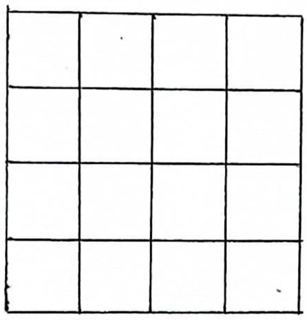
Geometrical Series

Proof by example - (Pt. 1)

Consider the problem of counting the total number of squares of any size that can be seen inside a bigger square with subdivisions of n subsquares per side, graphically:



Example for $n = 4$

How can we count easily the solicited quantity, specially for bigger amounts of subdivisions?

While analysing the problem, my first approach was to manually count them all for $n \in \{1, \ldots, 5\}$, and this data was gathered:

n	$f(x)$
1	1
2	5
3	14
4	30
5	55

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where $f(x)$ denotes how we are supposedly looking for a function.

If we were to make a scatter plot, we would notice how the function is notably a power function, where no value can be found. Maybe a value between two and three for the power of x, but guessing is not the way.

While manually counting, you would notice how, for squares of size $n-1$, the quantity is exactly 4.

That gives hints about the way to go, the place where the patterns can be.

May we return to the example with $n=4$, and count the amount of squares of sizes from 1×1 through $n\times n$. For $n \times n$, the amount is clearly 1, the biggest square. And for 1×1, n^2. Let's see for all other sizes, what happens.

n	x	Formula
2	4	$(n-2)^2$
3	9	$(n-1)^2$.

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Now we can see the absolute pattern, there are perfect squares. There is a notable characteristic; about to be described now:

We can notice an inversion in the "scale" from 1 through n, because there are 1 subsquares of size n, 4 squares of size $n-1$, ..., and n^2 squares of size 1.

In general, as logic also tells, the smaller the size of the subdivisions, the higher the quantity, being this quantity calculated as the square of the size of the subdivisions, as if we were calculating the areas in a weird way.

As an adjustment to this inversion and agreeing to our new approach, we can get this formula:

$$f(x) = [n - (x+1)]^2 : 1 \leq x \leq n$$

This way for $x = 1, \ldots, n$ and for n being the already known constant, $f(a)$ describes the amount of squares of size $a \times a$ inside a square with subdivisions of size n.

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Proof by example. (Pt 2)

With this new approach, the original solution we were looking for, of founding the total of squares of any size, turns into a sum of the integer values of the function:

$$\sum_{i=1}^{n} \left(f(i) \right) = n^2 + (n-1)^2 + \cdots + 1^2$$

Due to commutativity of addition, can we notice the equivalence of the sum above with our final solution:

$$\sum_{i=0}^{n} \left(i^2 \right)$$

Which, is actually, Faulhaber's formula, namely:

$$\frac{n(n+1)(2n+1)}{6} : n \in \mathbb{Z}$$