacobi identity implies:

$$[H^i, [E^\alpha, E^\beta]] = -[E^\alpha, [E^\beta, H^i]] - [E^\beta, [H^i, E^\alpha]] = (\alpha^i + \beta^i)[E^\alpha, E^\beta] \tag{1.9}$$

Hence, if $\alpha + \beta \in \Delta$ (the set of all roots), the commutator $[E^\alpha, E^\beta]$ must be proportional to $E^{\alpha+\beta}$ and it must vanish if $\alpha + \beta \notin \Delta$.

If $\alpha = -\beta$, $[E^\alpha, E^{-\alpha}]$ commutes with all H^i, which is possible only if it is a linear combination of the generators of the Cartan subalgebra; then, we require $[E^\alpha, E^{-\alpha}] = 2\frac{\alpha \cdot H}{|\alpha|^2}$.

Summarizing, the full set of commutation relations in the Cartan-Weyl basis is

$$\begin{aligned}
[H^i, H^j] &= 0 \\
[H^i, E^\alpha] &= \alpha^i E^\alpha \\
[E^\alpha, E^\beta] &= N_{\alpha,\beta} E^{\alpha+\beta} \quad \text{if} \quad \alpha + \beta \in \Delta \\
&= 2\frac{\alpha \cdot H}{|\alpha|^2} \qquad \text{if} \quad \alpha = -\beta \\
&= 0 \qquad\qquad \text{otherwise}
\end{aligned} \tag{1.10}$$

1.1.3 The Killing Form

The Killing form

$$\tilde{K}(X, Y) = \frac{1}{2g}\mathrm{Tr}(ad(X), ad(Y)) \tag{1.11}$$

where g is a constant, gives us a sort of scalar product for the Lie algebra; for semisimple Lie algebra, the Killing form is nondegenerate: if $\tilde{K}(X, Y) = 0 \;\forall\; Y$, implies that $X = 0$ (this can be considered as an alternative way of defining semisimplicity).

The standard basis $\{T^a\}$ is understood to be orthonormal with respect to K:

$$K(T^a, T^b) = \delta^{ab} \tag{1.12}$$

The same is true for Cartan subalgebra:

$$K(H^i, H^j) = \delta^{ij} \tag{1.13}$$

Since the Killing form defines a scalar product, it can be used to lower or raise the indices

$$c_{abc} = \sum_d c^{ad}{}_c [K(T^d, T^b)]^{-1} \tag{1.14}$$

The cyclic property of the trace yields the identity:

$$K([Z, X], Y) + K(X, [Z, Y]) = 0 \tag{1.15}$$

it follows that

$$[E^\alpha, E^{-\alpha}] = K(E^\alpha, E^{-\alpha})\alpha \cdot H \tag{1.16}$$

The fundamental role of the Killing form is to establish an isomorphism between the Cartan subalgebra h and its dual h^*, indeed, $K(H^i, \cdot)$ maps every element of the Cartan subalgebra onto a number; in this way the Killing form can be transferred into a positive definite scalar product in the dual space, in this way we define a scalar product between roots:

$$(\alpha, \beta) = K(H^\alpha, H^\beta) \tag{1.17}$$

with $H^\alpha = \alpha \cdot H$; from now on the scalar product between roots will be denoted as above, with the understanding that $|\alpha|^2 = (\alpha, \alpha)$.

1.1.4 Weights

For an arbitrary representation, a basis $\{|\lambda\rangle\}$ can always be found such that:

$$H^i |\lambda\rangle = \lambda^i |\lambda\rangle \tag{1.18}$$

The vector $\lambda = (\lambda^1, \ldots, \lambda^r)$ is called a weight and belong to the space $h^* : \lambda(H^i) = \lambda^i$; the scalar product between weights is also fixed by the Killing form. In the adjoint representation, the weights coincide with the roots.

An important property is that E^α changes the eigenvalue of a state by α:

$$H^i E^\alpha |\lambda\rangle = E^\alpha H^i |\lambda\rangle + [H^i, E^\alpha] |\lambda\rangle = (\lambda^i + \alpha^i) E^\alpha |\lambda\rangle \tag{1.19}$$

Hence, $E^\alpha |\lambda\rangle$, if nonzero, must be proportional to a state $|\lambda + \alpha\rangle$

For any state $|\lambda\rangle$ in a finite-dimensional representation, there are necessarily two positive integers p and q, such that:

$$(E^\alpha)^{p+1} |\lambda\rangle \sim E^\alpha |\lambda + p\alpha\rangle = 0 \tag{1.20}$$

$$(E^{-\alpha})^{q+1} |\lambda\rangle \sim E^\alpha |\lambda - q\alpha\rangle = 0 \tag{1.21}$$

for any root α. We notice also that $E^\alpha, E^{-\alpha}$ and $\frac{\alpha \cdot H}{|\alpha|^2}$ form an $su(2)$ subalgebra analogue to the set $\{J^+, J^-, J^3\}$ of Quantum Mechanics:

$$[J^+, J^-] = 2J^3, \quad [J^3, J^\pm] = \pm J^\pm \tag{1.22}$$

In this way, the state with highest $J^3 = \frac{\alpha \cdot H}{|\alpha|^2}$ projection ($m = j$) can be reached by a finite number (p) applications of $J^+ = E^\alpha$; while, the state with lowest value ($m = -j$) can be reached by a finite number (q) applications of $J^- = E^{-\alpha}$; in conclusion, we obtain:

$$j = \frac{(\alpha, \lambda)}{|\alpha|^2} + p \qquad -j = \frac{(\alpha, \lambda)}{|\alpha|^2} - q \tag{1.23}$$

Elimintating j from the above two equations, we obtain:

$$2\frac{(\alpha, \lambda)}{|\alpha|^2} = -(p - q) \tag{1.24}$$

where we see that any weight λ in a finite-dimensional representation is such that $(\alpha, \lambda)/|\alpha|^2$ is an integer.

1.1.5 Simple Roots

The number of roots is equal to the dimension of the algebra minus is rank and this number is in general much larger than the rank itself. Hence, the roots are linearly dependent; we fix a basis $\{\beta_1, \ldots, \beta_r\}$ in h^* so that $\alpha = \sum_{i=1}^r n_i \beta_i$

We can define an ordering, α is said to be positive if the first nonzero number in the sequence $(n_1, \ldots, n_r)$ is positive.

A simple root α_i is defined to be a root that cannot be written as the sum of two positive roots; there are necessarily r simple roots and their set $\{\alpha_1, \ldots, \alpha_r\}$ provides the most convenient basis for the r-dimensional space of roots.

1.1.6 Cartan Matrix

We define the Cartan matrix as the scalar product of simple roots:

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{\alpha_j^2} \tag{1.25}$$

The entries of this matrix are necesarily integers; its diagonal elements are all equal to 2 and it is not symmetric in general. In addition, $(\alpha_i, \alpha_j) \leq 0$ if $i \neq j$.

It is convenient to introduce $\alpha_i^\vee$, the coroot associated with the root α_i,

$$\alpha_i^\vee = \frac{2\alpha_i}{|\alpha_i|^2} \tag{1.26}$$

In this way

$$A_{ij} = (\alpha_i, \alpha_j^\vee) \tag{1.27}$$

1.1.7 Dynkin Diagrams

All the information contained in the Cartan matrix can be encapsulated in a simple planar diagram: the Dynkin diagram; to every simple root we associate a node and join the nodes i and j with $A_{ij}A_{ji}$ lines. Depending on the number of lines, we assign an angle between roots: orthogobal simple roots are disconnected, while those sustaining an angle of 120, 135 or 150 degrees are linked by one, two or three lines, respectevely.

The classifications of simple Lie algebra boils down to a classification of Dynkin diagrams. The complete list contains four infinte families, the algebras $A_r,\ B_r,\ C_r,\ D_r$ and five exceptional cases $E_6,\ E_7,\ E_8,\ F_4,\ G_2$.

1.1.8 Fundamental Weights

We define the fundamental weights ω_i as

$$(\omega_i, \alpha_j^\vee) = \delta_{ij} \tag{1.28}$$

We can expand a weight in terms of his fundamental weight:

$$\lambda = \sum_{i=1}^{r} \lambda_i \omega_i \Longleftrightarrow \lambda_i = (\lambda, \alpha_i^\vee) \tag{1.29}$$

The expansion coefficients λ_i are called Dynkin labels and are always integers for finite-dimensional irreducible representations; the elements of the Cartan matrix are the Dynkin labels of the simple roots

$$\alpha_j = \sum_j A_{ij}\omega_j \tag{1.30}$$

A very important weight is the one for which all Dynkin labels are unity:

$$\rho = \sum_i \omega_i = (1, 1, \dots, 1) \tag{1.31}$$

and is called the Weyl vector, it has also the following definition:

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+} \alpha \tag{1.32}$$

where Δ_+ is the set of positive roots.

The scalar product of weights can be expressed in terms of a symmetric quadratic form matrix F_{ij}:

$$(\omega_i, \omega_j) = F_{ij} \tag{1.33}$$

F_{ij} is the transformation matrix relating the two bases $\{\omega_i\}$ and $\{\alpha_i^\vee\}$

$$\omega_i = \sum_j F_{ij}\alpha_j^\vee \tag{1.34}$$

As can be seen by multiplicating of $\alpha_j^\vee$ with this equation. From eq. (1.30) we obtain:

$$\alpha_i^\vee = \sum_j \frac{2}{|\alpha_i|^2} A_{ij}\omega_j \tag{1.35}$$

This leads to an explicit relation between the quadratic form and the Casimir matrix:

$$F_{ij} = (A^{-1}_{ij})\frac{\alpha_j^2}{2} \tag{1.36}$$

In addition, we can now write the scalar product of two weights $\lambda = \sum \lambda_i \omega_i$ and $\mu = \sum \mu_i \omega_i$ as:

$$(\lambda, \mu) = \sum_{ij} \lambda_i \mu_j (\omega_i, \omega_j) = \sum_{ij} \lambda_i \mu_j F_{ij} \tag{1.37}$$

1.1.9 The Weyl Group

Given m the eigenvalue of the J^3 operator $\alpha \cdot H/|\alpha|^2$ on the state $|\beta\rangle$, we have:

$$2m = (\alpha^\vee, \beta) \tag{1.38}$$

If $m \neq 0$, this state is paired with another one with eigenvalue $-m$; hence, there must exists another state in the multiplet $|\beta + l\alpha\rangle$ such that

$$(\alpha^\vee, \beta + l\alpha) = (\alpha^\vee, \beta) + 2l = -(\alpha^\vee, \beta) \tag{1.39}$$

In this way, solving for l, we see that if β is a root, also $\beta - (\alpha^\vee, \beta)\alpha$ is a root.

The operation s_α defined by

$$s_\alpha \beta = \beta - (\alpha^\vee, \beta)\alpha \tag{1.40}$$

is the reflection respect to the hyperplane perpendicular to α. The set of all such reflection form a group, called Weyl group and is generated by the r elements $s_i = s_{\alpha_i}$, the simple Weyl reflections, in the sense that every element $w \in W$ can be decomposed as

$$w = s_i s_j \dots s_k \tag{1.41}$$

The simple Weyl reflections has the property that:

$$(s_i s_j)^{m_{ij}} = 1 \quad \text{where} \quad m_{ij} = \begin{cases} 2 & \text{if } i = j \\ \frac{\pi}{\pi - \theta_{ij}} & \text{if } i \neq j \end{cases} \tag{1.42}$$

where θ_{ij} is the angle between the simple root α_i and α_j. On the simple roots, the action of s_i takes the simple form

$$s_i \alpha_j = \alpha_j - A_{ji} \alpha_i \tag{1.43}$$

We have seen that W maps Δ into itself; moreover, we can generate the complete set Δ from the simple roots, by acting with all the elements of W on the set $\{\alpha_i\}$.

We can extend the action of the weyl group to weights:

$$s_\alpha \lambda = \lambda - (\alpha^\vee, \lambda)\alpha \tag{1.44}$$

which leaves the scalar product invariant.

In the end, we define the lenght of w, $l(w)$, as the minimum number of s_i, among all the possible decompositions of $w = \Pi_i s_i$, and the signature of w as $\epsilon(w) = (-1)^{l(w)}$

1.2 Highest-Weight Representations

Any finite-dimensional irreducible representation has a unique highest-weight state $|\lambda\rangle$. Being nondegenerate, $|\lambda\rangle$ is completely specified by its Dynkin labels.

The highest weight is the one for which the sum of the coefficient expansions in the basis of simple roots is maximal. Because of these properties, we have:

$$E^{\alpha}\,|\lambda\rangle = 0, \quad \forall \alpha > 0 \tag{1.45}$$

The highest weight of a finite-dimensional representation is necessarily dominant (with positive-integer Dynkin labels). In addition, to each dominant weight $|\lambda\rangle$ there corresponds a unique irreducible finite dimensional representation L_λ, whose highest weight is $|\lambda\rangle$.

1.2.1 Weights and Multiplicities

Starting from the highest-weight state λ, all the states in the representation space L_λ can be obtained by the action of the lowering operators of g as

$$E^{-\beta}E^{-\gamma}\dots E^{-\eta}\,|\lambda\rangle \quad \text{for} \quad \beta,\gamma,\eta \in \Delta_+ \tag{1.46}$$

We denote the set of eigenvalues of all the states in L_λ as Ω_λ. In order to find all the weights $\lambda' \in \Omega_\lambda$, we can rewrite equation (1.24) as:

$$(\lambda', \alpha_i^\vee) = \lambda'_i = -(p_i - q_i), \quad p_i, q_i \in \mathbb{Z}_+ \tag{1.47}$$

where we note that λ' is necessarily of the form $\lambda - \sum_i n_i\alpha_i$, with $n_i \in \mathbb{Z}_+$. We can construct all the weights in the representation starting from the highest weight $\lambda = (\lambda_1, \dots, \lambda_r)$; for each positive Dynkin label $\lambda_i > 0$, we construct the sequence of weights $\lambda - \alpha_i, \lambda - 2\alpha_i, \dots, \lambda - \lambda_i\alpha_i$ which all belongs to Ω_λ; we repeat the process with every weights that we obtain and until no more weights with positive Dynkin label are produced.

However, this procedure does not keep track of multiplicities; we can use the Freudenthal recursion formula, which gives the multiplicity of λ' in the representation λ, in terms of the multiplicity of all weights above it:

$$\left[|\lambda+\rho|^2 - |\lambda'+\rho|^2\right]\mathrm{mult}_\lambda(\lambda') = 2\sum_{\alpha>0}\sum_{k=1}^{\infty}(\lambda'+k\alpha, \alpha)\mathrm{mult}_\lambda(\lambda'+k\alpha) \tag{1.48}$$

In the end, counting all the generated weights with their multiplicity, we obtain the dimension of the representation, even if, better algorithm will be explained in the next sections.

1.3 Characters Theory

1.3.1 Weyl character formula

A character is a useful functional way of coding the whole content of a representation. The character of the representation of highest weight λ is formally defined as

$$\chi_\lambda = \sum_{\lambda' \in \Omega_\lambda} \mathrm{mult}_\lambda(\lambda') \exp\{\lambda'\} \tag{1.49}$$

Where the sum is over all the weights of the representation; e^λ denotes a formal exponential satisfying

$$e^\lambda e^\mu = e^{\lambda+\mu} \tag{1.50}$$

$$e^\lambda(\psi) = e^{(\lambda,\psi)} \tag{1.51}$$

Where in the last expression on the right hand side, e is a genuine exponential function because ψ is an arbitrary element of the dual Cartan subalgebra (a weight).

There is another definition for a character in the representation theory of group. Let G be the Lie group of g and H an element of the Cartan subgroup of G. The character of H in some representation is simply its trace evaluated in the corresponding module V:

$$\mathrm{Tr}_V H = \sum_\gamma \mathrm{mult}(\gamma)[\gamma(H)] \tag{1.52}$$

where $\gamma(H)$ denotes the eigenvalues of H. We know that H is associated with an element h of the Cartan subalgebra of g by $H = \exp(h)$ and spanning the full Cartan subgroup amounts to replacing the single element h by the vector $\bar{h} = (h^1, h^2, \ldots, h^r)$. Thus $\gamma(H)$ is replaced by $e^{\lambda'(h)} = e^{(\lambda'_1, \ldots, \lambda'_r)}$ where λ' is a weight.

The expression (1.49) for the character can be brought into a more manageable form in two steps; we define the auxiliary quantity

$$D_\rho = \prod_{\alpha>0} (e^{\alpha/2} - e^{-\alpha/2}) \tag{1.53}$$

which can be expressed as a sum of elements of the Weyl group

$$D_\rho = \sum_{w \in W} \epsilon(w) e^{w\rho} \tag{1.54}$$

Then, using Freudenthal formula, it is possible to show that:

$$D_\rho \chi_\lambda = D_{\lambda+\rho} \tag{1.55}$$

Hence, we obtain the Weyl character formula:

$$\boxed{\chi_\lambda = \frac{D_{\lambda+\rho}}{D_\rho} = \frac{\sum_{w\in W} \epsilon(w) e^{w(\lambda+\rho)}}{\sum_{w\in W} \epsilon(w) e^{w\rho}}} \tag{1.56}$$

For some manipulations, it is more convenient to work with the character evaluated at a special but arbitrary value ψ

$$\chi_\lambda(\xi) = \frac{\sum_{w\in W} \epsilon(w) e^{(w(\lambda+\rho),\xi)}}{\sum_{w\in W} \epsilon(w) e^{(w\rho,\xi)}} \tag{1.57}$$

1.3.2 The Dimension Formula

From eq. (1.49), the dimension of a representation correspond to evaluate the character at the point $\xi = 0$; this correspond to take the limit $\xi \to 0$ in (1.57), which leads to

$$\boxed{\dim|\lambda| = \prod_{\alpha>0} \frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}} \tag{1.58}$$

1.3.3 Tensor products and Branching Rules

In order to calculate the tensor product $L_\lambda \otimes L_\mu$, usually written as $\lambda \otimes \mu$, we can simply add together all pairs of weights λ', μ' (belonging respectively to the weight systems Ω_λ and Ω_μ), taking care of their multiplicities and reorganizing the full set of $dim\,\lambda d\, \times im\mu$ in irreducible representations, writing the result under the form $|equation \lambda \otimes \mu = \bigoplus_{\nu \in P_+} \mathcal{N}_{\lambda\mu}^{\nu} \nu$

where the sum is taken over all dominant weights and $\mathcal{N}_{\lambda\mu}^{\nu}$ is called tensor product coefficient and gives the multiplicity of the representation ν in the decomposition of the tensor product $\lambda \otimes \mu$.

However, this method in practice can be very complicated, a more efficient technique is obtained from manipulation of Weyl character formula; indeed, eq. (7.48) must hold in character form (indeed, the character of a tensor product is the product of the characters), obtaining

$$\chi_\lambda \chi_\mu = \sum_{\nu \in P_+} \mathcal{N}_{\lambda\mu}^{\nu} \chi_\nu \tag{1.59}$$

From this equation, we can obtain an expression for the tensor product coefficients

$$\mathcal{N}_{\lambda\mu}^{\nu} = \sum_{w \in W} \epsilon(w) \mathrm{mult}_\mu(w \cdot \nu - \lambda) \tag{1.60}$$

We now consider branching rules for an algebra embedding $p \subset g$. Viewed from the standpoint of the smaller algebra p, an irreducible representation of g usually breaks down into many irreducible representation of p; such decomposition are called branching rules and are noted as

$$\lambda \to \bigoplus_{\mu \in P_+} b_{\lambda\mu} \mu \tag{1.61}$$

The branching coefficient $b_{\lambda\mu}$ gives the multiplicity of the irreducible representation μ of p in the decomposition of the irreducible representation λ of g. Even in this case equation (1.61) must hold in the character form; however, the character variables are usually different from p to g; for this reason, it is necessary to find the relation between them analyzing, for instance, the fundamentals representaions.

In section 5 and 6 we evaluate specific branching rules using this method.

1.4 Evaluation of some Character Formulae

In this section, we will evaluate some character formulae of $SO(N)$ representations; these formulae are very important, because are the main tools that we will use to demonstrate the results of this thesis; in addition, the literature about $SO(N)$ characters are almost absent, at least for our purposes; for this reason, the derivation of these character formulae is a non-trivial task and an interesting result of this thesis.

1.4.1 Character of $SU(2)$

The evaluation of the $SU(2)$ character is a trivial task, we will use it just to show the procedure for the more complicated cases; we have only one simple root and the weyl group is simply:

$$\{1, s_\alpha\} \tag{1.62}$$

The Cartan matrix in this case is just the scalar 2 while the quadratic form matrix is $\frac{1}{2}$; hence, $|\alpha|^2 = (\alpha, \alpha) = 4(\omega, \omega) = 2$; the action of the simple reflection on a general weight $p = \lambda\omega$, where λ is the Dynkin label and ω the fundamental weight, is given by:

$$s(\lambda\omega) = \lambda\omega - \frac{8\omega\lambda}{|\alpha|^2}(\omega, \omega) = \lambda\omega - 2\omega\lambda = -\lambda\omega \tag{1.63}$$

In this way, the character formula is simply given by:

$$\chi_{SU(2)} = \frac{e^{\omega(\lambda+1)} - e^{-\omega(\lambda+1)}}{e^{\omega} - e^{-\omega}} = \frac{y^{\lambda+1} - y^{-\lambda-1}}{y - y^{-1}} \tag{1.64}$$

where we have defined $y = e^{\omega}$. Taking the limit $y \to 1$, we obtain the formula dimension for $SU(2)$ as $d_{SU(2)} = 1 + \lambda$.

1.4.2 Character of $SO(4)$

In this case, the rank of the group is 2; it is a bit more complicated than the previous case but the result can still be obtained by hand.

The Dynkin Dyagram of SO(4) is $\bigcirc \; \bigcirc$ and the angle between the two root is $\theta = \frac{\pi}{2}$; hence, the Weyl group simple reflection properties are: $s_1^2 = s_2^2 = (s_1 s_2)^2 = 1$; as a consequence, the Weyl group is formed by:

$$W = 1\ ,\ s_1\ ,\ s_2\ ,\ s_1 s_2 \tag{1.65}$$

The highest weight representation can be written as $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$, where λ_1, λ_2 are Dynkin labels, while ω_1, ω_2 are fundamental weights.

The Cartan matrix A_{ij} and the quadratic form matrix F_{ij} are given by

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \tag{1.66}$$

We can evaluate explicitly the action of s_1 on the first fundamental weight; using the Cartan matrix, we know that $\alpha_1 = 2\omega_1, \alpha_1^2 = 2$, in this way we find:

$$s_1(\omega_1) = \omega_1 - \frac{2}{2}(\alpha_1, \omega_1)\alpha_1 = \omega_1 - 2(\omega_1, \omega_1)2\omega_1 = \omega_1 - 2\omega_1 = -\omega_1 \tag{1.67}$$

In the same way we find

$$
\begin{aligned}
s_2\omega_1 &= \omega_1 \\
s_1\omega_2 &= \omega_2 \\
s_2\omega_2 &= -\omega_2 \\
s_1 s_2\omega_1 &= -\omega_1 \\
s_1 s_2\omega_2 &= -\omega_2
\end{aligned}
$$

Apllying Weyl character formula:

$$
\chi_{SO(4)} = \frac{e^{(\lambda_1+1)\omega_1+(\lambda_2+1)\omega_2} - e^{-(\lambda_1+1)\omega_1+(\lambda_2+1)\omega_2} - e^{(\lambda_1+1)\omega_1-(\lambda_2+1)\omega_2} + e^{-(\lambda_1+1)\omega_1-(\lambda_2+1)\omega_2}}{e^{w_1+w_2} - e^{-\omega_1+\omega_2} - e^{\omega_1-\omega_2} + e^{-\omega_1-\omega_2}} \tag{1.68}
$$

We apply the substitution $e^{\omega_i} \to y_i$ to get the final result:

$$
\chi_{SO(4)} = \frac{y_1^{\lambda_1+1} y_2^{\lambda_2+1} - y_1^{-\lambda_1-1} y_2^{\lambda_2+1} - y_1^{\lambda_1+1} y_2^{-\lambda_2-1} + y_1^{-\lambda_1-1} y_2^{-\lambda_2-1}}{y_1 y_2 - y_1^{-1} y_2 - y_1 y_2^{-1} + y_1^{-1} y_2^{-1}} \tag{1.69}
$$

Now, taking the limit in which the character variables are sent to one, we obtain:

$$
d_{SO[4]} = (1+\lambda_1)(1+\lambda_2) \tag{1.70}
$$

This formula can also be verified using directly the dimension Weyl formula. There is a simpler method to obtain the above results: noting that

$$
SO(4) = SU(2) \otimes SU(2) \tag{1.71}
$$

and using the character formula of $SU(2)$; however we give a general formulation, because it can easily be formulated for higher $SO(n)$ groups.

1.4.3 Character of $SO(5)$

For $SO(5)$ the Weyl group is bigger; for this reason, we will obatin the result with the aid of Mathematica [26].

For $SO(5)$ the Cartan matrix and the quadratic form matrix are:

$$
A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{1.72}
$$

The Dynkin diagram is $\bigcirc = \bigcirc$, with angle $\theta = \frac{3}{4}\pi$, wich implies the condition $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$, hence the Weyl group is:

$$
W = \ 1 \ , \ s_1 \ , \ s_2 \ , \ s_1 s_2 \ , \ s_2 s_1 \ , \ s_1 s_2 s_1 , \ s_2 s_1 s_2 \ , \ s_1 s_2 s_1 s_2 \tag{1.73}
$$

The action of the simple reflections on the fundamental weights is given by:

$$
\begin{aligned}
s_1(\omega_1) &= -\omega_1 + 2\omega_2 \\
s_2(\omega_1) &= \omega_1 \\
s_1(\omega_2) &= \omega_2 \\
s_2(\omega_2) &= \omega_1 - \omega_2
\end{aligned} \tag{1.74}
$$

Using Mathematica, we get a quite long expression which we write as Mathematica code in the Appendix; in the dimensional limit, we obtain:

$$
d_{SO(5)} = \frac{1}{6}\left(\lambda_1+1\right)\left(\lambda_2+1\right)\left(\lambda_1+\lambda_2+2\right)\left(2\lambda_1+\lambda_2+3\right) \tag{1.75}
$$

1.4.4 Character of $SO(6)$

We proceed as before, the weyl group has now 24 elements, given by:

$$
\begin{aligned}
W = \{\ 1\ ,\ s_1\ ,\ s_2\ ,\ s_3\ ,\ s_1s_2\ ,\ s_1s_3\ ,\ s_2s_1\ ,\ s_2s_3\ ,\ s_3s_1\ ,\ s_1s_2s_3\ ,\ s_2s_1s_3\ ,\ s_2s_3s_1\ ,\ s_3s_1s_2\ ,\dots \\
\dots,\ s_1s_2s_1\ ,\ s_3s_1s_3\ ,\ s_1s_2s_3s_1\ ,\ s_2s_1s_3s_1\ ,\ s_2s_1s_3s_2\ ,\ s_2s_3s_1s_2\ ,\ s_3s_1s_2s_3\ ,\ s_1s_2s_3s_1s_2\ ,\dots \\
\dots,\ s_1s_2s_3s_1s_3\ ,\ s_2s_1s_3s_1s_2\ ,\ s_1s_2s_3s_1s_2s_3\}
\end{aligned}
\tag{1.76}
$$

while the action of simple reflections on fundamental weights is given by:

$$
s_1 \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} -\omega_1 + \omega 2 + \omega 3 \\ \omega_2 \\ \omega_3 \end{pmatrix} \qquad s_2 \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_1 - \omega_2 \\ \omega_3 \end{pmatrix} \qquad s_3 \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_1 - \omega_3 \end{pmatrix} \tag{1.77}
$$

We obtain a very long expression for $\chi_{SO(6)}$, for this reason we only write dimension formula:

$$
d_{SO(6)} = \frac{1}{12} (\lambda_1 + 1) (\lambda_2 + 1) (\lambda_1 + \lambda_2 + 2) (\lambda_3 + 1) (\lambda_1 + \lambda_3 + 2) (\lambda_1 + \lambda_2 + \lambda_3 + 3) \tag{1.78}
$$

Note that also in this case, the evaluation could have be done in a different way, noting that:

$$
SO(6) \simeq SU(4) \tag{1.79}
$$

and using Schur's funtions of $SU(4)$.

