1. Question 2.1.2
   (a) Let (X, d_X) be discrete and $f : X \to Y$ be a function from X to Y, a metric space. Let $V \subseteq Y$ be open. Then $f^{-1}(V) \subseteq X$ clearly. But we know that all subsets of the discrete space is open. Thus, the preimage of open sets in Y are open in X. Hence, the f is continuous.
   (b) Assume that f is not constant. Then there are at least two distinct values in the range, say y_1, y_2. Since f is continous, we know that $f^{-1}(y)$ is open and so $f^{-1}(y_1)$ and $f^{-1}(y_2)$ must be open and disjoint sets. These sets would partition X which is a contradiction because X is connected but this implies that it is not. Thus, the function must be constant.
2. Question 2.1.3
   Let $f : \mathbb{R}^m \to \mathbb{R}^l$ be continuous. Then we know that for all $\varepsilon > 0$ there exists a δ such that for $a, b \in \mathbb{R}^m$, if $d(a, b) < \delta$ then $d(f(a), f(b)) < \varepsilon$. Assuming the Euclidean norms for each metric space, then $d(f(a), f(b)) = \sqrt{\sum_{i=1}^{l}(f_i(a) - f_i(b))^2}$. Since $d(f(a), f(b)) < \varepsilon$ we know that $\sqrt{\sum_{i=1}^{l}(f_i(a) - f_i(b))^2} < \varepsilon$. But through some algebraic manipulation, we see that $\sqrt{(f_i(a) - f_i(b))^2} < \sqrt{\sum_{i=1}^{l}(f_i(a) - f_i(b))^2} < \varepsilon$. Thus, for all a, b such that $d(a, b) < \delta$, we get that $d(f_i(a), f_i(b)) < \varepsilon$. Thus, if f is continuous, then so must be the component functions. Now, instead assume that all the f_i are continuous. This means that for all $a, b \in \mathbb{R}^m$, we see that if $d(a, b) < \delta$, we get that $d(f_i(a), f_i(b)) < \varepsilon$ for all i. Hence $\sqrt{\sum_{i=1}^{l}(f_i(a) - f_i(b))^2} < \sqrt{l \cdot \varepsilon}$ and thus function as a whole must be continous.
3. Question 2.1.4
   Let (X, d) be a metric space and $\varepsilon > 0$ be given. Now, suppose that $(a, b) \in X \times X$ and $(a', b') \in X \times X$ such that $\max d(a, a'), d(b, b') < 0.5\varepsilon = \delta$. Now we can see that $d(d(a, b), d(a', b')) \leq d(d(a, b), d(a', b)) + d(d(a', b), d(a', b')) \leq d(a, a') + d(b, b') \leq \varepsilon$. Thus, the metric is continuous.
4. Question 2.1.7
   Let $G \subseteq \mathbb{R}^m$ be the topologist sin curve. That is, $G = \{0\} \times [-1, 1] \cup \{(x, \sin(\frac{1}{x}) : x \in [0, 1])\}$. We know from class that this is a connected set. But let $a = (0, 0)$ and b be any other point in G. Then there is no continuous function from a to b since we know that the topologist sin curve is not connected. Hence this is a connected but not path connected set.
5. Question 2.1.9
   Let F be defined as in the question and let $a \in \mathbb{R}$ be arbitrary. Then there is a specific λ_0 such that $F(a) - \varepsilon < f_{\lambda_0}(a)$ for any $\varepsilon > 0$ since F is defined as the supremum. Since f_{λ_0} is continuous, we know there is a neighborhood G around a such that for all $x \in G$, we get $f_{\lambda_0}(x) - \varepsilon < f_{\lambda_0}(a)$ and thus
   $$F(a) - \varepsilon < f_{\lambda_0}(a) - \varepsilon < f_{\lambda_0}(a) < F(a)$$
   and hence F is lower semi-continuous.

