

MAS 5145 Lecture Notes - Fall 2024

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Contents

Chapter 1

Review

Page 2

- 1.1 08/20/2024
- 1.2 08/22/2024
- 1.3 08/27/2024
- 1.4 08/29/2024

2
2
4
6

Chapter 2

Eigen Things

Page 7

- 2.1 09/05/2024
- 2.2 09/10/2024

7
8

Chapter 1

Review

1.1 08/20/2024

Definition 1.1.1: Vector Spaces

A Vector Space is a nonempty set V with two operations, vector addition and scalar multiplication. These operations must satisfy a bunch of axioms, most important of which is $u, v \in V \implies u + v \in V$ and for $\alpha \in \mathbb{F}, \alpha v \in V$.

Definition 1.1.2: Subspace

A Subspace W of a vector space V is a nonempty subset of V with the same operations as V .

Proposition 1.1.1

The intersection of any collection of subspaces W_j of V is itself a subspace of V

Proof: In order to be a subspace, we must prove that the intersection is nonempty and that it is closed under the operations of V . Clearly, since each W_j is a subspace, they must each contain the zero element. Hence, the intersection must as well and hence, the intersection is nonempty. For any elements $u, v \in \bigcap W_j = W$, we know that a linear combination $\alpha u + \beta v \in W_j$ for each W_j since they are each subspaces and hence closed under the vector operations. This means that $\alpha u + \beta v \in W$ and hence W is a subspace. \odot

Definition 1.1.3: Direct Sum

Given two subspaces W_1 and W_2 of V , if $W_1 \cap W_2 = \{0\}$, then $W_1 + W_2$ is a direct sum of W_1 and W_2 . For a collection of subspaces, we have a direct sum of $W_i \cap_{j \neq i} W_j = \{0\}$

1.2 08/22/2024

Definition 1.2.1: Linear Combination

Let V be a vector space and $B = \{v_1, \dots, v_k\} \subset V$. A linear combination of B is a vector of the form $v = \sum c_i v_i (c_i \in \mathbb{F})$.

Definition 1.2.2: Spanning Set

$S \subseteq V$ is called a spanning set if $\text{span}(S) = V$.

Definition 1.2.3: Linear Independence and Dependence

$B \subset V$ is called linearly dependent if there exists c_1, \dots, c_n not all 0 such that $\sum b_i c_i = 0$. Otherwise we say that B is linearly independent.

Definition 1.2.4: Basis

$S \subseteq V$ is called a basis of V if it is linearly independent and spanning.

Theorem 1.2.1

Every Vector Space has a basis

Proof: In the finite dimensional case, we know that $V = \text{span}(\{v_1, \dots, v_n\})$ for some set v_1, \dots, v_n . If the spanning set is linearly independent, then we have a basis. Otherwise, remove linearly dependent vectors and recheck until we have a linearly independent set, which is thus a basis. In the infinite dimensional case, we must use Zorns Lemma but it is true that we can find the basis. ☺

Question 1

Every spanning set contains a basis

Proof: Let $B = \{v_1, \dots, v_n\}$ such that B is a spanning set. If B is linearly independent, the basis is itself. Otherwise, there must be some vector v_k that can be written as a linear combination of the other vectors. We can remove v_k and recheck the new B to see if it is linearly independent. This process must terminate and when it does, the final set will be linearly independent by construction. Hence, that final set will be a basis. ☺

Question 2

Every linearly independent set can be extended to a basis

Proof: Let $B = \{v_1, \dots, v_n\}$ such that B is linearly independent. If $\text{span}(B) = V$, then we have a basis. Otherwise, there must be some vector $v \in V$ such that $v \notin \text{span}(B)$. Append v to B and recheck if it is a spanning set. If not, repeat the process until we have a spanning set. At that point we will have a basis. ☺

Question 3

Suppose $A = \{v_1, \dots, v_k\}$ is linearly independent and $B = \{w_1, \dots, w_m\}$ is a spanning set. Then $k \leq m$.

