m248Week7_Action_of_SVD

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1 Math 248 Computers and Numerical Algorithms

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3 Week 7: Singular Value Decomposition in Action

4 Let's use the singular value decomposition to explore the action of a matrix A on space. We will work with two dimensional matrices because they are easy to visualize.

5 First we explore the action of A on the special vectors which are the columns of V:

$$A = \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}$$

The singular value decomposition of A is

$$A = U\Sigma V^t = \begin{pmatrix} 0.93788501 & 0.34694625 \\ 0.34694625 & -0.93788501 \end{pmatrix} \begin{pmatrix} 5.41565478 & 0 \\ 0 & 1.29254915 \end{pmatrix} \begin{pmatrix} 0.10911677 & 0.99402894 \\ 0.99402894 & -0.10911677 \end{pmatrix}$$

is equivalent to

$$AV = U\Sigma,$$

which means that the action of A on the orthonormal columns of V is the same as stretching/squeezing the columns of U by the singular values. That is,

$$Av_1 = \sigma_1 u_1$$

and

$$Av_2 = \sigma_2 u_2$$

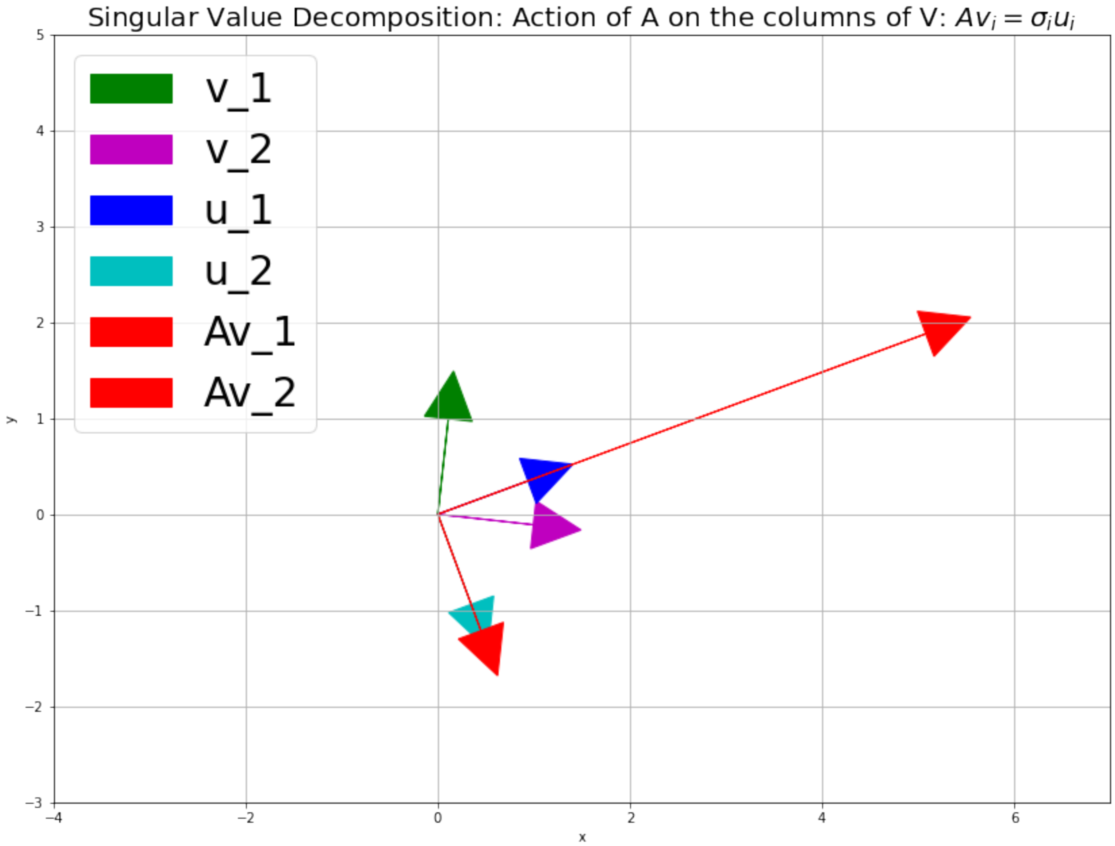
6 The following code shows that A sends the special vectors v's to multiples of the other special vectors u's