6. Question 2.2.1
Since f is continuous, we know that for any x in X and for any $\varepsilon > 0$, there exists a δ such that $f(a) \in B(x, \varepsilon)$ if $a \in B(x, \delta)$. This means that if we swap the roles of ε and δ, for the inverse function $f^{-1} : Y \to X$ we get that f^{-1} is continuous. If X is not compact, we aren't guarenteed that the inverse is continuous. For example, let $X = (0, 1)$ and $f = \frac{2x-1}{x-x^2}$. The inverse wouldn't be continuous.

7. Question 2.2.2
Since X is compact, it must be complete. This means we can use the Banach fixed point theorem and state that f has a fixed point. Assume that a, b are both fixed points of f. This means that $d(f(a), f(b)) = d(a, b) < d(a, b)$. But this is a contradiction unless $a = b$. Hence, the fixed point is also unique.

8. Question 2.2.4
   (a) Let $u = \frac{1}{x}$. Then we get $\limsup(\sin(u)) = \lim_{n\to\infty}(\sup_{k>n}(sin(u_k)))$ where u_k is some increasing sequence of input values. But since $\sin(x)$ oscillates between -1 and 1 infinetly often, we know that the supremums will all be 1 and hence the limit will also be. Thus the limsup is 1.
   (b) Let $u = \frac{1}{x}$. Then we get $\liminf(\sin(u)) = \lim_{n\to\infty}(\inf_{k>n}(sin(u_k)))$ where u_k is some increasing sequence of input values. But since $\sin(x)$ oscillates between -1 and 1 infinetly often, we know that the infimums will all be -1 and hence the limit will also be. Thus the liminf is -1.

9. Question 2.2.7
We know that f is 0 at all rational values between a and b. We must now note that between any two rational numbers, there exists an irrational. This can be seen by noting that for $a, b \in \mathbb{Q}$, $a + \frac{\sqrt{2}(b-a)}{2} \in [a, b]$ and this value is irrational. Now by way of contradiction assume that f is non zero on some irrational $x \in [a, b]$. Say $f(x) = k$. Since f is continuous, there is a $\delta > 0$ such that if $y \in B_1 = B(x, \delta)$, then $f(y) \in B_2 = B(f(x), \frac{k}{2})$. But there must be rational $y \in B_1$. This is a contradiction as these values are zero but they would be in B_2 and are hence nonzero. Thus, f is zero at all values of $x \in [a, b]$

10. Question 2.2.8
It is clear that $\frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n} \in [f(a), f(b)]$. Since f is continuous, we can apply the Intermediate Value Theorem and state that there is a $\zeta \in [a, b]$ such that $f(\zeta) = \frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}$

11. Question 2.2.9
Assume that there is some $x \in [a, b]$ such that $f(x) < 0$. Then we can apply the mean value theorem and state that there must be some point $\zeta \in [a, b]$ such that $f(\zeta) = 0$. But this is a contradiction to the assumption. Thus, f is never negative.

12. Question 2.2.11
Choose $\varepsilon > 0$ such that the maximum value of $f([a + \varepsilon, b - \varepsilon])$ is less than all values in

$f((a, a+\varepsilon) \cup (b-\varepsilon, b))$. We know this maximum exists and is attained because we can apply the Extreme Value Theorem to the compact set $[a+\varepsilon, b-\varepsilon]$. This also means that there is a minimum value in the set $f([a+\varepsilon, b-\varepsilon])$ and thus the proof is complete.

13. Question 2.2.12

    (a) Since f, g are uniformly continous, we know that for all $\varepsilon > 0$, ther exists a $\delta > 0$, for all x, y such that $d(x, y) < \delta$, we get that $d(f(x), f(y)) < \varepsilon$ and the same for g. Now examining $f \pm g$, we need to look at $d(f(x) \pm g(x), f(x) \pm g(y))$. I will focus on the sum but the difference comes with very similar logic. It is clear from the triangle inequality that $d(f(x)+g(x), f(y)+g(y)) < d(f(x), g(x)) + d(f(y), g(y)) < \varepsilon + \varepsilon$. Thus, the sum is uniformly continous as well.

