ME 7310 – Project 1
Finite Difference – Diffusion Equation

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Abstract

In this project a first-order forward-time and second-order central space (FTCS) scheme is used to discretize the governing partial differential equation (PDE). The governing PDE is a simplified form of the Navier-Stokes Equation, and is of the form of a parabolic PDE. The PDE is solved for a flow between two parallel plates extended to infinity.

To verify the accuracy of the numerical scheme developed, a mesh refinement study is conducted in order to obtain the grid independent solution. Both the truncation error and stability conditions for the FTCS scheme are investigated. In order to validate the solution, the results obtained from the numerical scheme are compared to those obtained from the analytical solution.

1 Problem Definition

In this project the one-dimensional, laminar, incompressible flow between two parallel plates extended to infinity is solved. The governing form of the simplified Navier-Stokes Equations is:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1}$$

where the kinematic viscosity, ν, of the fluid is given as 0.000217 m^2/s. The distance between the two plates, h, is also given, and is equal to 4 cm. The initial condition and boundary conditions are:

$$u(0,0) = U_0, \qquad u(0, y > 0) = 0 \tag{2}$$

$$u(t,0) = U_{0,}\, u(t,h) = 0 \tag{3}$$

where U_0 is the velocity of the lower plate once it is suddenly set in motion. The value of U_0 is given as 40 m/s.

2 Numerical Solution Procedure

For solving this problem, a first-order forward-time and second-order central space (FTCS) scheme is used. Starting with the left hand side of (2), the truncation error of the first-order forward-time scheme can be derived from the Taylor Series:

$$u_j^{n+1} = u_j^n + \left(\frac{\partial u}{\partial t}\right)\bigg|_j^n \Delta t + \left(\frac{\partial^2 u}{\partial t^2}\right)\bigg|_j^n \frac{(\Delta t)^2}{2!} + \left(\frac{\partial^3 u}{\partial t^3}\right)\bigg|_j^n \frac{(\Delta t)^3}{3!} + \cdots + \left(\frac{\partial^n u}{\partial t^n}\right)\bigg|_j^n \frac{(\Delta t)^n}{n!} \qquad (2)$$

Rearranging the terms in (2), the equation can be re-written as

$$\left(\frac{\partial u}{\partial t}\right)\bigg|_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \left(\frac{\partial^2 u}{\partial t^2}\right)\bigg|_j^n \frac{\Delta t}{2!} - \left(\frac{\partial^3 u}{\partial t^3}\right)\bigg|_j^n \frac{(\Delta t)^2}{3!} - \cdots - \left(\frac{\partial^n u}{\partial t^n}\right)\bigg|_j^n \frac{(\Delta t)^{n-1}}{n!} \qquad (3)$$

The terms dropped in the series can be replaced with the notation T.E., which is the truncation error of discretization of the partial differential equation.

$$\left(\frac{\partial u}{\partial t}\right)\bigg|_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + T.E. \qquad (4)$$

As Δx decreases, the higher order terms will drop out, so it can be said that the truncation error is of the order of Δx. Equation (5) can be re-written as

$$\left(\frac{\partial u}{\partial t}\right)\bigg|_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + O(\Delta t) \qquad (5)$$

The truncation error and discretized form of the PDE on the right hand side of (1) can be derived following the same procedure as above. The final form for the second-order central space scheme [1] used is:

$$\left(\frac{\partial^2 u}{\partial x^2}\right)\bigg|_j^n = \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + O((\Delta x)^2) \qquad (6)$$

The final form of the numerical scheme can now be written using (5) and (6).

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \nu \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} + O((\Delta t), (\Delta x)^2) \qquad (7)$$

As can be seen from the right side of (7), the Truncation Error is first-order in time and second-order in space. As $\Delta t \rightarrow 0$ and $\Delta x^2 \rightarrow 0$, the Truncation Error vanishes, thus the FTCS Scheme is consistent. The FTCS Scheme (7) is solved using a computer program whose algorithm marches forward in time, solving the explicit form of the equation:

$$u_j^{n+1} = u_j^n + \left(\frac{\nu \Delta t}{\Delta x^2}\right) \frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{\Delta x^2} \tag{8}$$

where the term $\left(\frac{\nu \Delta t}{\Delta x^2}\right) = d$ represents the stability conditions, $d \leq 0.5$.

