Geometrical Series (Part 1)
Proof by Example

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Abstract

This document is the textual representation of my ideas and way of thinking I used while researching the problem described here.

1 Proof by Test

Consider the question of counting the amount of squares, with any size, that can be seen a square divided in n sub-squares per side, including the biggest square itself, namely by the figure:

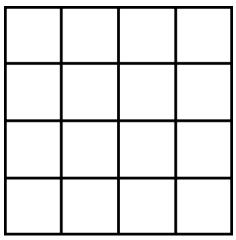

Figure 1: Example for $n = 4$.

If we wanted to exercise our brain and do it manually, we would see that, for bigger values of n, there is a big problem.

My first approach was to count them manually until a certain value of n, so I could look for patterns, maybe a function to describe this behavior. There is the table obtained for $n \in \{x \mid 1 \leq x \leq 5\}$:

x	$f(x)$
1	1
2	5
3	14
4	30
5	55

If we were to make a scatter plot, we would notice how the function is notably a polynomial of a grade between two and three. However, in mathematics, or at least in this very case, guessing is probably not the way.

While manual counting, one can notice how there are always exactly four sub-squares of size n-1. This is when we can start noticing patterns.

This new approach can help us find our equation. If we count the amount $y = f(x)$ of sub-squares with sizes $x = 1, \ldots, n$, possibly the main square, we may get a pattern.

Returning to the example for $n = 4$, we can notice that, for each size, there are the following amount of squares:

x	$f(x)$	f
1	16	n^2
2	9	$(n-1)^2$
3	4	$(n-2)^2$
4	1	$(n-3)^2$

Now we can see an actual deducible pattern: they are the perfect squares. However, there is notable characteristic in the correlation between the size of sub-squares x and the amount of them, $y = f(x)$. May seem obvious, but the bigger the size, the smaller the amount, thus, the third column shows the way to go respect to the equation that describes this relation:

$$f(x) = (n - x + 1)^2 : 1 \leq x \leq n;\ n\ is\ a\ constant$$

The function above describes the amount of sub-squares in a figure of the style already shown as a function of the size of them. If we sum these values, of course restricting to integer values of x, we get our desired equation:

$$\textstyle\sum_{x=1}^{n} f(x) : x \in \mathbb{Z}$$

However, developing on this equation:

$$\textstyle\sum_{x=1}^{n} f(x) = n^2 + (n-1)^2 + \ldots + 1^1 : x \in \mathbb{Z}$$

We notice how it is the sum of the first n perfect squares, and commutativity of addition allows us to do this:

$$\textstyle\sum_{x=1}^{n} x^2$$

Which is finally, our desired equation.