1. Question 1.1.4
   In order to check if $d(\cdot,\cdot)$ is a metric, we need to check that for $x, y \in \mathbb{R}^m$

   1. $d(x,y) > 0$ if $x \neq y$
   2. $d(x,x) = 0$
   3. $d(x,y) = d(y,x)$
   4. $d(x,y) \leq d(x,z) + d(z,y)$ for any $z \in \mathbb{R}^m$

   If $x \neq y$, then there must be at least one $i \in \{1, ..., m\}$ such that $x_i \neq y_i$. This means that $d(x,y) \geq 1 > 0$. Hence, the first condition is satisfied. Similarly, since $x = x$, there is no index where $x_i \neq x_i$ and hence $d(x,x) = 0$. Thus, the second condition is satisfied. Looking at the definition of $d(\cdot,\cdot)$, we see that

   $$d(x,y) = |\{k : x_k \neq y_k, k = 1, ..., m\}| = |\{k : y_k \neq x_k, k = 1, ..., m\}| = d(y,x)$$

   Hence the third condition is satisfied. If $z \neq x$ or $z \neq y$, then clearly $d(x,y) \leq d(x,z) + d(z,y)$. If $z = y$ or $z = x$, we would get equality. Now, let us define sets A, B, C as follows:

   $$A = \{k : x_k \neq y_k\}$$
   $$B = \{k : x_k \neq z_k\}$$
   $$C = \{k : z_k \neq y_k\}$$

   For any $k \in A$, this means that $x_k \neq y_k$. This means that either $x_k = z_k$ but $z_k \neq y_k$ OR $x_k \neq z_k$. The first case implies that $k \in C$ and the second implies that $k \in B$. This means that any index $k \in A$ must also be in either B or C. Thus, $A \subseteq B \cup C$. Hence, $d(x,y) = |A| \leq |B \cup C| \leq |B| + |C| = d(x,z) + d(z,y)$. Hence the final condition is satisfied.

2. Question 1.1.5
   In order to check if $d(\cdot,\cdot)$ is a metric, we need to check that for $x, y \in X$

   1. $d(x,y) > 0$ if $x \neq y$
   2. $d(x,x) = 0$
   3. $d(x,y) = d(y,x)$
   4. $d(x,y) \leq d(x,z) + d(z,y)$ for any $z \in X$

   Since $\hat{\rho}(x,y) = 0 \iff x = y$, then $d(x,x) = 0$. Hence the second condition is satisfied. Similarly, as $\hat{\rho}(x,y) \geq 0$, then we know that $d(x,y) \geq 0$ as it is the maximum of two numbers greater than 0. Hence the first conditin is met. Looking at the definition of $d(\cdot,\cdot)$, we see that

   $$d(x,y) = max\{\hat{\rho}(x,y), \hat{\rho}(y,x)\} = max\{\hat{\rho}(y,x), \hat{\rho}(x,y)\} = d(y,x)$$

Hence the third condition is satisfied. Finally, we must use the fact that $\hat{\rho}(x,y) \leq \hat{\rho}(x,z)+\hat{\rho}(z,y)$ and $\hat{\rho}(y,x) \leq \hat{\rho}(y,z)+\hat{\rho}(z,x)$ and state that

$$\begin{aligned} d(x,y) &= \max\{\hat{\rho}(x,y),\hat{\rho}(y,x)\} \\ &\leq \max\{\hat{\rho}(x,z)+\hat{\rho}(z,y),\hat{\rho}(y,z)+\hat{\rho}(z,x)\} \\ &\leq \max\{\hat{\rho}(x,z),\hat{\rho}(z,x)\}+\max\{\hat{\rho}(y,z),\hat{\rho}(z,y)\} \\ &= d(x,z)+d(z,y) \end{aligned}$$

Hence, the final condition is satisfied.

3. Question 1.1.16
   In order to check if $\bar{d}(\cdot,\cdot)$ is a metric, we must once again check that for $x, y \in X$

   1. $\bar{d}(x,y) > 0$ if $x \neq y$
   2. $\bar{d}(x,x) = 0$
   3. $\bar{d}(x,y) = d(y,x)$
   4. $\bar{d}(x,y) \leq d(x,z)+d(z,y)$ for any $z \in X$

   Since d is already known to be a metric, then we know that d satisfies that $d(f(x),f(y)) > 0$ when $f(x) \neq f(y)$. Since we know that f is one to one, we know that if $x \neq y$, then $f(x) \neq f(y)$ and hence $\bar{d}(x,y) > 0$ when $x \neq y$. Clearly $\bar{d}(x,x) = d(f(x),f(x)) = 0$ so the second condition is satisfied. Similarly, $\bar{d}(x,y) = d(f(x),f(y)) = d(f(y),f(x)) = \bar{d}(y,x)$ and so the third condition is satisfied. Now, let $z \in X$. Then $\bar{d}(x,y) = d(f(x),f(y)) \leq d(f(x),f(z)) + d(f(z),f(y)) = \bar{d}(x,z) + \bar{d}(z,y)$ since we know that d must satisfy the triangle inequality. Thus all conditions are met and $\bar{d}$ is a metric.