In order to validate the solution, the results from the numerical scheme are compared to those obtained by the analytical solution, given as:

$$u_a(y,t) = U_0 = \left(\sum_{n=0}^{\infty} erfc(2n\eta_1 + \eta) - \sum_{n=1}^{\infty} erfc(2n\eta_1 - \eta) \right) \tag{9}$$

where $\eta = \frac{y}{2\sqrt{\nu t}}$, $\eta_1 = \frac{h}{2\sqrt{\nu t}}$, and $erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-r^2} dr$ is the complementary error function.

3 Numerical Results

The analytical solution (9) is calculated for different values of t until steady state $(t \to \infty)$, for eleven equally spaced grid points ranging from y = 0 m to y = 0.04 m. Figure 1 shows the velocity at different distances from the bottom plate for different times.

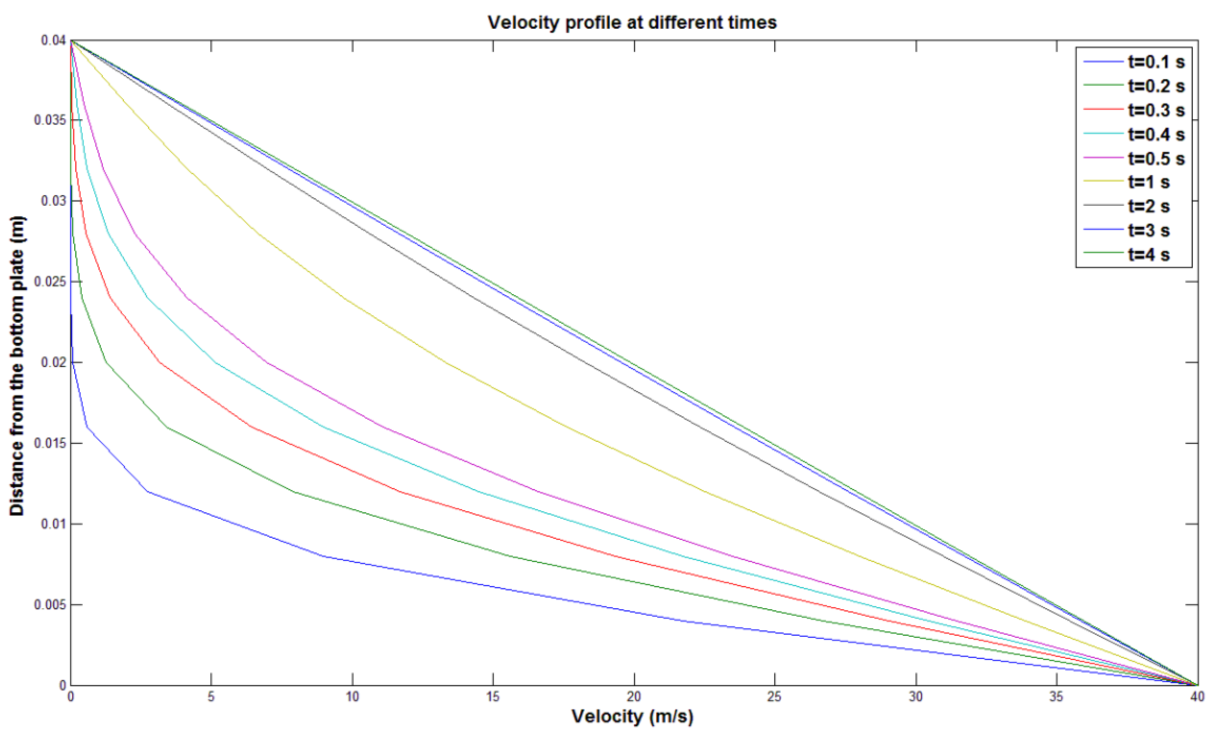

Figure 1: Velocity profile at different times

As expected, the velocity profile approaches a straight line as the solution approaches the steady state value. Once the solution reaches its steady state value around 4 seconds, the velocity profile is a straight line, where the velocity is 40 m/s at the bottom plate, and is 0 m/s at the top plate, which is consistent with what one would expect from the no-slip boundary condition of the flow.

The stability condition for the FTCS scheme is given by:

$$d = \nu \Delta t / \Delta y^2 \leq 0.5 \qquad (10)$$

The stability condition was investigated by varying Δt and Δy for various cases. The first case investigated varied Δt, but kept Δy constant, as shown in Table 1.