Proof: We know that every linearly independent set can be extended to form a basis. This means that one can turn the LI set into one that also spans only by adding vectors to it. Similarly, every spanning set contains a basis implies that one can turn a spanning set into one that also is LI only by removing vectors from it. Together, these imply that the cardinality of any spanning set must be greater than or equal to that of any LI set. Hence, $k \leq m$. ☺

Definition 1.2.5: Dimension of a Vector Space

Let S be a basis for a vector space V . Then $\dim(V)$ is the cardinality of S .

Lemma 1.2.1

Let $\dim(V) = n < \infty$. Every n LI vectors form a basis.

Proof: Let $\{v_1, v_2, \dots, v_n\}$ be a LI set. Then we can extend it to a basis $B = \{v_1, v_2, \dots, v_n, u_1, \dots, u_k\}$. But we know that the dimension of V is n and since B is a basis of V , then $\dim(V) = n + k$. Hence $n = n + k$ which implies $k = 0$ and the original set was a basis. ☺

Lemma 1.2.2

Let $\dim(V) = n < \infty$. Every n spanning vectors form a basis

Proof: Let $\{v_1, v_2, \dots, v_n\}$ be a spanning set. Then we can select vectors from it to form a basis $B = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$. But we know that the dimension of V is n and since B is a basis of V , then $\dim(V) = k$. Hence $n = k$ which implies the original set was a basis. \odot

If B is a basis of V , then every vector in V has a unique representation as a linear combination of vectors in B .

1.3 08/27/2024

Let W_1, W_2 be subspaces of V . Then the following are equivalent

1. $W_1 \oplus W_2$
2. For all $v \in W_1 + W_2$ there exists unique $w_1 \in W_1, w_2 \in W_2$ such that $v = w_1 + w_2$
3. $w_1 + w_2 = 0 \implies w_1 = w_2 = 0$
4. The union of a basis for W_1 and one for W_2 is basis of $W_1 + W_2$

Proof: The first three all follow from definition of the direct sum. The fourth equivalence can be seen by letting $A = \{u_1, u_2, \dots, u_k\}$ be a basis for W_1 and $B = \{v_1, v_2, \dots, v_n\}$ be a basis for W_2 . Clearly, $\text{span}(A \cup B) = W_1 + W_2$. Linear independence can be seen since the two sets themselves are basis and hence linearly independent. Thus, we have a basis of $W_1 + W_2$. \odot

Lemma 1.3.1

If W is a subspace of V , then there exists a subspace U of V such that $V = U \oplus W$. U is called a complement of W .

Proof: Let $\{w_1, w_2, \dots, w_k\}$ be a basis of W . Extend it to be a basis for $V, \{w_1, \dots, w_k, u_1, \dots, u_n\}$. Set $U = \text{span}(\{u_1, \dots, u_n\})$. Thus, $V = U \oplus W$. \odot

Theorem 1.3.1 Dimension Formula for Subspaces

Let W, U be finite dimensional subspaces of V . Then $\dim(W + U) = \dim(W) + \dim(U) - \dim(W \cap U)$.

Proof: Let $B_1 = \{v_1, \dots, v_n\}$ be a basis for $W \cap U$. Then we can extend B_1 to form a basis of W and get $B_2 = \{v_1, \dots, v_n, w_1, \dots, w_k\}$. Similarly, we can extend B_1 to form a basis for U and get $B_3 = \{v_1, \dots, v_n, u_1, \dots, u_l\}$. Thus $\dim(W) + \dim(U) - \dim(W \cap U) = n + k + n + l - n = n + k + l$. Now, we claim that $B = \{v_1, \dots, v_n, w_1, \dots, w_k, u_1, \dots, u_l\}$ is a basis for $W + U$. Clearly, $\text{span}(B) = W + U$. To verify linear independence, we first set $\sum a_i v_i + \sum b_j w_j + \sum c_m u_m = 0$. This means that $-\sum a_i v_i = \sum b_j w_j + \sum c_m u_m$. Since the left hand side is in W and the right hand side is in U , then they must be in the intersection. This means that there are some $d_i \in \mathbb{F}$ such that $\sum d_i v_i = -\sum c_m u_m$. Since B_1 is a linearly independent set, this implies that $c_m = 0$. From this, it follows $a_i, b_j = 0$ and hence the set B is linearly independent and a basis. Thus, $\dim(W + U) = n + k + l$. \odot

Definition 1.3.1: Linear Transformation

Let V, W be two vector spaces over \mathbb{F} . A map $\alpha : V \rightarrow W$ is linear if $\alpha(au + bv) = a\alpha(u) + b\alpha(v)$

Proposition 1.3.1

If α is linear, then:

- $\ker(\alpha) = \{v \in V : \alpha(v) = 0\}$ is a subspace of V .
- $\alpha(0) = 0$

- $im(\alpha) = \{\alpha(v) : v \in V\}$ is a subspace of W .