1.4.5 Character of $SO(8)$ and $SO(9)$

We have evaluated the character of $SO(8)$ and $SO(9)$ following the same procedure as before; in this case, the Weyl group dimension is equal to 192 and 384 respectevely; for this reason we have obtained the Weyl elements using the software Sagemath [1].

The obtained results are quite long and complicated, for this reason we prefer not to show them; however, these formulae was heavily used in this thesis job, especially for the computation in chapter 6, where we also explicit write some specific representations characters.

1.5 Racah Speiser Algorithm

The Racah Speiser algorithm [3],[8] provides a succint and simple way of decomposing tensor products. A representation $R_{\bar{\Lambda}}$ with Dynkin labels $\bar{\Lambda} = [\lambda_1, \ldots, \lambda_r]$ is characterised by a highest weight vector Λ

$$\Lambda = \sum_{i=1}^{r} \lambda_i w_i \qquad \lambda_i \in \mathbb{Z}, \lambda_i \geq 0 \tag{1.80}$$

where w_i are the fundamental weights. For a tensor product, $R_{\bar{\Lambda}} \otimes R_{\bar{\Lambda}'}$, we consider the set of weights $V_{\bar{\Lambda}'} = \{\lambda\}$ for all states in the representation space for $R_{\bar{\Lambda}'}$ (i.e. $\lambda = \sum_i \lambda_i w_i$ with λ_i positive or negative integers and allowing for multiplicity, there are $d_{\Lambda'}$ weights in $V_{\bar{\Lambda}'}$ for $d_{\Lambda'}$ the dimension of $R_{\bar{\Lambda}'}$).

Then the racah Speiser algorithm can be paraphrased by expressing the tensor product as

$$R_{\bar{\Lambda}} \otimes R_{\bar{\Lambda}'} = \sum_{\bar{\lambda} \in V_{\bar{\Lambda}'}} R_{\bar{\Lambda}+\bar{\lambda}} \tag{1.81}$$

where we require

$$R_{\bar{\lambda}} = \mathrm{sign}(\sigma) R_{\bar{\lambda}^\sigma} \quad \lambda^\sigma = \sigma(\lambda + \rho) - \rho \tag{1.82}$$

for σ an element of the Weyl group, while ρ is the Weyl vector.

We will use this algorithm in order to determine the superdescendants of the Konishi multiplet and $\frac{1}{2}$ BPS multiplet.

Chapter 2

Beyond Poincarè Symmetry

2.1 Symmetries of Minkowsky Space

2.1.1 The Lorentz Group

The Minkowsky space is a space $\mathbb{R}^{1,3}$ with metric

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1) \tag{2.1}$$

which is symmetrical with respect to Lorentz transformations $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, where $\Lambda^T \eta \Lambda = \eta$. The transformations that are continuously connected to the identity have $\det \Lambda = 1$ and $\Lambda^0{}_0 > 0$ and form the Lorentz group $SO^+(1,3)$ (which we will call $SO(1,3)$). We define the spinor group as the double cover of Lorentz group:

$$SO(1,3) \simeq \mathrm{Spin}(1,3)/\mathbb{Z}_2 \tag{2.2}$$

A Lorentz transformation acting on a 4-vector can be written as

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) \tag{2.3}$$

where $\omega_{\mu\nu}$ are six numbers that specify what Lorentz transformation we are doing, while $M^{\mu\nu}$ are a choice of six 4×4 anti-symmetric matrices that generates different Lorentz transformations (above matrix indices are suppresed).

The matrices $M^{\mu\nu}$ generates the algebra $so(1,3)$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma}) \tag{2.4}$$

The six different Lorentz transformations decompose into three rotations J_i and three boost K_i, defined by:

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk} \qquad K_i = M_{0i} \tag{2.5}$$

where the $j, k = 1, 2, 3$ indices are summed over and ϵ_{ijk} is the antisymmetric tensor.

The rotations matrices are hermitian $J_i^\dagger = J_i$ while the boost matrices are anti-Hermitian $K_i^\dagger = -K_i$, they obey:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad [J_i, K_j] = i\epsilon_{ijk} K_k \qquad [K_i, K_j] = -i\epsilon_{ijk} J_k \tag{2.6}$$

hence, the rotations form an $su(2)$ sub-algebra. However, we can find two mutually commuting $su(2)$ algebras inside $so(1,3)$, takin the linear combinations:

$$A_i = \frac{1}{2}(J_i + iK_i) \quad B_i = \frac{1}{2}(J_i - iK_i) \tag{2.7}$$

such that $A_i^\dagger = A_i$, $B_i^\dagger = B_i$ and

$$[A_i, A_j] = i\epsilon_{ijk}A_k \quad [B_i, B_j] = i\epsilon_{ijk}B_k \quad [A_i, B_j] = 0 \tag{2.8}$$

Hence, the representation can be defined by (j_i, j_2) with $j_1, j_2 \in \frac{1}{2}\mathbb{Z}$ and has dimension $(2j_i + 1)(2j_2 + 2)$

2.1.2 The Poincaré group

The continuous symmetries of Minkowsky space comprise of Lorentz transformations together with spacetime translations, forming the Poincaré group; spacetime translations are generated by the momentum 4-vector P^μ; we have

$$[P^\mu, P^\nu] = 0 \qquad [M^{\mu\nu}, P^\sigma] = i(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\nu\sigma}) \tag{2.9}$$

together with Lorentz algebra, those form the Poincaré algebra.

We can have others continuous symmetries, for instance a global $U(1)$ symmetry or a non abelian $SU(N)$ symmetry, if we denote with T any possible generator of those symmetries, we must have the T is always a Lorentz scalar, i.e. , we have $[P^\mu, T] = [M^{\mu\nu}, T] = 0$; because of the Coleman-Mandula theorem, the symmetry group of any interacting quantum field theory must factorise as

$$\text{Poincaré} \times \text{Internal} \tag{2.10}$$

In this way, it is not possible to enlarge the Poincarè group; however, if we relax the assumptions of this theorem, it is possible to enlarge the Poincarè group; for instance, we can consider only massless particles, for which we can have a conformal symmetry; another possibility is to consider supersymmetry.

2.2 Conformal Symmetry

The references for this section include [10] and [2].

2.2.1 Conformal algebra

We now consider Euclidean spacetime $\mathbb{R}^d$; Under change of coordinates $x \to x'$, we have

$$g_{\mu\nu} \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \tag{2.11}$$

The conformal group is the subgroup of coordinate transformations that leaves the metric invariant up to an arbitrary positive spacetime dependent scale factor:

$$g_{\mu\nu}(x) \to g'_{\mu\nu} = \Omega(x) g_{\mu\nu}(x) \tag{2.12}$$

These are the coordinate transformations which preserve angles. We note that the Poincarè group is a subgroup of the confromal group since it leaves the metric invariant.

The flat space infinitesimal generators of the conformal group can be determined by considering the infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu$, under which $ds^2 = \eta_{\mu\nu}(x) dx^\mu dx^\nu$ transforms as

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu \tag{2.13}$$

In order to be a conformal transformation, we require that $\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$ be proportional to $\eta_{\mu\nu}$, hence

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu} \tag{2.14}$$

This equation can be solved for $d > 2$, if $\epsilon(x)$ is at most quadratic in x; we can write:

$$\epsilon^\mu(x) = a^\mu + \omega^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2(b \cdot x)x^\mu \tag{2.15}$$

and we have:

- $\epsilon^\mu = a^\mu$ are translations.
- $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$ are Lorentz transformations.
- $\epsilon^\mu = \lambda x^\mu$ are scale transformations.
- $\epsilon^\mu = b^\mu x^2 - 2x^\mu b \cdot x$ are special conformal transformations.

We can define the generators of these transformations as:

$$P_\mu = \partial_\mu \tag{2.16}$$

$$J_{\mu\nu} = \frac{1}{2}(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{2.17}$$

$$D = x^\mu \partial_\mu \tag{2.18}$$

$$K_\mu = x^2 \partial_\mu - 2x_\mu x^\nu \partial_\nu \tag{2.19}$$

Having a total of $D + \frac{1}{2}D(D-1) + 1 + D = \frac{1}{2}(D+2)(D+1)$ generators of the conformal group; it is the same as for the group of rotations in $D+2$ dimensions.

The conformal algebra consists of the Poincaré algebra with the new commutation relations:

$$\begin{gathered} [J_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \\ [D, P_\mu] = iP_\mu \qquad [D, K_\mu] = -iK_\mu \qquad [D, J_{\mu\nu}] = 0 \\ [K_\mu, K_\rho] = 0 \qquad [K_\mu, P_\nu] = -2i(\eta_{\mu\nu}D - J_{\mu\nu}) \end{gathered} \tag{2.20}$$

The generators of conformal algebra can be grouped in such a way that the conformal algebra is the algebra $so(d, 2)$.

2.2.2 Conformal Field Theory

We define a theory with conformal invariance to satisfy the properties:

- There is a set of fields $\{A_i\}$, where i specifies the different fields, in general infinite, in which we consider also the derivative of the field.
- There is a subset of fields $\{\phi_j\} \subset \{A_i\}$, called "quasi-primary", that under global conformal transformations, $x \to x'$, transforms according to

$$\phi_j(x) \to \left| \frac{\partial x'}{\partial x} \right|^{\frac{\Delta_j}{d}} \phi_j(x') \tag{2.21}$$

  where Δ_j is the conformal dimension of ϕ_j (i.e. the eigenvalue of D), hence the correlation function transforms as

$$\langle \phi_1(x_1) \dots \phi_n(x_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\frac{\Delta_1}{d}} \dots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\frac{\Delta_n}{d}} \langle \phi_1(x'_1) \dots \phi_n(x'_n) \rangle \tag{2.22}$$

- The rest of the $\{A_i\}$'s can be expressed as linear combinations of the quasi-primary fields and their derivatives.
- There is a vacuum invariant under the global conformal group.

These fields transforms in irreducible representations of the conformal algebra; in order to construct the transformation representations for general dimensions, we use the method of induced representations: we analyse the transformation properties of the fields ϕ at $x = 0$ and with the momentum P^μ we may shift the argument of the field to an arbitrary point x, in order to obtain general transformation rule. We have:

$$[J_{\mu\nu}, \phi(0)] = -\mathcal{J}_{\mu\nu}\phi(0) \tag{2.23}$$

where $\mathcal{J}_{\mu\nu}$ is a finite dimensional representation of the Lorentz group determining the spin for the field $\phi(0)$. We also have

$$[D, \phi(0)] = -i\Delta\phi(0) \tag{2.24}$$

which implies that ϕ has scaling dimension Δ, i.e., under dilatations $x \to \lambda x$ it transforms as:

$$\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x) \tag{2.25}$$

Moreover, we can consider conformal primary fields, which satisfy:

$$[K_\mu, \phi(0)] = 0 \tag{2.26}$$

By applying the commutation relations of D with P_μ and K_μ , we see that P_μ increases the scaling dimension while K_μ decreases it. In an unitary CFT, there is a lower bound on the scaling dimension of the fields, which implies that any conformal representation must contain operators of lowest dimension, which are annihilated by K_μ at $x^\nu = 0$. All other fields, the conformal descendants of ϕ, are obtained by acting with P_μ on the conformal primary fields.

2.2.3 Correlation Functions

The n-point correlation function has some restrictions: translational invariance implies that the correlation function can only depend on the difference of the coordinates; if we consider spinless objects, rotational invariance implies that the correlation function can only depends on the distances $R_{ij} = |x_i - x_j|$; next, imposing scale invariance, we can only have dependence on the ratios r_{ij}/r_{kl}.

Finally, imposing invariance under special conformal transformations , we can have dependence only on $\frac{r_{ij}r_{kl}}{r_{ik}r_{jl}}$.

The 2-point function of two quasi primary fields must be:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{r_{12}^{2\Delta}} & \Delta_1 = \Delta_2 = \Delta \\ 0 & \Delta_1 \neq \Delta_2 \end{cases} \tag{2.27}$$

For the 3-point function we have

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{\Delta_2+\Delta_3-\Delta_1} r_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.28}$$

In four point function, we have less constraint because with four coordinates it is possible to construct two dimensionless invariants, the cross ratios

$$\eta = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 r_{24}^2} \qquad \xi = \frac{r_{14}^2 r_{23}^2}{r_{13}^2 r_{24}^2} \tag{2.29}$$

A four point function, of scalar conformal primary operators $\mathcal{O}_i$ with conformal dimension Δ_i, takes the general form:

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \frac{1}{x_{12}^{\Delta_1+\Delta_2} x_{34}^{\Delta_3+\Delta_4}} F(\eta, \xi) \tag{2.30}$$

2.3 The Supersymmetry algebra ($\mathcal{N} = 1$)

Supersimmetry can evad the Coleman-Mandula theorem, because this symmetry is not characterised by a Lie algebra but by a $\mathbb{Z}_2$ graded Lie algebra, where it is possible to have commutation and anti-commutation relations.

Supersymmetry enlarges the Poincaré algebra by including spinor supercharges:

$$a = 1, \dots, \mathcal{N} \qquad \begin{cases} Q^a_\alpha \quad \alpha = 1,2 & \text{left Weyl spinor} \\ \bar{Q}_{\dot{\alpha}a} = (Q^a_\alpha)^\dagger & \text{right Weyl spinor} \end{cases} \tag{2.31}$$

Where $\mathcal{N}$ is the number of independent supersymmetries of the algebra.

The $\mathcal{N} = 1$ supersymmetry algebra comprises the commutation relations of the Poincaré group toghether with the relations:

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \tag{2.32}$$

$$[M^{\mu\nu}, Q_\alpha] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \qquad [M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}} \tag{2.33}$$

$$[Q_\alpha, P^\mu] = \{Q_\alpha, Q_\beta\} = 0 \tag{2.34}$$

The first relation is telling us that the supercharges should be viewed as the square root of spacetime translations, while, in the second row, we see that the supercharges transform as Weyl fermions under Lorentz transformations, where $\sigma^{\mu\nu} = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$.

2.3.1 R-symmetry

All internal symmetries must commute with the spacetime symmetries of the Poincaré group, but this is not true for the supercharges; it may be theories that admit an internal $U(1)$ symmetry that act as

$$Q_\alpha \to e^{-i\lambda} Q_\alpha \quad \bar{Q}_{\dot{\alpha}} \to e^{i\lambda} \bar{Q}_{\dot{\alpha}} \tag{2.35}$$

this symmetry is known as R-symmetry $U(1)_R$; denoting the generator as R

$$[R, Q_\alpha] = -Q_\alpha \qquad [R, \bar{Q}_\alpha] = \bar{Q}_{\dot{\alpha}} \tag{2.36}$$

2.3.2 Energy and Number of States Condition

From the algebra alone we can derive

$$\langle\phi| P_0 |\phi\rangle \geq 0 \tag{2.37}$$

where $\langle\phi|$ is any physical state; hence, the energy of any state in a supersymmetric theory is necessarily positive.

The supercharge Q_α is a fermionic operator, this means:

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle \qquad Q\,|\text{fermion}\rangle = |\text{boson}\rangle \tag{2.38}$$

We introduce the fermionic number operator $(-1)^F$, such that

$$(-1)^F |B\rangle = |B\rangle \qquad (-1)^F |F\rangle = -|F\rangle \tag{2.39}$$

because Q_α swaps a bosonic and a fermionic state, we must have

$$\{(-1)^F, Q_\alpha\} = 0 \tag{2.40}$$

Suppose to have a finite collection of one-particle states that form a representation of the supersymmetry algebra, we can take the trace over elements of this multiplet and, from the the last equation, we find:

$$\mathrm{tr}[(-1)^F\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 0 \tag{2.41}$$

from which we derive

$$\sigma^\mu_{\alpha\dot{\alpha}}\mathrm{tr}[(-1)^F P_\mu] = 0 \tag{2.42}$$

We can choose the states to be momentum eigenstates, hence

$$\sigma^\mu_{\alpha\dot{\alpha}}p_\mu\mathrm{tr}[(-1)^F] = 0 \tag{2.43}$$

but $\mathrm{tr}[(-1)^F$ counts the number of bosonic states n_B minus the number of fermionic states n_F; hence, the number of bosonic states must be equal to the number of fermionic states.

However, it could happen that Q_α and $\bar{Q}_{\dot{\alpha}}$ annihilate states in the supersymmetry multiplet; this can only happen for states of zero energy, which must be ground states of the system.

2.3.3 Massless representations

Consider a state $|p_\mu, h\rangle$ of a massless particle, with helicity h (the eigenvalue of the $U(1)$ rotation in the (x_1, x_2) plane). We can boost to a frame in which $p_\mu = (E, 0, 0, E)$; in this way, supersymmetry algebra becomes

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu = 2E(1+\sigma^3)_{\alpha\dot{\alpha}} = 4E\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \tag{2.44}$$

From the positivity condition, Q_2 and $\bar{Q}_2$ necessarily annihilate this state

$$Q_2\,|p_\mu, h\rangle = \bar{Q}_2\,|p_\mu, h\rangle = 0 \tag{2.45}$$

To build a representation, we only need to consider Q_1 and $\bar{Q}_1$, wich act as fermionic creation and annihilation operators; indeed, defining

$$a = \frac{Q_1}{\sqrt{4E}},\; a^\dagger = \frac{\bar{Q}_1}{\sqrt{4E}} \quad \rightarrow \{a, a^\dagger\} = 1 \;\; \{a, a\} = \{a^\dagger, a^\dagger\} = 0 \tag{2.46}$$

we have have two states $|0\rangle$ and $|1\rangle$ such that $a\,|0\rangle = 0$ and $|1\rangle = a\dagger\,|0\rangle$; considering a state $|p_\mu, h\rangle$, such that $a\,|p_\mu, h\rangle = 0$, the full supersymmetry multiplet consists of $|p_\mu, h\rangle$ and $a^\dagger\,|p_\mu, h\rangle$. From the commutation relations we find that $a\dagger$ lower the helicity by $\frac{1}{2}$; then, we see that the massless representations of the supersymmetry algebra consists of two states

$$|p_\mu, h\rangle \qquad \left|p_\mu, h - \frac{1}{2}\right\rangle \tag{2.47}$$

However, in order to have CPT invariance, we must add the states with opposite helicity.

2.3.4 Massive Representations

For massive representation, in the rest frame, we have $p_\mu = (m, 0, 0, 0)$ and acting on such states, the supersymmetry algebra becomes

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu = 2m\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.48}$$

In this case, the fermionic creation/annihilation operators are

$$a_\alpha = \frac{Q_\alpha}{\sqrt{2m}} \quad a_{\dot\alpha}^\dagger = \frac{\bar{Q}_{\dot\alpha}}{\sqrt{2m}} \quad \rightarrow \quad \{a_\alpha, a_{\dot\alpha}^\dagger\} = \delta_{\alpha\dot\alpha} \tag{2.49}$$

Hence, starting with the state $|\Omega\rangle = |p_\mu; j, j_3\rangle$ such that $a_\alpha |\Omega\rangle = 0$, the full supermultiplet consist of four states

$$\begin{gathered} |\Omega\rangle \\ a_1^\dagger |\Omega\rangle \quad a_2^\dagger |\Omega\rangle \\ a_1^\dagger a_2^\dagger |\Omega\rangle \end{gathered}$$

In the end, a massive supermultiplet contains two particles of spin j, a particle of spin $j - \frac{1}{2}$ and a particle of spin $j + \frac{1}{2}$.

2.4 Extended Supersymmetry

We consider a collection of supercharges

$$Q_\alpha^I \text{ and } \bar{Q}_{\dot\alpha}^I \quad I = 1, \dots, \mathcal{N} \tag{2.50}$$

The commutation/anticommutation relations are the same for each supercharge, but we also have relations between the supercharges:

$$\{Q_\alpha^I, Q_\beta^J\} = \epsilon_{\alpha\beta} Z^{IJ} \qquad \{\bar{Q}_{\dot\alpha}^I, \bar{Q}_{\dot\beta}^J\} = \epsilon_{\dot\alpha\dot\beta} (Z^\dagger)^{IJ} \tag{2.51}$$

Here $Z^{IJ} = -Z^{JI}$ is a central charge, i.e. it commutes with all other elements of the algebra.

In addition, in extended supersymmetry, there is a larger R-symmetry group, which rotates the supercharges among themselves:

- $\mathcal{N} = 2$: The R-symmetry group is $U(2) \simeq U(1) \times SU(2)$
- $\mathcal{N} = 4$: The R-symmetry group is $SU(4) \simeq Spin(6)$

2.4.1 Massless Representations

Considering the states $|p^\mu, h\rangle$ of massless particles, in the frame with $p_\mu = (E, 0, 0, E)$, we have:

$$\{Q_\alpha^I, \bar{Q}_{\dot\alpha J}\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta^{IJ} \tag{2.52}$$

Hence, as before, $Q_2^I |p^\mu, h\rangle = \bar{Q}_2^I |p^\mu, h\rangle = 0$ and we obtain $Z^{IJ} |p^\mu, h\rangle = 0$; the central charge plays no role for massless states. Defining

$$a^I = \frac{Q_1^I}{\sqrt{4E}} \quad a^{I\dagger} = \frac{\bar{Q}_1^I}{\sqrt{4E}} \rightarrow \{a^I, a^{J\dagger}\} = \delta^{IJ} \tag{2.53}$$

Starting from some fiducial state $|\Omega\rangle = |p^\mu, h\rangle$ s.t. $a^I |\Omega\rangle = 0$, we build the full representation acting with successive creation operators:

$$|\Omega\rangle \tag{2.54}$$
$$a^{I\dagger} |\Omega\rangle \tag{2.55}$$
$$a^{I\dagger} a^{J\dagger} |\Omega\rangle \tag{2.56}$$
$$\dots \tag{2.57}$$
$$a^{1\dagger} \dots a^{\mathcal{N}\dagger} |\Omega\rangle \tag{2.58}$$

Our initial state $|\Omega\rangle$ has helicity h, if we act with p of the $a^\dagger$ excitation operators, we have $\begin{pmatrix} \mathcal{N} \\ p \end{pmatrix}$ different states, each of which has helicity $h - p/2$. The full multiplet has $2^{\mathcal{N}}$ different states; adding the CPT conjugate, we have $2^{\mathcal{N}+1}$ states.

2.4.2 Massive Representations

In the rest frame the superalgebra reads

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta^{IJ} \tag{2.59}$$

Working for simplicity with $\mathcal{N} = 2$, we can consider $\mathcal{Z}^{IJ} = 2\epsilon^{IJ} Z$. We can define a_α and b_α as a linear combination of $Q, \bar{Q}$ in such a way to disentangle the mass and central charge, obtaining

$$\{a_\alpha, a^\dagger_\beta\} = 2(m + Z)\delta_{\alpha\beta} \qquad \{b_\alpha, b^\dagger_\beta\} = 2(m - Z)\delta_{\alpha\beta} \tag{2.60}$$

the two anti-commutators are positive definite, hence, we must have;

$$m \geq |Z| \tag{2.61}$$

this is known as BPS bound; it is telling us that the masses of particles are bounded by some conserved charges. For $m > |Z|$, both $a^\dagger_\alpha$ and $b^\dagger_\alpha$ act as creation operators and we have $2^{2\mathcal{N}}$ (16) states; this is known as long multiplet.

For $m = |Z|$, half of the creation operators do nothing, and we have 8 states. This is the short multiplet.

For general $\mathcal{N}$ we can use the $SU(\mathcal{N})_R$ symmetry to diagonalize in block of 2×2 the anti-symmetric matrix Z^{ab}, where we find:

$$Z = \mathrm{diag}(\epsilon Z_1, \ldots, \epsilon Z_r, *) \tag{2.62}$$

where * equals 0 for $\mathcal{N}$ odd and is absent for $\mathcal{N}$ even. The BPS bound becomes:

$$M \geq |Z_{\bar{a}}| \quad \bar{a} = 1, \ldots, r = [\mathcal{N}/2] \tag{2.63}$$

Whenever one values $|Z_{\bar{a}}|$ equals $|M|$, the corresponding supercharges must vanish and we have multiplete shortening; in particular, if we have $M = |Z_{\bar{a}}|$ for $\bar{a} = 1, \ldots, r_0$, the corresponding representation is said to be $1/2^{r_0}$ BPS and has dimension $2^{2\mathcal{N}-2r_0}$.

2.4.3 Superspace

We want to write field theories that are manifestly invariant under supersymmetry transformations; for this reason, we will combine bosonic field and fermionic field into a single object called superfield. Superfields live on an extension of Minkowsky space known as superspace.

The coordinates of superspace are $x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}$, where x^μ are the coordinates of Minkowsy space and $\theta_\alpha, \bar{\theta}^{\dot{\alpha}}$ are Grassman valued spinors.

We can describe the Minkowsy space as a coset space:

$$\mathbb{R}^{1,3} = G/H = \frac{\text{Poincaré Group}}{\text{Lorentz Group}} \tag{2.64}$$

A general element of the Poincaré group G is comprised of Lorentz boosts, generated by $M^{\mu\nu}$, and translations, generated by P^μ:

$$g(\omega, a) = \exp\left\{\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} + i a_\mu P^\mu\right)\right\} \tag{2.65}$$

The Lorentz group H, instead, consists only of Lorentz boosts; in this way, coset space can be parameterised just by the a^μ, which can be thought as the coordinates $a^\mu = x^\mu$.