Table 1: Stability condition investigation for different time steps, final time = 0.5 s

Trial	Time Step Δt [s]	Spatial Discretization Δy [m]	Grid System (# time nodes x # spatial nodes	d	Final Time [s]
1	0.01	0.004	51x11	0.135625	0.5
2	0.037	0.004	14x11	0.501813	0.5
3	0.04	0.004	13x11	0.5425	0.5

As the time step increases, the numerical scheme becomes slightly unstable when Δt = 0.037 s, and becomes even more unstable when Δt = 0.04 s. These results are shown in Figure 2.

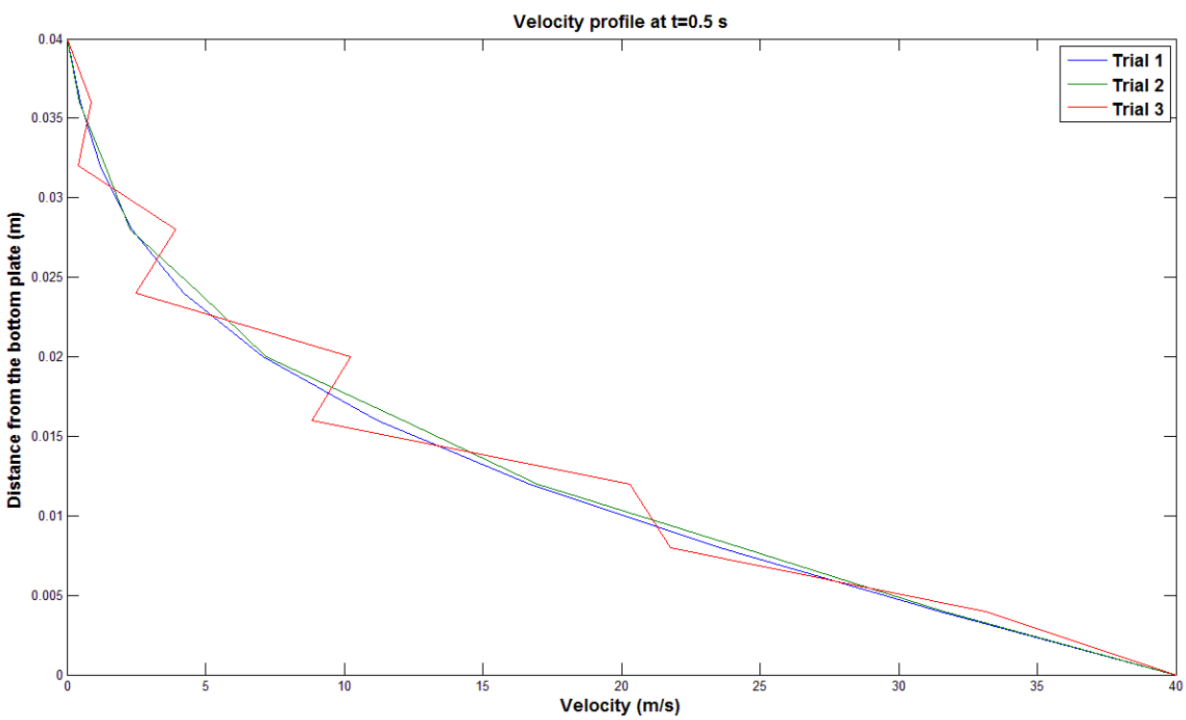

Figure 2: Velocity profile at t = 0.5 s for different values of Δt

It should be noted that although Trial 2 does not satisfy the stability condition, it appears from Figure 2 that the numerical scheme yields a fairly stable solution. It can be seen from Trial 3 displayed on Figure 2 that a time step of 0.04 s results in a very obvious unstable numerical solution.

The second case that was investigated varied Δy, but kept Δt constant, as shown in Table 2.

Table 2: Stability condition investigation for different spatial discretization, final time = 0.5 s

Trial	Time Step Δt [s]	Spatial Discretization Δy [m]	Grid System (# time nodes x # spatial nodes	d	Final Time [s]
1	0.01	0.004	51x11	0.135625	0.5
4	0.01	0.008	51x6	0.033906	0.5
5	0.01	0.002	51x21	0.5425	0.5

Varying the grid in the y-direction yields interesting results. Refining the grid in Trial 5 does not actually increase the accuracy of the solution, but rather, violates the stability condition. Increasing the grid size on the other hand, yields a more accurate solution, as shown in Trial 4. These results are also displayed by examining the velocity profiles in Figure 3 below.

