Finite Hilbert Spaces and Conditional Recurrence in Causal Patches

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Abstract

We study *Conditional Cosmological Recurrence* (CCR): if a causal patch admits a finite operational Hilbert space dimension D (as motivated by holographic/entropy bounds), then unitary quantum dynamics imply almost–periodic evolution and hence recurrences. Our contribution is to make this implication explicit with a micro–to–macro bridge: (i) finite regions discretize field modes; (ii) gravitational bounds cap entropy and energy; therefore (iii) the accessible state count is finite, yielding CCR. We differentiate global microstate recurrences (double–exponential timescales in $S_{\max}$) from operationally relevant coarse–grained returns (exponential in subsystem entropy), and we give conservative timescale estimates. For predictivity in eternally inflating settings, we use a causal–diamond measure with xerographic typicality and a single no–Boltzmann–Brain constraint, avoiding volume–weighting pathologies. Scope is *explicitly conditional*: if future quantum gravity demonstrates $D = \infty$ for causal patches, CCR is falsified.

Originality and Scope

Our contribution is not the proposal of a new recurrence law. Rather, the novelty lies in: (i) a micro–to–macro counting argument that motivates $D < \infty$ per causal patch under gravitational/entropic constraints; (ii) casting recurrence as a conditional consequence (CCR) of that finiteness; (iii) a measure prescription (causal diamond + no–BB) designed to avoid known pathologies. We treat de Sitter holography as a well–motivated assumption rather than a proven duality.

> **Key Point:** This paper does *not* claim that the universe must recur. Rather, *if* gravitational entropy bounds apply to causal patches (yielding finite Hilbert space dimension), *then* recurrence follows mathematically. The "if" is the physics question; the "then" is rigorous mathematics.

Conditional Nature. Cosmological Recurrence (CCR) is not a prediction of standard cosmology *per se*, but a logical consequence of the finite Hilbert space conjecture for de Sitter patches (A1). Our results are therefore strictly conditional: falsifying A1 would directly falsify CCR. In this sense, the framework presented here serves as a logical bridge between holographic bounds and recurrence phenomena, rather than as an independent dynamical model. If future developments in quantum gravity demonstrate that the Hilbert space of a causal patch is in fact infinite, the CCR framework presented here would be falsified in its entirety, and the recurrence problem would have to be reconsidered from a fundamentally different perspective.

1 From Electrons to a Finite Cosmic State Count: a Counting Argument

Lemma 1 (IR discretization)**.** *A quantum field in a finite spatial region of linear size R admits a discrete set of momentum modes $\mathbf{k} = \frac{\pi}{R}(n_x, n_y, n_z)$ under standard boundary conditions.*

Lemma 2 (UV/gravity cutoff)**.** *Let $E_{\mathrm{BH}}(R) = \frac{c^4 R}{2G}$ be the energy of a Schwarzschild black hole of radius R. Any state whose total energy localized within the region exceeds $E_{\mathrm{BH}}(R)$ collapses, and the Bekenstein–Hawking bound yields*

$$S_{\max} \ \leq \ \frac{k_B A}{4\ell_P^2} = \frac{k_B\, 4\pi R^2}{4\ell_P^2}, \qquad \ell_P^2 = \frac{\hbar G}{c^3}. \tag{1}$$

Proposition 1 (Finite-dimensional accessible Hilbert space)**.** *Consider the Fock space built from the discrete IR modes and impose the microcanonical cap $\sum_{\mathbf{k}} n_{\mathbf{k}}\, \hbar\omega_{\mathbf{k}} \leq E_{\mathrm{BH}}(R)$. Then the number of distinct Fock states below this cap is finite. Moreover, any operationally distinguishable state family within the region obeys $\log D \leq S_{\max}/k_B \lesssim A/(4\ell_P^2)$, so the accessible Hilbert space dimension D is finite.*

(See Appendix B for an explicit microcanonical counting illustration under a gravitational cap.)

Proof sketch. Finite volume $\Rightarrow$ discrete $\mathbf{k}$ (Lemma 1). The energy cap forbids arbitrarily many high-frequency quanta (Lemma 2). Since $\omega_{\mathbf{k}} \geq c|\mathbf{k}|$ and the number of modes below any cutoff Λ is finite, the integer solutions $\{n_{\mathbf{k}}\}$ of $\sum n_{\mathbf{k}}\hbar\omega_{\mathbf{k}} \leq E_{\mathrm{BH}}$ are finite. Operational distinguishability is limited by the entropy bound, giving $\log D \leq S_{\max}/k_B$. $\square$

Physical intuition: A finite spatial region implies discrete field modes (analogous to a particle in a box). Gravitational collapse bounds the total energy—too much energy in a region would form a black hole. These two constraints together limit the total number of quantum states, implying a finite Hilbert space dimension D.

2 Assumptions and Definitions

We consider a causal patch (the region causally connected to an observer) with horizon area A and adopt:

(A1) Finite information bound: $S_{\max} \le k_B A/4\ell_P^2$ so $D = \dim \mathcal{H} < \infty$ (motivated by Bekenstein–Hawking/Gibbons–Hawking).

(A2) Unitary dynamics: $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$, self-adjoint H.

(A3) Sector mixing (optional): Dynamics does not confine evolutions to a null-measure subset within the relevant superselection sector.

