1. Question 1
   Since $f(au+bv) \geq min(f(u), f(v))$, we know that in particular, $f(0_V) = f(v-v) = f(1v+(-1)v) \geq min(f(v), f(v)) = f(v)$. Thus, $f(0_V) \geq f(v)$. Now, in order to check that W is a subspace of V, we need to check first that it is nonempty. By definition, $0_V \in W$ and thus, W is nonempty. Next, let $u, v \in W$. Then $f(c \cdot u + v) \geq min(f(u), f(v)) \geq h$. Hence, it is closed under addition and scaler multiplication. Thus, W is a subspace of V.

2. Question 2
   Let $W_1 \cup W_2$ be a subspace of V. In particular, this means for any $v \in W_1 \setminus W_2$ and $u \in W_2 \setminus W_1$, $v + u \in W_1 \cup W_2$. Hence, $v + u \in W_1$ or W_2. But we know that both W_1 and W_2 are closed under addition since they are both subspaces. Hence, this implies that either $u \in W_1$ or $v \in W_2$. Which is a contradiction. The only way to avoid this contradiction is if either $W_1 \subseteq W_2$ or vice versa. The other direction is trivial.

3. Question 3
   Let $\{v_1, ..., v_k\}$ be a basis for W. Extend it to a basis of V by $\{v_1, ..., v_k, u_1, ..., u_n\}$. Let $U = span(\{u_1, ..., u_n\})$. By construction, U and W don't intersect and U is a subspace of V. Thus, we have a compliment of W.

4. Question 4

   1. The dimension formula for subspaces states that $dim(W+U) = dim(W)+dim(U)-dim(W \cap U)$. Let $B_1 = \{v_1, ..., v_n\}$ be a basis for $W \cap U$. Then we can extend B_1 to form a basis of W and get $B_2 = \{v_1, ..., v_n, w_1, ..., w_k\}$. Similarly, we can extend B_1 to form a basis for U and get $B_3 = \{v_1, ..., v_n, u_1, ..., u_l\}$. Thus $dim(W) + dim(U) + dim(W \cap U) = n + k + n + l - n = n + k + l$. Now, we claim that $B = \{v_1, ..., v_n, w_1, ..., w_k, u_1, ..., u_l\}$ is a basis for $W+U$. Clearly, $span(B) = W+U$. To verify linear independence, we first set $\sum a_i v_i + \sum b_j w_j + \sum c_m u_m = 0$. This means that $-\sum a_i v_i = \sum b_j w_j + \sum c_m u_m$. Since the left hand side is in W and the right hand side is in U, then they must be in the intersction. This means that there are some $d_i \in \mathbb{F}$ such that $\sum d_i v_i = -\sum c_m u_m$. Since B_1 is a linearly independent set, this implies that $c_m = 0$. From this, it follows $a_i, b_j = 0$ and hence the set B is linearly independent and a basis. Thus, $dim(W+U) = n + k + l$.
   2. First, assume that W_i are independent. This means that their sum is a direct sum. This means that $W_i \cap W_j = \{0\}$ for $i \neq j$. Thus, by the dimension formula for subspaces, we know that $dim(\sum W_i) = \sum dim(W_i) - \sum_{i \neq j} dim(W_i \cap W_j)$. But since the intersections are trivial, we get $dim(\sum W_i) = \sum dim(W_i)$. Going the other way, we first assume that $dim(\sum W_i) = \sum dim(W_i)$. Using the dimension formula for subspaces, we know that $dim(\sum W_i) = \sum dim(W_i) - \sum_{i \neq j} dim(W_i \cap W_j)$. Hence, in this case, $\sum_{i \neq j} dim(W_i \cap W_j) = 0$. By definition, this means that W_i are independent.

5. Question 5

1. A basis for W would be $B = \{x, x^3\}$. Clearly the set is linearly independent. To see that it spans W, we note that any element $p \in W$ must be odd, by definition. Hence, we know that any p must have the form $c_1x + c_2x^3$. Thus, clearly our set spans W.
2. We can extend B to $A = \{1, x, x^2, x^3, x^4\}$. This is the standard basis for P_4.
3. We know that such a set U must exist. It is clear that the basis for U would be $\{1, x^2, x^4\}$. Thus, we define $U = span(\{1, x^2, x^4\})$.

6. Question 6
Clearly $W = \{w_1, ..., w_n\}$ is still a spanning set. To see that it is linearly independent, we must check that $\sum w_ic_i = 0$ if and only if $c_i = 0$. By definition, we know that $\sum c_iw_i = \sum_{i=1}^n c_i \sum_{j=1}^i v_i$. Thus, $\sum c_iw_i = (c_1 + ... + c_n)v_1 + (c_1 + ... + c_{n-1})v_2 + ... + c_nv_n$. Since $\{v_1, ..., v_n\}$ is a basis for V, we know that it is linearly independent and hence $(c_1 + ... + c_n)v_1 + (c_1 + ... + c_{n-1})v_2 + ... + c_nv_n = 0$ if and only if $c_i = 0$. Thus, W is linearly independent and hence a basis.