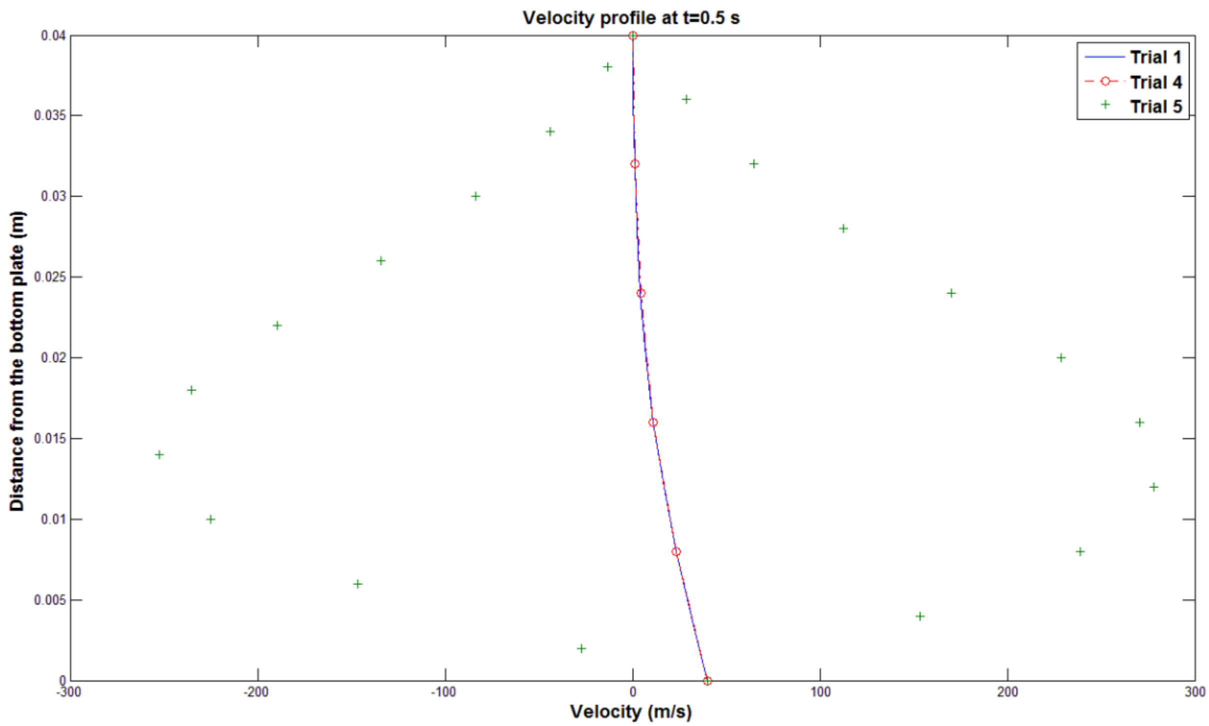

Figure 3: Velocity profile at t = 0.5 s for different values of Δy

The above cases also hold true for the steady-state solution when Δt and Δy are varied such that they violate the stability condition. Violation of the stability condition is even more prominent when evaluating the steady state solution, as shown below in Figure 4 and Figure 5, respectively.

Table 3: Stability condition investigation for different time steps, final time = 4 s

Trial	Time Step Δt [s]	Spatical Discretization Δy [m]	Grid System (# time nodes x # spatial nodes	d	Final Time [s]
1	0.01	0.004	51x11	0.135625	4
2	0.037	0.004	14x11	0.501813	4
3	0.04	0.004	13x11	0.5425	4