[130]:
```
import numpy as np
import matplotlib.pyplot as plt

# define A as a numpy array
A=np.array([[1,5],[-1,2]])



# perform SVD on A
U,sigma,Vt=np.linalg.svd(A)
print("U=\n",U)
print("sigma=\n",sigma)
# store sigma in a diagonal matrix that has the same shape as A
Sigma=np.diag(sigma)
print("Sigma=\n",Sigma)
print("Vt=\n",Vt)

# Check whether you can recover A
print('U.Sigma.Vt=\n',U.dot(Sigma.dot(Vt)))
print('A=\n',A)

# These are the columns of U
u_1=U[:,0]
u_2=U[:,1]

# These are the columns of V (not Vt, I transpose Vt first)
V=Vt.T
v_1=V[:,0]
v_2=V[:,1]

# This is A acting on the columns of V
Av_1=A.dot(v_1)
Av_2=A.dot(v_2)

# Plot the vectors using arrow in matplolib.pyplot.axes
# set the figure and labels
plt.figure(figsize=(14,14))
vec= plt.axes()
```

```
plt.axis('scaled') # the scale on the x-axis is the same as the y-axis
plt.xlim(-4,7)
plt.ylim(-3,5)
plt.title('Singular Value Decomposition: Action of A on the columns of V:␣
↪$Av_i=\sigma_i u_i$', fontsize=20)
plt.xlabel('x')
plt.ylabel('y')

# plot the vectors as arrows
arrow_v_1=vec.arrow(0, 0, *v_1, head_width=0.5, head_length=0.5, color='g',␣
↪label='v_1')
arrow_v_2=vec.arrow(0, 0, *v_2, head_width=0.5, head_length=0.5,␣
↪color='m',label='v_2')
arrow_u_1=vec.arrow(0, 0, *u_1, head_width=0.5, head_length=0.5, color='b',␣
↪label='u_1')
arrow_u_2=vec.arrow(0, 0, *u_2, head_width=0.5, head_length=0.5, color='c',␣
↪label='u_2')
arrow_Av_1=vec.arrow(0, 0, *Av_1,head_width=0.5, head_length=0.5, color='r',␣
↪label='Av_1')
arrow_Av_2=vec.arrow(0, 0, *Av_2,head_width=0.5, head_length=0.5, color='r',␣
↪label='Av_2')

# show the grid
plt.grid()

# set the legend
plt.legend([arrow_v_1,arrow_v_2,arrow_u_1,arrow_u_2,arrow_Av_1,arrow_Av_2],␣
↪['v_1','v_2','u_1','u_2','Av_1','Av_2'],loc=2, prop={'size': 30})
```

```
U=
 [[ 0.93788501  0.34694625]
 [ 0.34694625 -0.93788501]]
sigma=
 [5.41565478 1.29254915]
Sigma=
 [[5.41565478 0.        ]
 [0.         1.29254915]]
Vt=
 [[ 0.10911677  0.99402894]
 [ 0.99402894 -0.10911677]]
U.Sigma.Vt=
 [[ 1.  5.]
 [-1.  2.]]
A=
 [[ 1  5]
 [-1  2]]
```

[130]: <matplotlib.legend.Legend at 0x7fd1e4be97f0>



7 Let's use the SVD to explore the action of a matrix on the standard unit vectors in 2D space.

$$A = U\Sigma V^t$$

U and V could be rotation or reflection matrices.

The transpose of a rotation matrix is a rotation in the opposite direction. So if a matrix rotates clockwise its transpose rotates counterclockwise.

A rotation matrix clockwise by an angle θ is:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

A rotation matrix counterclockwise by an angle θ is the transpose of the above matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

A reflection matrix about a line L making an angle θ with the x-axis is:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