Proof: NEED TO DO



Definition 1.3.2: Injective or One-to-One

A linear transformation α is called injective or one to one if $\alpha(x) = \alpha(y)$ if and only if $x = y$

Question 4

A linear transformation α is injective if and only if $ker(\alpha) = \{0\}$.

Proof: Assume that α is injective. Now, suppose that $x \in ker(\alpha)$. This means that $\alpha(x) = 0$. But we know that $\alpha(0) = 0$. Since α is injective and $\alpha(x) = \alpha(0) = 0$, then $x = 0$. Hence $ker(\alpha) = \{0\}$. This has proved one direction. For the other direction, assume that $ker(\alpha) = \{0\}$. Now assume that $\alpha(x) = \alpha(y)$. This means that $\alpha(x) - \alpha(y) = 0$ and hence $x - y \in ker(\alpha)$. But this implies that $x = y$. Hence, when $\alpha(x) = \alpha(y)$, $x = y$. Thus, α is injective.



Definition 1.3.3: Surjective or onto

A linear transformation α is called surjective or onto if $im(\alpha) = W$.

Definition 1.3.4: Bijective

A linear transformation α is called bijective if it is both injective and surjective.

Proposition 1.3.2

Let $\alpha : V \rightarrow W$ be linear:

1. α is injective if and only if $ker(\alpha) = 0$
2. If α is bijective, α^{-1} is linear
3. If α is injective and S is LI, then $\alpha(S)$ is LI
4. If α is surjective and $span(S) = V$, $span(\alpha(S)) = W$.
5. If α is bijective, α maps a basis of V to a basis of W .

Proof: NEED TO DO



Proposition 1.3.3

If $\alpha : V \rightarrow W$ and $\beta : W \rightarrow U$ are both linear, then so is $\beta\alpha$.

Proof: NEED TO DO



Definition 1.3.5: Nullity

We define $nullity(\alpha) = dim(ker(\alpha))$.

Definition 1.3.6: Rank

We define $rank(\alpha) = dim(im(\alpha))$.

Theorem 1.3.2 Dimension Formula for Linear Transformations

If V and W are finite dimensional vector spaces, then if α is linear, $\dim(V) = \text{rank}(\alpha) + \text{nullity}(\alpha)$.

Proof: Since $\ker(\alpha)$ is a subspace, we know it must have a basis. Let $B = v_1, v_2, \dots, v_k$ be a basis for $\ker(\alpha)$. We know that the kernel is finite dimensional since it is a subspace of a finite dimensional space V . Now, extend B to form a basis for V . Let $\dim(V) = n$. Hence, we get $A = v_1, \dots, v_k, v_{k+1}, \dots, v_n$. Now, $\text{im}(\alpha) = \text{span}(\alpha(A)) = \text{span}(\alpha(v_1), \dots, \alpha(v_k), \alpha(v_{k+1}), \dots, \alpha(v_n)) = \text{span}(\alpha(v_{k+1}), \dots, \alpha(v_n)) = \text{span}(A \setminus B)$. We now know that $A \setminus B = C$ is a spanning set of $\text{im}(\alpha)$. To see that C is linearly independent, we set $\sum_{i=1}^{n-k} d_i \alpha(v_i) = 0$. Since α is linear, we can take out the transformation and get $\alpha(\sum_{i=1}^{n-k} d_i v_i) = 0$. But this means that $\sum_{i=1}^{n-k} d_i v_i \in \ker(\alpha)$ which cannot be the case unless all $d_i = 0$. Hence, C is linearly independent and thus a basis for $\text{im}(\alpha)$. Thus $\text{rank}(\alpha) + \text{im}(\alpha) = k + (n - k) = n = \dim(V)$. \odot

1.4 08/29/2024

Lemma 1.4.1

Suppose $\dim(V) = \dim(W) = n < \infty$ and $T : V \rightarrow W$ is a linear map. Then T is injective if and only if it is surjective.