Next, we want to construct a space on which the group of supersymmetry acts:

$$g(\omega, a, \theta, \bar{\theta}) = \exp\left\{\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} + ia_\mu P^\mu + i\theta^\alpha Q_\alpha + i\bar{\theta}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)\right\} \tag{2.66}$$

With Q_α and $\bar{Q}^{\dot{\alpha}}$ the supersymmetry generators; We have described an element of super-Poincaré group g, hence we define:

$$\text{Superspace} = \frac{\text{Super-Poicaré Group}}{\text{Lorentz Group}} \tag{2.67}$$

The momentum operator generates translations $x^\mu \to x^\mu + a^\mu$, while the supercharges shift the Grassmann coordinates and the point in Minkowsky space by a Grassmann bilinear:

$$\begin{aligned} x^\mu &\to x^\mu + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta} \\ \theta &\to \theta + \epsilon \\ \bar{\theta} &\to \bar{\theta} + \bar{\epsilon} \end{aligned}$$

2.4.4 Superfields

A superfield is a function of superspace $Y = Y(x, \theta, \bar{\theta})$.

Because of Grassmann property, we have $\theta_\alpha\theta_\beta = -\theta_\beta\theta_\alpha$ and $\theta_\alpha\theta_\beta\theta_\gamma = 0$; hence, a Taylor expansion of $Y(x, \theta, \bar{\theta})$ stops after terms quadratic in θ and $\bar{\theta}$. We find:

$$Y(x, \theta, \bar{\theta}) = \phi(x) + \theta^\alpha\psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \bar{\theta}^2 N(x) \tag{2.68}$$

$$+\theta^\alpha\bar{\theta}^{\dot{\alpha}}V_{\alpha\dot{\alpha}}(x) + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}(x) + \bar{\theta}^2\theta^\alpha\rho_\alpha(x) + \theta^2\bar{\theta}^2 D(x) \tag{2.69}$$

where $\theta^2 = \theta^\alpha\theta_\alpha$ and $\bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$; we have four complex scalars ϕ, M, N, D, two left-handed spinors ψ, ρ, two right handed spinors $\bar{\chi}, \bar{\lambda}$ and a vector $V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}V_\mu$.

Defining the generator of the superspace coordinates transformations as

$$V(\epsilon, \bar{\epsilon}) = \exp\left\{i\epsilon^\alpha Q_\alpha + i\bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right\} \tag{2.70}$$

we know that

$$VY(x, \theta, \bar{\theta})V^\dagger = Y(x + i\theta\sigma^\mu\bar{\epsilon} - i\epsilon\sigma^\mu\bar{\theta}, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) \tag{2.71}$$

We find, treating ϵ_α as an infinitesimal spinor:

$$[Q_\alpha, Y] = \left(-i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right)Y \tag{2.72}$$

$$[\bar{Q}_{\dot{\alpha}}, Y] = \left(+i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right)Y \tag{2.73}$$

Where the derivatives with respect to Grassmann coordinates are defined by:

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha} \quad \text{with } \partial_\alpha\theta^\beta = \delta^\beta_\alpha \ \text{ and } \partial_\alpha\bar{\theta}_{\dot{\beta}} = 0 \tag{2.74}$$

$$\bar{\partial}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \quad \text{with } \ \partial_{\dot{\alpha}}\theta^{\dot{\beta}} = \delta^{\dot{\beta}}_{\dot{\alpha}} \quad \text{and } \partial_{\dot{\alpha}}\theta_\beta = 0 \tag{2.75}$$

and are themselves Grassmann.

It is useful to define differential operators associated to generators:

$$\mathcal{P}_\mu = -i\partial_\mu \tag{2.76}$$

$$\mathcal{Q}_\alpha = -i\partial_\alpha - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{2.77}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = i\bar{\partial}_{\dot{\alpha}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{2.78}$$

We have $\{\mathcal{Q}_\alpha \bar{\mathcal{Q}}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{P}_\mu$ and $\{\mathcal{Q}_\alpha \mathcal{Q}_\alpha\} = \{\bar{\mathcal{Q}}_{\dot{\alpha}} \bar{\mathcal{Q}}_{\dot{\alpha}}\} = 0$; hence, $\mathcal{P}, \mathcal{Q}, \bar{\mathcal{Q}}$ furnish a representation of the supersymmetry algebra, now acting on field of superspace.

We define the covariant derivatives

$$\mathcal{D}_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \tag{2.79}$$
$$\bar{\mathcal{D}}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \tag{2.80}$$

We have that

$$\{\mathcal{D}_\alpha, \mathcal{Q}_\beta\} = \{\mathcal{D}_\alpha, \bar{\mathcal{Q}}_{\dot{\beta}}\} = \{\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{Q}_\beta\} = \{\mathcal{D}_{\dot{\alpha}}, \bar{\mathcal{Q}}_{\dot{\beta}}\} = 0 \tag{2.81}$$

and

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}\mathcal{P}_\mu \tag{2.82}$$

From the previous equations we find

$$[\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{D}}, \mathcal{D}_\alpha] = [\epsilon\mathcal{Q} + \bar{\epsilon}\bar{\mathcal{D}}, \bar{\mathcal{D}}_{\dot{\alpha}}] = 0 \tag{2.83}$$

Hence we have that $\mathcal{D}_\alpha Y$ and $\bar{\mathcal{D}}_{\dot{\alpha}} Y$ are superfields.

We can now discuss the various costraints that we can place on a superfield Y:

- A chiral superfield is defined by $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi$.
- An anti chiral superfield is defined by $\mathcal{D}_\alpha\Psi$.
- A real superfield V is defined by $V = V^\dagger$
- A linear superfield is defined by $J = J^\dagger$ and $\mathcal{D}^2 J = \bar{\mathcal{D}}^2 J = 0$

2.4.5 Integration

We know that, for a single Grassmann variable θ:

$$\int d\theta \; 1 = 0 \quad \text{and} \quad \int d\theta \; \theta = 1 \tag{2.84}$$

We define

$$\int d^2\theta = \frac{1}{2}\int d\theta^1 d\theta^2 \qquad \int d^2\bar{\theta} = -\frac{1}{2}\int d\bar{\theta}^1 d\bar{\theta}^2 = 1 \tag{2.85}$$
$$\int d^4\theta = \int d^2\theta d^2\bar{\theta} \tag{2.86}$$

We can construct an action as

$$S = \int d^4x d^4\theta K(x, \theta, \bar{\theta}) \tag{2.87}$$

Where K is a composite superfield, for which we require to be real in order to have a real action. Under a supersymmetry transformation, the variations of K depends on Grassmann derivatives, which vanish when integrated over superspace; hence, $\delta S = 0$. Moreover, we can expand K as

$$K(x, \theta, \bar{\theta}) = K_{\text{first}} + \ldots + \theta^2\bar{\theta}^2 K_{\text{last}}(x) \tag{2.88}$$

from Grassmann properties we have

$$S = \int d^4x K_{\text{last}}(x) \tag{2.89}$$

We refer to these as D-terms.

2.4.6 Action for chiral superfield

Because of the properties of a chiral superfield, we can expand it as:

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) \tag{2.90}$$

$$-\frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\phi(x) \tag{2.91}$$

In order to have a real superfield, we must take the combination $\Phi^\dagger\Phi$, integrating over superspace we obtain

$$S_{\text{chiral}} = \int d^4x d^4\theta \Phi^\dagger\Phi \tag{2.92}$$

Hence, we can express the action as

$$S_{\text{chiral}} = \int d^4x[\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F] \tag{2.93}$$

Where we see the standard kinetic terms for a complex scalar ϕ and a Weyl fermion ψ. F instead is an auxiliary field, without any kinetic terms.

We now want to include masses and Yukawa type interactions, which don't arise from D-terms; hence, given a chiral superfield, we can consider the function $W(\phi)$, which is also a chiral superfield, in components we have

$$W(\Phi) = W(\phi) + \sqrt{2}\frac{\partial W}{\partial\phi}\theta\psi + \theta^2\left(\frac{\partial W}{\partial\phi}F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi\right) + \ldots \tag{2.94}$$

where the ... terms include $\bar{\theta}$; however, each of these is a total derivative and will not contribute to the action. In the action, we can think of $W(\phi)$ as a function only of θ and not of $\bar{\theta}$

$$S_W = \int d^4x\left[\int d^2\theta W(\Phi) + \int d^2\bar{\theta} W^\dagger(\Phi^\dagger)\right] \tag{2.95}$$

this action picks out the θ^2 term in $W(\phi)$ and is known as $F - term$.

Hence, summing the two action we obtain:

$$S = S_{\text{chiral}} + S_W = \int d^4x\left[\partial_\mu\phi^\dagger\partial^\mu\phi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F + \left(F\frac{\partial W}{\partial\phi} - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi + \text{h.c.}\right)\right] \tag{2.96}$$

This is the Wess-Zumino action with $W(\phi)$ the superpotential. We can eliminate the auxiliary field F trough the equation of motion, $F + \frac{\partial W^\dagger}{\partial\phi^\dagger} = 0,\ F^\dagger + \frac{\partial W}{\partial\phi} = 0$

2.5 Supersymmetric Gauge theories

2.5.1 Abelian Gauge theories

Expanding a real superfield

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + i\theta^2 M(x) - i\bar{\theta}^2 M^\dagger(x) + \theta\sigma^\mu\bar{\theta}A_\mu(x) \\ &+ \theta^2\bar{\theta}\left(\bar{\lambda}(x) + \frac{1}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) + \bar{\theta}^2\theta\left(\lambda(x) + \frac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\ &+ \frac{1}{2}\theta^2\bar{\theta}^2\left(D(x) - \frac{1}{2}\Box C(x)\right) \end{aligned} \tag{2.97}$$

The real superfield contain two real scalars, C and D, a complex scalar M, two Weyl fermions χ_α and λ_α and a real vector field A_μ, who plays the role of gauge field.

We consider a chiral superfield

$$\Omega = \omega + \sqrt{2}\theta\rho + \theta^2 G + i\theta\sigma^\mu\bar{\theta}\partial_\mu\omega - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\rho\sigma^\mu\bar{\theta} - \frac{1}{4}\theta^2\bar{\theta}^2\Box\omega \tag{2.98}$$

Hence, $i(\Omega - \Omega^\dagger)$ is a real superfield; taking the Gauge transformations

$$V \to V + i(\Omega - \Omega^\dagger) \tag{2.99}$$

In this way the vector component of the real superfield shifts as

$$A_\mu \to A_\mu - 2\partial_\mu(\text{Re }\omega) := A_\mu + \partial_\mu\alpha \tag{2.100}$$

and this is the form of a gauge transformation; in addition, the other fields shifts as:

$$\begin{aligned} C &\to C - 2\text{Im }\omega \\ \chi &\to \chi + \sqrt{2}i\rho \\ M &\to M + G \\ \lambda &\to \lambda \\ D &\to D \end{aligned} \tag{2.101}$$

Using gauge transformation we can set $C = \chi = M = 0$ in the Wess-Zumino gauge. In this gauge the superfield is

$$V_{WZ} = \theta\sigma^\mu\bar{\theta}A_\mu + \theta^2\bar{\theta}\bar{\lambda} + \bar{\theta}^2\theta\lambda + \frac{1}{2}\theta^2\bar{\theta}^2 D \tag{2.102}$$

If we act with a supersymmetry transformation on V_{WZ}, we go out of Wess-Zumino gauge, we need a compensating transformation to go back in Wess-Zumino gauge; hence we should add $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

We will build the action by

$$W_\alpha = -\frac{1}{4}\bar{\mathcal{D}}^2\mathcal{D}_\alpha V \tag{2.103}$$

It is a chiral superfield, obeyng $\bar{\mathcal{D}}_{\dot{\alpha}}W_\alpha = 0$ (because $\bar{\mathcal{D}}^3 = 0$) and it is invariant under superfield gauge symmetry.

We can expand W_α, considering only θ terms:

$$W_\alpha(x,\theta) = \lambda_\alpha(x) + \theta_\alpha D(x) + (\sigma^{\mu\nu}\theta_\alpha)F_{\mu\ nu}(x) - i\theta^2\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(x) + \ldots \tag{2.104}$$

Hence, integrating

$$\int d^2\theta W^\alpha W_\alpha = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{i}{2}F_{\mu\nu}(*F^{\mu\nu}) - 2i\lambda\sigma^\mu\partial_m u\bar{\lambda}^{\dot{\alpha}}(x) + D^2 \tag{2.105}$$

Where we used the dual field strenght $*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$.

We can define

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{2.106}$$

Hence, we find the supersymmetric Maxwell action, as

$$\begin{aligned} S_{\text{Maxwell}} &= -\int d^4x \left[\int d^2\theta \frac{i\tau}{16\pi} W^\alpha W_\alpha + h.c.\right] = \\ &= \int d^4x \left[-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2}F_{\mu\nu} * F^{\mu\nu} - \frac{i}{e^2}\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \frac{1}{2e^2}D^2\right] \end{aligned} \tag{2.107}$$

2.5.2 $\mathcal{N} = 1$ Super Yang Mills

We work with a gauge group with Lie algebra

$$[T^A, T^B] = if^{ABC}T^C \tag{2.108}$$

with hermitian generators; we normalise the generator in the fundamental representation as:

$$\text{Tr}T^AT^B = \frac{1}{2}\delta^{AB} \tag{2.109}$$

We introduce a real superfield V in the adjoint of the gauge group

$$V = V^AT^A \quad A = 1, \dots, \text{dim}G \tag{2.110}$$

We focus on $G = SU(N)$, hence if T^A is in the fundamental representation, then V is an $N \times N$ matrix. Equivalentely, we have

$$A_\mu = A_\mu^AT^A \quad \lambda_\alpha = \lambda_\alpha^AT^A \quad D = D^AT^A \tag{2.111}$$

We want to generalise the non-Abelian gauge symmetry, we take $\Omega = \Omega^AT^A$, since Ω is in the Lie algebra, $e^{i\Omega} \in G$ and this act on the real superfield as

$$e^{2V} \to e^{-2i\Omega^\dagger}e^{2V}e^{2i\Omega} \tag{2.112}$$

From BCH formula, we get the transformation law for the superfield itself

$$V \to V + i(\Omega - \Omega^\dagger) - i[V, \Omega + \Omega^\dagger] + \dots \tag{2.113}$$

We can use the shift in the first term to go to Wess-Zumino gauge. The remaining gauge symmetry acts on A_μ in the usual way

$$A_\mu \to UA_\mu U^{-1} + iU\partial_\mu U^{-1} \tag{2.114}$$

The action is

$$S_{SYM} = \int d^4xTr\left[-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2}F_{\mu\nu} * F^{\mu\nu} - \frac{2i}{g^2}\lambda\sigma^\mu\mathcal{D}_\mu\bar{\lambda} + \frac{1}{g^2}D^2\right] \tag{2.115}$$

Chapter 3

$\mathcal{N} = 4$ Super Yang Mills theory

We need three chiral superfields $\phi_i, i = 1, 2, 3$, one gauge superfield V with field strenght W_α. The only field theory with $\mathcal{N} = 4$ supersiìymmetry is given by

$$S_{\mathcal{N}=4} = \int d^4 x Tr\Big[\int d^4\theta \phi^{i\dagger} e^V \phi^i e^{-V} + \frac{1}{8\pi} Im\left(\tau \int d^2\theta W_\alpha W^\alpha\right) + \\ + \left(i g_{YM} \frac{\sqrt{2}}{3!} \int d^2\theta \epsilon_{ijk} \phi^i [\phi^j, \phi^k] + h.c.\right)\Big] \tag{3.1}$$

where τ is complex gauge coupling and W_α is the chiral spinor field constructed from the vector field V. Writing out the superfields in component field, we have

$$\mathcal{L} = \mathrm{Tr}\bigg(-\frac{1}{2g_{YM}^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} - i\bar{\lambda}^a \bar{\sigma}^\mu D_\mu \lambda_a \\ - \sum_i D_\mu \phi^i D^\mu \phi^i + g_{YM} \sum_{a,b,i} C_i^{ab} \lambda_a [\phi^i, \lambda_b] \\ + g_{YM} \sum_{a,b,i} \bar{C}_{iab} \bar{\lambda}^a [\phi^i, \bar{\lambda}^b] + \frac{g_{YM}^2}{2} \sum_{i,j} [\phi^i, \phi^j]^2\bigg) \tag{3.2}$$

where the constants C_i^{ab} are related to the Clifford Dirac matrices for $SO(6)_R \simeq SU(4)_R$. The Lagrangian is invarinat under $\mathcal{N} = 4$ Poincarè supersymmetry, whose transformation laws are given by

$$\begin{aligned} \delta\phi^i &= [Q^a_\alpha, X^i] = C^{iab}\lambda_{ab} \\ \delta\lambda_b &= \{Q^a_\alpha, \lambda_{\beta b} = F^+_{\mu\nu} (\sigma^{\mu\nu})^\alpha{}_\beta \delta^a_b + [\phi^i, \phi^j]\epsilon_{\alpha\beta} (C_{if})^a{}_b \\ \delta\bar{\lambda}^b_\beta &= \{Q^a_\alpha, \bar{\lambda}^b_\beta\} = C^{ab}_i \bar{\sigma}^\mu_{\alpha\beta} D_\mu \phi^i \\ \delta A_\mu &= [Q^a_\alpha, A_\mu] = (\sigma_\mu)_\alpha{}^\beta \bar{\lambda}^a_\beta \end{aligned} \tag{3.3}$$

where $(C_{ij})^a{}_b$ are related to bilinears in Clifford Dirac matrices of $SO(6)_R$.

3.0.1 Properties

1. The coupling constant is dimensionless and all fields are massless, hence the theory is scale invariant on the classical level; the bare mass dimension of the fields are $[A_\mu] = 1$, $[\lambda] = 3/2$, $[\phi^i] = 1$.

2. The theory is also scale invariant after quantisation (the β function vanishes at all orders in perturbation theory). In fact, it is invariant under the conformal group $SO(4, 2)$; in addition, the Lagrangian is also invariant under $\mathcal{N} = 4$ supersymmetry, with R-symmetry group $SU(4)_R$. The combination of $\mathcal{N} = 4$ Poincaré supersymmetry and conformal invariance produces the superconformal symmetry, given by the supergroup $SU(2, 2|4)$.

3. The theory is believed to be UV finite, which implies that the symmetry group $SU(2,2|4)$ is a fully quantum mechanica symmetry.

4. The theory is invariant under S-duality group $SL(2,\mathbb{Z})$ acting on the complex coupling constant τ, where, as before, τ is given by $\tau = \frac{\theta_I}{2\pi} + \frac{4\pi i}{g^2}$, and the symmetry is

$$\tau \to \frac{a\tau + b}{c\tau + d} \quad ad - bc = 1 \quad a, b, c, d, \in \mathbb{Z} \tag{3.4}$$

note that for $\theta_I = 0$ the S-duality transformation amounts to $g \to \frac{1}{g}$, which is a strong-weak duality.

5. The theory has two different classes of vacua; indeed, considering the potential energy term

$$-g^2 \sum_{i,j} \int \mathrm{Tr}[\phi^i, \phi^j]^2 \tag{3.5}$$

each term in the sum is positive or zero (because of the positive definite behaviour of the Cartan-Killing form; consequently, the $\mathcal{N} = 4$ supersymmetric ground state is in the form:

$$[\phi^i, \phi^j] = 0 \qquad i, j = 1, \ldots, 6 \tag{3.6}$$

with two class of solutions to this equation:

- The superconformal phase, for which $\langle \phi^i \rangle = 0$ for all $i = 1, \ldots, 6$ with gauge algebra and superconformal symmetry $SU(2,2|4)$ unbroken. The physical states are gauge invariants and transforms under unitary representations of $SU(2,2|4)$.
- The Coulomb phase, $\langle \phi^i \rangle \neq 0$ for at least one i; in this case, the low energy behaviour is that of r copies of $\mathcal{N} = 4$ $U(1)$ theory, where r is the rank of the gauge group. Superconformal symmetry is also spontaneously broken, since the non-zero vacuum expectation value $\langle \phi^i \rangle$ sets a scale.

3.1 Superconformal symmetry

In this section we study a supersymmetric theory that is also conformal (which is the case of $\mathcal{N} = 4$ Super Yang Mills).

The generators of the superconformal algebra can be grouped into the generators of conformal group $(J_{\mu\nu}, P_\mu, D, K_\mu)$ as well as the Poincarè supercharges $(Q^a_\alpha,\ \bar{Q}^a_{\dot{\alpha}})$; however, to ensure closure of the superconformal algebra, we have to introduce further fermionic supercharges: $S^a_\alpha, \bar{S}^a_{\dot{\alpha}}$, which are the fermionic superpartners of K_μ:

$$\{S^a_\alpha, \bar{S}_{\dot{\beta} b}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} K_\mu \delta^a_b \tag{3.7}$$

$$\{Q^a_\alpha, S_{\beta b}\} = \epsilon_{\alpha\beta}(\delta^a_b D + R^a_b) + \frac{1}{2}\delta^a_b J_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta} \tag{3.8}$$

Now, we consider non vanishing local operators $\mathcal{O}(x)$, constructed from the elementary fields of the conformal theory, considering only gauge invariant operators ; they are only characterized by the conformal dimension Δ and the spin $\mathcal{J}_{\mu\nu}$

$$[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0) \qquad [J_{\mu\nu}, \mathcal{O}(0)] = -\mathcal{J}_{\mu\nu}\mathcal{O}(0) \tag{3.9}$$

A particularly important subset of operators are the superconformal primary operators $\mathcal{O}$, which in a superconformal multiplet of $su(2,2|\mathcal{N})$ have the lowest dimension Δ.

The existence of a lower dimension derive from unitarity, which imposes a lower bound on the dimension of the operators; otherwise, the successive applications of S, which has dimension $-\frac{1}{2}$, would generate operators with negative dimension; for this reason, we must have:

$$[S^a_\alpha, \mathcal{O}\} = 0 \qquad [\bar{S}_{a\dot{\alpha}}, \mathcal{O}\} = 0 \tag{3.10}$$

for all $a = 1, \ldots, \mathcal{N}$ and $\alpha \in \{1,2\}$. Commutation and anticommutation brackets have to be chosen depending on the bosonic of fermionic nature of the operator $\mathcal{O}$.

Starting from the superconformal primary operator, we may construct descendants by applying any generator of the superconformal algebra; for example, by applying P_μ to $\mathcal{O}$ we obtain a descendant $[P_\mu, \mathcal{O}(x)] = -i\partial_\mu \mathcal{O}(x)$, whose dimension Δ is increased by 1. The superconformal primary operator and its descendants correspond to an irreducible representation of $PSU(2,2|4)$ for $\mathcal{N} = 4$.

A special kind of descendants are superdescendants $\mathcal{O}'$, defined by

$$\mathcal{O}' = [Q, \mathcal{O}\} \tag{3.11}$$

with $\Delta_{\mathcal{O}'} = \Delta_{\mathcal{O}} + \frac{1}{2}$. They are conformal primary operators :

$$[K_\mu, \mathcal{O}'] = 0 \tag{3.12}$$

We can define the chiral primary operators, which are superconformal primary operators that are also annihilated by at least one of the Q^a_α

$$[Q^a_\alpha, \mathcal{O}\} = 0 \tag{3.13}$$

for at least one $a \in \{1, \ldots, \mathcal{N}\}$ and one $\alpha \in \{1,2\}$; they are BPS operators; the multiplet formed by chiral primary operators is smaller than the multiplets formed by superconformal primary operators, which are not chiral primary operators. Their conformal dimension does not receive any quantum corrections.

3.1.1 Superconformal operators in $\mathcal{N} = 4$ theory

The superconformal representations are realised in terms of gauge invariant composite operators involving the fields of the $\mathcal{N} = 4$ Super-Yang Mills theory Lagrangian. The elementary fields are the scalars ϕ^i, the fermions $\psi^a_\alpha, \bar{\psi}_{\dot{\alpha}a}$ and the gauge field A_μ; alternatively, we can consider the field strength tensor $F_{\mu\nu}$ instead of A_μ. Gauge invariant operators are obtained by taking the trace of a product of such covariant fields evaluated at the same space-time point.

In addition, superconformal primary operators can only contain the scalars ϕ; indeed, we know that the primary operator is not the $\mathcal{Q}$-commutator of another operator. in this way, considering schematically the $\mathcal{Q}$ transforms of the canonical fields, we have:

$$\begin{aligned} \{\mathcal{Q}, \lambda\} = F^+[\phi, \phi] \quad & [\mathcal{Q}, \phi] = \lambda \\ \{\mathcal{Q}, \bar{\lambda}\} = D\phi \quad & [\mathcal{Q}, F] = D\lambda \end{aligned} \tag{3.14}$$

We immediately see, from the rhs, that λ cannot be primary; in the same way, neither F or $D\phi$ can be primary and we only have ϕ. We can consider for example, the local operators

$$\mathcal{O}(x) = \mathrm{Tr}(\phi^i \ldots \phi^j)(x) \tag{3.15}$$

which are single trace operators. Of central importance are the single trace operators of the form:

$$\mathcal{O}(x) = \mathrm{Str}(\phi^{\{i_i}\phi^{i_2} \ldots \phi^{i_k\}})(x) \tag{3.16}$$

where $i_j,\ j = 1, \ldots, n$ stand for the $SO(6)_R$ fundamental representation indices; on the other side, "Str" stands for symmetrised trace of the gauge algebra, which, for the scalars $\phi^i = \phi^{ia} T_a$, in the adjoint representation, is given by the sum over all permutations:

$$\mathrm{Str}(T_{a_1} \ldots T_{a_n}) = \sum_{\text{all permutations } \sigma} \mathrm{Tr}(T_{\sigma(a_1)} \ldots T_{\sigma(a_n)}) \tag{3.17}$$

As a result, the above operator is totally symmetric in the $SO(6)_R$ indices i_j; in addition, the curly brackets denote that all the traces are removed. The resulting operators correspond to an irreducible representation of the superconformal algebra; for instance, the single trace $\frac{1}{2}$ BPS operator of dimension $\Delta = 2$, is given by

$$\mathrm{Str}(\phi^{\{i}\phi^{j\}}) = \mathrm{Tr}(\phi^i\phi^j) - \frac{1}{6}\delta^{ij}\mathrm{Tr}(\phi^k\phi^k) \tag{3.18}$$

The operators (3.16) are chiral primary operators; in particular, they are one half BPS states of the superconformal algebra; their dimension is $\Delta = k$, with k the number of scalar fields present; as explained, they do not acquire anomalous dimension.

Differently, a single trace operator, such as the Konishi operator

$$K = \mathrm{Tr}(\phi^i\phi^i) \tag{3.19}$$

is unprotected.

These operators belong to $SU(4) \simeq SO(6)$ representation. Unitary representations of the superconformal algebra $SU(2,2|4)$ are labelled by the quantum numbers of its maximal bosonic subalgebra, which, has said, is $SO(4,2) \times SO(6)_R$; in addition, the conformal group can be seen as the direct product of the Lorentz algebra $SO(1,3)$ and of the dilations $so(1,1)$. The corresponding quantum numbers are the spin labels s_+, s_- for the Lorentz algebra, the scale dimension Δ for dilations and the three Dynkin labels $[r_1, r_2, r_3]$ for $SO(6)_R$.