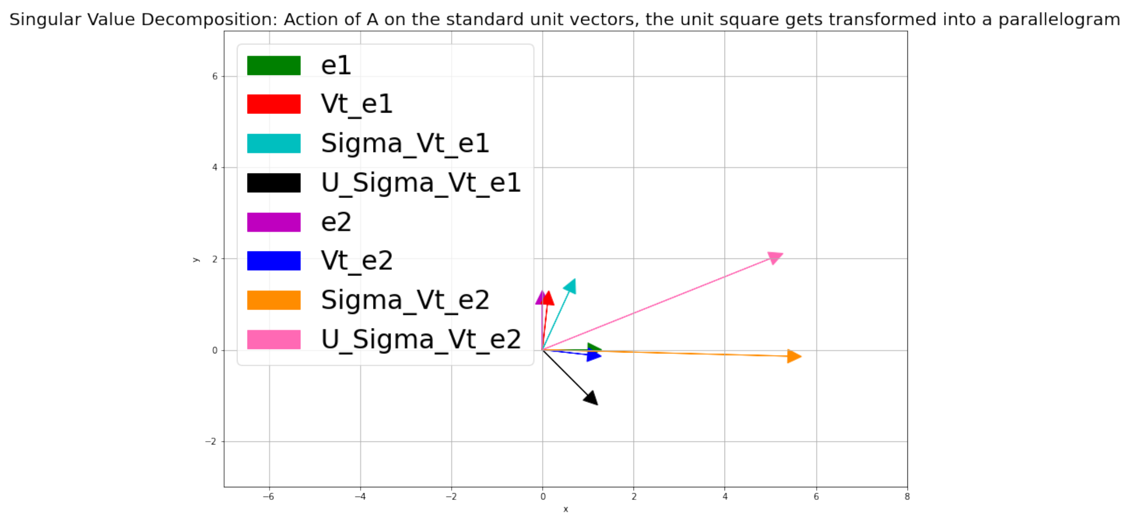
> Note: The equation of L is $y = (\tan\theta)x$ since it has slope $\tan\theta$ and passes through the origin.

The determinant of a rotation matrix is 1 and the determinant of the reflection matrix is -1. Notice that both rotations and reflections have orthonormal rows and columns. Also, their inverse is their transpose.

[249]:
```
e1=[1,0]
e2=[0,1]

Vt_e1=(Vt).dot(e1)
Vt_e2=(Vt).dot(e2)
Sigma_Vt_e1=Sigma.dot(Vt_e1)
Sigma_Vt_e2=Sigma.dot(Vt_e2)
U_Sigma_Vt_e1=U.dot(Sigma_Vt_e1)
U_Sigma_Vt_e2=U.dot(Sigma_Vt_e2)

# Plot the vectors using arrow in matplolib.pyplot.axes
# set the figure and labels
plt.figure(figsize=(14,14))
vec= plt.axes()
plt.axis('scaled') # the scale on the x-axis is the same as the y-axis
plt.xlim(-7,8)
plt.ylim(-3,7)
plt.title('Singular Value Decomposition: Action of A on the standard unit␣
↪vectors, the unit square gets transformed into a parallelogram',fontsize=20)
plt.xlabel('x')
plt.ylabel('y')

# plot the vectors as arrows
arrow_e1=vec.arrow(0, 0, *e1, head_width=0.3, head_length=0.3, color='g',␣
↪label='e1')
arrow_e2=vec.arrow(0, 0, *e2, head_width=0.3, head_length=0.3,␣
↪color='m',label='e2')
arrow_Vt_e1=vec.arrow(0, 0, *Vt_e1, head_width=0.3, head_length=0.3,␣
↪color='r',label='Vt_e1')
arrow_Vt_e2=vec.arrow(0, 0, *Vt_e2, head_width=0.3, head_length=0.3,␣
↪color='b',label='Vt_e2')
arrow_Sigma_Vt_e1=vec.arrow(0, 0, *Sigma_Vt_e1, head_width=0.3, head_length=0.
↪3, color='c',label='Sigma_Vt_e1')
arrow_Sigma_Vt_e2=vec.arrow(0, 0, *Sigma_Vt_e2, head_width=0.3, head_length=0.
↪3, color='darkorange',label='Sigma_Vt_e2')
arrow_U_Sigma_Vt_e1=vec.arrow(0, 0, *U_Sigma_Vt_e1, head_width=0.3,␣
↪head_length=0.3, color='k',label='U_Sigma_Vt_e1')
```

```
arrow_U_Sigma_Vt_e2=vec.arrow(0, 0, *U_Sigma_Vt_e2, head_width=0.3,␣
↪head_length=0.3, color='hotpink',label='U_Sigma_Vt_e2')

# show the grid
plt.grid()

# set the legend
plt.legend([arrow_e1,arrow_Vt_e1,arrow_Sigma_Vt_e1, arrow_U_Sigma_Vt_e1,␣
↪arrow_e2,arrow_Vt_e2, arrow_Sigma_Vt_e2, arrow_U_Sigma_Vt_e2],␣
↪['e1','Vt_e1','Sigma_Vt_e1','U_Sigma_Vt_e1','e2','Vt_e2','Sigma_Vt_e2','U_Sigma_Vt_e2'],loc
↪prop={'size': 30})
```

[249]: <matplotlib.legend.Legend at 0x7fd1b9c8c2b0>



8 There are three ways to multiply two matrices $A_{m\times n}$ and $B_{n\times s}$ together:

1.