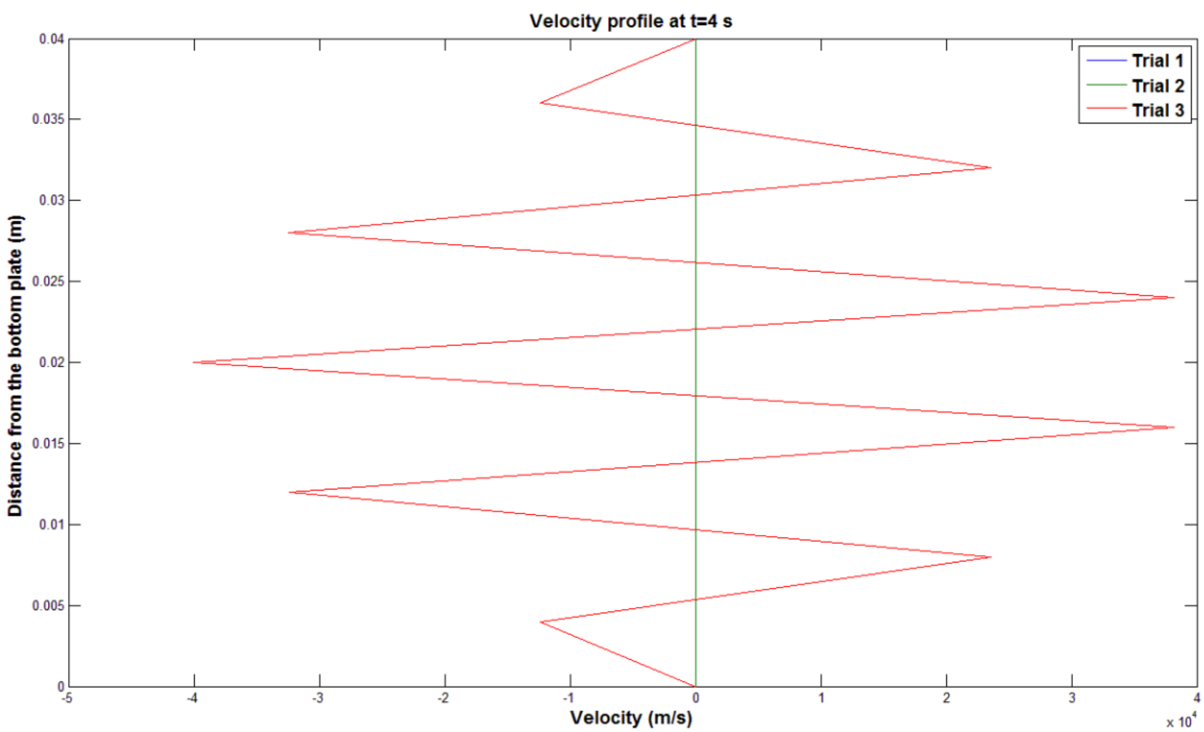

Figure 4: Velocity profile at t = 4 s for different values of Δt

Table 4: Stability condition investigation for different spatial discretization, final time = 4 s

Trial	Time Step Δt [s]	Spatical Discretization Δy [m]	Grid System (# time nodes x # spatial nodes	d	Final Time [s]
1	0.01	0.004	51x11	0.135625	4
4	0.01	0.008	51x6	0.033906	4
5	0.01	0.002	51x22	0.5425	4