The chiral primary (one half BPS) operators are built from a symmetric product of ϕ^i and the scalars ϕ^i transforms in the $[1,0,0]$ representation of $SO(6)_R$; hence, the chiral primary operator of the form (3.16), has to be in $[k,0,0]$, which has dimension

$$\dim[k,0,0] = \frac{1}{12}(k+1)(k+2)^2(k+3) \tag{3.20}$$

The simplest example of a chiral primary operator is the case $k = 2$ (because, trivially, $\mathrm{tr}\phi^i = 0$), with representation $[2,0,0]$ and dimension 20. The associated operator in $\mathcal{N} = 4$ is given by (3.18).

It is also possible to have multiple trace $\frac{1}{2}$ BPS operators; they are built as follow: considering $SO(6)$ Dynkin labels, the tensor product of n representations $[k_1,0,0]\otimes,\ldots,\otimes[k_n,0,0]$ always contain the representation $[k,0,0]$, $k = k_1 + \ldots + k_n$, with multiplicity 1. the most general $\frac{1}{2}$ BPS gauge invariant operators are given by the projection onto the representation $[0,k,0]$ of the product operator $[\mathcal{O}_{k_1}(x)\ldots\mathcal{O}_{k_n}(x)]_{[k,0,0]}$, with $k = k_1 + \ldots k_n$.

We can also have 1/4 and 1/8 BPS operators, but they are not single trace operators; the $\frac{1}{4}$ BPS operators has the simple construction in terms of double trace operators; in general, they are of the form

$$[\mathcal{O}_{k_1}(x)\ldots\mathcal{O}_{k_n}(x)]_{[k,l,l]} \qquad k + 2l = k_1 + \ldots + k_n \tag{3.21}$$

On the other hand, the series of $\frac{1}{8}$ BPS operators starts with triple trace operators and are generally of the form

$$[\mathcal{O}_{k_1}(x)\ldots\mathcal{O}_{k_n}(x)]_{[k,l,l+2m]} \qquad k + 2l + 3m = k_1 + \ldots + k_n \tag{3.22}$$

In unitary representations, the dimension Δ of the operators are bounded from below by the spin and $SO(6)_R$ quantum numbers; indeed, we can consider only primary operators, which have the lowest dimension in a given irreducible multiplet. It is possible to show [7] that exist 4 distinct series:

$$\begin{aligned}
&\Delta = r_1 + r_2 + r_3 \\
&\Delta = \frac{3}{2} = r_1 + r_2 + \frac{1}{2}r_3 \geq 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3 \quad \text{ with } r_1 \geq r_3 + 2 \\
&\Delta = \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3 \geq 2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3 \quad \text{ with } r_3 \geq r_1 + 2 \\
&\Delta \geq \mathrm{Max}\left[2 + \frac{3}{2}r_1 + r_2 + \frac{1}{2}r_3; 2 + \frac{1}{2}r_1 + r_2 + \frac{3}{2}r_3\right]
\end{aligned}$$

for the $SU(4)$ representation $[r_1, r_2, r_3]$; the first three cases correspond to BPS multiplets, while the last is valid in the general case of long multiplets.

Chapter 4

Superstring theory

4.1 The Superparticle

Let us consider the motion of a point particle of mass m in Minkowski space; we always use units in which $c = \hbar = 1$. The action must be proportional to the invariant length of the worldline, i.e.

$$S = -m \int ds \tag{4.1}$$

where the invariant interval is given by (we adopt in this chapter opposite convention for the metric)

$$ds^2 = -\eta_{\mu\nu}(x) dx^\mu dx^\nu \tag{4.2}$$

Suppose that a classical trajectory is written as $x^\mu(\tau)$, where τ is an arbitrary parameter that labels the points along the world line, then (4.2) can be written in the form

$$S = -m \int d\tau \sqrt{-\dot{x}^2} \tag{4.3}$$

where

$$\dot{x}^2 = g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{4.4}$$

The action (4.3) is invariant under reparametrizations $\tau \to \tilde{\tau}(\tau)$. However, formula (4.3) does not apply to massless particles and, in addition, the square root is not suitable when we will go to the quantum theory. We introduce an auxiliary coordinate $e(\tau)$, which can be interpreted as an einbein for the one dimensionale geometry of the worldline; the action can be expressed equivalently as

$$S = \int (e^{-1}\dot{x}^2 - em^2) d\tau \tag{4.5}$$

Solving the equations of motion for e we have $\dot{x}^2 + e^2 m^2 = 0$ and substituting in (4.5) we recover (4.3); in addition, this action is still invariant under $\tau \to \tilde{\tau}(\tau)$, knowing that the einbein field transforms as $\tilde{e}(\tilde{\tau})d\tilde{\tau} = e(\tau)d\tau$; moreover, for action (4.5), the limit $m \to 0$ exist and there are no roots. In the superstring generalization we will take $m = 0$ since, in that case, the mass term is not relevant.
The action (4.5) is also invariant under global space-time Poincaré transformations, generated by

$$\delta x^\mu = a^\mu + b^\mu_\nu x^\nu \tag{4.6}$$

$$\delta e = 0 \tag{4.7}$$

where $b_{\mu\nu}$ is antisymmetric.

We achieve space-time supersymmetry by generalizing Minkowsky space with its "bosonic" coordinates x^μ to a superspace with fermionic coordinates as well as bosonic coordinates. If there are to be N supersymmetries, we introduce N anticommuting spinor coordinates $\theta^{Aa}(\tau),\ A = 1, 2, \ldots, N$. The index a is that of a space-time spinor appropriate to D dimensions. For a general Dirac spinor $a = 1, \ldots, 2^{D/2}$ (D even).

Supersymmetry is realized introducing infinitesimal Grassmann parameters ϵ^A, spinors of the same type as the corresponding θ^A coordinates, the transformation formulas are

$$\delta\theta^a = \epsilon^A \qquad \delta x^\mu = i\bar{\epsilon}^A\Gamma^\mu\theta^A \tag{4.8}$$

$$\delta\bar{\theta}^A = \bar{\epsilon}^A \qquad \delta e = 0 \tag{4.9}$$

with ϵ^A a constant spinor and Γ^μ the Dirac gamma matrices in general dimension. Many supersymmetric actions can be written, since both $\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A$ and $\dot{\theta}^{Aa}$ are invariant under supersymmetry, indeed

$$\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A \to \dot{x}^\mu + i\bar{\epsilon}^A\Gamma^\mu\dot{\theta}^A - i(\bar{\theta}^A + \bar{\epsilon}^A)\Gamma^\mu(\dot{\theta}^A) = \dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A \tag{4.10}$$

The simplest generalization of (4.3) utilizes the first invariant, giving

$$S = \frac{1}{2}\int e^{-1}(\dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A)^2 d\tau \tag{4.11}$$

This action has the super-Poincaré symmetry; the equations of motion are;

$$\frac{\partial\mathcal{L}}{\partial e} = \frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{e}} \to p^2 = 0 \tag{4.12}$$

$$\frac{\partial\mathcal{L}}{\partial x^\mu} = \frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} \to \dot{p}^\mu = 0 \tag{4.13}$$

$$\frac{\partial\mathcal{L}}{\partial\bar{\theta}} = \frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{\bar{\theta}}} \to \Gamma\cdot p\dot{\theta} = 0 \tag{4.14}$$

where we define

$$p^\mu = \dot{x}^\mu - i\bar{\theta}^A\Gamma^\mu\dot{\theta}^A \tag{4.15}$$

Since $(\Gamma\cdot p)^2 = -p^2 = 0$, the matrix $\Gamma\cdot p$ has half the maximum possible rank; furthermore, θ always appears multiplied by $\Gamma\cdot p$, as a result, half of its components are decoupled from theory. This is a consequence of an additional local fermionic symmetry of (4.11): let $k^{Aa}(\tau)$ denote N infinitesimal Grassmann spinor parameters which can depend on τ and consider the transformation

$$\delta\theta^A = i\Gamma\cdot pk^A \tag{4.16}$$

$$\delta x^\mu = i\bar{\theta}^A\gamma^\mu\delta\theta^A \tag{4.17}$$

$$\delta e = 4e\dot{\bar{\theta}}^A k^A \tag{4.18}$$

The above transformations are actually a symmetry of (4.11). Indeed, considering

$$\delta p^\mu = \delta\dot{x}^\mu - i\delta\bar{\theta}^A\Gamma_\mu\dot{\theta}^A - i\bar{\theta}^A\Gamma^\mu\delta\dot{\theta}^A = i\dot{\bar{\theta}}^A\Gamma^\mu\delta\theta^A - i\delta\bar{\theta}^A\Gamma^\mu\dot{\theta}^A = 2i\dot{\bar{\theta}}^A\Gamma^\mu\delta\theta^A \tag{4.19}$$

Thus

$$\delta p^2 = 2p^\mu\delta p_\mu = 4i\dot{\bar{\theta}}^A\Gamma\cdot p\delta\theta^A = 4p^2\dot{\bar{\theta}}^A k^A \tag{4.20}$$

It follows that $e^{-1}p^2$ (the integrand of the action) is invariant for

$$\delta e^{-1} = -4e^{-1}\dot{\bar{\theta}}^A k^A \tag{4.21}$$

which is equivalent to (4.18)

There is no on-shell conserved charge associated with k, the conserved quantities that one might attempt to derive from k invariance all vanish by the equation of motion.

4.2 The Supersymmetric String Action

Consider the bosonic string action

$$S_{bos} = -\frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X \tag{4.22}$$

where $h^{\alpha\beta}$ describes the worldsheet metric of the string and has Minkowsky signature; the functions $X^\mu(\sigma)$ give a map of the worldsheet into the physical space-time and $g_{\mu\nu}$ describe the background space-time.

Comparing with the superparticle action, there is an obvious guess for a supersymmetric superstring action:

$$S_1 = -\frac{1}{2\pi}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\Pi_\alpha\cdot\Pi_\beta \tag{4.23}$$

where

$$\Pi_\alpha^\mu = \partial_\alpha X^\mu - i\bar{\theta}^A\Gamma^\mu\partial_\alpha\theta^A \tag{4.24}$$

This action possess local reparametrization invariance and N global supersymmetries; however the locl k symmetry of superparticle action is lost in this generalization and θ describes twice as many degrees of freedom as it should; however, it is possible to add a second term S_2 so that the resulting action has local k symmetry; in this way, half of the component of θ are decoupled and the equations of motion can be solved, at least in a particular gauge.

We need to consider at most two supersymmetries

$$S_2 = \frac{1}{\pi}\int d^2\sigma\{-i\epsilon^{\alpha\beta}\partial_\alpha X^\mu(\bar{\theta}^1\Gamma_\mu\partial_\beta\theta^1 - \bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2) + \epsilon^{\alpha\beta}\bar{\theta}^1\Gamma^\mu\partial_\alpha\theta^1\bar{\theta}^2\Gamma_\mu\partial_\beta\theta^2\} \tag{4.25}$$

this second term is independent from $h^{\alpha\beta}$ and dows not contribute to the energy momentum tensor. In addition, this term has local reparametrization symmetry and global Lorentz symmetry; moreover it also has global $N = 2$ supersymmetry in special cases, indeed, considering the transformations $\delta\theta^A = epsilon^A$ and $\delta X^\mu = i\bar{\epsilon}^A\Gamma^\mu\theta^A$, the action is invariant for

- $D = 3$ and θ is Majorana;
- $D = 4$ and θ is Majorana or Weyl;
- $D = 6$ and θ is Weyl;
- $D = 10$ and θ is Majorana-Weyl;

Hence, the classical superstring theory exists only in these four cases.

We now wish to show that the sum $S_1 + S_2$ has a local fermionic symmetry that is not possessed by either term separately. In the superparticle case this symmetry involved $\delta\theta^A = i\Gamma\cdot pk^A$; in the string case the analog of p^μ is Π_α^μ, therefore the parameter k has to carry a world-sheet vector index as well.

The infinitesimal parameter k now carry three indices $k^{A\alpha a}$: the index $A = 1, 2$ correspond to the label θ^A, the index $\alpha = 0, 1$ is a world-sheet vector index and a is a D-dimensional space-time index corresponding to one of the four poossible type of spinors (the spinor index $a$is suppressed in most formulas). In analogy to the superparticle case, the supersymmetry is

$$\delta\theta^A = 2i\Gamma\cdot\Pi_\alpha k^{A\alpha}\delta X^\mu = i\bar{\theta}^A\Gamma^\mu\delta\theta^A \tag{4.26}$$

in addition, there is a further local bosonic symmetry.

4.2.1 Type I and Type II Superstrings

In ten dimensions, the θ coordinates in the superstring action must be chosen to be Majorana-Weyl spinors; this means that θ^1 and θ^2 must each be assigned a definite handedness, hence θ^1 and θ^2 are chosen to have the same handedness or the opposite handedness. In the case of closed strings we only impose perioidicity in σ, which is possible in either case, since it does not relate θ^1 and θ^2; on the other hand for open strings, θ^1 and θ^2 must be equated at the end of the strings, which is possible only when θ^1 and θ^2 have the same handedness.

A superstring theory based on superstrings is called a type I superstring theory, indeed the open-string boundary conditions reduce the space-time supersymmetry to $N = 1$.

Considering closed superstring, if θ^1 and θ^2 have opposite handedness, the resulting theory necessarily involves oriented strings since θ^1 and θ^2 describes modes that propagates in opposite directions; this theory has two conserved $D = 10$ supersymmetries of opposite chirality and is called type IIA.

On the other case, we can have a theory of closed superstrings on two θ coordinates of the same handedness, this is type IIB superstring theory.

As a last case, we can use only one θ coordinate, which led to heterotic strings.

4.2.2 Quantization

Assuming that $D = 10$ and θ^1, θ^2 are Majorana-Weyl. The local reparametrization, Weyl invariances and the local k symmetries allow for a number of gauge choices to be made; we can set $h_{\alpha\beta} = \eta_{\alpha\beta}$; however, to linearize the equation of motion we also need the residual reparametrization invariance and k symmetry to make light cone gauge choice.

We can use k symmetry to impose

$$\Gamma^+\theta^1 = \Gamma^+\theta^2 = 0 \tag{4.27}$$

where

$$\Gamma^{\pm} = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9) \tag{4.28}$$

it follows that Γ^+ and $Gamma^-$ are nilpotent, in this way half the eigenvalues of each must be zero; in this way, our gauge choice sets half of the components of θ to zero.

The equations for X^+ and X^i are simply free wave equations, then the residual conformal invariance can be used to impose the condition

$$X^+(\sigma, \tau) = x^+ + p^+\tau \tag{4.29}$$

which amounts to setting all the α_n^+ modes with $n \neq 0$ equal to zero.

A generic spinor in ten dimension has 32 components, the Majorana condition makes them real and the Weyl condition sets half equal to zero, leaving 16 real components; the light-cone gauge condition reduces the count by another factor of two leaving just eight real components. The only manifest symmetry inn the light cone gauge is the rotational invariance of the eight transverse dimensions; hence the eight surviving components of each θ form an eight-dimensional spinor representation of the transverse $SO(8)$ group. This group possesses the triality symmetry which is an automorphism; for example the fundamental vector representation 8_v and the two spinor representations 8_s and 8_c are real beacuse of this symmetry.

Using the symbol S for the eight surviving components of θ in the light cone gauge, we have

$$\sqrt{p^+}\theta^1 \to S^{1a}\text{or}S^{1\dot{a}} \tag{4.30}$$

$$\sqrt{p^+}\theta^2 \to S^{2a}\text{or}S^{2\dot{a}} \tag{4.31}$$

The determination of whether the 8_s or 8_c representation occurs in each case is controlled by the chirality of the corresponding θ in ten dimension; we may choose S^1 to belong to 8_s as a convention, then S^2 belongs to 8_s for type I and IIB thories, but to 8_c for type IIA.

The equations of motion collapses dramatically in the light cone gauge, which can be obtained from the simplified action:

$$S_{l.c.} = -\frac{1}{2}\int d^2\sigma(T\partial_\alpha X^i\partial^\alpha X^i - \frac{i}{\pi}\bar{S}^a\rho^\alpha\partial_\alpha S^a \tag{4.32}$$

where S^{1a} and S^{2a} have been combined into a two-component Majorana world-sheet spinor S^a.

The quantization of the X^i coordinate is the same as in the bosonic case, while the S^{Aa} coordinates have canonical anticommutation relations

$$\{S^{Aa}(\sigma,\tau), S^{Bb}(\sigma',\tau)\} = \pi\delta^{ab}\delta^{AB}\delta(\sigma-\sigma') \tag{4.33}$$

Specifying the boundary condition, we write the mode expansion, which in the open case are:

$$S^{1a}(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{\infty}^{\infty} S_n^a \exp\{-in(\tau-\sigma)\} \tag{4.34}$$

$$S^{2a}(\sigma,\tau) = \frac{1}{\sqrt{2}}\sum_{\infty}^{\infty} S_n^a \exp\{-in(\tau+\sigma)\} \tag{4.35}$$

The canonical anticommutation relation becomes

$$\{S_m^a, S_n^b\} = \delta^{ab}\delta_{m+n} \tag{4.36}$$

Applying, instead, the closed strings boundary condition we have:

$$S^{1a}(\sigma,\tau) = \sum S_n^a \exp\{-2in(\tau-\sigma)\} \tag{4.37}$$

$$S^{2a}(\sigma,\tau) = \sum \tilde{S}_n^a \exp\{-2in(\tau+\sigma)\} \tag{4.38}$$

with independent sets of modes for right and left movers. If S^1 and S^2 belong to different representations (8_s and 8_c), this describes type IIA, which are necessarily oriented; otherwise, we have IIB if strings are oriented od type I if they are not.

4.2.3 Analysis of the Spectrum

Open Superstrings

The massless sector is the ground state since there are no tachyons; the ground state represents the algebra $\{S_0^a, S_0^b\} = \delta^{ab}$. We have a 16-dimensional multiplet of massless ground states which we denote by $|\phi_0\rangle$; this consists of eight Bose states in the 8_v representation of spin(8) and eight Fermi states in the 8_c representation.

The massive open-string states are most easily obtained by applying α_{-n}^i and S_{-n}^a excitations to the ground state $|\phi_0\rangle$. At the first excited level the physical states are

$$\alpha_{-1}^i\,|\phi_0\rangle \qquad S_{-1}^a\,|\phi_0\rangle \tag{4.39}$$

describing a total of 128 bosonic and 128 fermionic modes. These states fit into a spin(9) multiplet, as must happen for massive sates; we have representations $(44+84)$ for the bosons and 128 for the fermions.

The second excited level we have

$$\begin{aligned}\alpha^i_{-2}\left|\phi_0\right\rangle \quad S^a_{-2}\left|\phi_0\right\rangle \quad \alpha^i_{-1}\alpha^j_{-1}\left|\phi_0\right\rangle \\ S^a_{-1}S^b_{-1}\left|\phi_0\right\rangle \quad \alpha^i_{-1}S^a_{-1}\left|\phi_0\right\rangle\end{aligned} \tag{4.40}$$

These content can be summarized in terms of spin(9) multiplets in the form

$$9 \otimes (44 + 84 + 128) \tag{4.41}$$

At the third level we have

$$(44 + 16) \otimes (44 + 84 + 128) \tag{4.42}$$

Closed Superstring

We require two sets of modes, one for right movers and one for left movers. The massless states are described by direct product states $\left|\phi_0\right\rangle \otimes \left|\tilde{\phi}_0\right\rangle$; there are two cases to be distinguished, according to wether the original two majorana-Weyl spinors have the same chirality or opposite chirality; if they are distinct we find

$$(8_v + 8_c) \otimes (8_v + 8_s) = (1 + 28 + 35_v + 8_v + 56_v)_B + (8_s + 8_c + 56_s + 56_c)_F \tag{4.43}$$

which correspond to the particle content of type IIA supergravity in $D = 10$.

If both spinor belong to the same chirality representation, we have

$$(8_v + 8_c) \otimes (8_v + 8_c) = (1 + 28 + 35_v + 1 + 28 + 35_c)_B + (8_s + 8_s + 56_s + 56_s)_F \tag{4.44}$$

which correspond to chiral type IIB supergravity theory in $D = 10$.

When two spinors are the same type, it is possible to impose a symmetrization restriction keeping only terms that are invariants under interchange of $\left|\phi_0\right\rangle$ and $\left|\tilde{\phi}_0\right\rangle$, which correspond to a graded symmetrization

$$[(8_v{+}8_c)\times(8_v{+}8_c)]_{\text{graded sym}} = (8_v\times 8_v)_{\text{sym}}+(8_v\times 8_c)+(8_c\times 8_c)_{\text{antisym}} = (1{+}28{+}35_v)_B+(8_s{+}56_s)_F \tag{4.45}$$

which is the particle content of chiral type I supergravity in $D = 10$.

4.3 D-Branes

Let ξ^a denote the coordinates for the worldvolume of a Dp-brane; for the case of the fundamental string this reduces to $\xi^0 = \tau$ and $\xi^1 = \sigma$

Chapter 5

The Maldacena AdS/CFT Correspondence

5.1 The Maldacena limit

The space-time metric of N coincident D3-branes may be recast in the following form:

$$ds^2 = \left(1 + \frac{L^4}{y^4}\right)^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + \left(1 + \frac{L^4}{y^4}\right)^{\frac{1}{2}} (dy^2 + y^2 d\Omega_5^2) \tag{5.1}$$

where the radius L of the D3- brane is given by

$$L^4 = 4\pi g_s N (\alpha')^2 \tag{5.2}$$

If we take the limit of $y >> L$, we recover flat space-time $\mathbb{R}^{10}$. On the other hand, when $y < L$, the geometry is often referred to as the throat and would appear to be singular as $y << L$; however, the redefinition $u = \frac{L^2}{y}$ in the large u limit, transform the metric in the following form

$$ds^2 = L^2 \left[\frac{1}{u^2} \eta_{ij} dx^i dx^j + \frac{du^2}{u^2} + d\Omega_5^2 \right] \tag{5.3}$$

which corresponds to a product geometry: one component is the five-sphere S^5 with metric $L^2 d\Omega_5^2$. The remaining component is the hyperbolic space AdS_5, with constant negative curvature metric $L^2 u^{-2}(du^2 + \eta_{ij} dx^i dx^j)$. In cocnlusion, the geometry close to the brane ($y \simeq 0$) is regular and highly symmetrical and may summarized as $AdS_5 \times S^5$, where both components have identical radius L.

The Maldacena limit corresponds to keeping fixed g_s and N as well as all physical length scales, while letting $\alpha' \to 0$. In this limit, only the $AdS_5 \times S^5$ region of the D3-brane geometry survives the limit and contributes to the string dynamics of physical processes, while the dynamics in the asymptotically flat region decouples from the theory.

To see this decoupling, consider the effective action $\mathcal{L}$ and carry out its α' expansion in an arbitrary background with Riemannn tensor R. The expansion takes on the schematic form

$$\mathcal{L} = a_1 \alpha' R + a_2 (\alpha')^2 R^2 + a_3 (\alpha')^3 R^3 \tag{5.4}$$

Now, physical objects and legth scales in the asymptotically flat region are characterized by a scale $y >> L$, so that by simple scaling arguments we have $R \simeq 1/y^2$; hence, the expansion of the effective action becomes

$$\mathcal{L} = a_1 \alpha' \frac{1}{y^2} + a_2 (\alpha')^2 \frac{1}{y^4} + a_3 (\alpha')^3 \frac{1}{y^6} + \dots \tag{5.5}$$

Keeping the physical size y fixed, the entire contribution to the effective action from the limit $\alpha' \to 0$ is then seen to vanish.

5.2 The AdS/CFT Conjecture

The AdS/CFT conjecture states the equivalence between the following thories:

- Type IIB superstring theory on $AdS_5 \times S^5$ where both AdS_5 and S^5 have the same radius L, where the 5-form F_5^+ has integer flux $N = \int_{S^5} F_5^+$ and where the string coupling is g_s.
- $\mathcal{N} = 4$ super-Yang-Mills theory in 4-dimensions, with gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in its superconformal phase.

with the following identifications between the parameters of both theories

$$g_s = g_{YM}^2 \qquad L^4 = 4\pi g_s N(\alpha')^2 \tag{5.6}$$

The Maldacena equvalence, includes a precise map between the states on the superstring side and the local gauge invariant operators on the $\mathcal{N} = 4$ SYM side, as well as a correspondence between the correlators in both theories.

The above statement of the conjecture is referred to as the strong form, as it is to hold for all values of N and of $g_s = g_{YM}^2$; however, string theory quantization on a general curved manifold appears to be difficult; therefore, it is natural to seek limits in which the maldacena conjecture becomes more tractable but still remain non trivial.

5.2.1 The 't Hooft Limit

This limit consists in keeping the 't Hooft coupling $\lambda = g_{YM}^2 N = g_s N$ fixed and letting $N \to \infty$. In Yang Mills theory this limit correspond to a topological expansion of the field theory's Feynmann diagrams. On the AdS side, we can express the string coupling has $g_s = \lambda/N$; hence, the 't Hooft limit, correspond to weak coupling string perturbation theory.

5.2.2 The Large λ Limit

After the 't Hooft limit, the only parameter left is λ on the field theory, while we have the radius of curvatura $L/\sqrt{alpha'}$ on yhe string theory; the two parameters are related by $L^4/\alpha'^2 = 2\lambda$.

Since we are interested in strongly coupled field theories, we take the limit $\lambda \to \infty$ on the field theory side, which correspond to $\sqrt{\alpha'}/L \to 0$. The string length is then very small compared to the radius of curvature; therefore, for $\sqrt{\alpha'}/L \to 0$ we obtain the point-particle limit of type IIB string theory, which is given by IIB supergravity on $AdS_5 \times S^5$.

In this way, we have a strong/weak duality in the sense that strongly coupled $\mathcal{N} = 4$ Super Yang Mills is mapped to type IIB supergravity on weakly curved $AdS_5 \times S^5$ space; due to the special limits taken, this is referred to as the weak form of the AdS/CFT conjecture.

5.3 Mapping Global Symmetries

A key necessary ingredient for the AdS/CFT to hold is that the global unbroken symmetries of the two theories be identical.

The continuous global symmetry of $\mathcal{N} = 4$ SYM was previously shown to be the superconformal group $PSU(2,2|4)$, whose maximal bosonic subgroup is $SU(2,2) \times SO(6)_R$.

The same bosonic group is immediately recognized on the AdS side as the isometry group of the $AdS_5 \times S^5$ background; the completion into the full supergroup arises on the AdS side because 16 of the 32 Poincarè supersymmetries are preserved by the array of N parallel D3-branes, and in the AdS limit, are supplemented by another 16 conformal supersymmetries.