4. Question 1.2.5
   Since $x_n < y_n$ for all $n \geq 1$, we know that $\sup_{n>k}(x_n) \leq \sup_{n>k}(y_n)$. Hence, by definition, $\limsup(x_n) \leq \limsup(y_n)$. Similarly, $\inf_{n>k}(x_n) \leq \inf_{n>k}(y_n)$. Hence, by definition, $\liminf(x_n) \leq \limsup(y_n)$. An example of the first equality holding is letting $x_n = 1-\frac{2}{n}$ and $y_n = 1-\frac{1}{n}$. Then $y_n > x_n$ but $\limsup(x_n) = \limsup(y_n) = 1$. Similarly, for an example of the second equality holding is $x_n = \frac{1}{n}$ and $y_n = \frac{2}{n}$. Then $y_n > x_n$ but $\liminf(x_n) = \liminf(y_n) = 0$

5. Question 1.2.6
   First we must note that $x_n \leq \sup(x_n)$ and $y_n \leq \sup(y_n)$. This means that $x_n + y_n \leq \sup(x_n)+\sup(y_n)$ and hence $\sup(x_n+y_n) \leq \sup(x_n)+\sup(y_n)$. Similarly, $\inf(x_n+y_n) \geq \inf(x_n)+\inf(y_n)$. Now, taking the limit of both sides of $\sup(x_n+y_n) \leq \sup(x_n)+\sup(y_n)$, we get
   $$\lim_{k\to\infty} sup_{n\geq k}(x_n+y_n) \leq \lim_{k\to\infty} sup_{n\geq k}(x_n) + \lim_{k\to\infty} sup_{n\geq k}(y_n)$$
   which by definition gives us
   $$\limsup(x_n+y_n) \leq \limsup(x_n)+\limsup(y_n)$$

Taking the limit of both sides for the infimum inequality gets us

$$\liminf(x_n + y_n) \leq \liminf(x_n) + \liminf(y_n)$$

Now An example of the first strict inequality holding is letting $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then $\limsup(x_n + y_n) = 0$ but $\limsup(x_n) = \limsup(y_n) = 1$ so $\limsup(x_n) + \limsup(y_n) = 2$. An example of the second strict inequality holding is letting x_n and y_n be defined as above. Then $\liminf(x_n + y_n) = 0$ but $\liminf(x_n) = \liminf(y_n) = -1$ so $\limsup(x_n) + \limsup(y_n) = -2$.

6. Question 1.2.7
   Since $x_n \in c$, we know that x_n is cauchy and since $\mathbb{R}$ is complete, convergent. As it is convergent, $\limsup(x_n) = \lim_{n\to\infty} x_n = L$. Now, we know from the previous problem that $\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n)$. But this means that for large enough n, we can approximate x_n as L and see that $\sup_{k\geq n}(x_n + y_n) \approx \sup_{k\geq n}(L + y_n) = L + \sup_{k\geq n}(y_n)$. Taking the limit, we get

   $$\lim_{k\to\infty} sup_{n\geq k}(x_n + y_n) = L + \lim_{k\to\infty} sup_{n\geq k}(y_n) = \lim_{n\to\infty} x_n + \limsup(y_n)$$

7. Question 1.2.8
   (a) Using 1.2.5, we know that $\limsup(x_n) \leq \limsup(y_n)$ and $\limsup(y_n) \leq \limsup(z_n)$. Thus,

   $$\bar{L} = \limsup(x_n) \leq \limsup(y_n) \leq \limsup(z_n) = \bar{L}$$

   Hence, $\limsup(y_n) = \bar{L}$. Similarly, we know that $\liminf(x_n) \leq \liminf(y_n)$ and $\liminf(y_n) \leq \liminf(z_n)$. Thus,

   $$\underline{\mathrm{L}} = \liminf(x_n) \leq \liminf(y_n) \leq \liminf(z_n) = \underline{\mathrm{L}}$$

   Hence, $\liminf(y_n) = \underline{\mathrm{L}}$.

   (b) The Squeeze Theorem states that if $x_n \leq y_n \leq z_n$ and $x_n \to a, z_n \to a$, then $y_n \to a$. To prove it, we must first note that **NEED TO FINISH**

8. Question 1.2.9
   (a)

9. Question 1.2.10

10. Question 1.2.11