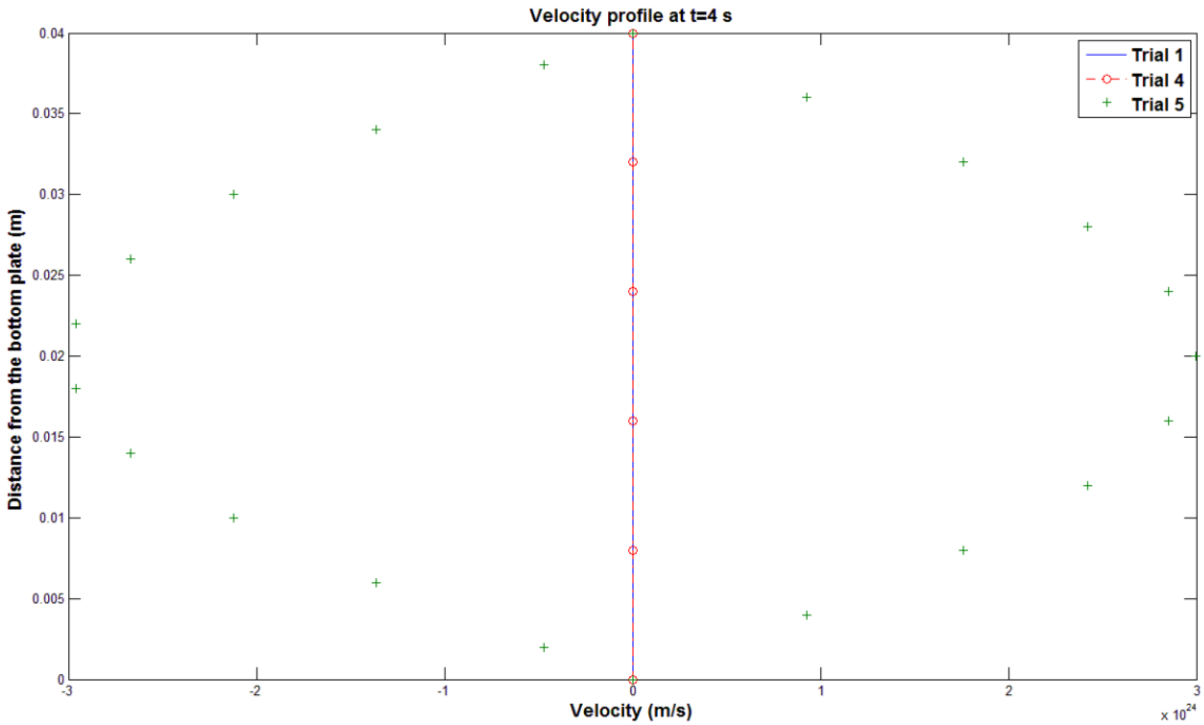

Figure 5: Velocity profile at t = 4 s for different values of Δy

Verification of the numerical scheme was achieved by refining the grid such that further refinement in Δy did not change the solution. This indicates the most accurate solution that can be obtained from the numerical scheme, known as the grid independent solution. The grid refinement study is displayed in Table 5 and Figure 6.

Table 5: Grid refinement study

Trial	Time Step Δt [s]	Spatical Discretization Δy [m]	Grid System (# time nodes x # spatial nodes	d	Final Time [s]
1	0.01	0.004	19x11	0.135625	0.18
2	0.0025	0.002	73x21	0.135625	0.18
3	0.000625	0.001	289x41	0.135625	0.18

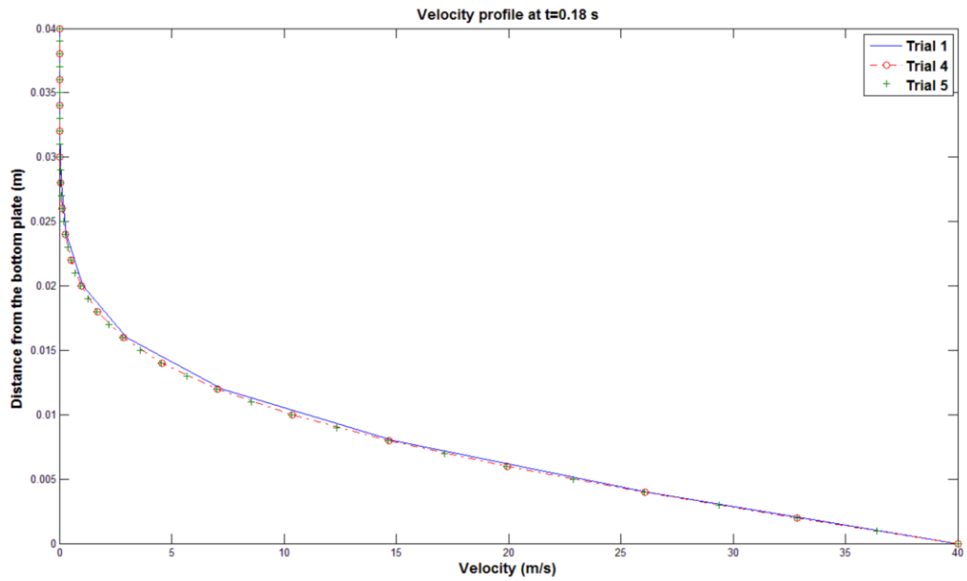

Figure 6: Velocity profile at t = 0.18 s for different grids

Another verification of the numerical scheme was obtained by refining the grid, and plotting the error versus Δy at t = 0.18 s, where error is calculated as:

$$\frac{\|u(y,t_1) - u_a(y,t_1)\|}{\|u_a(y,t_1)\|} \tag{11}$$

As shown in Figure 7, as the mesh is refined, the accuracy of the solution improves.