8.1 Row-column approach: Produce one entry $(ab)_{ij}$ at a time by taking the dot product of a row with a column:

$$(ab)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

2.

8.2 Column-columns approach: Produce one column $(AB)_l$ at a time by linearly combining the columns of A with the entries in the columns of B:

$$(AB)_l = b_{1l}A_1 + b_{2l}A_2 + \cdots + b_{nl}A_n$$

3.

8.3 Column-row approach: Produce rank one pieces of the product one at a time by multiplying a column of A with the corresponding row of B, then add all these rank one matrices together to get the final product AB:

$$AB = A_1B_1^r + A_2B_2^r + \cdots + A_nB_n^r$$

where A_l is the lth column of A and B_l^r is the lth row of B.

9 Multiplying by a diagonal matrix Σ:

1. If you multiply A by Σ from the right $A\Sigma$ then you scale the columns of A by the σ's.
2. If you multiply A by Σ from the left ΣA then you scale the rows of A by the σ's.

10 How does this help us understand the usefulness of the Singular Value Decomposition?

Recall that $A = U\Sigma V^t$. We can expand this product using the sum of rank one matrices method to multiply $U\Sigma$ with V^t (note that $U\Sigma$ scales each colum U_i of U by σ_i):

$$A = U\Sigma V^t = \sigma_1 U_1 V_1^t + \sigma_2 U_2 V_2^t + \cdots + \sigma_r U_r V_r^t$$

where r is the rank of the matrix A. The great thing about this expression is that it splits A into a sum of rank one matrices arranged according to their order of importance. Moreover, it provides a straightforward way to approximate A by lower rank matrices by setting lower sigular values to zero.

11 The matrices U and V above are both reflection matrices:

A reflection matrix about a line L making an angle θ with the x-axis is:

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

The matrices U and V in the singular value decomposition of the above $A = U\Sigma V^t$ are both reflections. 1. Find the angle of the straight lines L_U and L_V^t that acts as a mirror for these reflections. 2. Find the equations of the lines of reflection. 3. Use python to plot these lines. 4. On the same plot, plot a general vector x, $V^t x$, $\Sigma V^t x$, and $U\Sigma V^t x = Ax$

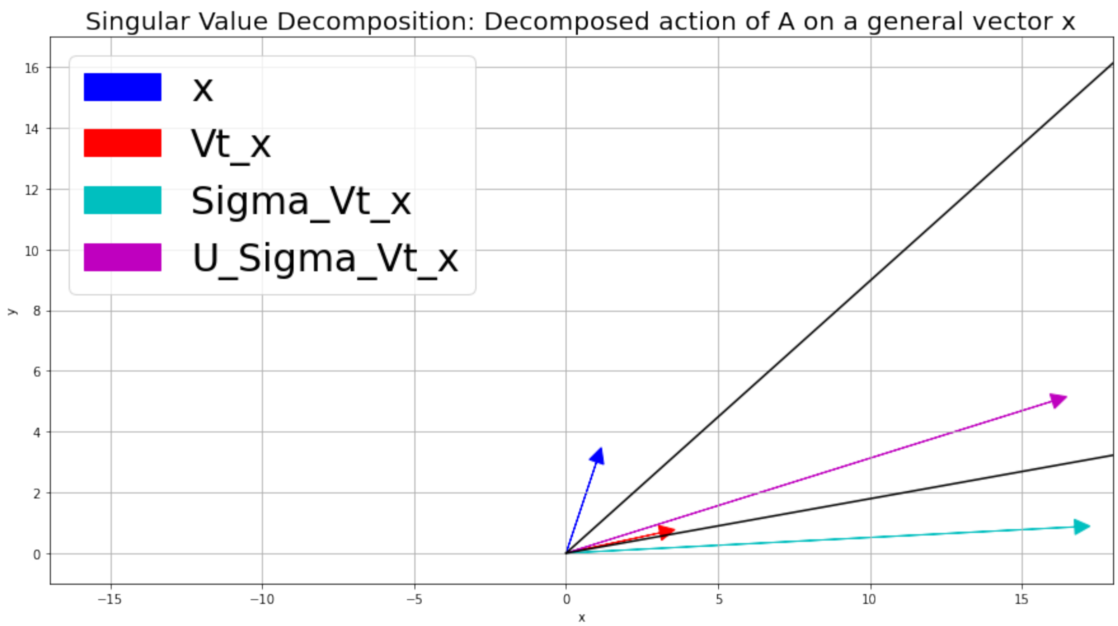
$$A = U\Sigma V^t = \begin{pmatrix} 0.93788501 & 0.34694625 \\ 0.34694625 & -0.93788501 \end{pmatrix} \begin{pmatrix} 5.41565478 & 0 \\ 0 & 1.29254915 \end{pmatrix} \begin{pmatrix} 0.10911677 & 0.99402894 \\ 0.99402894 & -0.10911677 \end{pmatrix}$$