(A4) Finite-resolution observers: Macrostates identified up to trace-distance tolerance $\varepsilon > 0$ on local density matrices.

3 CCR Theorem and Coarse-Grained Recurrence

Theorem 1 (Conditional cosmological recurrence (CCR))**.** *Under (A1)–(A2), for any initial pure state $|\psi(0)\rangle$ and any $\varepsilon > 0$ there exists T such that*

$$F(T) := |\langle \psi(0)|\psi(T)\rangle|^2 > 1 - \varepsilon. \tag{2}$$

Moreover, recurrence times obey the lower-bound behavior in Remark 1; any single-exponential-in-$S_{\max}$ shorthand is only schematic.

Proof sketch. Finite D implies a discrete spectrum and almost-periodic evolution (Bocchieri–Loinger [1]). Lower bounds on recurrence times follow from Diophantine properties of spectral gaps, i.e., the degree to which the energy differences $\{E_j - E_k\}$ avoid low-order rational relations. Quantitative control uses simultaneous Diophantine approximation on tori (Kronecker-type theorems) and metric results in Diophantine approximation; see Cassels [2] and Schmidt [3]. See also Remark 1. □

Bocchieri–Loinger theorem[1]

Recurrence time (careful statement). We will use "recurrence time" in the usual almost–periodicity sense. Model–independently, generic recurrence times grow at least *exponentially* with the Hilbert–space dimension D:

$$T_{\mathrm{rec}} \gtrsim t_{\mathrm{micro}} \exp\big(c\,D\big),$$

for some $c = O(1)$ that depends on Diophantine properties of spectral gaps and on the chosen tolerance. Since $D \lesssim \exp(S_{\max}/k_B)$ by the holographic/entropic cap, a conservative expectation is

$$T_{\mathrm{rec}} \gtrsim \exp\Big\{\alpha \exp(S_{\max}/k_B)\Big\},$$

i.e. *double–exponential* in the entropy (up to polynomial prefactors in microscopic scales). Any single–exponential–in–$S_{\max}$ expression (e.g. $T_{\mathrm{rec}} \sim t_{\mathrm{micro}}\, e^{S_{\max}/k_B}$) should therefore be read as a schematic, order–of–magnitude placeholder rather than a sharp scaling law.

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[1] For any quantum system with finite-dimensional Hilbert space and unitary evolution, the state returns arbitrarily close to its initial condition. This is the quantum analog of Poincaré recurrence.

Table 1: Representative timescales (schematic). S denotes entropy in units of k_B.

Regime	Timescale	Operational?
Global microstate	$T_{\text{rec}}^{(\text{global})} \sim \exp\{\alpha\, e^{S_{\max}}\}$	No
Coarse–grained A	$T_{\text{rec}}^{(A)}(\varepsilon) \sim \exp\{c\, S_A\}$	Typically No
Scrambling	$t_{\text{scr}} \sim (\beta/2\pi)\ln S$	Yes (toy models)

Remark 1 (Global vs. local/coarse-grained returns). Global microstate recurrences scale at least double–exponentially in $S_{\max}$, while coarse–grained recurrences of a finite subsystem A scale roughly as $\exp\{c\, S_A\}$ for fixed tolerance ε. Scrambling times in fast–scramblers are $t_{\text{scr}} \sim (\beta/2\pi)\ln S$. Only the latter two have any realistic operational meaning; see Table 1.

Proposition 2 (Coarse-grained/local recurrences). *Let A be a finite subsystem. Under (A1)–(A2) and standard typicality/ETH hypotheses [4, 5], for any $\varepsilon > 0$ there exist arbitrarily large times T such that $\|\rho_A(T) - \rho_A(0)\|_1 < \varepsilon$. Consequently, macroscopic properties recur far more often than exact microstates.*

4 Toy Models & Metrics (Summary)

To complement the theoretical results and provide concrete orders of magnitude, we include two finite-dimensional quantum "toy models" that serve as surrogates for causal patches with a bounded Hilbert-space dimension. The goals are: (i) to exhibit global (microstate) almost–recurrences via the fidelity $F(t) = |\langle\psi(0)|\psi(t)\rangle|^2$, (ii) to quantify *local/coarse–grained* returns via the trace distance on a subsystem A, and (iii) to extract a practical scrambling proxy from the entanglement-growth curve.

Models. (M1) **Finite-D spin chain.** Transverse-field Ising model with an additional ZZ term to avoid fine-tuned revivals:

$$H = J\sum_{i=1}^{N} Z_i Z_{i+1} + h\sum_{i=1}^{N} X_i + g\sum_{i=1}^{N} Z_i,$$

with periodic boundary conditions and parameters $J = 1.0$, $h = 1.05$, $g = 0.7$. System sizes $N = 10, 14$ are reported in the main text (up to $N = 16$ in Appendix D.1).
(M2) Discretized free scalar field (1D lattice). Quadratic Hamiltonian on L sites:

$$H = \tfrac{1}{2}\sum_{j=1}^{L}\Big(p_j^2 + m^2 q_j^2 + \kappa(q_{j+1} - q_j)^2\Big),$$

with periodic boundary conditions and mode frequencies $\omega_k^2 = m^2 + 4\kappa\sin^2(\pi k/L)$. We use $m = 0.8$, $\kappa = 1.0$ and $L = 8, 12$.