Proof: Assume that T is injective. Then $\ker(T) = \{0\}$ and hence $\text{nullity}(T) = 0$. Thus, by the Dimension Formula for Linear Transformations, we know that $\text{rank}(T) = \dim(V) - \text{nullity}(T) = n - 0 = n$. Hence $\text{rank}(T) = n$ and thus, T is surjective. Now, assume instead that T is surjective. Similarly, we know that $\text{nullity}(T) = 0$ and thus, T is injective. \odot

Theorem 1.4.1

Say $T : V \rightarrow W$ and $S : W \rightarrow Y$ with T, S linear. Then

1. $\text{nullity}(ST) \leq \text{nullity}(T) + \text{nullity}(S)$
2. $\text{rank}(T) + \text{rank}(S) - \dim(W) \leq \text{rank}(ST) \leq \min(\text{rank}(S), \text{rank}(T))$

Proof: Since $\ker(ST)$ is a subspace, we know it has a basis and hence, let $B = \{c_1, \dots, c_g\}$ to be a basis for the kernel. This is the set of all vectors $c \in V$ such that $S(T(c)) = 0$. These specific $T(v)$ form a subset of $\ker(S)$. Call that set G . Then $G \subseteq \ker(S)$ and hence $\ker(ST) \subseteq \ker(S)$. This implies that $\text{nullity}(ST) \leq \text{nullity}(S) \leq \text{nullity}(S) + \text{nullity}(T)$.

To prove the second item, NEED TO DO \odot

Definition 1.4.1: Homomorphisms

Let V, W be two vector spaces. The set of all linear transformations from V to W is called $\text{Hom}(V, W)$ or $L(V, W)$.

: $\text{Hom}(V, W)$ is a vector space and $\dim(\text{Hom}(V, W)) = \dim(V) \cdot \dim(W)$.

Chapter 2

Eigen Things

2.1 09/05/2024

Definition 2.1.1: Eigenvalue

Let $\alpha \in \text{End}(V)$. Then λ is an eigenvalue if there exists $v \neq 0$ such that $\alpha(v) = \lambda v$. We call v an eigenvector of α .

Definition 2.1.2: Spectrum

We call the set of all eigenvalues of α the spectrum of α and denote it $\text{spec}(\alpha)$.

Definition 2.1.3: Eigenspace

We call $E_\lambda(\alpha) = \{v : \alpha(v) = \lambda v\}$ the eigenspace of α with respect to λ . It is a subspace of V .

Definition 2.1.4: Spectral Radius

The spectral radius of α is $\rho(\alpha) = \sup\{|\lambda| : \lambda \in \text{spec}(\alpha)\}$.

Definition 2.1.5: Characteristic Polynomial

Let $\alpha \in \text{End}(V)$ and $\dim(V) = n$ be finite. Then the characteristic polynomial of α is defined by $p(t) = \det(tI - A)$ where A is a matrix representation of α with respect to a basis D . $p(t) = t^n + a_{n-1}t^{n-1} + \dots + a_1t + a_0$ and is independent of the choice of basis.

Question 5

Prove that $p(t)$ is independent of the choice of basis

: If B and C are two representations of α with respect to basis D, E respectively, then $B = S^{-1}CS$ for some invertible matrix S (the change of basis matrix). Then $\det(tI - B) = \det(tS^{-1}S - S^{-1}CS) = \det(S^{-1}(tI - C)S) = \det(tI - C)$ ☺

Question 6

Prove that if $\lambda \in \text{spec}(\alpha)$, then $p_\alpha(\lambda) = 0$.