We have seen that in $\mathcal{N} = 4$ SYM theory, single color trace operators has a special role, because, out of them, all higher trace operators may be constructed using the OPE. Thus one should expect single trace operators on the SYM side to correspond to single particle states on the AdS side. Multiple trace states should then be interpreted as bound states of these one particle states.

Chapter 6

Strings Spectrum from Flat to Curved Space

6.1 Open Superstring Spectrum in Flat Space

We start from the results of section 4.2.3: in GS formulation [17] the chiral string excitations are created by the raising modes α^I_{-n}, S^a_{-n} acting on the vacuum $|\mathcal{Q}_s\rangle$:

$$S^a_n\,|\mathcal{Q}_c\rangle = \alpha^I_n\,|\mathcal{Q}_c\rangle = 0 \qquad n > 0 \tag{6.1}$$

with $I = 1, \ldots, 8_v$, $a = 1, \ldots, 8_c$, $\dot{a} = 1, \ldots, 8_c$, run over the vector, spinor left and spinor right representations of the $SO(8)$ little Lorentz group. We have introduced the notation:

$$\mathcal{Q} = 8_v + 8_s \quad \mathcal{Q}_c = 8_v + 8_c \tag{6.2}$$

to describe chiral worldsheet supermultiplets.

For the first few open string excitation levels, one finds:

$$\begin{aligned}
&l = 0 \quad |\mathcal{Q}_c\rangle \\
&l = 1 \quad (S^a_{-1} + \alpha^I_{-1})\,|\mathcal{Q}_c\rangle \\
&l = 2 \quad (S^a_{-2} + \alpha^I_{-2})\,|\mathcal{Q}_c\rangle \quad S^{[a}_{-1}S^{b]}_{-1}\,|\mathcal{Q}_c\rangle \quad \alpha^{(I}_{-1}\alpha^{J)}_{-1}\,|\mathcal{Q}_c\rangle \quad S^a_{-1}\alpha^I_{-1}\,|\mathcal{Q}_c\rangle
\end{aligned} \tag{6.3}$$

The level $l = 1$ has a total of 256 states, 128 bosons, created from the α^I_{-1}, and 128 fermions, created from the S^a_{-1}.

However, we know that these states, being massive states, must admit a decomposition in $SO(9)$ representation; indeed, the bosonic part can be written as **44** + **84**, which, in Dynkin labels, reads $[2, 0, 0, 0] + [0, 0, 1, 0]$, on the other hand, the representation **128** ($[1, 0, 0, 1]$) is a fermionic representation with spin 3/2.

The level $l = 2$ is more complicated, the first term written in (6.3) contributes as before with 256 states, the second term is composed by two S^a_{-1} and for the anticommutation rules it must be antisymmetric, obtaining $\frac{8\cdot7}{2}\cdot16 = 448$ terms; on the other hand, the third term must be symmetric, beacuse of its commutation relations, giving $\frac{8\cdot9}{2}\cdot16 = 576$ terms; for the last term, there are no commutation or anticommutation relations, therefore it contributes with $8\cdot8\cdot16 = 1024$ states. The total dimension is therefore 2304, we see that this content can be succinctly written in terms of $spin(9)$ representations as

$$\mathbf{9} \otimes (\mathbf{44{+}84{+}128}) \qquad \text{at } l = 2 \tag{6.4}$$

where we see that the firs term appears in the factorization.

Indeed, the multiplet corresponding to level $l = 1$ is a multiplet of supercharges which includes all massive representations in ten dimension; any higher order massive supermultiplet is a tensor product of this multiplet with another representation of $SO(9)$.

At higher levels, the complexity of the evaluation increases exponentially, for this reason we need a better method for the evaluation of states.

6.1.1 Unrefined Partition Function

We can evaluate the asymptotic density of states by the computation of the generating function, which expansion gives us the dimension of the representations for each string level. This is given by:

$$G(q) = \text{tr } q^N \tag{6.5}$$

where q is the variable of the generating function, with respect we expand, and N is the number operator; for the easier case of bosonic string, we have $N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n$ and the generating function is given by

$$Z(q)_{\text{bos}} = \text{tr}q^N = \prod_{n=0}^{\infty} \text{tr}q^{\alpha_{-n}\cdot\alpha_n} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{24}} \tag{6.6}$$

In our case, with superstring, we have

$$N = \sum_{m=1}^{\infty} (\alpha_{-m}^i \alpha_m^i + m S_{-m}^a S_m^a) \tag{6.7}$$

$$Z(q)_{\text{superstring}} = \text{tr } q^N = 16 \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^8 \tag{6.8}$$

where the factor in the denominator comes in the same way as it comes in the bosonic case, while the numerator with the plus factor is due to the fermionic modes; the factor of 16 is the degeneracy of the ground state.

The expansion of this function furnish the number of physical polarization modes at any given mass, it is called unrefined because there is no information about the fugacities (the character variables), but only about the dimension; expanding the generating function (with the help of a symbolic software, in our case Mathematica [26]) we have:

$$Z(q) = 16 + 256q + 2304q^2 + 15360q^3 + 84224q^4 + 400896q^5 + 1711104q^6 + 6690816q^7 + O(q^8) \tag{6.9}$$

Of course, we can check that the massless state and the first two excited states have the dimension that we have evaluated in the precedent section; moreover, we have argued that any massive state can be written as

$$(\mathbf{44} + \mathbf{84} + \mathbf{128}) \otimes \mathbf{R}_{SO(9)} \tag{6.10}$$

Where $\mathbf{R}_{SO(9)}$ is a specific representation of $SO(9)$.

This means that we can write our generating function as

$$G(q) = 16 + 256qZ_m(q) \tag{6.11}$$

Where $Z_m(q)$ is the generating function for the specific representation written above; recasting (6.9) in the form (6.11) we find

$$Z_m(q) = \frac{G(q) - 16}{256q} = 1 + 9q + 60q^2 + 329q^3 + 1566q^4 + 6684q^5 + 26136q^6 + O(q)^7 \tag{6.12}$$

We see that equation (6.10) is correct because the obtained generating function contains only integer coefficients. From equation (6.12) we can obtain other states for the open string spectrum;

indeed, at low dimension, the dimension of the representation defines uniquely the decomposition in irreducible representation.

We find that the dimension 60 can be obtained in $SO(9)$ only by $[2,0,0,0]+[0,0,01]$, in this way we find the level $l=3$ of open superstrings:

$$(\mathbf{44}+\mathbf{16})\otimes(\mathbf{44}+\mathbf{84}+\mathbf{128}) \quad \text{at } l=3 \tag{6.13}$$

However, for higher level states, the total dimension at a certain mass level is not sufficient to characterize the irreducible representations of that level; we need to include in our generating function the fugacities, in order to completely characterizes the representations.

6.1.2 Refined Partition Function

In order to discuss the refined partition function, we need to define the formalism of plethystics [6],[24]: for a function of m variables, $g(t_1,\ldots,t_m)$ that vanish at the orìgin $g(0,\ldots,0)$, the plethystic exponential is defined to be

$$PE[g(t_1,\ldots,t_m)] = \exp\left(\sum_{k=1}^{\infty}\frac{g(t_i^k,\ldots,t_m^k)}{k}\right) \tag{6.14}$$

While the fermionic plethystic exponential contains extra minus signs

$$PE_F[g(t_1,\ldots,t_m)] = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}g(t_i^k,\ldots,t_m^k)}{k}\right) \tag{6.15}$$

The general solution of a plethystic exponential can be obtained from Taylor series-expansion; for instance, in the easier case of a function with one variable, we know that

$$g(t)=\sum_{n=0}^{\infty}a_n t^n \tag{6.16}$$

$$PE[g(t)] = \exp\left(\sum_{n=0}^{\infty}a_n\sum_{k=1}^{\infty}\frac{t^{nk}}{k}\right) = \exp\left(-\sum_{n=1}^{\infty}a_n\log\left(1-t^n\right)\right) = \frac{1}{\prod_{n=1}^{\infty}(1-t^n)^{a_n}} \tag{6.17}$$

Otherwise, for a fermionic plethystic exponential, we have:

$$PE[g(t)]_F = \exp\left(\sum_{n=0}^{\infty}a_n\sum_{k=1}^{\infty}\frac{(-1)^{k+1}t^{nk}}{k}\right) = \exp\left(\sum_{n=1}^{\infty}a_n\log\left(1+t^n\right)\right) = \prod_{n=1}^{\infty}(1+t^n)^{a_n} \tag{6.18}$$

Upon expansion of $PE[f(t)]$, we would see that the coefficient fot t^m is the number of ways of partitioning m, each weighted by a_n.

We recall that the perturbative Type II spectrum is made out of a tower of eight bosonic and eight fermionic oscillators, that transform as vector and spinor under $SO(8)$ representaion; the tower of bosonic string states is formed by symmetrization of the bosonic oscillators and the plethystic exponential precisely keeps track of the symmetrization procedure:

$$Z_B(q;z_1,z_2,z_3,z_4) = PE\left[\frac{q}{1-q}[1,0,0,0]_8\right] \tag{6.19}$$

Where the eight bosonic oscillators are described by $[1,0,0,0]_8$, while the prefactor $\frac{q}{1-q}$ is necessary to obtain the correct generating function in the blind limit, indeed:

$$Z_B(q;1,1,1,1) = PE\left[\frac{8q}{1-q}\right] = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^8} \tag{6.20}$$

For the fermionic partition function, the antisymmetrization can be treated by the fermion plethystic exponential; in addition, to implement GSO projection, we also wish to keep track of the fermion number of all states, for which we introduce an extra fugacity:

$$Z_F(q;f;z_1,z_2,z_3,z_4) = PE_F\left[\frac{f}{1-q}[1,0,0,0]_8\right] \tag{6.21}$$

Again, sending to one all the fugacities, we find the infinite numerator of (6.8) (supplemented with the fermion number fugacity)

$$Z_F(q;f;1,1,1,1) = PE_F\left[\frac{8f}{1-q}\right] = \prod_{n=0}^{\infty}(1+fq^n)^8 \tag{6.22}$$

We define the GSO projected partition function in the Neveu-Schwartz

$$Z_{NS} = \frac{1}{2\sqrt{2}}(Z_F(\sqrt{q}) - Z_F(-\sqrt{q})) \tag{6.23}$$

and in the Ramond sector

$$Z_{\pm} = \frac{1}{2}(Z_F(q) \pm Z_F(-q)) \tag{6.24}$$

We can write a single partition function which collects the boundary conditions, the GSO projection and the fugacities associated to the vacua

$$Z_{RNS}(q;z_1,z_2,z_3,z_4) = Z_{NS} + [0,0,0,1]_8 Z_+ + [0,0,1,0]_8 Z_- \tag{6.25}$$

In this way, the open string, or left moving partition function, takes the form

$$Z_{left}(q;z_1,z_2,z_3,z_4) = Z_B(Z_{NS} + [0,0,0,1]_8 Z_+ + [0,0,1,0]_8 Z_-) \tag{6.26}$$

this is the refined partition function as a function of $SO(8)$ fugacities. We want to rewrite it in terms of $SO(9)$ characters; for this reason, we explicitly write some character function of $SO(8)$ and $SO(9)$

$$\begin{aligned}
[1,0,0,0]_8 &= z_1 + \frac{z_2}{z_1} + \frac{z_3 z_4}{z_2} + \frac{z_4}{z_3} + \frac{z_3}{z_4} + \frac{z_2}{z_3 z_4} + \frac{z_1}{z_2} + \frac{1}{z_1} \\
[0,0,1,0]_8 &= z_3 + \frac{z_2}{z_3} + \frac{z_4 z_1}{z_2} + \frac{z_1}{z_4} + \frac{z_4}{z_1} + \frac{z_2}{z_4 z_1} + \frac{z_3}{z_2} + \frac{1}{z_3} \\
[0,0,0,1]_8 &= z_4 + \frac{z_2}{z_4} + \frac{z_1 z_3}{z_2} + \frac{z_3}{z_1} + \frac{z_1}{z_3} + \frac{z_2}{z_1 z_3} + \frac{z_4}{z_2} + \frac{1}{z_4} \\
[1,0,0,0]_9 &= y_1 + \frac{y_2}{y_1} + \frac{y_3}{y_2} + \frac{y_4^2}{y_3} + \frac{y_3}{y_4^2} + \frac{y_2}{y_3} + \frac{y_1}{y_2} + \frac{1}{y_1} + 1 \\
[0,0,0,1]_9 &= \frac{y_3}{y_4} + \frac{y_2 y_4}{y_3} + \frac{y_4 y_1}{y_2} + \frac{y_1}{y_4} + \frac{y_4}{y_1} + \frac{y_2}{y_4 y_1} + \frac{y_3}{y_2 y_4} + \\
&+ \frac{y_4}{y_3} + y_4 + \frac{y_2}{y_4} + \frac{y_1 y_3}{y_4 y_2} + \frac{y_3}{y_1 y_4} + \frac{y_1 y_4}{y_3} + \frac{y_4 y_2}{y_1 y_3} + \frac{y_4}{y_2} + \frac{1}{y_4}
\end{aligned} \tag{6.27}$$

In order to understand the lifting of $SO(8) \to SO(9)$, we can compare the fundamental vector representation of $SO(8)$ and $SO(9)$; indeed, knowing that

$$[1,0,0,0]_9 = [1,0,0,0]_8 + 1 \tag{6.28}$$

we can relate the fugacities of the two groups as:

$$y_1 = z_1 \quad y_2 = z_2 \quad y_3 = z_3 z_4 \quad y_4 = z_4 \tag{6.29}$$

Generically, the reconstruction of $SO(9)$ representations from their $SO(8)$ reductions is ambiguous but, for low dimensional representations, it is unique; in this way, for the vector and spinors representations of $SO(8)$ we can lift uniquely to:

$$[1,0,0,0]_8 = [1,0,0,0]_9 - 1 \tag{6.30}$$

$$[0,0,1,0]_8 = \frac{y_3}{y_4} + \frac{y_2 y_4}{y_3} + \frac{y_4 y_1}{y_2} + \frac{y_1}{y_4} + \frac{y_4}{y_1} + \frac{y_2}{y_4 y_1} + \frac{y_3}{y_2 y_4} + \frac{y_4}{y_3} \tag{6.31}$$

$$[0,0,0,1]_8 = y_4 + \frac{y_2}{y_4} + \frac{y_1 y_3}{y_4 y_2} + \frac{y_3}{y_1 y_4} + \frac{y_1 y_4}{y_3} + \frac{y_4 y_2}{y_1 y_3} + \frac{y_4}{y_2} + \frac{1}{y_4} \tag{6.32}$$

From the lasts expressions, we can write the left moving partition function in term of $SO(9)$ fugacities, obtaining

$$Z_{left}(q,z_1,z_2,z_3,z_4) = \frac{1}{2}([0,0,0,1]_8 - [0,0,1,0]_8) + Z_B\left(Z_{NS} + \frac{1}{2}[0,0,0,1]_9 Z_F(q)\right) \tag{6.33}$$

The first term in (6.33) has no q dependence and contributes only at the massless level; in the last part, the $\frac{1}{2}$ is necessary because the arguments of Z_B and $Z_F(q)$ are equal; then, the contribution to the plethystic exponential gets a factor of 2 from odd powers of the argument and it is removed by the factor $\frac{1}{2}$.

In order to obtain an expression for the massive spectrum, with factored massive supermultiplet, we define:

$$Z_Q = [2,0,0,0]_9 + [1,0,0,1]_9 + [0,0,1,0]_9 \tag{6.34}$$

which is the character of the first massive state and

$$Z_0 = [1,0,0,0]_8 + [0,0,0,1]_8 \tag{6.35}$$

which correspond to the massless state; in this way, the massive spectrum generating function is given by:

$$Z_m(q,y_i,y_2,y_3,y_4) = \frac{Z_{left} - Z_0}{q Z_Q} \tag{6.36}$$

Using this formula it is possible to write the character contribution at each massive level; then, using an Erastotene's sieve agorithm [16], it is possible to obtain the specific states theorically at each strin level (with a limit imposed by numerical computation time). These results are written in [6] and we have explicitly verified them using (6.36) and a Mathematica code up to order 8 in mass level (which is order 9 in string level). In addition, we see that these states coincide with the states written in appendix of [16], where they were evaluated explicitly in $SO(8)$ representations and then reorganized in the $SO(9)$ group.

In this way, a general state is written as:

$$T_l^{open} = T_1^{open} \otimes \mathrm{vac}_l \tag{6.37}$$

where T_1^{open} is the first level open massive state and l is the string state, we have found that:

$$
\begin{aligned}
\mathrm{vac}_1 =&[0,0,0,0] = \mathbf{1}\\
\mathrm{vac}_2 =&[1,0,0,0] = \mathbf{9}\\
\mathrm{vac}_3 =&[2,0,0,0]+[0,0,0,1] = \mathbf{44}+\mathbf{16}\\
\mathrm{vac}_4 =&[3,0,0,0]+[1,0,0,1]+[1,0,0,0]+[0,1,0,0] = \mathbf{156}+\mathbf{128}+\mathbf{9}+\mathbf{36}\\
\mathrm{vac}_5 =&[4,0,0,0]+[2,0,0,1]+[2,0,0,0]+[1,1,0,0]+[1,0,0,1]+[0,1,0,0]+\\
&+[0,0,1,0]+[0,0,0,1]+[0,0,0,0] =\\
&= \mathbf{450}+\mathbf{576}+\mathbf{44}+\mathbf{231}+\mathbf{128}++\mathbf{36}+\mathbf{84}+\mathbf{16}+\mathbf{1}\\
\mathrm{vac}_6 =&[5,0,0,0]+[3,0,0,1]+[3,0,0,0]+[2,1,0,0]+[2,0,0,1]+[2,0,0,0]+\\
&+2[1,1,0,0]+[1,0,1,0]+2[1,0,0,1]+2[1,0,0,0]+[0,1,0,1]+\\
&+2[0,1,0,0]+[0,0,0,2]+2[0,0,0,1] =\\
&= \mathbf{1122}+\mathbf{1920}+\mathbf{156}+\mathbf{910}+\mathbf{576}+\mathbf{44}+2\times\mathbf{231}+\mathbf{594}+2\times\mathbf{128}+\\
&+2\times\mathbf{9}+\mathbf{432}+2\mathbf{36}+\mathbf{126}+2\mathbf{16}\\
\mathrm{vac}_7 =&[6,0,0,0]+[4,0,0,1]+[4,0,0,0]+[3,1,0,0]+[3,0,0,1]+[3,0,0,0]+\\
&+2[2,1,0,0]+[2,0,1,0]+3[2,0,0,1]+3[2,0,0,0]+[1,1,0,1]+2[1,1,0,0]+\\
&+[1,0,1,0]+[1,0,0,2]+4[1,0,0,1]+2[1,0,0,0]+[0,2,0,0]+2[0,1,0,1]\\
&+2[0,1,0,0]+3[0,0,1,0]+[0,0,0,2]+2[0,0,0,1]+2[0,0,0,0] =\\
&= \mathbf{2508}+\mathbf{5280}+\mathbf{450}+\mathbf{2772}+\mathbf{1920}+\mathbf{156}+2\times\mathbf{910}+\mathbf{2457}+3\times\mathbf{576}+\\
&+3\times\mathbf{44}+\mathbf{2560}+2\times\mathbf{231}+\mathbf{924}+4\times\mathbf{128}+2\times\mathbf{9}+\mathbf{495}+2\times\mathbf{432}+\\
&+2\times\mathbf{36}+3\times\mathbf{84}+\mathbf{126}+2\times\mathbf{16}+\mathbf{21}
\end{aligned}
\tag{6.38}
$$

We note that, at least until the order considered, any vacuum state contains a specific representation:

$$
[l-1,0,0,0] \subset \mathrm{vac}_l \tag{6.39}
$$

6.2 Closed IIB Superstring Spectrum in Flat Space

In order to extract the spectrum of the closed IIB superstring theory, we need to tensor two open string spectrum states at the same chirality; However this differernce can be seen only in the massless state which is chiral and belong to $SO(8)$. The massive states, being non-chiral, make no distinction between IIA and IIB theory.

In representation of $SO(8)$, the ground state for the IIB theory, can be written as:

$$(\mathbf{8}_v + \mathbf{8}_s) \otimes (\mathbf{8}_v + \mathbf{8}_s) \tag{6.40}$$

but knowing that a general matrix decomposes in a symmetric part, antisymmetric part and a trace

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} + \mathbf{28} + \mathbf{35}_v \tag{6.41}$$

For the tensor product of the spinor and vector representation, we have:

$$\mathbf{8}_v \otimes \mathbf{8}_s = \mathbf{8}_s + \mathbf{56}_s \tag{6.42}$$

while for the product of the two spinor representation

$$\mathbf{8}_s \otimes \mathbf{8}_s = \mathbf{1} + \mathbf{28} + \mathbf{35}_- \tag{6.43}$$

where $\mathbf{56}_s$ is a third rank antisymmetric tensor and $\mathbf{35}_-$ is the anti-self-dual part of a fourth-rank antisymmetric tensor.

We obtain in the end

$$(\mathbf{8}_v + \mathbf{8}_s) \otimes (\mathbf{8}_v + \mathbf{8}_s) = 2(\mathbf{1}) + 2(\mathbf{8}_s) + 2(\mathbf{28}) + \mathbf{35}_v + \mathbf{35}_- + 2(\mathbf{56}_s) \tag{6.44}$$

which differs from the IIA type spectrum.

The massive states are given by the tensor product of the correspoding open string state with themeselves; we have derived the tensor product using the software Sagemath [1] and we have verified the obtained using character theory; these are some results:

$$\begin{aligned} T_1 \otimes T_1 =& ([2,0,0,0] + [1,0,0,1] + [0,0,1,0])^2 = 3[0,0,0,0] + 4[0,0,0,1] + [1,0,0,0] + \\ &+ 4[0,1,0,0] + 5[0,0,1,0] + 3[0,0,0,2] + 6[1,0,0,1] + 6[0,1,0,1] + 2[0,0,1,1] + \\ &+ 2[0,0,0,3] + 3[2,0,0,0] + 4[1,1,0,0] + 3[1,0,1,0] + 5[1,0,0,2] + 3[0,2,0,0] + \\ &+ [0,1,1,0] + 2[0,1,0,2] + [0,0,2,0] + 4[2,0,0,1] + 4[1,1,0,1] + 2[1,0,1,1] + \\ &+ [3,0,0,0] + 2[2,1,0,0] + 3[2,0,1,0] + [2,0,0,2] + +2[3,0,0,1] + [4,0,0,0] \end{aligned} \tag{6.45}$$

$$\begin{aligned} \mathrm{vac}_1^2 =& [0,0,0,0] \\ \mathrm{vac}_2^2 =& [0,0,0,0] + [0,1,0,0] + [2,0,0,0] \\ \mathrm{vac}_3^2 =& 2[0,0,0,0] + [1,0,0,0] + 2[0,1,0,0] + [0,0,1,0] + [0,0,0,2] + 2[1,0,0,1] + \\ &+ [2,0,0,0] + [0,2,0,0] + 2[2,0,0,1] + [2,1,0,0] + [4,0,0,0] \\ &\dots \end{aligned}$$

6.3 IIB Superstring Spectrum in Curved Space

We have evaluated the superstring spectrum in curved space, breaking the representations from $SO(9)$ to $SO(5) \times SO(4)$; we will see that this breaking is propedeutic to express the string states on the space $AdS_5 \times S^5$.

We express the results for the open states, because we will start with these states in the subsequent analysis:

$$\begin{aligned} T_1^{open} = [2,0,0,0] + [1,0,0,1] + [0,0,1,0] \rightarrow & [2,2]_{[00]} + [2,0]_{[0,0]} + [0,2]_{[0,0]} + 2[0,0]_{[0,0]} \\ & + 2[1,1]_{[1,0]} + [0,0]_{[2,0]} + [0,0]_{[0,2]} \end{aligned} \tag{6.46}$$

$$\mathrm{vac}_2 = [1,0,0,0] \rightarrow [1,0]_{[0,0]} + [0,0]_{[1,1]} \tag{6.47}$$

$$\begin{aligned} \mathrm{vac}_3 = [2,0,0,0] + [0,0,1,1] \rightarrow & [0,0]_{[0,0]} + [0,1]_{[1,0]} + [0,1]_{[0,1]} + [1,0]_{[1,1]} + \\ & [2,0]_{[0,0]} + [0,0]_{[2,2]} \end{aligned} \tag{6.48}$$

$$\begin{aligned} \mathrm{vac}_4 = [3,0,0,0] + [1,0,0,1] + [1,0,0,0] + [0,1,0,0] \rightarrow & [0,1]_{[1,0]} + [0,1]_{[0,1]} + [0,1]_{[2,1]} + [0,1]_{[1,2]} + \\ & 2[1,0]_{[0,0]} + [0,2]_{[0,0]} + [1,0]_{[1,1]} + [1,0]_{[2,2]} \\ & [1,1]_{[1,0]} + [1,1]_{[0,1]} + [2,0]_{[1,1]} + [3,0]_{[0,0]} \\ & 2[0,0]_{[1,1]} + [0,0]_{[2,0]} + [0,0]_{[2,2]} + [0,0]_{[3,3]} \end{aligned} \tag{6.49}$$

...

Where on the r.h.s. the represantions are written in terms of $SO(5)_{SO(4)}$.

Chapter 7

Kaluza Klein Expansion

7.1 Kaluza Klein Theory

In theories of the Kaluza-Klein type [23], one starts with the hypothesis that spacetime has $d+K$ dimensions. Next, one supposes that because of some dynamical mechanism, the ground state of the system is partially compactified $M^d \times B^K$, where B^K is a compact K-dimensional space; in certain theories, where M^d is the 4-dimensional Minkowsky space, we require that the size of B^K must be sufficiently small to render it unresolvable at the currently available energies.

We assume that the compact space B^K can be written as a coset space G/H; one way to obtain the M^d theory is to expand all the fields variables in complete set of harmonics on the homogeneous space G/H.

To distinguish the compact dimension from the remaining ordinary spacetime, we can adopt a notational convention:

$$z^M = (x^m, y^\mu) \tag{7.1}$$

where m has values in $0, 1, \ldots, d$, while μ on $1, \ldots, K$.

Suppose that the coordinates y^μ label the cosets of G with respect to the subgroup H. That is, from each coset, let there be chosen a representative element L_y; multiplication from the left by an arbitrary element $g \in G$ will generally carry L_y into another coset, one for which the reresentative element is $L_{y'}$; this defines the so-called left translation

$$gL_y = L_{y'}h \tag{7.2}$$

where h is an element of H. Both $y^{'\mu}$ and h are determined by this equation as function of y^μ ang g.