    (b) Looking at the product, we need to examine $d(f(x) \cdot g(x), f(x) \cdot g(y))$. For the purpose of notation, we will assume the Euclidean norm. Thus, we can rewrite $d(f(x) \cdot g(x), f(x) \cdot g(y)) = |f(x)g(x) - f(y)g(y)|$. This can be manipulated to be

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\
&\leq |f(x)|\varepsilon + |g(y)|\varepsilon
\end{aligned}$$

    Thus, the product will be uniformly continous only if the functions are bounded.

14. Question 2.2.15

    (a) $f(x) = \sin(\frac{1}{x})$ is not uniformly continous. Assume that it was by way of contradiction. Then for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-y| < \delta$, then $|\sin(\frac{1}{x}) - \sin(\frac{1}{y})| < \varepsilon$. Now, choose $\varepsilon = \frac{1}{100}$ and set $y = x + \frac{\delta}{2}$. Then we find that $|\sin(\frac{1}{x}) - \sin(\frac{1}{x+\frac{\delta}{2}})| < \frac{1}{100}$. But for x small enough, this might not be true as $\sin(\frac{1}{x})$ oscillates very quickly between 1 and -1 and hence it is not uniformly continous.

    (b) $f(x) = \sin(x)$ is uniformly continous. To see this, assume that x and y satisfy $|\sin(x) - \sin(y)| < \varepsilon$. This implies that $|sin(x) - sin(y)| = |2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2})| \leq 2|\sin(\frac{x-y}{2})| < \varepsilon$. Now, if $|x-y| < \delta$, this implies $\frac{|x-y|}{2} < \delta$. Now, since for small $|\sin(x)| \leq |x|$, we know that $2|\sin(\frac{x-y}{2})| \leq 2\delta < \varepsilon$. Thus, if we choose $\delta = \frac{\varepsilon}{2}$ then we get a uniformly continous function.

    (c) $f(x) = x^2$ is not uniformly continous. Assume that it was by way of contradiction. Then for any $\varepsilon > 0$, there is a $\delta > 0$ such that if $|x-y| < \delta$, then $|x^2 - y^2| < \varepsilon$. Now, choose $\varepsilon = \frac{1}{100}$ and set $y = x + \frac{\delta}{2}$. Then we find that $|x^2 - (x+\frac{\delta}{2})^2| < \frac{1}{100}$. But for x large enough, this might not be true and hence it is not uniformly continous.

15. Question 2.2.16
    We will first examine f on the closed interval $[-1, 1]$. Since f is uniformly continous, it must be continous and hence we can find $M = \sup_{|x| \leq 1} |f(x)|$. Now, for any x outside

of the interval, choose x_0 such that $x_0 = -1$ if $x < 0$ and $x_0 = 1$ if $x > 1$. Since f is uniformly continous, we know that there exists a $\delta > 0$ such that if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < L|x - x_0|$ where L is chosen to be a constant that satisfies that inequality. We know it exists since f is uniformly continous. Now,

$$|f(x)| \leq M + L|x - x_0|$$

but $|x - x_0| \leq |x| + 1$ so

$$|f(x)| \leq M + L(|x| + 1)$$

Thus, choosing $K = L + M$, we get the inequality we seek.

16. Question 3.1.4
First, we must ensure that f is continous at 0. This means $\lim_{x \to 0} f(x) = f(0) = 0$. Since $\sin \frac{1}{|x|}$ oscillates between -1 and 1, we know that the limit will be dominated by the $|x|^\alpha$ term. By the squeeze theorem, the limit will tend to 0 as long as $\alpha > 0$. We also need the function to be differentiable at 0 and hence need the limit $f'(0) = \lim_{x \to 0} \frac{|x|^\alpha \sin \frac{1}{|x|^\beta}}{x} = \lim_{x \to 0} |x|^{\alpha - 1} \sin \frac{1}{|x|^\beta}$ to converge. Like before, the sinusoidal term will be dominated by the linear term and hence as long as $\alpha - 1 > 0$, then the limit tends to 0 and hence exists. Thus the conditions are that $\beta \in \mathbb{R}$ and $\alpha > 1$.