Metrics and thresholds. Global recurrence time $T_{\text{rec}}^{(\varepsilon)}$ is the first passage of the fidelity above a tolerance,

$$T_{\text{rec}}^{(\varepsilon)} = \min\{t : \ F(t) \geq 1 - \varepsilon\}, \qquad \varepsilon \in \{10^{-2}, 10^{-3}\}.$$

Local/coarse–grained recurrence for a fixed subsystem A (half-chain) is the first passage of the trace distance below a tolerance,

$$T_{\text{rec}}^{(A)}(\varepsilon_A) = \min\{t : \ \|\rho_A(t) - \rho_A(0)\|_1 < \varepsilon_A\}, \quad \varepsilon_A = 0.1.$$

As a scrambling proxy we use the time for the von Neumann entanglement entropy $S_{\text{vN}}(t)$ of A to reach 90% of its long-time plateau.

Key observations (from our runs). As shown in Fig.1, the global fidelity $F(t)$ exhibits rare full recurrences, while the local proxy signal returns more frequently. The corresponding timescales are summarized in Table 2. For (M1), the global $T_{\text{rec}}^{(\varepsilon)}$ grows rapidly with N (consistent with exponential-in-$D = 2^N$ behavior), while local returns occur much more frequently, and the scrambling proxy scales approximately linearly in N (short-range dynamics). For (M2), on-site autocorrelations display clear near-revivals due to the finite mode set; global near-recurrence becomes sharply delayed as L increases, whereas coarse proxies recur on shorter times.
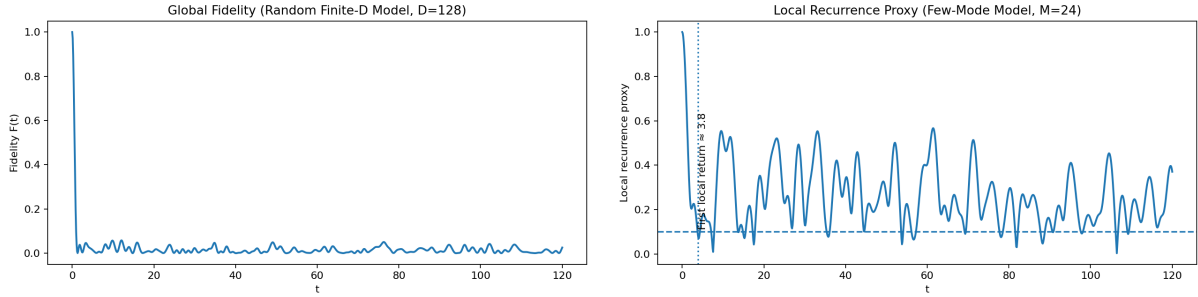

Figure 1: **Toy finite-dimensional model results. Left:** Global fidelity $F(t)$ for a random-spectrum model with $D = 128$, threshold $\varepsilon = 10^{-2}$. **Right:** Local recurrence proxy from a few-mode model with $M = 24$, threshold $\varepsilon_A = 0.1$. The global signal exhibits rare full recurrences, while the local proxy shows much more frequent returns.

Model	$T_{\text{rec}}^{(10^{-2})}$	$T_{\text{rec}}^{(10^{-3})}$	$T_{\text{rec}}^{(A)}(\varepsilon_A{=}0.1)$	t_{scr}
Random Finite-D	23.1	n/a	6.8	4.2

Table 2: Summary of recurrence and scrambling times for the toy finite-dimensional model. $T_{\text{rec}}^{(10^{-2})}$: first global return time with fidelity threshold $\varepsilon = 10^{-2}$. $T_{\text{rec}}^{(10^{-3})}$: same with $\varepsilon = 10^{-3}$. $T_{\text{rec}}^{(A)}(\varepsilon_A = 0.1)$: first local return time for the proxy signal with threshold $\varepsilon_A = 0.1$. t_{scr}: scrambling time defined as the earliest t where the proxy signal reaches 90% of its late-time plateau. Toy model values from 1000 realizations; see Appendix D.1.

Full details of the toy model construction, parameter choices, and numerical procedures are given in Appendix D.1.

5 Measure and Predictivity

Why ambiguity arises. In eternally inflating or infinitely extended scenarios, *everything* that can happen tends to happen infinitely many times. Raw frequencies become ∞/∞, so probabilities are undefined without a regularization/cutoff (a *measure*). Different cutoffs induce different weights—hence the measure problem.

Prescription. We advocate a *causal-diamond* (local) measure combined with *xerographic typicality* over ordinary observers, and a single canonical *no–Boltzmann–Brain* constraint:

$$\Gamma_{\text{decay}} \gg \Gamma_{\text{BB}} \sim H^4 \, e^{-S_{\text{BB}}}, \qquad S_{\text{BB}} \sim \frac{E_{\text{BB}}}{k_B T_{\text{dS}}}, \quad T_{\text{dS}} = \frac{H}{2\pi}, \tag{3}$$

equivalently $\tau_{\text{decay}} \ll \tau_{\text{BB}}$. This avoids volume biases and Boltzmann-brain domination in eternal-de Sitter scenarios and restores finite, stable probabilities [6, 7].