To define a covariant basis we can consider the 1-form:

$$e(y) = L_y^{-1} dL_y \tag{7.3}$$

which belongs to the infinitesimal algebra of G and therefore can be expressed as a linear combination of the generators $Q_{\dot{\alpha}}$

$$e(y) = e^{\dot{\alpha}}(y) Q_{\dot{\alpha}} = dy^\mu e_\mu^{\dot{\alpha}}(y) Q_{\dot{\alpha}} \tag{7.4}$$

The generators $Q_{\dot{\alpha}}$ fall in two categories, the set $Q_{\bar{\alpha}}$ which generates the subgroup H and the remainder $Q_\alpha, \alpha = 1, 2, \ldots, K$ associated with the cosets G/H; correspondingly, one writes

$$e(y) = e^\alpha(y) Q_\alpha + e^{\bar{\alpha}}(y) Q_{\bar{\alpha}} \tag{7.5}$$

The behaviour of $e(y)$ under left transitions is given by:

$$e(y) \to e(y') = hL_y^{-1}g^{-1}d(gL_yh^{-1}) = he(y)h^{-1} + hdh^{-1} + hL_y^{-1}g^{-1}dgL_yh^{-1} \tag{7.6}$$

We introduce the matrices $D_{\dot{\alpha}}^{\ \dot{\beta}}$ of the adjoint representation of G, defined by:

$$g^{-1}Q_{\dot{\alpha}}g = D_{\dot{\alpha}}^{\ \dot{\beta}}(g)Q_{\dot{\beta}} \tag{7.7}$$

In addition, we see that

$$g^{-1}dg = (g^{-1}dg)^{\dot{\alpha}}Q_{\dot{\alpha}} \tag{7.8}$$
$$hdh^{-1} = (hdh^{-1})^{\dot{\alpha}}Q_{\dot{\alpha}} = (hdh^{-1})^{\bar{\alpha}}Q_{\bar{\alpha}} \tag{7.9}$$

since hdh^{-1} belongs to the algebra of H, we must have $(hdh^{-1})^{\bar{\alpha}}Q_{\bar{\alpha}}$.

From the previous equations, we see that (7.6) becomes:

$$e^{\dot{\alpha}}(y') = e^{\dot{\beta}}(y)D_{\dot{\beta}}^{\ \dot{\alpha}}(h^{-1} + (hdh^{-1})^{\dot{\alpha}} + (g^{-1}dg)^{\dot{\beta}}D_{\dot{\beta}}^{\ \dot{\alpha}}(L_yh^{-1} \tag{7.10}$$

Now consider the question of harmonic expansions on the internal space; to begin with, if the internal space were simply the group G itself (i.e. $H = 1$) it would be natural to employ the full set of matrices of the unitary irreducible representations of G. For the function $\phi(g)$, one would write:

$$\phi(g) = \sum_n \sum_{p,q} \sqrt{d^n} D^n_{pq}(g)\phi^n_{qp} \tag{7.11}$$

where D^n_{pq} is a unitary matrix of dimension d_n and the sum includes all matrix elements of all the unitary irreducible representations $g \to D^n(g)$.

For functions on the coset space G/H, the expansion is somewhat restricted; typically, one is concerned with functions $\phi_i(g)$ that are subjected to the auxiliary symmetry

$$\phi_i(hg) = \mathbb{D}_{ij}(h)\phi_j(g) \tag{7.12}$$

where $h \in H$ and $\mathbb{D}(h)$ is some particular representation of H. This means that $\phi(g)$ is related by a linear rule

$$\phi_i(g_1) = \mathbb{D}_{ij}(g_1g_2^{-1})\phi_j(g_2) \tag{7.13}$$

for g_1 and g_2 in the same coset; than, the appropriate restriction on the expansion (7.11) is given by those terms for which

$$\mathbb{D}^n(hg) = \mathbb{D}(h)D^n(g) \tag{7.14}$$

For irreducible $\mathbb{D}(h)$ one should write

$$\phi_i(g) \sum_n \sum_{l,q} \sqrt{\frac{d_n}{d_{\mathbb{D}}}} D^n_{il,q}(g)\phi^n_{ql} \tag{7.15}$$

where the sum includes all unitary irreducible representations og G for which $D^n_{il,q}$ make sense, i.e., all those which include $\mathbb{D}(h)$ on a restriction to H. The dimension of $\mathbb{D}(h)$ is denoted $d_{\mathbb{D}}$.

7.2 Scalar and Vector Spherical Harmonics for the compactification of IIB Supergravity on S^5

7.2.1 Scalar Spherical Harmonics

In the Kaluza-Klein reduction [19], [11], a scalar field $B(x,z)$ is expanded in terms of harmonics on a compact space:

$$B(x,z) = \sum_{I_1} B_{I_1}(x) Y^{I_1}(z) \tag{7.16}$$

The $Y^{I_1}(z)$ are eigenfunctions of the Laplacian on S^n ($\Box_S$) , $B_{I_1}(x)$ are fields in d-dimensional (usually anti de Sitter) space with coordinates x. The symbol I_1 indicates that these fields have one component on S^n.

What are the $Y^{I_1}(z)$ in (7.16)? We begin with a homogenous polynomial in $\bar{x}^\mu$, where $\bar{x}^\mu(\mu = 1, ..., n+1)$, are the Cartesian coordiantes of the flat embedding space $\mathbb{R}^{n+1}$. We can formally write these polynomials as:

$$\bar{P}^{(l)}(\bar{x}) = c_{\mu_1,...,\mu_l} \bar{x}^{\mu_1} ... \bar{x}^{\mu_l} \tag{7.17}$$

We require that $\bar{\Box}\bar{P} = 0$, where $\bar{\Box} = \frac{\partial}{\partial \bar{x}^\mu}\frac{\partial}{\partial \bar{x}^\nu}\delta^{\mu\nu}$, this implies that c is traceless: $\delta^{\mu\nu} c_{\mu\nu\mu_3...\mu_l} = 0$. Going to polar coordinates $\hat{y} = (r, \theta^\alpha)$, where r is the radius of S^n and θ^α are any angles on S^n, the metric is of the form:

$$\hat{g}_{\mu\nu}(\hat{y}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 g_{\alpha\beta}(\theta) \end{pmatrix} \tag{7.18}$$

The Laplacian becomes:

$$\bar{\Box} = \hat{\Box} = \frac{1}{\sqrt{\hat{g}}} \frac{\partial}{\partial \hat{y}^\mu} \sqrt{\hat{g}} \hat{g}^{\mu\nu} \frac{\partial}{\partial \hat{y}^\nu} = \frac{1}{r^n} \partial_r r^n \partial_r + \frac{1}{r^2} \Box_S(\theta) \tag{7.19}$$

Since $\hat{P}^{(l)}(\bar{x})$ is homogeneous in $\bar{x}$, it factorizes as:

$$\bar{P}^{(l)}(\bar{x}) = \hat{P}^{(l)}(\hat{y}) = r^l Y(\theta) \tag{7.20}$$

substituting into $\bar{\Box}\bar{P}^{(l)} = \hat{\Box}\hat{P}^{(l)} 0$ we have:

$$\left[\frac{1}{r^2} l(l+n-1) + \frac{1}{r^2} \Box_S(\theta) \right] r^l Y(\theta) = 0 \tag{7.21}$$

Hence the eigenvalues λ_s of $-\Box_S(\theta)$ on S^n are given by

$$\lambda_s(n,l) = l(n+l-1); l = 0, 1, 2. \tag{7.22}$$

The degeneracy $d_s(n,l)$ of these eigenvalues is given by the number of polynomials $\bar{P}^{(l)}(\bar{x})$, that is equal to the number of symmetric polynomials of order l in $n+1$ dimensions, minus the number of such polynomials of order $l-2$ (because of the trace); the number of symmetric polynomials of order l, in $n+1$ dimensions, is equal to the number of symmetric tensor with l index in $n+1$ dimensions; hence we find:

$$d_s(n,l) = \binom{n+l}{l} - \binom{n+l-2}{l-2} \tag{7.23}$$

On S^5, the case of interest, we find the following spectrum:

l	0	1	2	3	4	5	6	7	8	9
d	1	6	20	50	105	196	336	540	825	1210

7.2.2 Vector spherical harmonics

We consider transversal vector spherical harmonics $Y_\alpha(\theta)$ on S^n, i.e. $D^\alpha(\theta)T_\alpha(\theta) = 0$, where D^α is the covariant derivative on S^n; those are obtained from vector fields $\bar{P}^{(l)}_\mu(\bar{x})$ whose components $\mu = 1, ..., n+1$ are homogeneous polynomials in $\bar{x}^\mu$. Now the costraint $\bar{\Box}\bar{P}^{(l)}_\mu(\bar{x}) = 0$ implies that each component is traceless. We choose a basis in which only the last component of $\bar{P}^{(l)}_\mu$ is nonzero. We go to polar coordinates $(\hat{y})^\nu = (r, \theta^\alpha)$, with $\alpha = 1, ..., n$:

$$\begin{aligned} \bar{\Box}\bar{P}^{(l)}_\mu = \frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu}\hat{\Box}\hat{P}^{(l)}_\nu = 0 \Rightarrow \hat{\Box}\hat{P}^{(l)}_\nu = \frac{\partial \bar{x}^\mu}{\partial \hat{y}^\nu}\bar{\Box}\bar{P}^{(l)}_\mu = 0 \\ \Rightarrow \hat{\Box}\hat{P}^{(l)}_r = 0, \quad \hat{\Box}\hat{P}^{(l)}_\alpha = 0 \end{aligned} \tag{7.24}$$

where:

$$\begin{aligned} \hat{P}^{(l)}_r(\hat{y}) = \frac{\partial \bar{x}^\mu}{\partial r}\bar{P}^{(l)}_\mu = \frac{\bar{x}^\mu}{r}\bar{P}^{(l)}_\mu(\bar{x}) = r^l \rho(\theta) \\ \hat{P}^{(l)}_\alpha(\hat{y}) = \frac{\partial \bar{x}^\mu}{\partial \theta^\alpha}\bar{P}^{(l)}_\mu = r^{l+1}V_\alpha(\theta) \end{aligned} \tag{7.25}$$

where we have defined $\rho(\theta)$ and $V_\alpha(\theta)$. Here $\hat{\Box}$ is the laplacian in $n+1$ dimension in polar coordinates.

The degeneracy in this case is given by

$$d_v(n,l) = (n+1)d_s(n,l) - d_s(n,l+1) - d_s(n,l-1) \tag{7.26}$$

In the case of interest, with $n = 5$, we have:

l	0	1	2	3	4	5	6	7	8	9
d	1	15	64	175	384	735	1280	2079	3200	4719

In addition, one can form vectors on S^n by considering the gradient of scalars; such vectors, has the same degeneracies of scalars [5].

7.3 Kaluza Klein Expansion of a General $SO(5)$ Representation

In this section we will demonstrate one of the main result of this thesis, the Kaluza Klein Expansion of a general $SO(5)$ representation. In order to do the evaluation, we will evaluate the branching rule for $SO(6) \to SO(5)$ decomposition; then, applying the decomposition to a general state, we will obtain the formula for the Kaluza Klein expansion; this will be a peculiar result, because different from the equation written on papers [16] [20],[15];then, we will motivate this result, showing the reasons for which the paper's equation is wrong and our equation is right.

7.3.1 Branching Rule for $SO(6) \to SO(5)$

We propose an ansatz for the branching rule of $SO(6) \to SO(5)$ in terms of Dynkin labels:

$$[p, j, l] \xrightarrow[SO(6)\to SO(5)]{} \sum_{m=0}^{p} \sum_{k=0}^{\min(j,l)} [m + k, j - 2k + l] \tag{7.27}$$

this ansatz was obtained observing the decomposition to the sub-algebra at low Dynkin labels representation; however, we still have to demonstrate this formula.

A sanity check is given by the comparison of the dimensions on the two side of this branching; indeed, using the dimension formulae for $SO(5)$ and $SO(6)$ representations, as given in (1.75) and (1.78), we can check, using a software [26], that:

$$d_{SO(6)}[p, j, l] = \sum_{m=0}^{p} \sum_{k=0}^{\min(j,l)} d_{SO(5)}[m + k, j - 2k + l] \tag{7.28}$$

The sanity check is confirmed; however, it is not enough to demonstrate the validity of formula (7.27), because, especially at high dimension, there are ambiguities in the identification of representations from dimension; hence, we have to analyze the two side of (7.27) using character formula of these groups; however, in order to have comparable equations, we have to find a relation between the fugacities of the two groups; for this reason, we compare the vector representations of the two character group in order to find a relation between the fugacities.

Identifying as y_1, y_2, y_3 the fugacities of $SO(6)$ and x_1, x_2 the fugacities of $SO(5)$, we find, using our derived character formulae:

$$\chi_{SO(6)}[1, 0, 0] = y_1 + \frac{y_1}{y_2 y_3} + \frac{y_2}{y_3} + \frac{y_3}{y_2} + \frac{1 + y_2 y_3}{y_1} \tag{7.29}$$

$$\chi_{SO(5)}[1, 0, 0] = 1 + \frac{1}{x_1} + x_1 + \frac{x_1}{x_2^2} + \frac{x_2^2}{x_1} \tag{7.30}$$

Now requiring that

$$\chi_{SO(6)}[1, 0, 0] = \mathbf{1} + \chi_{SO(5)}[1, 0, 0] \tag{7.31}$$

we find the relations between fugacities as:

$$x_2 = y_2 = y_3 \qquad x_1 = y_1 \tag{7.32}$$

we also note that the branching imposes a constraints on the $SO(6)$ character's variables; this is natural, because the rank of the group decreases by one in this branching.

Using (7.32) and Mathematica for the computations, we can verify that:

$$\chi_{SO(6)}[p, j, l](x_1, x_2, x_2) = \sum_{m=0}^{p} \sum_{k=0}^{\min(j,l)} \chi_{SO(5)}[m + k, j - 2k + l](x_1, x_2) \tag{7.33}$$

This result fully demonstrate (7.27).

7.3.2 Kaluza Klein Expansion Formula

We now want to obtain the formula for the Kaluza Klein expansion of a $SO(5)$ representation, using Kaluza Klein theory as previously described.

We start considering the most general representation of $SO(6)$:

$$[a, b, c] \quad \text{with } a, b, c \in \mathbb{N} \tag{7.34}$$

Now we ask ourselves, what is the most general representation which gives the state $[m, n]$ in the decomposition to the subalgebra $SO(5)$? To answer, we parametrize (7.34) with two new variables:

$$\begin{cases} s = \frac{n+c-b}{2} \\ k = \frac{c-n+b}{2} \end{cases} \quad \text{if } n + c - b \text{ is even} \tag{7.35}$$

$$\begin{cases} s = \frac{n+c-b+1}{2} \\ k = \frac{c-n+b-1}{2} \end{cases} \quad \text{if } n + c - b \text{ is odd} \tag{7.36}$$

In this way, we mantain the original number of degrees of freedom (no restriction are imposed on our general state). In the case (7.35), (7.34) becomes

$$[a, n + k - s, k + s] \ \text{ with } a \in \mathbb{N}, s, k \in \mathbb{Z} \tag{7.37}$$

Now we apply (7.27) to obtain the decomposition on $SO(5)$:

$$\sum_{t=0}^{a} \sum_{l=0}^{\min(k+s,k+n-s)} [t + l, n + 2k - 2l] \tag{7.38}$$

and we require that the state $[m, n]$ is contained, which implies

$$t = m - l \qquad k = l \tag{7.39}$$

but the summation implies that this variables must be contained inside some range of values:

$$\begin{aligned} 0 \le t \le a \quad &\Rightarrow \quad \boxed{a \ge m - k} \ , \ \boxed{k \le m} \\ 0 \le l \le \min(k + s, k + n - s) \quad &\Rightarrow \quad \boxed{k \ge 0} \ , \ \boxed{s \ge 0} \ , \ \boxed{s \le n} \end{aligned} \tag{7.40}$$

Hence, applying the restrictions (7.40) on (7.34), the most general state in the Kaluza-Klein, can be expressed as:

$$KK_{[m,n]} = \sum_{k=0}^{m} \sum_{s=0}^{n} \sum_{a=m-k}^{\infty} [a, n + k - s, k + s] \tag{7.41}$$

Now, for what concerns the states generated by (7.36), we would obtain the general state:

$$[a, n + k - s + 1, k + s] \tag{7.42}$$

but applying (7.27) we obtain:

$$\sum_{t=0}^{a} \sum_{l=0}^{\min(k+s,k+n-s)} [t + l, n + 2k - 2l + 1] \tag{7.43}$$

we should satisfy $2k - 2l + 1 = 0$ which is impossible, because k, l are integers, hence those states do not contribute to Kaluza-Klein states.

In this way, we have shown that the Kaluza Klein expansion is given by:

$$\boxed{KK_{[m,n]} = \sum_{r=0}^{m} \sum_{s=0}^{n} \sum_{p=m-r}^{\infty} [p, r + n - s, r + s]} \tag{7.44}$$

This is a peculiar result because it is in contrast with the formula written on the cited papers; in addition to the given proof, we will show in the last subsection other proofs of the inconsistency of the paper's formula.

7.3.3 Comparison of Kaluza Klein Formula with Spherical and Vector Harmonics

We can make a comparison between formula (7.44) and the states obtained from scalars and vector spherical harmonics; using Kaluza Klein formula, the scalar spherical harmonics are given by

$$KK_{[0,0]} = \sum_{p=0}^{\infty} [p, 0, 0] \tag{7.45}$$

Using the formula of the dimension, we see that the dimesion of the generated representaions is

$$d\left(KK_{[0,0]}\right) = 1 + 6 + 20 + 50 + 105 + 196 + 336 + 540 + 825 + 1210 + \dots \tag{7.46}$$

In perfect agreement with the table of section 7.2.1. On the other hand, we can compare with the transversal vector harmonics of section 7.2.2; the degeneracy of transversal vector harmonics must be equal to the dimension of $KK_{[1,0]} - KK_{[0,0]}$; we find:

$$\begin{aligned} d\left(KK_{[1,0]} - KK_{[0,0]}\right) = 1 + \sum_{p=0}^{\infty} d([p, 1, 1]) = \\ = 1 + 15 + 64 + 175 + 384 + 735 + 1280 + 2079 + 3200 + 4719 \end{aligned} \tag{7.47}$$

7.4 Another Proof about Inconsistency of Kaluza Klein Paper's Formula

We want to check the validity of formula 3.6 of [1], for some specific representations, which states:

$$KK_{\mathcal{R}_{SO(5)}} = \sum_{n=0}^{\infty}[0,n,0] \times \hat{\mathcal{R}}_{SO(5)} \tag{7.48}$$

where the $\mathcal{R}_{SO(5)}$ is a (in general reducible) representation of $SO(5)$ which can be lifted to a representation $\hat{\mathcal{R}}_{SO(5)}$ of $SO(6)$, while the r.h.s. describes a tensor product in $SU(4)$.
Moreover, assuming the validity of this formula, we want to check which is the correct expression for the Kaluza Klein formula, between expression 3.4 of [1], which contains two different sums, and the expression that we derived in the previous section, which contains only the first sum of 3.4; we explictly write both of the formulae:

$$KK_{[m,n]} = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=m-r}^{\infty}[r+s,p,r+n-s] + \sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\sum_{p=m-r-1}^{\infty}[r+s+1,p,r+n-s] \tag{7.49}$$

$$KK_{[m,n]} = \sum_{r=0}^{m}\sum_{s=0}^{n}\sum_{p=m-r}^{\infty}[r+s,p,r+n-s] \tag{7.50}$$

Now and in the rest of this section, we will use Dynkin labels of $SU(4)$.

Analysis of representation [1,1,1]

We assume the easiest examples described in formula 3.7 of [1] to be true; we require a bigger representation in order to have different results from formula (7.49) and (7.50).
The easiest suitable representation is $[1,1,1]$ of $SU(4)$; the reducible representation of $SO(5)$, which is lifted to this representation of $SO(6)$, is $[1,0]+[0,2]+[2,0]+[1,2]$; the term $[1,2]$, will produce, as required, two different results as Kaluza Klein towers.
The Kaluza Klein towers are:

$$KK_{[1,0]} = \sum_{p=1}^{\infty}[0,p,0] + \sum_{p=0}^{\infty}[1,p,1] \tag{7.51}$$

$$KK_{[0,2]} = \sum_{p=0}^{\infty}([0,p,2]+[1,p,1]+[2,p,0]) \tag{7.52}$$

$$KK_{[2,0]} = \sum_{p=2}^{\infty}[0,p,0] + \sum_{p=1}^{\infty}[1,p,1] + \sum_{p=0}^{\infty}[2,p,2] \tag{7.53}$$

$$KK_{[1,2]} = \sum_{p=1}^{\infty}([0,p,2]+[1,p,1]+[2,p,0]) +$$

$$+\sum_{p=0}^{\infty}([1,p,3]+[2,p,2]+[3,p,1]+[1,p,2]+[2,p,1]) \tag{7.54}$$

Where we have written in red the ambigous terms, i.e., terms that are present in (7.49) but not in (7.50).
Summing and organizing all the terms, we obtain:

$$KK_{([1,0]+[0,2]+[2,0]+[1,2])} =$$

$$= \sum_{p=0}^{\infty}([0,p,2]+2[1,p,1]+[2,p,0]+[2,p,4]+[1,p,3]+[2,p,2]+[3,p,1]+[1,p,2]+[2,p,1])+$$

$$+\sum_{p=1}^{\infty}([0,p,0]+[0,p,2]+[1,p,1]+[2,p,0]+[1,p,3]) + \sum_{p=2}^{\infty}[0,p,0] \tag{7.55}$$

Now, we explicitly write the sum over p to a certain arbitrary big value, let's say 10, in order to compare to the tensor product of (7.48); we have used Sage Math to save the sets of states in two variables: "old", which contains all the terms written in (7.4) and "new", which contains all the terms in (7.4) except the red ones:

```
sage: new=sum(A3(0,p,2)+2*A3(1,p,1)+A3(2,p,0)+A3(2,p,2)+A3(1,p,3)+
    A3(2,p,2)+A3(3,p,1) for p in [0,1,2,3,4,5,6,7,8,9,10])+
    sum(A3(0,p,0)+A3(0,p,2)+2*A3(1,p,1)+A3(2,p,0) for p in
    [1,2,3,4,5,6,7,8,9,10])+sum(A3(0,p,0) for p in [2,3,4,5,6,7,8,9,10])

sage: old=sum(A3(0,p,2)+2*A3(1,p,1)+A3(2,p,0)+A3(2,p,2)+A3(1,p,3)+
    A3(2,p,2)+A3(3,p,1)+A3(1,p,2)+A3(2,p,1) for p in
    [0,1,2,3,4,5,6,7,8,9,10])+sum(A3(0,p,0)+A3(0,p,2)+2*A3(1,p,1)
    +A3(2,p,0) for p in [1,2,3,4,5,6,7,8,9,10])+sum(A3(0,p,0) for
    p in [2,3,4,5,6,7,8,9,10])
```

Now, we save in the variable "sumN" the tensor product $\sum_{n=0}^{12}[0,n,0] \times [1,1,1]$:

```
sage: sumN=sum(A3(0,n,0)*A3(1,1,1) for n in [0,1,2,3,4,5,6,7,8,9,10,11,12])
```

We want to check which set of states between "new" and "old" is more similar to "sumN"; for this reason, we compute the differences `sumN-new` and `sumN-old`:

```
sage: sumN-new
    2*A3(0,11,0) - A3(1,10,1) + A3(0,12,0) + A3(0,11,2) + 3*A3(1,11,1) +
    A3(0,13,0) + A3(0,12,2) + A3(2,11,0) - A3(2,10,2) + A3(1,12,1) +
    A3(2,12,0) + A3(2,11,2) + A3(1,13,1)

sage: sumN-old
    -A3(1,0,2) - A3(2,0,1) - A3(1,1,2) - A3(2,1,1) - A3(1,2,2) -
    A3(2,2,1) - A3(1,3,2) - A3(2,3,1) - A3(1,4,2) - A3(2,4,1) -
    A3(1,5,2) - A3(2,5,1) - A3(1,6,2) - A3(2,6,1) - A3(1,7,2) -
    A3(2,7,1) - A3(1,8,2) + 2*A3(0,11,0) - A3(2,8,1) - A3(1,9,2) -
    A3(1,10,1) + A3(0,12,0) + A3(0,11,2) - A3(2,9,1) - A3(1,10,2) +
    3*A3(1,11,1) + A3(0,13,0) + A3(0,12,2) - A3(2,10,1) + A3(2,11,0) -
    A3(2,10,2) + A3(1,12,1) + A3(2,12,0) + A3(2,11,2) + A3(1,13,1)
```

We see that the difference between "sumN" and "new" contains only terms with high weights, which would disappear taking the sum over p to infinity; they arise because equation (7.48) is not valid for a specific value of p and n. In an infinite sum, "sumN" and "new" are equal; hence, we show the validity of (7.48) assuming formula (7.50) to be correct.
Differently, the additive terms present in (7.49), are not present in formula (7.50), as we can see from the difference `sumN-old`, where we have terms with low weights.

