1. Question 1 Let $\alpha\beta$ be invertible. Since $dim(V) < \infty$, we know that $\alpha\beta$ is injective and hence, $ker(\alpha\beta) = \{0\}$. For any $v \in ker(\beta)$, we know that $(\alpha\beta)(v) = \alpha(\beta(v)) = \alpha(0) = 0$ since the kernal of $\alpha\beta$ is trivial. But this means that β is injective and hence invertible since otherwise, the kernal wouldn't be trivial. Similarly, α is injective and thus invertible.

2. Question 2
   We can easily apply the dimension formula for subspaces and see that $dim(Im(\alpha) + Im(\beta)) = dim(Im(\alpha)) + dim(Im(\beta)) - dim(Im(\alpha) \cap Im(\beta))$ and thus $rank(\alpha + \beta)$

3. Question 3
   (a) To show that α is linear, we need to show that $\alpha(f+kg) = \alpha(f)+k\alpha(g)$ for all $f, g \in V$ and $k \in \mathbb{R}$. It is easy to see that $\alpha(f+kg) = \begin{bmatrix} f'(0) + kg'(0) & 2(f(1) + kg(1)) \\ 0 & f''(3) + kg''(3) \end{bmatrix} = \begin{bmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{bmatrix} + k \begin{bmatrix} g'(0) & 2g(1) \\ 0 & g''(3) \end{bmatrix} = \alpha(f) + k\alpha(g)$ And thus it is linear.

   (b) We need $\alpha(f(x)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This means that $f'(0) = 0, 2f(1) = 0$, and $f''(3) = 0$. Since $f \in P_2(\mathbb{R})$, we know that $f = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$. This means that $f'(x) = 2ax + b$ and $f''(x) = 2a$. Thus, in order to satisfy the last condition, $a = 0$. Hence $b = 0$ to satisfy the first condition. This means that $c = 0$ to satisfy the middle one. Thus, α is injective since the kernal is the zero function. Hence $rank(\alpha) = 3$.

   (c) We can write $B_1 = 1E_{11} + 4E_{12}, B_2 = 1E_{11} - 4E_{12}, B_3 = 2E_{11} + 2E_{12} + 2E_{22}$.

4. Question 4
   It is clear that $spec(\alpha^{-1}\alpha)$