1. Question 1.1.4
   In order to check if $d(\cdot, \cdot)$ is a metric, we need to check that for $x, y \in \mathbb{R}^m$

   1. $d(x, y) > 0$ if $x \neq y$
   2. $d(x, x) = 0$
   3. $d(x, y) = d(y, x)$
   4. $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in \mathbb{R}^m$

   If $x \neq y$, then there must be at least one $i \in \{1, ..., m\}$ such that $x_i \neq y_i$. This means that $d(x, y) \geq 1 > 0$. Hence, the first condition is satisfied. Similarly, since $x = x$, there is no index where $x_i \neq x_i$ and hence $d(x, x) = 0$. Thus, the second condition is satisfied. Looking at the definition of $d(\cdot, \cdot)$, we see that

   $$d(x, y) = |\{k : x_k \neq y_k, k = 1, ..., m\}| = |\{k : y_k \neq x_k, k = 1, ..., m\}| = d(y, x)$$

   Hence the third condition is satisfied. If $z \neq x$ or $z \neq y$, then clearly $d(x, y) \leq d(x, z) + d(z, y)$. If $z = y$ or $z = x$, we would get equality. Now, let us define sets A, B, C as follows:

   $$A = \{k : x_k \neq y_k\}$$
   $$B = \{k : x_k \neq z_k\}$$
   $$C = \{k : z_k \neq y_k\}$$

   For any $k \in A$, this means that $x_k \neq y_k$. This means that either $x_k = z_k$ but $z_k \neq y_k$ OR $x_k \neq z_k$. The first case implies that $k \in C$ and the second implies that $k \in B$. This means that any index $k \in A$ must also be in either B or C. Thus, $A \subseteq B \cup C$. Hence, $d(x, y) = |A| \leq |B \cup C| \leq |B| + |C| = d(x, z) + d(z, y)$. Hence the final condition is satisfied.

2. Question 1.1.5
   In order to check if $d(\cdot, \cdot)$ is a metric, we need to check that for $x, y \in X$

   1. $d(x, y) > 0$ if $x \neq y$
   2. $d(x, x) = 0$
   3. $d(x, y) = d(y, x)$
   4. $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in X$

   Since $\hat{\rho}(x, y) = 0 \iff x = y$, then $d(x, x) = 0$. Hence the second condition is satisfied. Similarly, as $\hat{\rho}(x, y) \geq 0$, then we know that $d(x, y) \geq 0$ as it is the maximum of two numbers greater than 0. Hence the first conditin is met. Looking at the definition of $d(\cdot, \cdot)$, we see that

   $$d(x, y) = max\{\hat{\rho}(x, y), \hat{\rho}(y, x)\} = max\{\hat{\rho}(y, x), \hat{\rho}(x, y)\} = d(y, x)$$

Hence the third condition is satisfied. Finally, we must use the fact that $\hat{\rho}(x,y) \leq \hat{\rho}(x,z) + \hat{\rho}(z,y)$ and $\hat{\rho}(y,x) \leq \hat{\rho}(y,z) + \hat{\rho}(z,x)$ and state that

$$\begin{aligned} d(x,y) &= \max\{\hat{\rho}(x,y), \hat{\rho}(y,x)\} \\ &\leq \max\{\hat{\rho}(x,z) + \hat{\rho}(z,y), \hat{\rho}(y,z) + \hat{\rho}(z,x)\} \\ &\leq \max\{\hat{\rho}(x,z), \hat{\rho}(z,x)\} + \max\{\hat{\rho}(y,z), \hat{\rho}(z,y)\} \\ &= d(x,z) + d(z,y) \end{aligned}$$

Hence, the final condition is satisfied.

3. Question 1.2.5
Since $x_n < y_n$ for all $n \geq 1$, we can state that $\{y_n\}$ is an upperbound of $\{x_n\}$. Hence, $\sup{(x_n)} \leq \sup{(y_n)}$. Thus $\limsup_{n\to\infty}(x_n) \leq$