Analysis of representation [1,2,1]

We analyze, for completeness, another state with higher weight: [1,2,1], which decompose in $SO(5)$ as [1,0]+[0,2]+[2,0]+[1,2]+[3,0]+[2,2], writing only the Sage code, we obtain, proceeding exactly as before:

```
sage:new2=sum(A3(2,p,4)+A3(3,p,3)+A3(4,p,2)+A3(3,p,3)+A3(0,p,2)+
    2*A3(1,p,1)+A3(2,p,0)+A3(2,p,2)+A3(1,p,3)+A3(2,p,2)+A3(3,p,1)
    for p in [0,1,2,3,4,5,6,7,8,9,10])+sum(A3(1,p,3)+A3(2,p,2)
    +A3(3,p,1)+A3(2,p,2)+A3(0,p,0)+A3(0,p,2)+A3(1,p,1)+A3(2,p,0)
    +A3(1,p,1) for p in [1,2,3,4,5,6,7,8,9,10])+sum(A3(0,p,2)
```

```
    +A3(1,p,1)+A3(2,p,0)+A3(1,p,1)+A3(0,p,0) for p in
    [2,3,4,5,6,7,8,9,10])+sum(A3(0,p,0) for p in [3,4,5,6,7,8,9,10])

sage: old2=sum(A3(1,p,2)+A3(2,p,1)+A3(2,p,4)+A3(3,p,3)+A3(4,p,2)+
    A3(2,p,3)+A3(3,p,2)+A3(3,p,3)+A3(0,p,2)+2*A3(1,p,1)+A3(2,p,0)
    +A3(2,p,2)+A3(1,p,3)+A3(2,p,2)+A3(3,p,1) for p in
    [0,1,2,3,4,5,6,7,8,9,10])+sum(A3(1,p,2)+A3(2,p,1)+A3(1,p,3)
    +A3(2,p,2)+A3(3,p,1)+A3(2,p,2)+A3(0,p,0)+A3(0,p,2)+A3(1,p,1)
    +A3(2,p,0)+A3(1,p,1) for p in [1,2,3,4,5,6,7,8,9,10])+sum(A3(0,p,2)
    +A3(1,p,1)+A3(2,p,0)+A3(1,p,1)+A3(0,p,0) for p in
    [2,3,4,5,6,7,8,9,10])+sum(A3(0,p,0) for p in [3,4,5,6,7,8,9,10])

sage: sumN2=sum(A3(0,n,0)*A3(1,2,1) for n in
    [0,1,2,3,4,5,6,7,8,9,10,11,12,13])
```

Now, we evaluate, as before, the differencies:

```
sage: new2-sumN2
-3*A3(0,11,0) + A3(1,10,1) - 2*A3(0,12,0) - 2*A3(0,11,2) - 5*A3(1,11,1)
- 2*A3(0,13,0) - 2*A3(0,12,2) - 2*A3(2,11,0) + A3(2,10,2) - 3*A3(1,12,1)
- A3(1,11,3) - A3(0,14,0) - A3(0,13,2) - 2*A3(2,12,0) - 3*A3(2,11,2) -
3*A3(1,13,1) - A3(1,12,3) - A3(0,15,0) - A3(0,14,2) - A3(3,11,1) +
A3(3,10,3) - A3(2,13,0) - A3(2,12,2) - A3(1,14,1) - A3(3,12,1) -
A3(3,11,3) - A3(2,14,0) - A3(2,13,2) - A3(1,15,1)

sage: old2-sumN2
A3(1,0,2) + A3(2,0,1) + 2*A3(1,1,2) + 2*A3(2,1,1) + A3(2,0,3) +
2*A3(1,2,2) + A3(3,0,2) + 2*A3(2,2,1) + A3(2,1,3) + 2*A3(1,3,2) +
A3(3,1,2) + 2*A3(2,3,1) + A3(2,2,3) + 2*A3(1,4,2) + A3(3,2,2) +
2*A3(2,4,1) + A3(2,3,3) + 2*A3(1,5,2) + A3(3,3,2) + 2*A3(2,5,1) +
A3(2,4,3) + 2*A3(1,6,2) + A3(3,4,2) + 2*A3(2,6,1) + A3(2,5,3) +
2*A3(1,7,2) + A3(3,5,2) + 2*A3(2,7,1) + A3(2,6,3) + 2*A3(1,8,2) -
3*A3(0,11,0) + A3(3,6,2) + 2*A3(2,8,1) + A3(2,7,3) + 2*A3(1,9,2) +
A3(1,10,1) - 2*A3(0,12,0) - 2*A3(0,11,2) + A3(3,7,2) + 2*A3(2,9,1) +
A3(2,8,3) + 2*A3(1,10,2) - 5*A3(1,11,1) - 2*A3(0,13,0) - 2*A3(0,12,2) +
A3(3,8,2) + 2*A3(2,10,1) + A3(2,9,3) - 2*A3(2,11,0) + A3(2,10,2) -
3*A3(1,12,1) - A3(1,11,3) - A3(0,14,0) - A3(0,13,2) + A3(3,9,2) +
A3(2,10,3) - 2*A3(2,12,0) - 3*A3(2,11,2) - 3*A3(1,13,1) - A3(1,12,3) -
A3(0,15,0) - A3(0,14,2) + A3(3,10,2) - A3(3,11,1) + A3(3,10,3) -
A3(2,13,0) - A3(2,12,2) - A3(1,14,1) - A3(3,12,1) - A3(3,11,3) -
A3(2,14,0) - A3(2,13,2) - A3(1,15,1)
```

We obtain the same result as before, "new2" and "sumN2" are equal if we consider the infinite sum, while "old2" is different.

We have shown that formula (7.48) can be correct only if we assume for the Kaluza Klein expansion the formula (7.50); hence, formula (7.49) (i.e. 3.4 of [1]) must be incorrect.

Chapter 8

$\mathcal{N} = 4$ Super Yang Mills States

The spectrum of KK descendants of fundamental strings states can be precisely tested against that of gauge invariant operators in $SU(N)$ $\mathcal{N} = 4$ SYM theory. In this chapter we apply Polya theory [16], [18], [21], [25] to determine the spectrum of superconformal primaries in $\mathcal{N} = 4$ SYM theory.

The spectrum of gauge invariant $\mathcal{N} = 4$ SYM operators is organized under the supergroup $PSU(2,2|4)$. Single trace operators are associated to 'words' built with the 'alphabet' consisting of the 'letters' ϕ^i with $i = 1,\ldots,6$, λ^A_α and $\lambda^A_{\dot\alpha}$, with $A = 1,\ldots,4$ and $\alpha, \dot\alpha = 1,2$, $F_{\mu\nu}$, with $\mu,\nu = 0,\ldots,3$ and derivative thereof.

The quantum numbers of a given operator in free theory can be read off from those of the building letters collected in the table:

Fields	$SU(4)$	$(j,\bar{j})$	Δ
ϕ^i	$[010]$	$(0,0)$	1
$\partial_{\alpha\dot\alpha}$	$[000]$	$(\frac{1}{2},\frac{1}{2})$	1
λ^A_α	$[100]$	$(\frac{1}{2},0)$	$3/2$
$\lambda_{\dot\alpha A}$	$[001]$	$(0,\frac{1}{2})$	$3/2$
$F_{\alpha\beta}$	$[000]$	$(1,0)$	2
$F_{\dot\alpha\dot\beta}$	$[000]$	$(0,1)$	2

8.1 Enumerating SYM operators: Polya(kov) Theory

In Polya theory we consider a set of words $A, B, \ldots$ made out of n letters chosen within the alphabet $\{a_i\}$, with $i = 1,\ldots,p$. Let g be a group action defining the equivalence relation $A \approx B$ for $A = gB$ with g an element of $G \subset S_n$. Elements $g \in S_n$ can be divided into conjugacy classes $[g] = (1)^{b_1}\ldots(n)^{b_n}$, according to the numbers $\{b_k(g)\}$ of cycles of length k. Polya theorem states that the set of inequivalent words are generated by the formula

$$P_G(\{a_i\}) = \frac{1}{|G|}\sum_{g\in G}\prod_{k=1}^{n}(a_1^k + a_2^k + \ldots + a_p^k)^{b_k(g)} \tag{8.1}$$

In particular, for $G = Z_n$, the cyclic permutation subgroup of S_n, the elements $g \in G$ belong to one of the conjugacy classes $[g] = (d)^{\frac{n}{d}}$ for each divisor d of n; in this way, the number of elements in a given conjugacy class labeled by d is given by Euler's totient function $\varphi(d)$, equal to the number of numbers relatively prime to n, smaller than n; in this case, we can write:

$$P_G(\{a_i\}) = \frac{1}{n}\sum_{d|n}\varphi(d)(a_1^d + a_2^d + \ldots + a_p^d)^{\frac{n}{d}} \tag{8.2}$$

The number of inequivalent words can be read by letting $a_i \to 1$.

We want to use this theory to count the numbers of gauge invariant words (single trace operators) in four dimensional $SU(N)$ SYM in the free theory; starting with the scalar operator ϕ, we know that it has $\Delta = 1$, while its derivatives $\partial_\mu \phi$ has dimension $\Delta = 2$ and degeneracy 4 due to the indices μ; at the next level we have $\partial_\mu \partial_\nu \phi$ with $\Delta = 3$ and the degeneracy is $\frac{5 \cdot 4}{2}$ because of symmetrization of derivatives indices; in addition, because in SYM the fields are massless, we have $\Box \phi = 0$, which gives us a total of 9 degeneracy. We find at every level a degeneracy of $(n+1)^2$.

In terms of the representations, the derivatives act only on the $SO(4)$ Dynkin labels; in addition, instead of considering the combinatorics at arbitrary level, it is simpler to infer the complet tower of $SO(4)$ representations from the existence of a partial wave expansion; this reasoning can be justified as follows: at level $\Delta = n+1$ the field transforms in the $[n, n]$ of $SO(4)$ (we are using Dynkin label index instead of spin index, which are given by $n = 2j$); acting with the derivative ∂_μ yields:

$$[1,1] \otimes [n,n] = [n-1, n-1] \oplus [n-1, n+1] \oplus [n+1, n-1] \oplus [n+1, n+1] \tag{8.3}$$

which was verified using character theory; however, we want only the symmetric contribute, in order to consider the contribution of multiple derivatives; for $SO(4)$, the symmetric contribution is given by considering the representations with the same Dynkin labels, i.e.

$$[1,1] \otimes [n,n]_{\text{sym}} = [n-1, n-1] \oplus [n+1, n+1] \tag{8.4}$$

On the other hand, the field equation removes exactly the contribute $[n-1, n-1]$ and we are left only with $[n+1, n+1]$.

If we consider, instead, the Maxwell field, we know that the field strength has dimension $\Delta = 2$ and there are 6 components; at the next level $\partial_\lambda F_{\mu\nu}$ with $\Delta = 3$ there are 24 components; using the constraints from the field equations $\partial^\mu F_{\mu\nu} = 0$ and $\partial^{\mu *} F_{\mu\nu} = 0$ yield net degeneracy of 16; at the next level, similar computation give a degeneracy of 40. These degeneracies agree with those of two $SO(4)$ towers of the form $[n, n+2]$ and $[n+2, n]$.

In the same way a Weyl fermion at level n has $\Delta = n + \frac{3}{2}$ and an $SO(4)$ representation given by $[n, n+1] + [n+1, n]$.

These results can be written in terms of the partition function [13]:

$$Z = \sum_\Delta t^\Delta \tag{8.5}$$

For instance, for the scalar field, we would have $Z = \sum_\Delta t^\Delta = \prod_{n=0}^\infty \left(\frac{1}{1-t^{n+1}}\right)^{(n+1)^2}$, where we have used the bosonic nature of ϕ, $\Delta = n+1$ with degeneracy $(n+1)^2$.

In general, we find:

$$\begin{aligned} Z_S &= \prod_{n=0}^\infty \left(\frac{1}{1-t^{n+1}}\right)^{(n+1)^2} \\ Z_W &= \prod_{n=0}^\infty \left(1 + t^{n+\frac{1}{2}}\right)^{2n(n+1)} \\ Z_V &= \prod_{n=0}^\infty \left(\frac{1}{1-t^{n+1}}\right)^{2n(n+2)} \end{aligned} \tag{8.6}$$

Returning to Polya theory, we can write the field content of single-letter words as

$$\mathcal{Z}_1(t, y_i) = \sum_{s=0}^\infty \left[t^{s+1}\ \partial^s \phi + t^{s+\frac{3}{2}}\ \partial^s \lambda + t^{s+\frac{3}{2}} \partial^s \bar{\lambda} + t^{s+2} \partial^s F + t^{s+2} \partial^s \bar{F}\right] \tag{8.7}$$

Now, using the previous decomposition in $SO(4)$ towers and describing uniquely the representations in terms of their characters, we obtain:

$$\mathcal{Z}_1(t, y_i) = \sum_{s=0}^{\infty} \Big[\chi_{[1,0,0]}\chi_{[s,s]}t^{s+1} + \chi_{[0,0,0]}\chi_{[s+2,s]}t^{s+2} + \chi_{[0,0,0]}\chi_{[s,s+2]}t^{s+2} - \\ -\chi_{[0,1,0]}\chi_{[s+1,s]}t^{s+\frac{3}{2}} - \chi_{[0,0,1]}\chi_{[s,s+1]}t^{s+\frac{3}{2}}\Big] \tag{8.8}$$

now, using Polya formula (8.2) we can obtain cyclic words with $n > 1$ letters:

$$\mathcal{Z}_{\text{SYM}}(t, y_i) = \sum_{n=2}^{\infty} \mathcal{Z}_n(t, y_i) = \sum_{n,n|d} \frac{\varphi(d)}{n} \mathcal{Z}_1(t^d, y^d)^{\frac{n}{d}} \tag{8.9}$$

with the sum running over all integers n and their divisors d. The omission of the $n = 1$ term in the sum is due to the fact that we are considering SYM with gauge group $SU(N)$ rather than $U(N)$.

Considering the limit in which all the character's variables go to 1, we obtain the unrefined partition function, which gives us the dimension at any conformal dimension.

For instance, we can easily evaluate the blind partition function of $\mathcal{Z}_1$ using the dimension formula that we have derived for $SO(4)$ and $SO(6)$, obtaining:

$$\mathcal{Z}_1^0 = \sum_{s=0}^{\infty} \Big[6(1+s)^2\ t^{s+1} + 2(3+s)(1+s)t^{s+2} - 8(2+s)(1+s)t^{s+\frac{3}{2}}\Big] \tag{8.10}$$

We can simplify the above expression using Mathematica, obtaining:

$$\mathcal{Z}_1^0 = \frac{2t(3+\sqrt{t})}{(1+\sqrt{t})^3} \tag{8.11}$$

Using Polya formula, we obtain the Super Yang Mills blind generating function, which we can expand in powers of t

$$\mathcal{Z}_{SYM}^0(t) = \sum_{n=2}^{\infty} \sum_{n|d} \frac{\phi(d)}{n} \left[\frac{2t(3+t^{\frac{d}{2}}}{(1+t^{\frac{d}{2}})^3}\right]^{\frac{n}{d}} \tag{8.12}$$

$$= 21t^2 - 96t^{5/2} + 376t^3 - 1344t^{\frac{7}{2}} + 4605t^4 - 15456t^{\frac{9}{2}} + 52152t^5 - \\ - 177600t^{\frac{11}{2}} + 608365t^6 - 2095584t^{\frac{13}{2}} + 7262256t^7 + \mathcal{O}(t^{\frac{15}{2}}) \tag{8.13}$$

However, in the above partition function, are included all the descendendants and $\frac{1}{2}$ BPS states; we need to decompose the multiplet in order to mantain only long superconformal primary states.

8.2 Decomposition of Long Multiplets

The set of states (8.9) organizes into multiplets of the $\mathcal{N}=4$ superconformal algebra $PSU(2,2|4)$; in order to obtain only the spectrum of superconformal primaries, we must remove all the descendants from superconformal primaries.

We observe that the highest weight states are characterized by their $SU(2)_L \times SU(2)_R \simeq SO(4)$ spins $(j,\bar{j})$ and $SO(6)$ Dynkin labels, that determines the dimension of the supermultiplet to be $\dim\left(\mathcal{A}^{\Delta}_{[k,p,q](j,\bar{j})}\right) = 2^{16} \times \dim[k,p,q]_{(j,\bar{j})}$, and by their scaling dimension $\Delta = \Delta_0 + \gamma$, which is constrained by unitarity lower bounds.

The $SO(6)\times SO(4)$ content of long supermultiplets can be read off from:

$$\mathcal{A}^{\Delta_0}_{[k,p,q](j,\bar{j})} = T_1^{(2)} \times [k,p,q]^{\Delta_0-2}_{[2j,2\bar{j}]} \tag{8.14}$$

where $T_1^{(2)}$ can be identified with the Konishi multiplet, whose highest weight state is a scalar of bare scaling dimension $\Delta_0 = 2$. Its 2^{16} components arise from the unconstrained action of 16 supercharges Q.

$$T_1^{(2)} = (1 + Q + Q\wedge Q + \ldots) \times [000]^2_{[0,0]} \qquad Q = [010]^{\frac{1}{2}}_{[1,0]} + [001]^{\frac{1}{2}}_{[0,1]} \tag{8.15}$$

the dimension of its super descendants can be read off by assigning a $\Delta_Q = \frac{1}{2}$ any time we act with a supercharge Q.

The precise representation content may be found using the Racah-Speiser algorithm:

$$\mathcal{A}^{\Delta_0}_{[k,p,q](j,\bar{j})} = \sum_{\epsilon_{i\alpha},\bar{\epsilon}^i_{\dot{\alpha}}\in\{0,1\}} [(k,p,q)_{(j,\bar{j})} + \epsilon_{i\alpha}q^{i\alpha} + \bar{\epsilon}^i_{\dot{\alpha}}\bar{q}^{\dot{\alpha}}_i] \tag{8.16}$$

with the sum running over the 2^{16} combinations of the 16 weights, $q^{i\pm}, \bar{q}^{\pm}_i, i = 1,\ldots,4$ of the supersymmetry charges

$$\begin{aligned}
q^{1\pm} &= [0,1,0]_{[\pm1,0]} & q^{2\pm} &= [1,-1,0]_{[\pm1,0]}\\
q^{3\pm} &= [-1,0,1]_{[\pm1,0]} & q^{4\pm} &= [0,0,-1]_{[\pm1,0]}\\
\bar{q}^{\pm}_1 &= [0,0,1]_{[0,\pm1]} & \bar{q}^{\pm}_2 &= [1,0,-1]_{[0,\pm1]}\\
\bar{q}^{\pm}_3 &= [-1,1,0]_{[0,\pm1]} & \bar{q}^{\pm}_4 &= [0,-1,0]_{[0,\pm1]}
\end{aligned} \tag{8.17}$$

where we have written the weights of the supercharges, which are given by the commutators of the third generator of the corresponding simple root subalgebra $SU(2)$, with the supercharges.

As we have said, every $q,\bar{q}$ raises the conformal dimension by $\frac{1}{2}$; in addition, in order to fully apply the Racah Speiser algorithm, we need to count the representations with negative weights, according to the identifications

$$\begin{aligned}
&[p,k,q]_{[2j,2\bar{j}]} = -[p+k+1,-k-2,q]_{[2j,2\bar{j}]} = -[p+q+1,k,-q-2]_{[2j,2\bar{j}]} =\\
&= -[-p-2,k+p+1,q+p+1]_{[2j,2\bar{j}]} = -[k,p,q]_{[-2j-2,2\bar{j}]} = -[k,p,q]_{[2j,-2\bar{j}-2]}
\end{aligned} \tag{8.18}$$

which derive from the fact that the character formula has this kind of symmetry (we have explicitly verified it using Mathematica and our derived character formulae). This implies that $[p,k,q]_{[2j,2\bar{j}]}$ is counted as zero whenever any of the weights k,p,q equals -1 or one of the spins $j,\bar{j}$ equals $-\frac{1}{2}$.

In addition to the algorithm, we should also consider the superalgebra constraints; for instance, the anticommutativity of the q's is telling us that we cannot act on a state two times wit the same operator.

We can explicitly construct the superdescendendants in the Konishi multiplet, from Racah Speiser algorithm, starting with the highest state $[000]^2_{[0,0]}$:

$$\begin{aligned}
&[000]^2_{[0,0]} \xrightarrow[q^{1+},\bar{q}^{1+}]{} [001]^{\frac{5}{2}}_{[0,1]} + [010]^{\frac{5}{2}}_{[1,0]} \xrightarrow[q^{4+},\bar{q}^{1-},\bar{q}^{2+},q^{2+},q^{1+},\bar{q}^{1+},q^{1-}]{} \\
&\to [000]^3_{[1,1]} + [002]^3_{[0,0]} + [100]^3_{[0,2]+[2,0]} + [011]^3_{[1,1]} + [020]^3_{[0,0]} \to \\
&\to [001]^{\frac{7}{2}}_{[1,0]+[1,2]+[3,0]} + [101]^{\frac{7}{2}}_{[0,1]+[2,1]} + [0,1,0]^{\frac{7}{2}}_{[0,1]+[0,3]+[2,1]} + \\
&\quad + [012]^{\frac{7}{2}}_{[1,0]} + [110]^{\frac{7}{2}}_{[1,0]+[1,2]} + [021]^{\frac{7}{2}}_{[0,1]} \to \dots
\end{aligned} \tag{8.19}$$

We see that, due to the antisymmetry of the q's, the total dimension of the representations at each conformal level l is given by $\begin{pmatrix} l \\ 16 \end{pmatrix}$

Now, we want to construct the generating function of the Konishi multiplet, in order to write all the states, with total dimension 2^{16}, in a compact form, keeping also the information on the conformal dimension; in general, for this and also other similar case, we can write the generating function as

$$\prod_{s=1}^{n} (1 - t^{\delta} \mathbf{y}^{\mathbf{w_s}}) \tag{8.20}$$

Where n is the dimension of the supercharges, δ is the conformal dimension of the supercharge and $\mathbf{w_s}$ is a vector running over the weights of the generator; moreover, we have $\mathbf{y}^{\mathbf{w_s}} = \prod_{i=1}^{5} y_i^{w_i}$, where the y^i are the character's variables for $SO(6) \times SO(4)$ representations.

In the case at hand, the dimension of Q is 16, $\delta = \frac{1}{2}$ and $\mathbf{w_s}$ run over the 16-spinor representation of Q:

$$\mathcal{T}_{\text{Konishi}}(t, y_i) = \prod_{s=1}^{16} (1 - t^{\frac{1}{2}} \mathbf{y}^{\mathbf{w_s}}) \tag{8.21}$$

The problem is now to find the weight vector which, when inserted in (8.21), furnish the states in (8.19) when expanded in t.

The solution is given by:

$$\begin{aligned}
\mathcal{T}_{\text{Konishi}}(t, y_i) = &\left(\frac{\sqrt{t}y_1}{y_2 z_1} + 1\right)\left(\frac{\sqrt{t}y_2}{z_1} + 1\right)\left(\frac{\sqrt{t}}{y_3 z_1} + 1\right)\left(\frac{\sqrt{t}y_3}{y_1 z_1} + 1\right)\left(\frac{\sqrt{t}y_1 z_1}{y_2} + 1\right) \\
&\left(\sqrt{t}y_2 z_1 + 1\right)\left(\frac{\sqrt{t}z_1}{y_3} + 1\right)\left(\frac{\sqrt{t}y_3 z_1}{y_1} + 1\right)\left(\frac{\sqrt{t}}{y_2 z_2} + 1\right)\left(\frac{\sqrt{t}y_2}{y_1 z_2} + 1\right) \\
&\left(\frac{\sqrt{t}y_1}{y_3 z_2} + 1\right)\left(\frac{\sqrt{t}y_3}{z_2} + 1\right)\left(\frac{\sqrt{t}z_2}{y_2} + 1\right)\left(\frac{\sqrt{t}y_2 z_2}{y_1} + 1\right)\left(\frac{\sqrt{t}y_1 z_2}{y_3} + 1\right)
\end{aligned} \tag{8.22}$$

where y_1, y_2, y_3 are the character's variables of $SO(6)$, while z_1, z_2 are the variables of $SO(4)$.

As explained before, we have checked this result up to $\Delta_0 = \frac{7}{2}$, expanding (8.22) up to order $t^{\frac{3}{2}}$.

On the other hand, the superconformal multiplet is also generated by the supertranslations:

$$\begin{aligned}
\mathcal{T}_{\text{sconf}} =& (1 + Q + Q \wedge Q + \dots)(1 + P_4 + (P_4 \times P_4)_s + \dots) = \\
&\mathcal{T}_{\text{Konishi}} \underbrace{(1 + P_4 + (P_4 \times P_4)_s + \dots)}_{\mathbf{P}}
\end{aligned} \tag{8.23}$$

where P_4 is the supertranslation generator; we have seen before that P_4 brings a variation in the conformal dimension equal to 1 and acts only on the $SO(4)$ Dynkin labels; for these reasons, the generating function is given by

$$\mathbf{P} = \frac{1}{\left(1 - \frac{t}{z_1 z_2}\right)\left(1 - \frac{t z_1}{z_2}\right)\left(1 - \frac{t z_2}{z_1}\right)(1 - t z_1 z_2)} \tag{8.24}$$

8.3 Decomposition of $\frac{1}{2}$ BPS multiplets

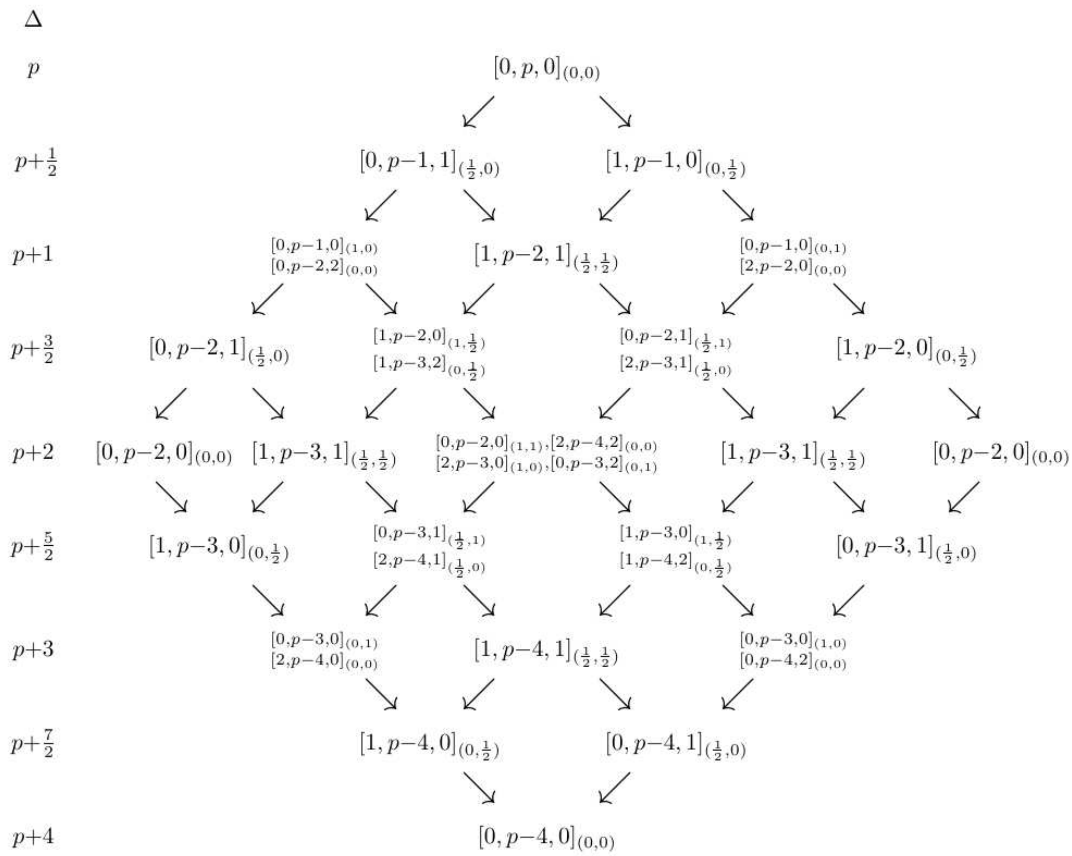
Short multiplets are obtained by acting on the ground state only with a limited number of supercharges, which means that the sum in (8.16) is restricted to a sum over partial subsets of weights $q^{i\pm}$'s, $\bar{q}_i^{\pm}$'s. This requires certain conditions on the ground state $[k,p,q]_{[2j,2\bar{j}]}$ and its conformal dimension Δ_0. The general multiplet is obtained by combining a subset of $q^{i\pm}$'s with a subset of $\bar{q}_i^{\pm}$'s provided the ground state $[k,p,q]_{[2j,2\bar{j}]}$ satisfies both constraints.