: If $\lambda \in \text{spec}(\alpha)$, then $(\lambda I - A)\hat{x} = 0$ has a nontrivial solution. Thus $(\lambda I - A)$ is non invertible and hence $\det(\lambda I - A) = p_\alpha(\lambda) = 0$. ☺

Theorem 2.1.1 Spectral Mapping Theorem for Polynomials

Let $\alpha \in \text{End}(V)$ and $p(t)$ be a polynomial. Then $\text{spec}(p(\alpha)) = \{p(\lambda) : \lambda \in \text{spec}(\alpha)\}$ where $p(\alpha) = \sum a_i \alpha^i = a_0 I + a_1 \alpha + \dots + a_n \alpha^n \in \text{End}(V)$.

: Let $\lambda \in \text{spec}(\alpha)$. Then there exists $v \neq 0$ such that $\alpha(v) = \lambda v$. This means that for all $k \in \mathbb{N}$, $\alpha^k(v) = \lambda^k v$. Thus $p(\alpha)(v) = \sum a_i \alpha^i(v) = \sum a_i \lambda^i v = v \sum a_i \lambda^i = p(\lambda)v$. Since $v \neq 0$, $p(\lambda) \in \text{spec}(p(\alpha))$. This proves one direction. To show the other, let $\mu \in \text{spec}(p(\alpha))$. Consider $q(t) = p(t) - \mu = a_n(t - \lambda_1)(t - \lambda_2)\dots(t - \lambda_n)$. This means that $q(\alpha) = p(\alpha) - \mu I$. Since $\mu \in \text{spec}(p(\alpha))$, we know that $p(\alpha) - \mu I$ is not invertible. But this means for some i_0 , $\alpha - \lambda_{i_0} I$ is not invertible. Thus, $\lambda_{i_0} \in \text{spec}(\alpha)$. Since $q(\lambda_{i_0}) = 0$, $\mu = p(\lambda_{i_0})$. \odot

2.2 09/10/2024

Theorem 2.2.1

Let $\alpha \in \text{End}(V)$ and $p(t) \in P(\mathbb{F})$. Then $p(\text{spec}(\alpha)) = \{p(\lambda) : \lambda \in \text{spec}(\alpha)\} \subseteq \text{spec}(p(\alpha))$

Question 7

Let $\alpha_1, \dots, \alpha_k \in \text{End}(V)$ such that $\alpha_i \cdot \alpha_j = \alpha_j \cdot \alpha_i$ for all i, j . Then $\alpha = \prod_j \alpha_j$ is invertible if and only if every α_i is invertible.

Proposition 2.2.1

Suppose $\dim(V) < \infty$ and $\alpha, \beta \in \text{End}(V)$. Then $\text{spec}(\alpha\beta) = \text{spec}(\beta\alpha)$. If $\dim(V) = \infty$ then $\text{spec}(\alpha\beta) \cup \{0\} = \text{spec}(\beta\alpha) \cup \{0\}$

Theorem 2.2.2 Cayley Hamilton Theorem

Suppose $\dim(V) = n < \infty$ and $\alpha \in \text{End}(V)$. Then $p(\alpha) = 0$ where $p(t)$ is the characteristic polynomial of α , $p(t) = \det(tI - A)$.

Definition 2.2.1: Minimal Polynomial

Let $\alpha \in \text{End}(V)$ and $\dim(V) = n < \infty$. Then the minimal polynomial, $m(t)$ of α is the monic polynomial of smallest positive degree such that $m(\alpha) = 0$.

Proposition 2.2.2 Minimal Polynomial divides the characteristic polynomial

Let $m(t)$ be the minimal polynomial of α and $p(t)$ such that $p(\alpha) = 0$. Then $m(t) | p(t)$

: Clearly, $\deg(m) \leq \deg(p)$. Doing long division, we see that $p(t) = m(t)g(t) + h(t)$. So this means that $p(\alpha) = h(\alpha)$. But this means that $\deg(h) > \deg(m)$, which is a contradiction. Thus $h(t) = 0$ and hence, $m | p$. \odot

Proposition 2.2.3

Let $m(t)$ be the minimal polynomial of $\alpha \in \text{End}(V)$. Then $\text{spec}(\alpha) = \{\lambda \in \mathbb{F} : m(\lambda) = 0\}$.