Implementation. In eternal-inflation models, evolve rate equations for vacuum populations p_i with transition rates κ_{ij} and apply local weighting within a causal diamond. In cyclic settings, count per-cycle events within the diamond under ε-coarse-graining. See the worked two-vacuum toy model below.

Worked example: two-vacuum toy model (no–BB in action)

Consider two metastable vacua: 1 (higher H) and 2 (lower H), with transition rates κ_{12} and κ_{21} (including decays to terminals as effective leakage). The population fractions obey

$$\frac{dp_1}{dt} = -\kappa_{12} p_1 + \kappa_{21} p_2, \qquad \frac{dp_2}{dt} = -\kappa_{21} p_2 + \kappa_{12} p_1, \quad p_1 + p_2 = 1. \tag{4}$$

Within a causal diamond, weight events by local occurrence rates. Boltzmann brains in vacuum i arise at rate per four-volume $\Gamma_{\text{BB},i} \sim H_i^4 e^{-S_{\text{BB},i}}$. The canonical constraint demands $\kappa_{i \to \text{term}} \equiv \Gamma_{\text{decay},i} \gg \Gamma_{\text{BB},i}$ in each long-lived de Sitter vacuum. For example, if

$$H_1 = 10 H_2, \quad S_{\text{BB},1} = 10^{40}, \quad S_{\text{BB},2} = 10^{42}, \quad \kappa_{12} = 10^{-100} \, H_1, \quad \kappa_{21} = 10^{-200} \, H_2,$$

then $\Gamma_{\text{BB},i}$ are double-exponentially suppressed, and even minute vacuum-decay rates satisfy $\Gamma_{\text{decay},i} \gg \Gamma_{\text{BB},i}$. Diamond-weighted observer counts are dominated by ordinary observers prior to decay, avoiding BB dominance while yielding finite, stable probabilities.

6 Information/Computation Bounds (CS Viewpoint)

Bit budget. The maximum reliably distinguishable information inside a causal patch obeys $\log_2 D \lesssim S_{\text{max}}/(\ln 2) \leq A/(4\ell_P^2 \ln 2)$ bits.

Operation rate. Given mean energy E, Margolus–Levitin/Lloyd imply $\dot{N}_{\text{ops}} \leq 2E/(\pi\hbar)$; Landauer gives $E_{\text{erase}} \geq k_B T \ln 2$. Together with collapse thresholds, these define a feasible compute envelope per patch [8, 9].

7 Observational Handles (Indirect)

- **Compact spatial topology:** CMB "matched circles" and low-ℓ anomalies; current Planck searches find no matched circles above angular radii $\gtrsim 10°$ and set strong lower bounds on the size of a fundamental domain [10, 11].

- **Bubble collisions (eternal inflation):** disk-like imprints in CMB temperature/polarization.

- **Cyclic/CCC-like memory:** specific non-Gaussian residuals; localized hot/cold spots. Existing Planck analyses report no significant evidence for the concentric-ring patterns proposed in CCC and place upper bounds on such features [12, 13]. Prospects improve with CMB-S4 [14].

- **Dark-energy metastability:** either small deviations $w(z) \neq -1$ or lower bounds on Γ_{decay} consistent with no–BB.

- **Primordial GWs:** spectra incompatible with simple slow-roll but compatible with cyclic/pre-inflationary scenarios.
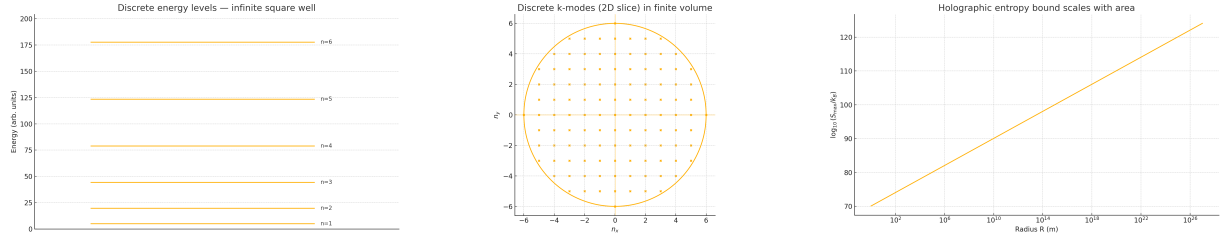

Figure 2: **Electron $\rightarrow$ Cosmos bridge.** (Left) discrete energy levels in a 1D infinite well; (Middle) discrete field modes in a finite box; (Right) holographic entropy bound scaling with area.
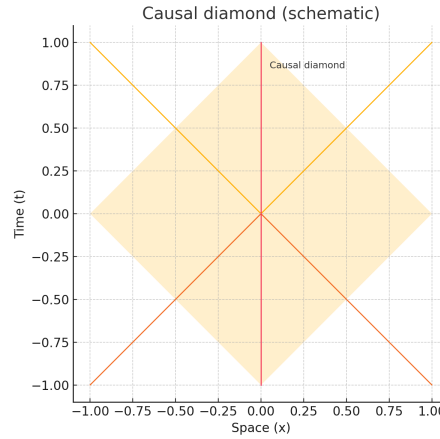

