MAA 5237 - Mathematical Analysis

Nickolas Arustamyan

September 7, 2024

Contents

Chapter 1 — **Page 2**

1.1 08/21/2024 — 2
1.2 08/22/2024 — 3
1.3 08/27/2024 — 3

Chapter 1

1.1 08/21/2024

Definition 1.1.1: Metric Space

A metric space is a nonempty set X with a function $d : X \times X \to [0, \infty)$ such that d has the following properties:

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) + d(y, z) = d(x, z)$

We call d a metric and (X, d) a matric space.

Question 1

Is $\mathbb{R}^n$ with the Euclidean norm a metric space?

Proof: YES! NEED TO do 😄

Definition 1.1.2: Convergent Sequence

Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X. We say $\lim_{n\to\infty} x_n = a$ if for all $\varepsilon > 0$, there exists an N_ε such that $d(x_n, a) < \varepsilon$ for all $n > N_\varepsilon$.

Proposition 1.1.1

A sequence of vectors converges to a if and only if it converges component wise.

Definition 1.1.3: Neighborhood

a neighborhood of $p \in X$ is a set that contains $B(p, \varepsilon) = \{x \in X : d(p, x) < \varepsilon\}$ for some $\varepsilon > 0$.

Definition 1.1.4: Limit Point

A point $p \in X$ is a limit point for a set if there exists a sequence x_n such that $x_n \to p$ but $x_n \neq p$.

Definition 1.1.5: Isolated Point

A point $p \in X$ is called an isolated point if it is not a limit point.

Definition 1.1.6: Closed Set

A set $E \subseteq X$ is closed if it contains all of its limit points. Another definition would be that if $x_n \to x$ and $x_n \in E$, then $x \in E$.

Definition 1.1.7: Interior Point

A point $p \in E \subseteq X$ is an interior point of E if there exists some positive ε such that $B(p, \varepsilon) \subseteq E$.

Definition 1.1.8: Open Set

A set $E \subseteq X$ is open if all of its point are interior points. Another definition would be if E^c is closed.

Definition 1.1.9: Dense Sets

A set $E \subseteq X$ is dense in X if $\bar{E} = X$. Another definition would be if for any point $p \in X$, there is a sequence in E such that $x_n \to p$.

1.2 08/22/2024

Definition 1.2.1: Convexity

Let $\varphi : I \to \mathbb{R}$ for some interval I. We call φ convex if

$$\varphi(\lambda x_1 + (1-\lambda)x_2) \leqslant \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$$

for all $x_1, x_2 \in I$ and $\lambda \in [0,1]$. Thanks to this definition, we know that φ is below any of its secant chords. One can easily extend this to the n-dimensional case using induction.

NEED TO ADD STUFF ABOUT JENSENS INEQUALITY, YONGS, AND HOLDERS

1.3 08/27/2024

Definition 1.3.1: Cauchy Sequence

A sequence (x_n) is a Cauchy sequence if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Proposition 1.3.1 Convergence implies Cauchy

If x_n is convergent, then it must be cauchy. The other direction is not always the case.

: First, we will show that a cauchy sequence is not always convergent. Take x_n to be a sequence such that $x_n \to \sqrt{2}$. This is a cauchy sequence in $\mathbb{Q}$ but not convergent in $\mathbb{Q}$ since $\sqrt{2} \notin \mathbb{Q}$. Now, assume that x_n is convergent to a. Then we know that for any $\varepsilon > 0$, there is an N such that $d(x_n, a) < \varepsilon$ for all $n > N$. Take $m, k > N$. Then by the triangle inequality, we know $d(x_m, x_k) < d(x_m, a) + d(a, x_k) < \varepsilon + \varepsilon = 2\varepsilon$. Hence, it must be cauchy as well. 😃

Definition 1.3.2: Completeness

A metric space (X, d) is complete if every cauchy sequence converges.

Example 1.3.1

Both $\mathbb{R}$ and $\mathbb{R}^n$ are complete.

: NEED TO DO 😊

Proposition 1.3.2

If (X, d) is complete and $F \subseteq X$, then F is complete if and only if it is closed.

: NEED TO DO 😊