The $\frac{1}{2}$ BPS multiplets are obtained requiring, for instance, $q^{1,2}, \bar{q}_{1,2}$ equal zero on the ground state; they are realized in the KK tower of the $l=0$ supergravity sector, with total dimension

$$dim\mathcal{BB}^{\frac{1}{2},\frac{1}{2}}_{[n00][0,0]} = 2^8\dim[n-2,0,0] \tag{8.25}$$

In particular, for $n=2$ this is the massless supergravity multiplet of dimension 256.

The complet set of superdcendants can be obtained acting with the q and $\bar{q}$ on the ground state $[p,0,0]$; we report here the result, written for instance in [3], with $SU(4)$ conventions instead of our $SO(6)$ conventions:



Where with the right arrows we act with $\bar{q}$ and with the left arrows we act with q. We will call the character of this total $\frac{1}{2}$ BPS multiplet as $\chi_{\frac{1}{2}BPS}(p,t,y_1,y_2,y_3,z_1,z_2)$; we have a dependence of t, because every descendant at conformal level Δ is moltiplicated by t^{Δ}, in order to consider the information about the conformal dimension; differently, the $y_{1,2,3}$ and $z_{1,2}$ are the characters of $SO(6)$ and $SO(4)$ respectevely.

The total contribute of $\frac{1}{2}$ BPS generating function to the SYM generating function, can be expressed as

$$\mathcal{Z}_{BPS} = \sum_{n=2}^{\infty} \chi_{\frac{1}{2}BPS}(n,t,y_1,y_2,y_3,z_1,z_2) \cdot \mathbf{P}(t,z_1,z_2) \tag{8.26}$$

we have written **P** because it is necessary to consider also the derivative descendants.

We can write the unrefined limit of $\mathcal{Z}_{BPS}$, obtaining

$$\mathcal{Z}^0_{BPS} = \frac{t^2(20 + 80t^{\frac{1}{2}} + 146t + 144t^{\frac{3}{2}} + 81t^2 + 24t^{\frac{5}{2}} + 3t^3}{(1-t)(1+t^{\frac{1}{2}})^8} \tag{8.27}$$

8.4 Superconformal Primaries

In order to compare with the superstring spectrum, we have to identify the superconformal primaries of Super Yang Mills; this reduction can be obtained noting that single trace operators falls into two classes of $PSU(2,2|4)$ multiplets:

$$\mathcal{Z}_{SYM}(t, y_i) = \mathcal{Z}_{BPS}(t, y_i) + \mathcal{Z}_{long}(t, y_i) \tag{8.28}$$

States in $\mathcal{Z}_{long}(t, y_i)$ sit in long multiplets of the superconformal algebra $PSU(2,2|4)$ and we want to verify that they match the multiplet structure of the string states; in addition, the comparison to the string theory can be restricted to superconformal primaries, which can be found by factoring out $\mathcal{T}_{sconf}$ in $\mathcal{Z}_{long}$:

$$\mathcal{Z}_{sconf}(t, y_i) = \mathcal{Z}_{long}(t, y_i)/\mathcal{T}_{sconf}(t, y_i) \tag{8.29}$$

In the unrefined limit, we find:

$$\mathcal{Z}^0_{sconf} = (\mathcal{Z}^0_{SYM} - \mathcal{Z}^0_{BPS})/\mathcal{T}^0_{sconf} = \tag{8.30}$$

$$t^2 + 6t^3 + 120t^4 + 886t^5 - 1728t^{\frac{11}{2}} + 8464t^6 + \mathcal{O}(t^{\frac{13}{2}}) \tag{8.31}$$

which gives us the dimension at the corresponding conformal level.

We have also performed the evaluation for the refined partition function and, using character theory, we have found the first superconformal states of SYM:

$$\begin{aligned}
(\Delta = 2) \quad &\rightarrow \quad [000]_{[0,0]} \\
(\Delta = 3) \quad &\rightarrow \quad [100]_{[0,0]} \\
(\Delta = 4) \quad &\rightarrow \quad 2[000]_{[0,0]} + [000]_{[0,2]} + [000]_{[2,0]} + [000]_{[2,2]} + [011]_{[0,0]} + 2[2,0,0]_{[0,0]} + 2[100]_{[1,1]} \\
(\Delta = 5) \quad &\rightarrow \quad 4[100]_{[0,0]} + 2[100]_{[0,2]} + [100]_{[2,2]} + 2[100]_{[2,0]} + 2[002]_{[0,0]} + [002]_{[2,0]} + 2[020]_{[0,0]} + \\
&\quad + [020]_{[0,2]} + 2[111]_{[0,0]} + 2[300]_{[0,0]} + 2[000]_{[1,1]} + 4[011]_{[1,1]} + 2[200]_{[1,1]} \\
\left(\Delta = \frac{11}{2}\right) \quad &\rightarrow \quad 4[010]_{[0,1]} + 2[010]_{[2,3]} + 4[001]_{[1,0]} + 2[001]_{[3,2]} + 4[101]_{[0,1]} + 2[210]_{[0,1]} + 2[120]_{[1,0]} + \\
&\quad + 2[001]_{[1,2]} + 2[010]_{[2,1]} + [110]_{[1,2]} + [110]_{[2,1]} \\
(\Delta = 6) \quad &\rightarrow \quad 11[000]_{[0,0]} + 9[011]_{[0,0]} + 10[200]_{[0,0]} + 3[102]_{[0,0]} + 3[120]_{[0,0]} + 3[022]_{[0,0]} + 3[211]_{[0,0]} + \\
&\quad + 3[400]_{[0,0]} + 5[000]_{[0,2]} + 5[000]_{[2,0]} + 6[011]_{[0,2]} + 6[011]_{[2,0]} + 4[200]_{[0,2]} + 4[200]_{[2,0]} + \\
&\quad + 4[200]_{[0,2]} + [120]_{[0,2]} + [102]_{[2,0]} + 16[100]_{[1,1]} + 4[002]_{[1,1]} + 4[020]_{[1,1]} + 8[111]_{[1,1]} + \\
&\quad + 4[300]_{[1,1]} + 2[000]_{[4,0]} + 2[000]_{[0,4]} + 4[100]_{[1,3]} + 4[100]_{[3,1]} + 7[000]_{[2,2]} + 2[011]_{[2,2]} + \\
&\quad + 4[200]_{[2,2]} + [200]_{[2,2]} + [000]_{[2,4]} + [000]_{[4,2]} + 2[100]_{[3,3]} + [000]_{[4,4]}
\end{aligned} \tag{8.32}$$

Chapter 9

Comparison of States in AdS/CFT Correspondence

9.1 Comparison between Superconformal Multiplet and the first massive string

As a first step in the comparison between long superconformal multiplets and superstring states we compare the Konishi multiplet with the first massive string level; for this reason, we consider the character of the superconformal multiplet (8.22);however, in order to compare this character with the character of the first massive string level, we need to remove the conformal dimension information, sending $t \to 1$ and, in addition, we need to break the representation from $SO(6) \times SO(4)$ to $SO(5) \times SO(4)$; this last requirement means, as shown in (7.32), a relation between character variables: $y_2 = y_3$, obtaining:

$$\lim_{t\to 0} \mathcal{T}_{Konishi}(y_1, y_2, y_2, z_1, z_2, t) =$$
$$= \frac{(y_2+z_1)^2 (y_1 z_1 + y_2)^2 (y_2 z_1 + 1)^2 (y_2 z_1 + y_1)^2 (y_2 + z_2)^2 (y_1 z_2 + y_2)^2 (y_2 z_2 + 1)^2 (y_2 z_2 + y_1)^2}{y_1^4 y_2^8 z_1^4 z_2^4} \tag{9.1}$$

We note that it is a quadratic form, in this way we can relate the square root of $\mathcal{T}_{Konishi}$ directly with T_1^{open}, evaluating the character of the branched representation as given by (6.46).

We find that the two chracter are equivalent, without the need of redefining the character's variables; in the end, we have verified

$$\boxed{\lim_{t\to 0} \chi_{\mathcal{T}_{Konishi}}(y_1, y_2, y_2, z_1, z_2, t) = \chi_{T_1^{closed}}(y_1, y_2, z_1, z_2)} \tag{9.2}$$

9.2 Comparison Beyond Konishi Multiplet

We have seen that at level $l = 1$ the $SO(5) \times SO(4)$ content of T_1^2 coincides with the Konishi multiplet $\hat{T}_1^{(2)}$ upon breaking $SO(6)$ down to $SO(5)$. In this way, we can apply formula (7.48), in order to evaluate the KK tower at $l = 1$, which results in

$$\mathcal{H}_1 = \sum_{n=0}^{\infty} [n, 0, 0]^n_{[0,0]} \times \hat{T}_1^{(2)} \tag{9.3}$$

The resulting multiplets have bare dimensions $\Delta = 2 + n$. Recall that for $l > 1$ the flat space spectrum is given by

$$T_l^{closed} = T_1^{closed} \times (\mathrm{vac}_l)^2 \tag{9.4}$$

In order to obtain the KK tower in closed form using the above result for T_1, it remains to lift $(\mathrm{vac}_l)^2$ to $SO(6) \times SO(4)$. For this reason, we have to consider the string spectrum on curved space $SO(5) \times SO(4)$ previously derived and lift the $SO(5)$ component to $SO(6)$; for instance, at level $l = 2$, in the decomposition $SO(9) \to SO(5) \times SO(4)$, we have

$$\mathrm{vac}_2 = \mathbf{9} \to [1, 0]_{[0,0]} + [0, 0]_{[1,1]} \tag{9.5}$$

now inverting formula (7.27), we obtain the $SO(6) \times SO(4)$ lift:

$$\mathbf{9} \to [1, 0, 0]^1_{[0,0]} + [0, 0, 0]^1_{[1,1]} - [0, 0, 0]^{\delta}_{[0,0]} \tag{9.6}$$

we note that the states with negative multiplicity should be unphysical; however, we choose the conformal dimension δ in such a way that, when we insert (9.6) in the KK tower, the negative signs should be deleted; in this case we have:

$$\begin{aligned}\Big([1,0,0]^{(1)}_{[0,0]} + [0,0,0]^{(1)}_{[1,1]} - [0,0,0]^{(\delta)}_{[0,0]}\Big)^2 &= [0,0,0]^{(2)}_{2[0,0]+[2,0]+[0,2]+[2,2]} + [0,1,1]^{(2)}_{[0,0]} + \\ + [2,0,0]^{(2)}_{[0,0]} + 2[1,0,0]^{(2)}_{[1,1]} &- 2[0,0,0]^{1+\delta}_{[1,1]} - 2[1,0,0]^{1+\delta}_{[0,0]} + [0,0,0]^{2\delta}_{[0,0]}\end{aligned} \tag{9.7}$$

and inserting the states in (9.6) we have

$$\begin{aligned}\mathcal{H}_2 = \sum_{n=0}^{\infty} [n,0,0]^n_{[0,0]} \times \hat{T}_1^{(2)} \times \Big([0,0,0]^{(2)}_{2[0,0]+[2,0]+[0,2]+[2,2]} + [0,1,1]^{(2)}_{[0,0]} + \\ + [2,0,0]^{(2)}_{[0,0]} + 2[1,0,0]^{(2)}_{[1,1]} - 2[0,0,0]^{1+\delta}_{[1,1]} - 2[1,0,0]^{1+\delta}_{[0,0]} + [0,0,0]^{2\delta}_{[0,0]}\Big)\end{aligned} \tag{9.8}$$

Now, expanding the above product to the first n levels, we have:

$$\begin{aligned}\mathcal{H}_2 = \sum_{n=0}^{\infty} \hat{T}_1^{(2)} \times \Big(&[0,0,0]^{(2)}_{(2[0,0]+[2,0]+[0,2]+[2,2])} + [0,1,1]^{(2)}_{[0,0]} + [2,0,0]^{(2)}_{[0,0]} + 2[1,0,0]^{(2)}_{[1,1]} + \\ &+ [1,0,0]^{(3)}_{(3[0,0]+[0,2]+[2,0]+[2,2])} + [0,2,0]^{(3)}_{[0,0]} + [0,0,2]^{(3)}_{[0,0]} + [1,1,1]^{(3)}_{[0,0]} + \\ &+ [3,0,0]^{(3)}_{[0,0]} + 2[0,1,1]^{(3)}_{[1,1]} + 2[2,0,0]^{(3)}_{[1,1]} + 2[0,0,0]^{(3)}_{[1,1]} - 2[0,0,0]^{(1+\delta)}_{[1,1]} - \\ &- 2[1,0,0]^{(1+\delta)}_{[0,0]} + \\ &+ [2,0,0]^{(4)}_{(4[0,0]+[2,0]+[0,2]+[2,2])} + [0,1,1]^{(4)}_{2[0,0]} + [1,2,0]^{(4)}_{[0,0]} + [1,0,2]^{(4)}_{[0,0]} \\ &+ [2,1,1]^{(4)}_{2[0,0]} + [0,0,0]^{(4)}_{[0,0]} + [0,2,2]^{(4)}_{[0,0]} + [4,0,0]_{[0,0]} + 2[1,0,0]^{(4)}_{[1,1]} + \\ &+ 2[1,1,1]^{(4)}_{[1,1]} + 2[3,0,0]^{(4)}_{[1,1]} - 2[1,0,0]^{(2+\delta)}_{[1,1]} - 2[0,0,0]^{(2+\delta)}_{[0,0]} - 2[0,1,1]^{(2+\delta)}_{[0,0]} - \\ &- 2[2,0,0]^{(2+\delta)}_{[0,0]} + [0,0,0]^{(2\delta)}_{[0,0]} + \dots\Big)\end{aligned}$$

where we see that every term with a negative coefficient is cancelled when we choose $\delta = 2$. In this way, we obtain

$$\mathcal{H}_2=\sum_{n=0}^{\infty}\hat{T}_1^{(2)}\times\Big([0,0,0]^{(2)}_{(2[0,0]+[2,0]+[0,2]+[2,2])}+[0,1,1]^{(2)}_{[0,0]}+[2,0,0]^{(2)}_{[0,0]}+2[1,0,0]^{(2)}_{[1,1]}+$$
$$+[1,0,0]^{(3)}_{(2[0,0]+[0,2]+[2,0]+[2,2])}+[0,2,0]^{(3)}_{[0,0]}+[0,0,2]^{(3)}_{[0,0]}+[1,1,1]^{(3)}_{[0,0]}+$$
$$+[3,0,0]^{(3)}_{[0,0]}+2[0,1,1]^{(3)}_{[1,1]}+2[2,0,0]^{(3)}_{[1,1]}+$$
$$+[2,0,0]^{(4)}_{(2[0,0]+[2,0]+[0,2]+[2,2])}+[1,2,0]^{(4)}_{[0,0]}+[1,0,2]^{(4)}_{[0,0]}+[2,1,1]^{(4)}_{2[0,0]}+$$
$$+[0,2,2]^{(4)}_{[0,0]}+[4,0,0]_{[0,0]}+2[1,1,1]^{(4)}_{[1,1]}+2[3,0,0]^{(4)}_{[1,1]}+\dots\Big)$$

We can proceed in the same way with the level $l=3$ states, which are given by $[2,0,0,0]+[0,0,0,1]$; considering $[2,0,0,0]$, it decompose in $SO(5)\times SO(4)$ as:

$$[2,0,0,0]\to[0,0]_{[0,0]}+[1,0]_{[1,1]}+[2,0]_{[0,0]}+[0,0]_{[2,2]} \tag{9.9}$$

Now, we lift the $SO(5)$ representations, obtaining:

$$[0,0]\to[0,0,0] \tag{9.10}$$
$$[1,0]\to[1,0,0]-[0,0,0] \tag{9.11}$$
$$[2,0]\to[2,0,0]-[1,0,0] \tag{9.12}$$

and we obtain

$$[2,0,0,0]\to[0,0,0]^{(2)}_{[0,0]}+[1,0,0]^{(2)}_{[1,1]}+[2,0,0]^{(2)}_{[0,0]}-[0,0,0]^{(\delta)}_{[1,1]}-[1,0,0]^{(\delta)}_{[0,0]} \tag{9.13}$$

On the other hand, for $[0,0,0,1]$ we have

$$[0,0,0,1]\to[0,1]_{[1,0]+[0,1]}\to[0,1,0]^{(\frac{3}{2})}_{[0,1]}+[0,0,1]^{(\frac{3}{2})}_{[1,0]} \tag{9.14}$$

On the other hand, we can consider the $l=4$ string level:

$$\mathrm{vac}_4=[3,0,0,0]+[1,0,0,1]+[1,0,0,0]+[0,1,0,0] \tag{9.15}$$

of them, only the state $[0,1,0,0]$ will produce a conformal dimension appropriate for our purposes:

$$\begin{aligned}&[0,1,0,0]\to[0,2]_{[0,0]}+[1,0]_{[1,1]}+[0,0]_{[2,0]}+[0,0]_{[0,2]}\to\\&\to[0,2,0]^{(3)}_{[0,0]}+[1,0,0]^{(3)}_{[1,1]}-[0,0,0]^{(\delta)}_{[1,1]}+[0,0,0]^{(3)}_{[2,0]}+[0,0,0]^{(3)}_{[0,2]}\end{aligned} \tag{9.16}$$

while for string level $l=5$, we have that only $[0,0,0]_{[0,0]}$ is produced with dimension $\Delta=3$.

when we insert these states in the formula

$$\mathcal{H}_l=\sum_{n=0}^{\infty}[n,0,0]^{(n)}_{[0,0]}\times\hat{T}_1^{(2)}\times v\hat{a}c_l^2 \tag{9.17}$$

we reproduce the states of SYM, in accordance with the AdS/CFT conjecture; in particular, we can find what SYM state is given by a certain string level, obtaining:

$$(\Delta = 2) \quad \rightarrow \quad [000]_{[0,0]}$$

$$(\Delta = 3) \quad \rightarrow \quad [100]_{[0,0]}$$

$$\begin{aligned}(\Delta = 4) \quad \rightarrow \quad & [200]_{[0,0]} + 2[000]_{[0,0]} + [000]_{[0,2]} + [000]_{[2,0]} + [000]_{[2,2]} + [011]_{[0,0]} + [200]_{[0,0]} + \\ & + 2[100]_{[1,1]}\end{aligned}$$

$$\begin{aligned}(\Delta = 5) \quad \rightarrow \quad & [300]_{[0,0]} + 2[100]_{[0,0]} + [100]_{[0,2]} + [100]_{[2,2]} + [100]_{[2,0]} + +[002]_{[0,0]} + [020]_{[0,0]} + \\ & + 2[111]_{[0,0]} + [300]_{[0,0]} + 2[011]_{[1,1]} + 2[200]_{[1,1]} + 2[000]_{[1,1]} + 2[011]_{[1,1]} + \\ & + 2[100]_{[0,0]} + [100]_{[0,2]} + [002]_{[2,0]} + [020]_{[0,0]} + [020]_{[0,2]} + [100]_{[2,0]} + [002]_{[0,0]} +\end{aligned}$$

$$\begin{aligned}\left(\Delta = \frac{11}{2}\right) \quad \rightarrow \quad & 4[010]_{[0,1]} + 4[101]_{[0,1]} + 2[210]_{[0,1]} + 4[001]_{[1,0]} + 4[110]_{[1,0]} + 2[201]_{[1,0]} + \\ & + 4[001]_{[1,2]} + 2[110]_{[1,2]} + 4[010]_{[2,1]} + 2[101]_{[2,1]} + 2[010]_{[2,3]} + 2[001]_{[3,2]}\end{aligned}$$

$$\begin{aligned}(\Delta = 6) \quad \rightarrow \quad & [400]_{[0,0]} + 2[200]_{[0,0]} + [102]_{[0,0]} + [120]_{[0,0]} + [022]_{[0,0]} + 2[211]_{[0,0]} + [400]_{[0,0]} + \\ & + [200]_{[0,2]} + [200]_{[2,0]} + 2[111]_{[1,1]} + [200]_{[2,2]} + 2[300]_{[1,1]} + 5[000]_{[0,0]} + \\ & + 6[200]_{[0,0]} + [102]_{[0,0]} + [120]_{[0,0]} + [022]_{[0,0]} + [211]_{[0,0]} + [400]_{[0,0]} + 2[000]_{[0,2]} + \\ & + 2[000]_{[2,0]} + 3[011]_{[0,2]} + 3[011]_{[0,2]} + 2[200]_{[0,2]} + 2[200]_{[2,0]} + [120]_{[0,2]} + \\ & + 8[100]_{[1,1]} + 2[002]_{[1,1]} + 2[020]_{[1,1]} + 4[111]_{[1,1]} + 2[300]_{[1,1]} + +[200]_{[2,2]} + \\ & + [011]_{[2,2]} + 6[011]_{[0,0]} + [102]_{[2,0]} + 5[000]_{[0,0]} + 2[020]_{[1,1]} + 2[111]_{[1,1]} + \\ & + 4[100]_{[3,1]} + [011]_{[2,2]} + 3[200]_{[2,2]} + [000]_{[2,4]} + [000]_{[4,2]} + 2[000]_{[0,4]} + \\ & + 2[100]_{[3,3]} + [000]_{[4,4]} + 2[200]_{[0,0]} + 3[000]_{[2,0]} + 3[011]_{[0,0]} + [102]_{[0,0]} + \\ & + [120]_{[0,0]} + 3[000]_{[0,2]} + [022]_{[0,0]} + 3[011]_{[2,0]} + 3[011]_{[2,0]} + [200]_{[0,2]} + \\ & + [200]_{[2,0]} + 8[100]_{[1,1]} + 2[002]_{[1,1]} + 4[100]_{[1,3]} + 7[000]_{[2,2]} + 2[000]_{[4,0]} + \\ & + [000]_{[0,0]}\end{aligned} \tag{9.18}$$

where the red color means the $l = 1$ KK tower, green color for $l = 2$ KK tower, blue for $l = 3$, orange for $l = 4$ and violet for $l = 5$.

We can compare these states with the states analyzed by Gromov [20], where an indipendent analysis were done, based on integrability and a bootsrap method; in that case were evaluated the conformal dimensions for the 219 states of SYM, up to bare conformal dimension $\Delta = 6$.

Using the bootsrap methos, they were able to evaluate the evolution of the conformal dimension from weak to strong coupling, obtaining, for instance in fig. 9.1, are present 45 of the 219 states analyzed, with the running conformal dimension from weak to strong coupling.

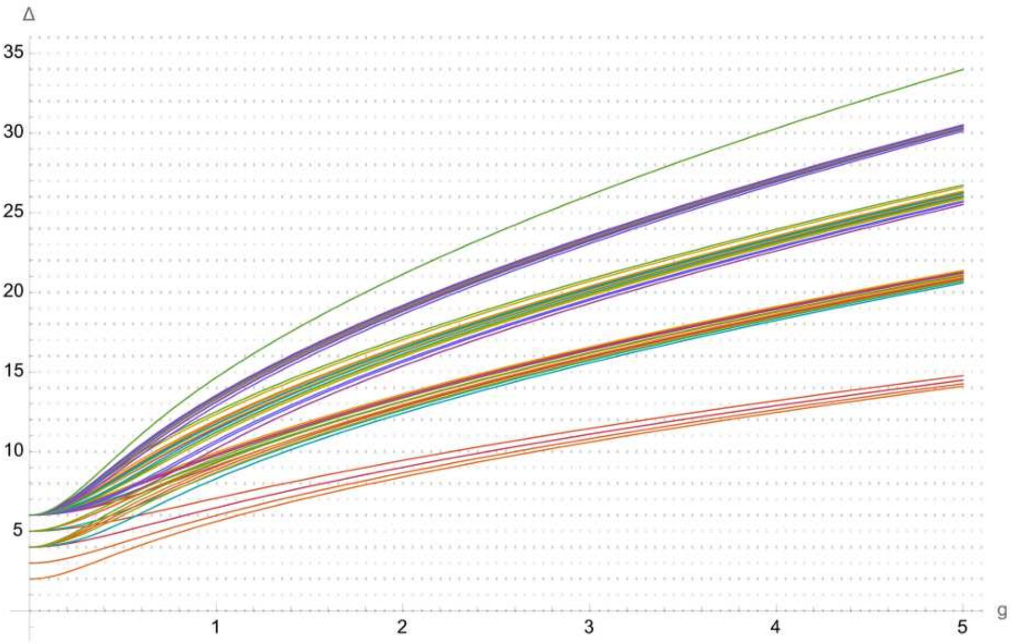
We know, from the analysis of [22], that at strong coupling and in first approximation, the behaviour of conformal dimension depends only by the corresponding string level.

$$\Delta \simeq 2\sqrt{l}\lambda^{\frac{1}{4}} \qquad \text{At strong coupling} \tag{9.19}$$

From the work of Gromov, it is then possible to associate the string level to the bare conformal dimension; in this way, we can give a justification to the assignment of conformal dimensions of string states in curved space.

Our analysis is perfectly consistent with the analysis of Gromov, confirming the attribution of conformal dimensions.

Figure 9.1: Running conformal dimension from weak to strong couplin for 45 states in SYM



Chapter 10

Conclusion

In this thesis, we have evaluated the massive Superstring states up to string level $l = 8$, using the method of plethystic exponential; from the tensor product of string states at the same string level, we have obtained the IIB Superstring spectrum, obtaining as a result:

$$T_l^{IIB} = T_1^2 \otimes (vac_l \otimes vac_l) \tag{10.1}$$

where T_1 is the first massive open string state.

We have evaluated the branched representation $SO(5) \times SO(4)$ for the Superstring states, starting from their $SO(9)$ representation; the $SO(5)$ component were expanded using Kaluza Klein theory, were we have demonstrated a formula for this expansion:

$$KK_{[m,n]} = \sum_{k=0}^{m} \sum_{s=0}^{n} \sum_{q=m-k}^{\infty} [a, n+k-s, k+s] \tag{10.2}$$

This formula can be a starting point for future analysis and it is different from formula used in several papers; for instance, it would be possible to analyze the paper [20], in order to understand how the "wrong" formula is used.

In addition, we have demonstrated the consistency of our KK formula with

$$KK_{\mathcal{R}_{SO(5)}} = \sum_{n=0}^{\infty} [n, 0, 0] \times \hat{\mathcal{R}}_{SO(5)} \tag{10.3}$$

On the CFT side, we have evaluated the spectrum of long single trace operators in $\mathcal{N} = 4$ SYM using Polya theory up to order $\Delta = 6$.

We have obtained from the first 5 string levels KK towers all the SYM states up to conformal dimension $\Delta = 6$. However, in this analysis, we have assigned arbitrarily the conformal dimensions to the string states, usually saturating their unitary bounds; however, we have demonstrated that this assignment is correct, comparing with the independent analysis of Gromov et. al. [20] in which they have evaluated the conformal dimension running from weak to strong coupling; knowing the behaviour of conformal dimension string states at strong coupling, we were able to evaluate as correct our assignment.

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