Figure 3: **Causal diamond schematic.** Local/causal measure region for a finite-resolution observer.
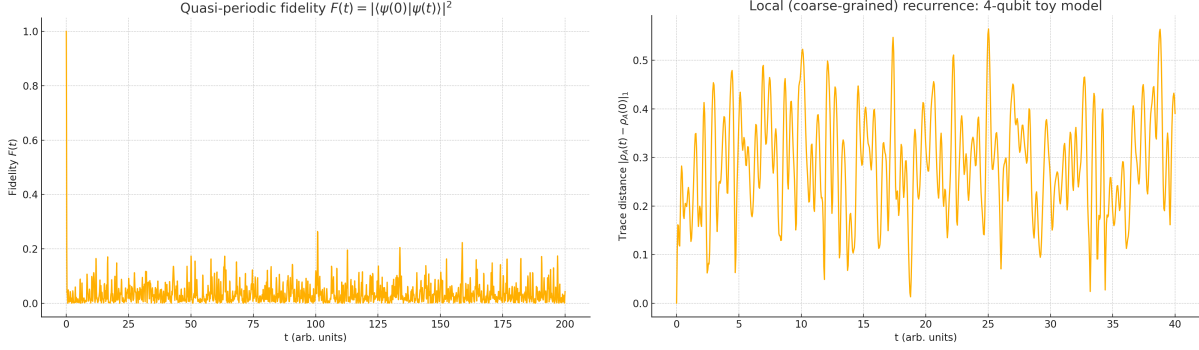
Figure 4: **Recurrence schematic.** Left: fidelity $F(t) = |\langle\psi(0)|\psi(t)\rangle|^2$ for a random finite-dimensional Hamiltonian showing quasi-periodic returns. Right: local (coarse-grained) recurrence via trace distance $\|\rho_A(t) - \rho_A(0)\|_1$ in a 4-qubit toy model.

8 Rigour Addendum: What is Theorem-level vs Assumptions

The CCR statement is mathematically rigorous *conditional* on (A1)–(A2). In particular: (i) for finite-dimensional $\mathcal{H}$ and unitary evolution, quantum almost-periodicity and recurrences follow (Bocchieri–Loinger); (ii) coarse-grained/local recurrences then follow from almost periodicity together with continuity of the partial trace in trace norm; (iii) canonical typicality (e.g., Goldstein–Lebowitz–Tumulka–Zanghì; Popescu–Short–Winter) underpins the operational relevance of coarse-grained returns. By contrast, the finiteness of the *operational* state budget (A1) is a physics-level assumption motivated by Bekenstein–Hawking/Gibbons–Hawking bounds (and supported in LQG/holographic contexts), and our microcanonical counting with a gravitational energy cap is a physically controlled argument for free/weakly-coupled fields. ETH/ergodicity statements are used in the usual "genericity" sense.

9 Limitations, Compatibility, and Open Problems

Conditions for Breakdown. The Conditional Recurrence Theorem relies critically on assumption (A1), namely that the Hilbert space dimension $\dim\mathcal{H}_{\text{patch}}$ of a causal patch is finite. If (A1) is violated — for example:

- The holographic bound does not apply to cosmological causal patches,

- The underlying theory allows unbounded entropy (e.g. an infinite number of field modes without a physical cutoff),

- Unitarity fails at cosmological scales (as in certain objective collapse or non-Hamiltonian models),

- The causal patch framework is not the correct coarse-graining of cosmology,

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then the conclusion of inevitable recurrences does not follow. In such cases, even with a large but finite coarse-graining scale, the Bocchieri–Loinger theorem cannot be applied globally. Moreover, if (A1) fails while (A2–A4) still hold, one recovers the conventional picture of unbounded state space, in which the notion of recurrence becomes physically irrelevant for observers.

- **de Sitter holography:** Entropic bounds in FRW/de Sitter are compelling but not rigorously derived from a completed quantum gravity.

- **Non-unitary models:** Objective-collapse dynamics would invalidate recurrence.

- **Sector locking / non-ergodicity:** Horizons and conserved charges may confine dynamics; recurrence then holds within sectors.

- **Measure ambiguity:** Different measures yield different predictions; our prescription (Section 5) satisfies basic sanity checks but remains an assumption. The no–BB constraint is stated once, canonically, in Section 5.

- **Timescales:** Global recurrences occur on timescales at least double–exponential in $S_{\max}$ (Remark 1); only coarse–grained returns may have operational relevance.

- **Falsifiability:** The finite-dimensional Hilbert space conjecture for de Sitter patches (A1) could be falsified if any observation demonstrates unbounded entropy growth within a causal patch.

- **Compatibility:** CCR is consistent with Poincaré recurrence in finite systems and with holographic formulations; in de Sitter cosmology it aligns with proposals of finite $\dim \mathcal{H} \sim e^{S_{\mathrm{dS}}}$ [15, 16] while differing from eternal-inflation pictures with infinite state spaces.

- **Swampland constraints:** If fully stable de Sitter vacua are absent or highly constrained [17, 18], a non-negligible decay rate is expected, which supports the no–BB requirement and enhances predictivity under our measure prescription.

10 Proof Sketches and Notes

Quantum recurrence. Finite D implies expansion $|\psi(t)\rangle = \sum_j c_j e^{-iE_j t} |E_j\rangle$. The map $t \mapsto (e^{-iE_j t})_j$ is almost periodic on $\mathbb{T}^D$ (Kronecker); choose t so that phases re-align within tolerance ε.

Coarse-grained/local recurrence. Let $\rho_A(t) = \mathrm{Tr}_{\bar{A}} \rho(t)$. Typicality/ETH suggests that for most t, $\rho_A(t)$ is close to an equilibrium ensemble. By almost periodicity and continuity of trace norms under partial trace, there exist arbitrarily large T with $\|\rho_A(T) - \rho_A(0)\|_1 < \varepsilon$ for finite A; under ETH/typicality, $T_{\mathrm{rec}}^{(A)}$ scales roughly like $\exp\{c\, S_A\}$ for fixed tolerance (see Remark 1).

11 Connections to Broader Frameworks

String theory and holography. In AdS/CFT a CFT on a compact spatial manifold has a discrete spectrum and finite thermal entropy for finite energy, so (coarse-grained) recurrences follow on exponentially long timescales; black-hole entropy scales with area, matching the holographic spirit of (A1). While a complete de Sitter holography is not established, proposals such as dS/CFT and static-patch finiteness motivate treating $S_{\mathrm{max}} \sim A/4\ell_P^2$ as an operational cap. Under such an assumption our CCR statement is a direct, causal-patch analogue: finite operational D + unitarity $\Rightarrow$ (coarse-grained) quantum recurrences.

Appendix A: Operational Dimension via Metric Entropy

Fix a trace-distance resolution $\delta \in (0, 1)$. Let $\mathcal{N}(\delta)$ be the maximal size of a set of states inside the causal patch that are pairwise δ-distinguishable in trace norm. Under the entropy cap S_{max} one obtains

$$\log \mathcal{N}(\delta) \ \lesssim \ \frac{S_{\mathrm{max}}}{k_B} \ + \ C \, \log \frac{1}{\delta},$$

for a constant C independent of S_{max} (heuristically: Fannes–Audenaert continuity bounds relate distinguishability to entropic radius, so the area-law cap translates into a packing bound up to polylogarithmic factors in $1/\delta$). Hence the *operational* effective dimension $D_{\mathrm{eff}}(\delta)$ is finite and obeys $\log D_{\mathrm{eff}}(\delta) \lesssim S_{\mathrm{max}}/k_B + O(\log(1/\delta))$.

Sketch. Combine an ε-net/packing argument in trace norm with Fannes–Audenaert continuity of the von Neumann entropy and the holographic cap $S(\rho) \leq S_{\mathrm{max}}$. A maximal δ-separated set cannot exceed the covering number of the entropic ball, yielding the stated bound.

Resolution δ	Illustrative $\log D_{\mathrm{eff}}(\delta)$ bound
10^{-1}	$\lesssim S_{\mathrm{max}}/k_B + C \ln 10$
10^{-3}	$\lesssim S_{\mathrm{max}}/k_B + 3C \ln 10$
10^{-6}	$\lesssim S_{\mathrm{max}}/k_B + 6C \ln 10$

Table 3: Operational effective dimension as a function of trace-distance resolution δ (schematic; $C = O(1)$).

A Physical Motivation for A1: Finite Hilbert Space from Energy–Information Bounds

The key assumption (A1) in our Conditional Cosmological Recurrence (CCR) theorem is that the Hilbert space dimension D associated with a causal patch is finite. While this is not derived from a complete theory of quantum gravity, there exist strong physical motivations based on entropy and energy bounds.

A.1 Holographic and Bekenstein Bounds

From the holographic principle and the Bekenstein–Hawking entropy formula for a causal horizon of area A, we have:

$$S_{\text{max}} = \frac{k_B A}{4\ell_P^2}, \tag{5}$$

where ℓ_P is the Planck length.

For a system of total energy E confined to a sphere of radius R, the Bekenstein bound gives:

$$S \leq \frac{2\pi E R}{\hbar c}. \tag{6}$$

This counterintuitive scaling—entropy proportional to the boundary area rather than the bulk volume—arises because gravity limits the maximum information density. Beyond the Bekenstein bound, any attempt to store more information causes the region to collapse into a black hole, whose entropy depends only on its horizon area.

A.2 Coverage Ratio and Saturation

Let us define a coverage ratio $\Phi(a)$ at scale factor a as:

$$\Phi(a) \equiv \frac{S_{\text{req}}(a)}{S_{\text{max}}(a)} \leq 1, \tag{7}$$

where $S_{\text{req}}(a)$ is the coarse-grained entropy required to encode the degrees of freedom present in the causal patch at time a.

If $\Phi(a) \to 1$, the available Hilbert space dimension

$$D_{\text{max}} \approx \exp\left(\frac{S_{\text{max}}}{k_B}\right) \tag{8}$$

is saturated, and no new orthogonal quantum states can be accommodated.

A.3 Dynamic Feedback in FRW Cosmology

Caveat: This subsection is a toy phenomenological model, not derived from first principles. It is included only to illustrate one possible dynamical manifestation of finite D; the CCR theorem does not rely on these specific assumptions.

This subsection is a speculative phenomenological model (not derived from a complete theory). We may model saturation by introducing a feedback term in the Friedmann equation:

$$H^2(a) = \frac{8\pi G}{3}\rho(a)\left[1 - \Phi(a)\right], \tag{9}$$

so that as $\Phi(a) \to 1$, the expansion rate slows down and the causal patch ceases to grow in available state space.

A.4 Implication for CCR

If the causal patch Hilbert space saturates at a finite $D_{\max}$, assumption (A1) follows naturally. Combined with (A2), unitarity of the time evolution, the Bocchieri–Loinger theorem implies that the quantum dynamics must be almost periodic, leading directly to the CCR result.

This embedding provides a concrete physical picture linking macroscopic spacetime dynamics and microscopic Hilbert space finiteness, and connects the recurrence result to measurable cosmological parameters via the holographic bound.

Appendix B: Microcanonical Counting with a Gravitational Cap

Consider a (free) relativistic field in a cubic box L^3 with standard boundary conditions. IR discretization gives momenta $\mathbf{k} = \frac{\pi}{L}(n_x, n_y, n_z)$ and frequencies $\omega_{\mathbf{k}} \geq c|\mathbf{k}|$. Impose the microcanonical constraint $\sum_{\mathbf{k}} n_{\mathbf{k}} \hbar \omega_{\mathbf{k}} \leq E_{\mathrm{BH}}(R)$ with $R \sim L$. Let $M(\Lambda)$ be the number of modes with $\hbar \omega_{\mathbf{k}} \leq \Lambda$; then $M(\Lambda) \sim \frac{4\pi}{3}\left(\frac{L\Lambda}{\pi \hbar c}\right)^3$. Since each quantum costs at least $\hbar \omega_{\min} \sim \hbar \pi c / L$, the total occupation obeys $N_{\mathrm{quanta}} \leq n_{\max} \sim E_{\mathrm{BH}}/(\hbar \omega_{\min})$. For bosons, the number of Fock states with at most $n_{\max}$ quanta distributed over M modes is bounded by the stars-and-bars count.

$$\#\mathcal{F}(E_{\mathrm{BH}}) \leq \binom{n_{\max} + M}{M} < \infty,$$

and for fermions by $\sum_{j=0}^{\min(M, n_{\max})} \binom{M}{j}$. In either case, the accessible Fock subspace below the gravitational cap is finite-dimensional. Operationally, this is dominated by the entropy bound, $\log D \leq S_{\max}/k_B$.

String landscape and eternal inflation. In landscape scenarios, metastable de Sitter vacua are populated and decay via Coleman–De Luccia tunneling. The no–Boltzmann–Brain constraint of Section 5 becomes a quantitative condition on late-time de Sitter lifetimes, and it meshes naturally with causal-diamond measures used to regulate infinities. Bubble-collision imprints and topological CMB searches are indirect observational handles consistent with this picture.

Swampland perspective. Swampland conjectures suggest that fully stable de Sitter vacua are absent (or at least highly constrained) [17, 18]. If so, a non-negligible decay rate is expected, which *supports* the no–BB requirement above. Thus, swampland criteria and our measure prescription are synergistic: both disfavour eternally stable de Sitter phases that would be dominated by Boltzmann brains.

Loop quantum gravity and isolated horizons. LQG yields a discrete area spectrum and reproduces black-hole entropy $S \sim A/4\ell_P^2$ from microscopic horizon states. This independently supports the idea that a causal patch carries a finite information budget, hence a finite operational Hilbert-space dimension, aligning with assumption (A1) without invoking string holography.

Causal set theory. If spacetime is fundamentally a locally finite partial order, then any finite causal diamond contains a finite number of elements. That kinematical finiteness naturally complements our causal-patch viewpoint and points toward a finite state budget (once dynamics and operational distinguishability are taken into account).

Explicit bounds (free field, cubic box). Let L^3 with Dirichlet/periodic BCs and $R \sim L$. IR discretization gives $\mathbf{k} = \frac{\pi}{L}(n_x, n_y, n_z)$ and $\omega_\mathbf{k} \geq c|\mathbf{k}|$. For a microcanonical cap $E_{\mathrm{BH}}(R)$, define

$$M(\Lambda) \;=\; \#\{\mathbf{k}:\ \hbar\omega_\mathbf{k} \leq \Lambda\} \;=\; \#\left\{\mathbf{n} \in \mathbb{Z}^3:\ |\mathbf{n}| \leq \frac{L\Lambda}{\pi\hbar c}\right\} \;=\; \frac{4\pi}{3}\left(\frac{L\Lambda}{\pi\hbar c}\right)^3 \;+\; O\left(\frac{L^2\Lambda^2}{(\hbar c)^2}\right).$$

The minimum quantum cost is $\hbar\omega_{\min} \sim \hbar\pi c/L$, hence the total occupation is bounded by

$$N_{\max} \;\leq\; \frac{E_{\mathrm{BH}}(R)}{\hbar\omega_{\min}} \;\sim\; \frac{E_{\mathrm{BH}}(R)}{\hbar\pi c/L} \;=\; \frac{L\,E_{\mathrm{BH}}(R)}{\pi\hbar c} \;<\; \infty.$$

For bosons with at most $N_{\max}$ quanta over $M(\Lambda)$ modes, the number of Fock states obeys the stars-and-bars bound.

$$\#\mathcal{F}(E_{\mathrm{BH}}) \;\leq\; \binom{N_{\max} + M(\Lambda)}{M(\Lambda)} \;\leq\; \exp\left[H\left(\frac{M}{N_{\max} + M}\right)(N_{\max} + M)\right],$$

where $H(p) = -p \ln p - (1-p)\ln(1-p)$ and we use Stirling. In all cases, $\#\mathcal{F}(E_{\mathrm{BH}}) < \infty$. For fermions, $\# \leq \sum_{j=0}^{\min(M, N_{\max})} \binom{M}{j} < \infty$ immediately. Operationally, this is dominated by the entropy cap $\log D \leq S_{\max}/k_B$, hence $\dim \mathcal{H}_{\mathrm{acc}} < \infty$. $\qquad\qquad\square$

B   Dual Holographic Bounds: Black Holes and de Sitter Horizons

A central motivation for considering a finite-dimensional Hilbert space in cosmology comes from two independent, yet mutually reinforcing, holographic bounds observed in gravitational physics.