[132]:
```
# choose a vector x
x=[1,3]

# apply V^t then Sigma then U to x
Vt_x=(Vt).dot(x)
Sigma_Vt_x=Sigma.dot(Vt_x)
U_Sigma_Vt_x=U.dot(Sigma_Vt_x)

# Calculate the slopes of the reflection lines: slope= tan(theta)=tan(0.
↪5*cos^-1(R(0,0))) where R is the reflection matrix
slope_L_Vt=np.tan(0.5*np.arccos(Vt[0,0]))
print(f'Equation of line along which Vt reflects is:\n y={slope_L_Vt}x')
slope_L_U=np.tan(0.5*np.arccos(U[0,0]))
print(f'Equation of line along which U reflects is:\n y={slope_L_U}x')

# The equation of the line is y=tan(theta)x. Discretize in order to plot the␣
↪straightlines.
x_discrete = np.linspace(0,18,100)
y_L_Vt = slope_L_Vt*x_discrete
y_L_U = slope_L_U*x_discrete

# Plot the vectors using arrow in matplolib.pyplot.axes
# set the figure and labels
plt.figure(figsize=(15,15))
vec= plt.axes()
plt.axis('scaled') # the scale on the x-axis is the same as the y-axis
plt.xlim(-17,18)
plt.ylim(-1,17)
plt.title('Singular Value Decomposition: Decomposed action of A on a general␣
↪vector x',fontsize=20)
plt.xlabel('x')
plt.ylabel('y')

# plot the straight lines
plt.plot(x_discrete, y_L_Vt, 'k')
plt.plot(x_discrete, y_L_U, 'k')

# show the grid
plt.grid()

# plot the vectors as arrows
arrow_x=vec.arrow(0, 0, *x, head_width=0.5, head_length=0.5, color='b',␣
↪label='x')
arrow_Vt_x=vec.arrow(0, 0, *Vt_x, head_width=0.5, head_length=0.5,␣
↪color='r',label='Vt_x')
arrow_Sigma_Vt_x=vec.arrow(0, 0, *Sigma_Vt_x, head_width=0.5, head_length=0.5,␣
↪color='c',label='Sigma_Vt_x')
```

```
arrow_U_Sigma_Vt_x=vec.arrow(0, 0, *U_Sigma_Vt_x, head_width=0.5, head_length=0.
↪5, color='m',label='U_Sigma_Vt_x')

# set the legend
plt.legend([arrow_x,arrow_Vt_x,arrow_Sigma_Vt_x, arrow_U_Sigma_Vt_x],␣
↪['x','Vt_x','Sigma_Vt_x','U_Sigma_Vt_x'],loc=2, prop={'size': 30})
```

```
Equation of line along which Vt reflects is:
 y=0.8962347008436108x
Equation of line along which U reflects is:
 y=0.17903345403184898x
```

[132]: <matplotlib.legend.Legend at 0x7fd1e55270a0>



12 The linear transformation A takes the unit circle to an ellipse and the standard unit square to a parallelogram.

We can easily see the above action from the Sigular Value Decomposition.