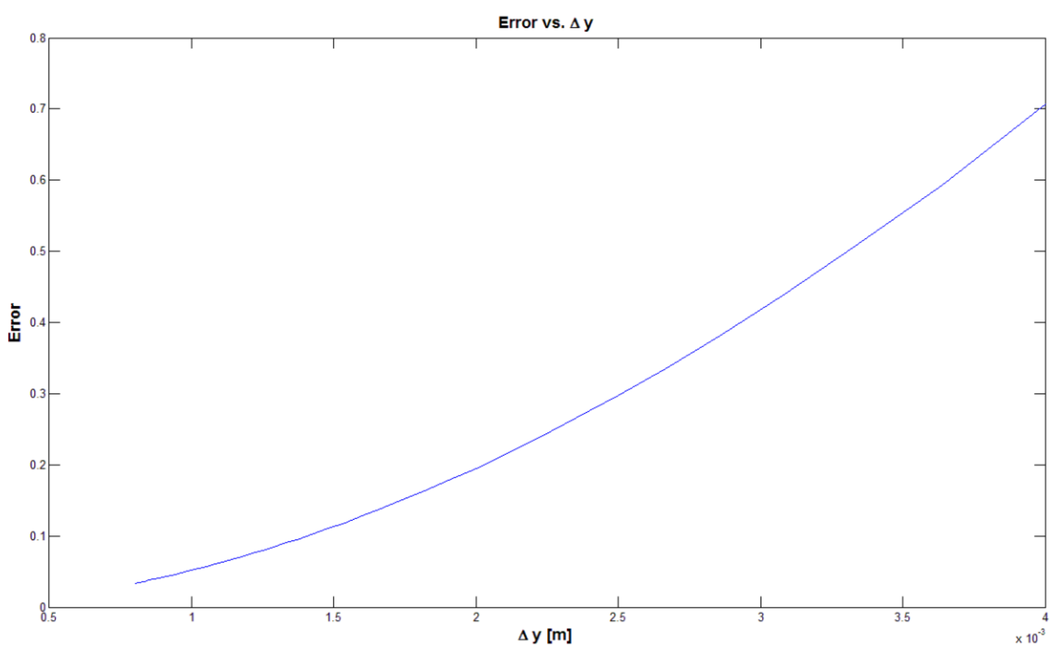

Figure 7: Error versus grid refinement at t = 0.18 s

In order to validate the model, the velocity profile from the FTCS scheme was compared with the velocity profile for the analytical solution at t = .18 s, as well as t = 1.08 s. The grid implemented was that obtained from the grid independent solution of Trial 3 listed in Table 5. The numerical scheme used 289 nodes for the time grid ($\Delta t = 0.000625$ s), and 41 nodes for the spatial grid ($\Delta y = 0.001$ m). The results are displayed in Figure 8.

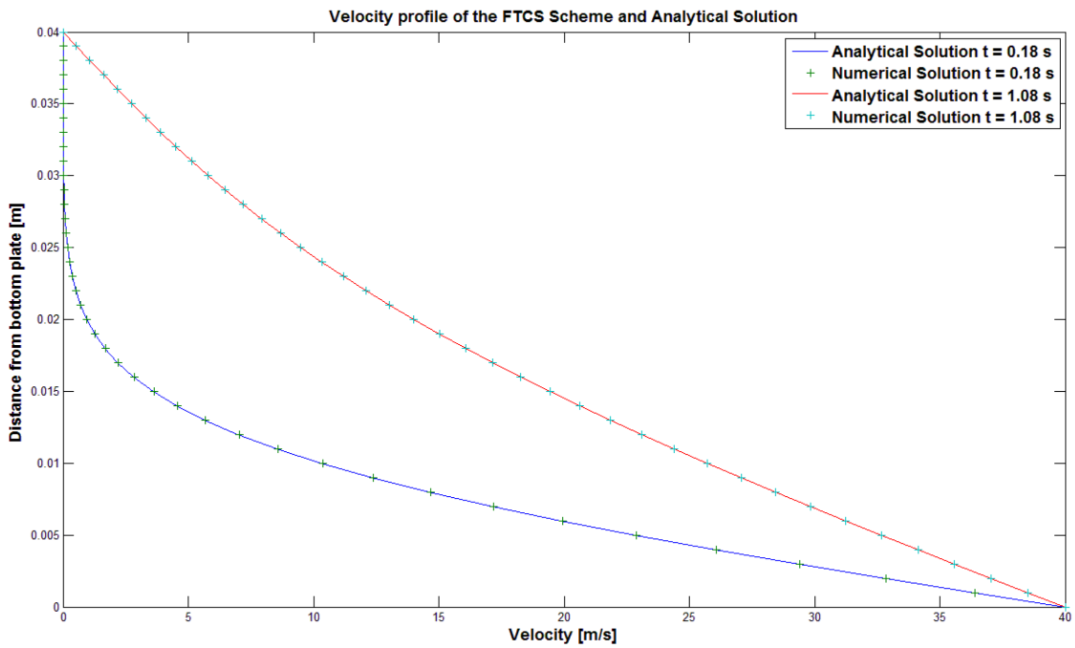

Figure 8: Velocity profile of the FTCS Scheme and Analytical Solution

A validation of the steady state solution for the numerical scheme can be compared with the solution obtained by solving the PDE directly for $\left(\frac{\partial^2 u}{\partial y^2} = 0\right)$. Given the boundary conditions, the resulting form of the analytical solution is:

$$u = -1000y + 40 \tag{12}$$

A comparison of the two solutions is plotted in Figure 9. Again, the mesh from the grid independent solution was used.

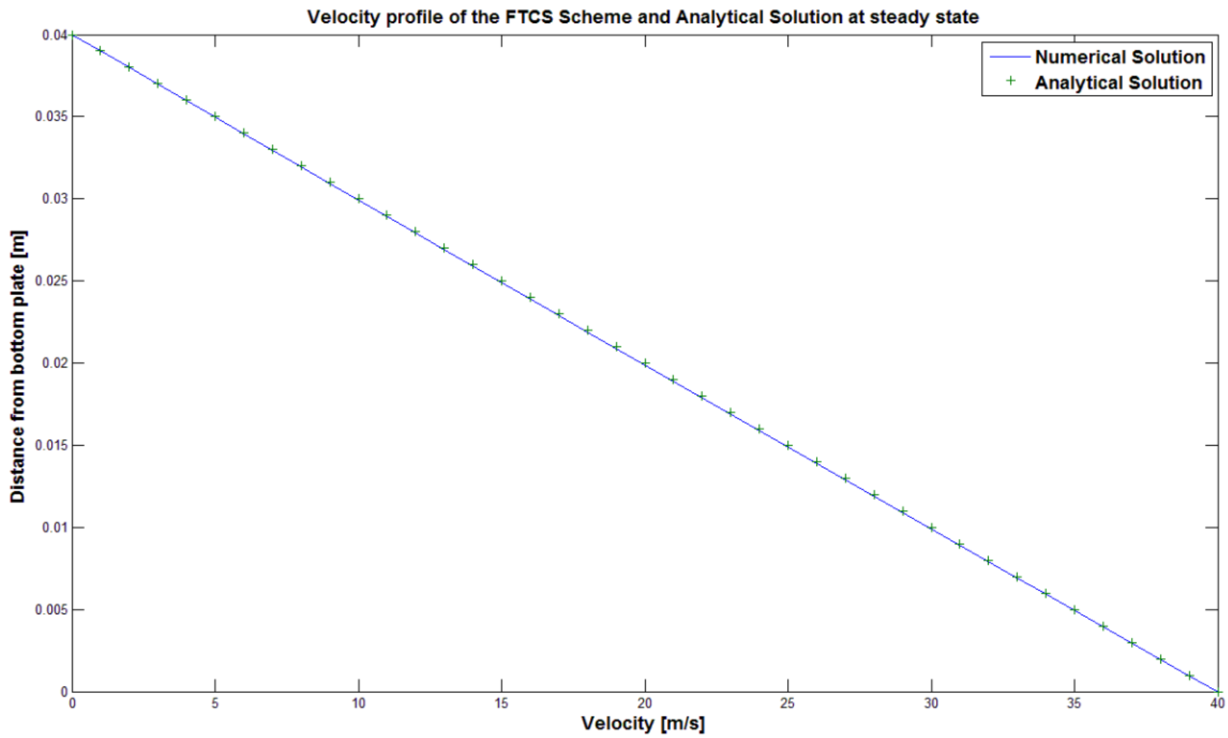

Figure 9: Velocity profile of the FTCS Scheme and Analytical Solution at Steady State

As seen from Figure 9, the FTCS Scheme closely matches that of the analytical solution, thus it can be said that the numerical solution has been validated.

4 References

[1] Essential Computational Fluid Dynamics, by O. Zikanov, John Wiley & Sons, Inc., 2010