17. Question 3.1.5
   (a) Since f is even $f(-x) = f(x)$. Thus, $f(-x) - f(x) = 0$. Taking the derivative, we get that $-f'(-x) - f'(x) = 0 \implies f(-x) = -f(x)$. Thus, the derivative is odd.
   (b) Since f is odd $f(-x) = -f(x)$. Thus, $f(-x) + f(x) = 0$. Taking the derivative, we get that $-f'(-x) + f'(x) = 0 \implies f(-x) = f(x)$. Thus, the derivative is even.

18. Question 3.1.8
It is clear that f is continous because $\lim_{x \to 0} f(x) = f(0) = 0$ since the limit is dominated by the quadratic term. To show the derivative exists, we need to show that $f'(0) = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x}$ exists. But it is clear that again, the sinusoidal term is dominated by the linear term and hence the limit goes to 0. Thus the function is differentiable. But we know that $\lim_{x \to 0} x \sin \frac{1}{x} = 1 \neq f'(0) = 0$. Thus, the derivative isn't continous.

19. Question 3.1.9

   (a) To be Gateaux differentiable, the function must have derivatives from every direction. This means that

$$D_v(0,0) \lim_{t \to 0} \frac{f(tv_1, tv_2) - f(0,0)}{t}$$

   must exist and where $v = (v_1, v_2)$ is a direction vector. Substituting that in, we get $\frac{(tv_1)^p (tv_2)^q}{|tv_1|^n + |tv_2|^m} = \frac{t^{p+q} v_1^p v_2^q}{t^n |v_1|^n + t_m |v_2|^m}$ in the numerator. Thus, we would need $p + q > \max n, m$ to ensure that the power on the t is nonnegative and the limit converges.

(b) In order for f to have a gradient, we need the partials to exist and be continous. Taking the partials, we get

$$\frac{\partial f}{\partial x} = \frac{px^{p-1}y^q}{|y|^m + |x|^n} - \frac{nx^{p+1}y^q|x|^{n-2}}{(|y|^m + |x|^n)^2},$$
$$\frac{\partial f}{\partial y} = \frac{qx^p y^{q-1}}{|y|^m + |x|^n} - \frac{mx^p y^{q+1}|y|^{m-2}}{(|y|^m + |x|^n)^2}.$$

Clearly, in order for the functions to exist as we tend to 0, we would need $p+1+n-2>0$ or $p+n-1>0$ or $p>n+1$. Similarly, we would need $q+1+m-2>0$ or $q+m-1>0$ or $q>m+1$.

(c) In order for f to be Frechet differentiable, we need the partials to be continuous. Thus, we need

$$\frac{\partial f}{\partial x} = \frac{px^{p-1}y^q}{|y|^m + |x|^n} - \frac{nx^{p+1}y^q|x|^{n-2}}{(|y|^m + |x|^n)^2},$$
$$\frac{\partial f}{\partial y} = \frac{qx^p y^{q-1}}{|y|^m + |x|^n} - \frac{mx^p y^{q+1}|y|^{m-2}}{(|y|^m + |x|^n)^2}.$$

To be continuous.

20. Question 3.1.11
We can apply the Mean Value Theorem and see that there exists a point $c \in [x, x+h]$ such that $f(x+h) - f(x) = f'(c)(x+h-x)$ and similarly there exists a point $d \in [x, x+h]$ such that $g(x+h) - g(x) = g'(d)(x+h-x)$. Thus, $\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(c)}{g'(d)}$ We know that $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$. But, as $x \to \infty$, we know that $f(x) \to 0$ and hence $f(x+h) - f(x) \approx f(x+h)$ and $g(x+h) - g(x) \approx g(x+h)$. Thus, $\frac{f(x+h)-f(x)}{g(x+h)-g(x)} \approx \frac{f(x+h)}{g(x+h)} \approx \frac{f'(c)}{g'(d)}$. As $x \to \infty$, so will c and d. Thus, $\frac{f(x+h)}{g(x+h)} \approx \frac{f'(c)}{g'(d)} \to L$. As x grows, the approximation will go to an equality. Thus, we have proved the statement.