The Polar Decomposition

$$A = QS$$

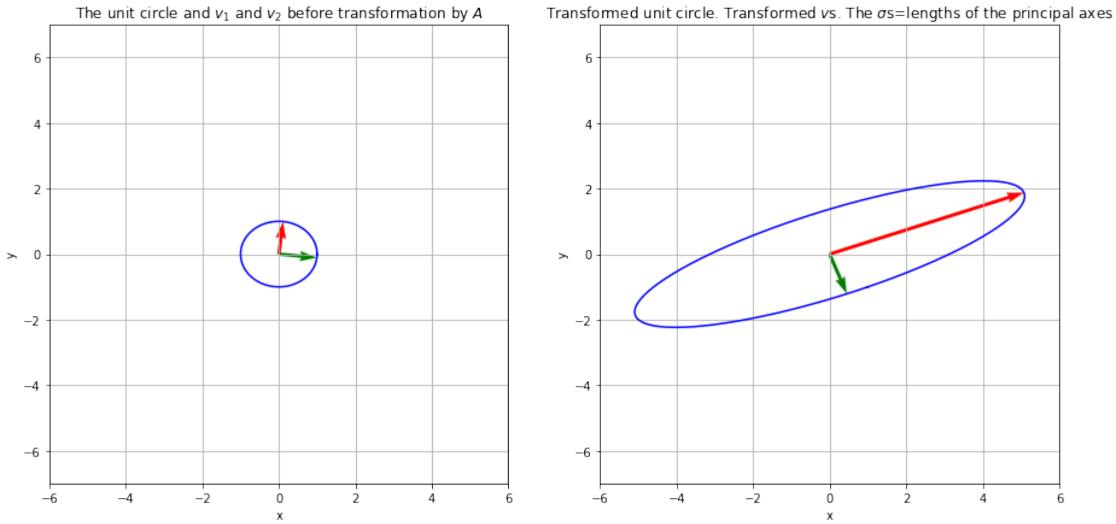
is a very easy way that geometrically shows how a circle gets transformed into an ellipse.

[250]:
```
# set the figure and labels
fig, subs=plt.subplots(nrows=1, ncols=2, figsize=(15,7))
subs[0].grid()
```

```
subs[0].set_xlabel('x')
subs[0].set_ylabel('y')
subs[0].set_xlim(-6,6)
subs[0].set_ylim(-7,7)
subs[0].set_title('The unit circle and $v_1$ and $v_2$ before transformation by␣
↪$A$')

subs[1].grid()
subs[1].set_xlabel('x')
subs[1].set_ylabel('y')
subs[1].set_xlim(-6,6)
subs[1].set_ylim(-7,7)
subs[1].set_title('Transformed unit circle. Transformed $v$s. The␣
↪$\sigma$s=lengths of the principal axes')


# plot v_1 and v_2: I will use the quiver function instead of the arrow function
# The parameters passed to quiver are chosen so as the vector arrows have␣
↪length= Euclidean length
subs[0].
↪quiver(0,0,v_1[0],v_1[1],scale=1,scale_units='xy',angles='xy',color=['r'])
subs[0].quiver(0,0,v_2[0],v_2[1],scale=1, scale_units='xy',angles='xy',␣
↪color=['g'])

print(Av_1,Av_2)
# plot Av_1=sigma_1 u_1 and Av_2=sigma_2 u_2
subs[1].
↪quiver(0,0,Av_1[0],Av_1[1],scale=1,scale_units='xy',angles='xy',color=['r'])
subs[1].
↪quiver(0,0,Av_2[0],Av_2[1],scale=1,scale_units='xy',angles='xy',color=['g'])

# plot the unit circle in the first subplot
t=np.linspace(0,2*np.pi,100)
x=np.cos(t)
y=np.sin(t)
subs[0].plot(x,y,'b')

# plot the tranformed unit circle in the second plot. It becomes an ellipse␣
↪with major and minor axes along the u's
Ax=A[0,0]*x+A[0,1]*y
Ay=A[1,0]*x+A[1,1]*y
subs[1].plot(Ax,Ay,'b')
```

```
[5.07926146 1.8789411 ] [ 0.44844508 -1.21226248]
```

[250]:
```
[<matplotlib.lines.Line2D at 0x7fd1ba315a90>]
```



13 Time to finally understand the ingredients of the singular value decomposition of any matrix $A = U\Sigma V^t$:

1. The columns of V are the orthonormal eigenvectors of the symmetric matrix A^tA
2. The columns of U are the orthonormal eigenvectors of the symmetric matrix AA^t
3. The singular values σ_1, σ_2, $\ldots$ σ_r are the square roots of the eigenvalues of A^tA or AA^t. The singular values are non-negative and arranged in decreasing order.
4. Recall $Av_i = \sigma_i u_i$

   Note: Every real symmetric positive semi-definite (non-negative eigenvalues) matrix is diagonalizable $S = PDP^{-1}$. A^tA and AA^t happen to both be symmetric positive semi-definite (meaning their eigenvalues are non-negative).

[ ]: