1. Question 1.1.4
In order to check if $d(\cdot,\cdot)$ is a metric, we need to check that for $x, y \in \mathbb{R}^m$

   1. $d(x, y) > 0$ if $x \neq y$
   2. $d(x, x) = 0$
   3. $d(x, y) = d(y, x)$
   4. $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in \mathbb{R}^m$

   If $x \neq y$, then there must be at least one $i \in \{1, ..., m\}$ such that $x_i \neq y_i$. This means that $d(x, y) \geq 1 > 0$. Hence, the first condition is satisfied. Similarly, since $x = x$, there is no index where $x_i \neq x_i$ and hence $d(x, x) = 0$. Thus, the second condition is satisfied. Looking at the definition of $d(\cdot,\cdot)$, we see that

$$d(x, y) = |\{k : x_k \neq y_k, k = 1, ..., m\}| = |\{k : y_k \neq x_k, k = 1, ..., m\}| = d(y, x)$$

   Hence the third condition is satisfied. If $z \neq x$ or $z \neq y$, then clearly $d(x, y) \leq d(x, z) + d(z, y)$. If $z = y$ or $z = x$, we would get equality. Now, let us define sets A, B, C as follows:

$$A = \{k : x_k \neq y_k\}$$
$$B = \{k : x_k \neq z_k\}$$
$$C = \{k : z_k \neq y_k\}$$

   For any $k \in A$, this means that $x_k \neq y_k$. This means that either $x_k = z_k$ but $z_k \neq y_k$ OR $x_k \neq z_k$. The first case implies that $k \in C$ and the second implies that $k \in B$. This means that any index $k \in A$ must also be in either B or C. Thus, $A \subseteq B \cup C$. Hence, $d(x, y) = |A| \leq |B \cup C| \leq |B| + |C| = d(x, z) + d(z, y)$. Hence the final condition is satisfied.

2. Question 1.1.5
In order to check if $d(\cdot,\cdot)$ is a metric, we need to check that for $x, y \in X$

   1. $d(x, y) > 0$ if $x \neq y$
   2. $d(x, x) = 0$
   3. $d(x, y) = d(y, x)$
   4. $d(x, y) \leq d(x, z) + d(z, y)$ for any $z \in X$

   Since $\hat{\rho}(x, y) = 0 \iff x = y$, then $d(x, x) = 0$. Hence the second condition is satisfied. Similarly, as $\hat{\rho}(x, y) \geq 0$, then we know that $d(x, y) \geq 0$ as it is the maximum of two numbers greater than 0. Hence the first conditin is met. Looking at the definition of $d(\cdot,\cdot)$, we see that

$$d(x, y) = max\{\hat{\rho}(x, y), \hat{\rho}(y, x)\} = max\{\hat{\rho}(y, x), \hat{\rho}(x, y)\} = d(y, x)$$

Hence the third condition is satisfied. Finally, we must use the fact that $\hat{\rho}(x,y) \leq \hat{\rho}(x,z) + \hat{\rho}(z,y)$ and $\hat{\rho}(y,x) \leq \hat{\rho}(y,z) + \hat{\rho}(z,x)$ and state that

$$\begin{aligned} d(x,y) &= \max\{\hat{\rho}(x,y), \hat{\rho}(y,x)\} \\ &\leq \max\{\hat{\rho}(x,z) + \hat{\rho}(z,y), \hat{\rho}(y,z) + \hat{\rho}(z,x)\} \\ &\leq \max\{\hat{\rho}(x,z), \hat{\rho}(z,x)\} + \max\{\hat{\rho}(y,z), \hat{\rho}(z,y)\} \\ &= d(x,z) + d(z,y) \end{aligned}$$

Hence, the final condition is satisfied.

3. Question 1.1.16

   In order to check if $\bar{d}(\cdot,\cdot)$ is a metric, we must once again check that for $x, y \in X$

   1. $\bar{d}(x,y) > 0$ if $x \neq y$
   2. $\bar{d}(x,x) = 0$
   3. $\bar{d}(x,y) = d(y,x)$
   4. $\bar{d}(x,y) \leq d(x,z) + d(z,y)$ for any $z \in X$

   Since d is already known to be a metric, then we know that d satisfies that $d(f(x), f(y)) > 0$ when $f(x) \neq f(y)$. Since we know that f is one to one, we know that if $x \neq y$, then $f(x) \neq f(y)$ and hence $\bar{d}(x,y) > 0$ when $x \neq y$. Clearly $\bar{d}(x,x) = d(f(x), f(x)) = 0$ so the second condition is satisfied. Similarly, $\bar{d}(x,y) = d(f(x), f(y)) = d(f(y), f(x)) = \bar{d}(y,x)$ and so the third condition is satisfied. Now, let $z \in X$. Then $\bar{d}(x,y) = d(f(x), f(y)) \leq d(f(x), f(z)) + d(f(z), f(y)) = \bar{d}(x,z) + \bar{d}(z,y)$ since we know that d must satisfy the triangle inequality. Thus all conditions are met and $\bar{d}$ is a metric.

4. Question 1.2.5

   Since $x_n < y_n$ for all $n \geq 1$, we know that $\sup_{n>k}(x_n) \leq \sup_{n>k}(y_n)$. Hence, by definition, $\limsup(x_n) \leq \limsup(y_n)$. Similarly, $\inf_{n>k}(x_n) \leq \inf_{n>k}(y_n)$. Hence, by definition, $\liminf(x_n) \leq \limsup(y_n)$.
   An example of the first equality holding is letting $x_n = 1 - \frac{2}{n}$ and $y_n = 1 - \frac{1}{n}$. Then $y_n > x_n$ but $\limsup(x_n) = \limsup(y_n) = 1$. Similarly, for an example of the second equality holding is $x_n = \frac{1}{n}$ and $y_n = \frac{2}{n}$. Then $y_n > x_n$ but $\liminf(x_n) = \liminf(y_n) = 0$

5. Question 1.2.6

   First we must note that $x_n \leq \sup(x_n)$ and $y_n \leq \sup(y_n)$. This means that $x_n + y_n \leq \sup(x_n) + \sup(y_n)$ and hence $\sup(x_n + y_n) \leq \sup(x_n) + \sup(y_n)$. Similarly, $\inf(x_n + y_n) \geq \inf(x_n) + \inf(y_n)$. Now, taking the limit of both sides of $\sup(x_n + y_n) \leq \sup(x_n) + \sup(y_n)$, we get

   $$\lim_{k\to\infty} \sup_{n\geq k}(x_n + y_n) \leq \lim_{k\to\infty} \sup_{n\geq k}(x_n) + \lim_{k\to\infty} \sup_{n\geq k}(y_n)$$

   which by definition gives us

   $$\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n)$$

Taking the limit of both sides for the infimum inequality gets us

$$\liminf(x_n + y_n) \leq \liminf(x_n) + \liminf(y_n)$$

Now An example of the first strict inequality holding is letting $x_n = (-1)^n$ and $y_n = (-1)^{n+1}$. Then $\limsup(x_n + y_n) = 0$ but $\limsup(x_n) = \limsup(y_n) = 1$ so $\limsup(x_n) + \limsup(y_n) = 2$. An example of the second strict inequality holding is letting x_n and y_n be defined as above. Then $\liminf(x_n + y_n) = 0$ but $\liminf(x_n) = \liminf(y_n) = -1$ so $\limsup(x_n) + \limsup(y_n) = -2$.

6. Question 1.2.7
   Since $x_n \in c$, we know that x_n is cauchy and since $\mathbb{R}$ is complete, it must be convergent. As it is convergent, $\limsup(x_n) = \lim_{n\to\infty} x_n = L$. Now, we know from the previous problem that

   $$\limsup(x_n + y_n) \leq \limsup(x_n) + \limsup(y_n) = \lim_{n\to\infty} x_n + \limsup(y_n) = L + \limsup(y_n)$$

   But since $x_n \to L$, we know that for any $\varepsilon > 0$, there must be an N such that for all $n > N$, $|x_n - L| < \varepsilon$. Thus, taking $n > N$, we can reasonably approximate x_n as L and see that $\sup_{k\geq n}(x_n + y_n) \approx \sup_{k\geq n}(L + y_n) = L + \sup_{k\geq n}(y_n)$. Taking the limit, we get

   $$\lim_{k\to\infty} sup_{n\geq k}(x_n + y_n) = L + \lim_{k\to\infty} sup_{n\geq k}(y_n) = \lim_{n\to\infty} x_n + \limsup(y_n)$$

7. Question 1.2.8
   (a) Using 1.2.5, we know that $\limsup(x_n) \leq \limsup(y_n)$ and $\limsup(y_n) \leq \limsup(z_n)$. Thus,

   $$\bar{L} = \limsup(x_n) \leq \limsup(y_n) \leq \limsup(z_n) = \bar{L}$$

   Hence, $\limsup(y_n) = \bar{L}$. Similarly, we know that $\liminf(x_n) \leq \liminf(y_n)$ and $\liminf(y_n) \leq \liminf(z_n)$. Thus,

   $$\underline{\mathrm{L}} = \liminf(x_n) \leq \liminf(y_n) \leq \liminf(z_n) = \underline{\mathrm{L}}$$

   Hence, $\liminf(y_n) = \underline{\mathrm{L}}$.

   (b) The Squeeze Theorem states that if $x_n \leq y_n \leq z_n$ and $x_n \to a, z_n \to a$, then $y_n \to a$. To prove it, we must first note that in order for $\lim x_n = a$, then $\limsup x_n = \liminf x_n = a$. Applying part a, we see that $\limsup y_n = a$ and $\liminf y_n = a$. Thus, $\lim y_n = a$.

8. Question 1.2.9
   (a) We first note that $x_n < y_n$ for all n since $a < b$. This means that $x_{n+1} = \sqrt{x_n y_n} \leq x_n$ and hence non-decreasing. Similarly, $y_{n+1} = \frac{x_n + y_n}{2} \geq y_n$ and hence non-increasing. Since $0 < a \leq x_n \leq y_n \leq b$, we know that both sequences are bounded and monotonic. Thus, each sequence is convergent by the monotone convergence theorem. Say $x_n \to L_x$ and $y_n \to L_y$. Taking the limit of the first formula, we get

   $$\lim_{n\to\infty} x_{n+1} = \sqrt{\lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n} \implies L_x = \sqrt{L_x L_y} \implies L_x^2 = L_x L_y \implies L_x = L_y$$

We can safely divide by L_x since $L_x > 0$. Similarly, taking the limit of the second formula and substituting in $L_x = L_y$, we get

$$\lim_{n\to\infty} y_{n+1} = \frac{\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n}{2} \implies L_y = \frac{L_x + L_y}{2} = L_x$$

9. Question 1.2.10
Since x_n converges to a, we know that for any ε, there must be an $N \in \mathbb{N}$ such that $d(x_n, a) < \varepsilon$ for all $n > N$. We can split the sum up as $S_n = \frac{1}{n}((x_1 + ... + x_N) + (x_{N+1} + ... + x_n))$.Written another way, we get $S_n = \frac{1}{n}\sum_{k=1}^{N} x_k + \frac{1}{n}\sum_{k=N+1}^{n} x_k$. Now, we examine $d(S_n, a)$.

$$\begin{aligned} |S_n - a| &= \left|\frac{1}{n}\sum_{k=1}^{N} x_k + \frac{1}{n}\sum_{k=N+1}^{n} x_k - a\right| \\ &\leq \left|\frac{1}{n}\sum_{k=1}^{N} x_k\right| + \left|\frac{1}{n}\sum_{k=N+1}^{n} x_k - a\right| \\ &\leq \left|\frac{1}{n}\sum_{k=1}^{N} x_k\right| + \left|\frac{1}{n}(n-N)\varepsilon\right| \end{aligned}$$

Now we can take the limit as $n \to \infty$. Since N is fixed, the first sum will tend to 0 as $n \to \infty$. The coefficent in the second term would tend to 1 so we get $\lim_{n\to\infty} |S_n - a| \leq \varepsilon$ and so $\lim_{n\to\infty} S_n = a$.

10. Question 1.2.11
First, notice that $\frac{x_n}{1+x_n} < 1$ for all n and so $x_n < 2$. Thus it is bounded. To show the sequence is increasing, we must show that $x_{n+1} - x_n = \frac{(1+x_{n-1})x_n - (1+x_n)x_{n-1}}{(1+x_{n-1})(1+x_n)} > 0$. We will do so by induction. To verify the base case, let $n = 0$. Then $x_{0+1} - x_0 = x_1 - x_0 = (1 + \frac{1}{1+1}) - 1 = 0.5 > 0$. Now, assume that for some positive integer k, the inequality holds and $x_k - x_{k-1}$. This means that

$$x_k - x_{k-1} = \frac{(1 + x_{k-2})x_{k-1} - (1 + x_{k-1})x_{k-2}}{(1 + x_{k-2})(1 + x_{k-1})} > 0$$

. Now, examining $x_{k+1} - x_k$, we get

$$\frac{x_k}{1 + x_k} - \frac{x_{k-1}}{1 + x_{k-1}} = \frac{(1 + x_{k-1})x_k - (1 + x_k)x_{k-1}}{(1 + x_{k-1})(1 + x_k)} = \frac{x_k - x_{k-1}}{(1 + x_{k-1})(1 + x_k)}$$

Since the bottom is always greater than 0 and the top we assumed to be greater than 0, we get a positive ratio and hence $x_k - x_{k-1} > 0 \implies x_{k+1} - x_k > 0$ and thus by induction, we get that x_n is increasing. Thus, by the monotone convergence theorem, we know that x_n converges, say to L. Taking the limit, we get $\lim_{n\to\infty} x_{n+1} = 1 + \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} x_n + 1} \implies L = 1 + \frac{L}{1+L}$. This can be simplified and we get $L^2 - L - 1 = 0$. Solving this quadratic, we get two roots, $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Since L must be positive, we know that $L = \frac{1+\sqrt{5}}{2}$