: Recall that $\text{spec}(\alpha) = \{\lambda \in \mathbb{F} : p(\lambda) = 0\}$ where p is the characteristic polynomial of α . Since $m | p$, if $m(\lambda) = 0$, then $p(\lambda) = 0$. So this means that $\{\lambda \in \mathbb{F} : m(\lambda) = 0\} \subseteq \text{spec}(\alpha)$. Conversely, if $\lambda \in \text{spec}(\alpha)$, then by the spectral mapping theorem, $m(\lambda) \in \text{spec}(m(\alpha)) = \text{spec}(0) = \{0\}$. This means that $m(\lambda) = 0$. Hence $\text{spec}(\alpha) \subseteq \{\lambda \in \mathbb{F} : m(\lambda) = 0\}$ and so they are equal. \odot

Corollary 2.2.1

If $\lambda_1, \dots, \lambda_k$ are all the distinct eigenvalues of α , then $m(t) = (t - \lambda_1) \cdot (t - \lambda_2) \cdot \dots \cdot (t - \lambda_k)h(t)$ for some polynomial $h(t)$. If $\dim(V) = k$, then $m(t) = (t - \lambda_1) \cdot (t - \lambda_2) \cdot \dots \cdot (t - \lambda_k)$.

Definition 2.2.2

Let $\alpha \in \text{End}(V)$ and $\dim(V) = n < \infty$. Then α is diagonalizable if there exists an ordered basis $B = \{v_1, \dots, v_k\}$ of V such that $[\alpha]_B$ is diagonal.

2.3 09/12/2024

Let $\alpha \in \text{End}(V)$

1. α is diagonalizable if there exists a basis B for V such that $[\alpha]_B$ is diagonal
2. α is diagonalizable if and only if B is only eigenvectors
3. α is diagonalizable implies that $p(t)$ splits
4. $A \in M_{n \times n}(\mathbb{F})$ is diagonalizable if and only if there exists an invertible matrix S such that $S^{-1}AS$ is diagonal

Proposition 2.3.1

Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of α and v_1, \dots, v_k be their corresponding eigenvectors. Then $\{v_1, \dots, v_k\}$ is linearly independent

: NEED TO DO

☺

Corollary 2.3.1

Let $\lambda_1, \dots, \lambda_k$ be distinct eigenvalues of α . Then $E_{\lambda_1} + \dots + E_{\lambda_k}$ is a direct sum

: Assume that $w_i \in E_{\lambda_i}$ such that $\sum w_i = 0$. Without loss of generality, assume that $w_1, \dots, w_L \neq 0$ and $w_{L+1} = \dots = w_k = 0$. Then w_i is an eigenvector of α with respect to λ_i . But $\sum w_i = 0$ which implies that the set $\{w_1, \dots, w_L\}$ is linearly dependent. This is a contradiction to the proposition above. ☺

Let $\alpha \in \text{End}(V)$ and $\{\lambda_1, \dots, \lambda_k\}$ all distinct eigenvalues. Then the following are equivalent:

1. α is diagonalizable
2. $V = \sum E_{\lambda_i}$
3. $\sum \dim(E_{\lambda_i}) = n = \dim(V)$

: NEED TO DO

☺

Definition 2.3.1: Geometric and Algebraic Multiplicity

Let $\alpha \in \text{End}(V)$ and $\lambda \in \text{spec}(\alpha)$ and $p(t)$ be the characteristic polynomial of α . Then the algebraic multiplicity of λ is the largest m_λ such that $p(t) = (t - \lambda)^{m_\lambda} g(t)$ and the geometric multiplicity of λ is $\dim(E_\lambda) = d_\lambda$.

Proposition 2.3.2

$d_\lambda \leq m_\lambda$ for all $\lambda \in \text{spec}(\alpha)$.

: Let $\alpha \in \text{End}(V)$ and $\{v_1, \dots, v_d\}$ be a basis for E_λ . Extend it to a basis $B = \{v_1, \dots, v_d, v_{d+1}, \dots, v_n\}$ of V . Then $[\alpha]_B = A$ and $p(t) = \det(tI - A) = (t - \lambda)^d h(t)$. This implies that $d \leq m$. ☺