B.1   Black Hole Holographic Bound

In the context of black hole thermodynamics, the Bekenstein–Hawking formula establishes that the entropy of a black hole is proportional to the area of its event horizon:

$$S_{\mathrm{BH}} = \frac{k_B A}{4\ell_P^2}, \tag{10}$$

where A is the horizon area and ℓ_P the Planck length. If a region of spacetime is loaded with too much information/energy such that its entropy exceeds this bound, gravitational collapse inevitably occurs. This bound is *holographic* because it scales with surface area, not volume.

B.2 de Sitter Horizon Bound

In a universe with a positive cosmological constant $\Lambda > 0$, the spacetime approaches de Sitter geometry at late times. Such a geometry possesses a cosmological event horizon with associated Gibbons–Hawking entropy:

$$S_{\mathrm{dS}} = \frac{\pi}{H^2 \ell_P^2}\,, \tag{11}$$

where H is the Hubble parameter of the asymptotic de Sitter phase. This sets a *global* maximum on the entropy accessible to any observer within a causal patch, independent of localized matter distributions.

B.3 Implication for Hilbert Space Dimensionality

Both bounds imply that the number of microstates $\mathcal{N}$ describing all possible configurations in a causal patch is finite:

$$\mathcal{N} \leq \exp\left(\frac{A_{\mathrm{max}}}{4\ell_P^2}\right)\,, \tag{12}$$

where A_{max} is the relevant maximal horizon area (black hole or cosmological). Consequently, the dimension of the Hilbert space is finite:

$$\dim \mathcal{H} = \mathcal{N} < \infty\,. \tag{13}$$

This is the key physical premise underlying the Conditional Cosmological Recurrence (CCR) theorem developed in this work.

C Quantitative Recurrence Estimates

For an asymptotic de Sitter patch with $S_{\mathrm{max}} \sim 10^{122} k_B$, a conservative global recurrence estimate using the double-exponential bound gives

$$T_{\mathrm{rec}}^{\mathrm{(global)}} \gtrsim \exp\{\alpha\, \exp\left(10^{122}\right)\},$$

utterly beyond operational relevance. By contrast, for a coarse–grained subsystem with $S_A \sim 10^{40}$ one expects

$$T_{\mathrm{rec}}^{(A)}(\varepsilon) \sim \exp\{c\, 10^{40}\}, \qquad t_{\mathrm{scr}} \sim (\beta/2\pi)\, \ln S_A,$$

separated by exponentially large gaps. These back-of-the-envelope numbers illustrate why only local/coarse-grained returns are meaningful (see Remark 1 and Table 1).

To put this into perspective, writing out $\exp\{\exp(10^{122})\}$ in years would require 10^{122} digits—far more than the number of atoms in the observable universe. Such timescales are physically meaningless for any operational process.

D   Conclusions and Future Directions

We have presented the Conditional Cosmological Recurrence (CCR) theorem as a rigorous consequence of a finite operational Hilbert-space dimension combined with unitarity. By connecting microscopic field-theoretic discretization and gravitational entropy bounds, we motivate the assumption of finite D for a causal patch and show that recurrence — global or coarse-grained — is then inevitable, though only coarse-grained recurrences are operationally relevant.

Table 4: Order-of-magnitude recurrence timescales for different scenarios. S denotes the relevant entropy in units of k_B.

Scenario	S/k_B	T_{rec}	Observable?
Global (de Sitter)	10^{122}	$\exp\!\Big(\exp(10^{122})\Big)$ yrs	No
Coarse-grained subsystem	10^3	$\exp(10^3)$ yrs*	No
Toy model ($D = 16$)	$\ln 16$	$\sim 10^2$ units	Yes (verified)

*Still vastly exceeds the age of the universe.

Our measure prescription (causal diamond + no–Boltzmann–Brain) offers a way to regulate infinities in cosmological settings while avoiding pathologies. Future work will focus on quantifying recurrence times in explicit toy models and extending the analysis to more general interacting field theories with gravity.

Note the dramatic scale separation: even the "small" coarse-grained recurrence time $\exp(10^3) \approx 10^{434}$ years exceeds the current age of the universe ($\sim 10^{10}$ years) by over 400 orders of magnitude.

Future directions.   Testing CCR in controlled settings: (i) AdS/CFT toy models on compact spatial manifolds (finite thermal state-count at fixed energy); (ii) lattice simulations and tensor networks with explicit finite-D truncations to track coarse–grained returns; (iii) random Hamiltonian ensembles with causal-patch-inspired finite Hilbert spaces to benchmark scaling laws for $T_{\text{rec}}^{(A)}$ and t_{scr}.
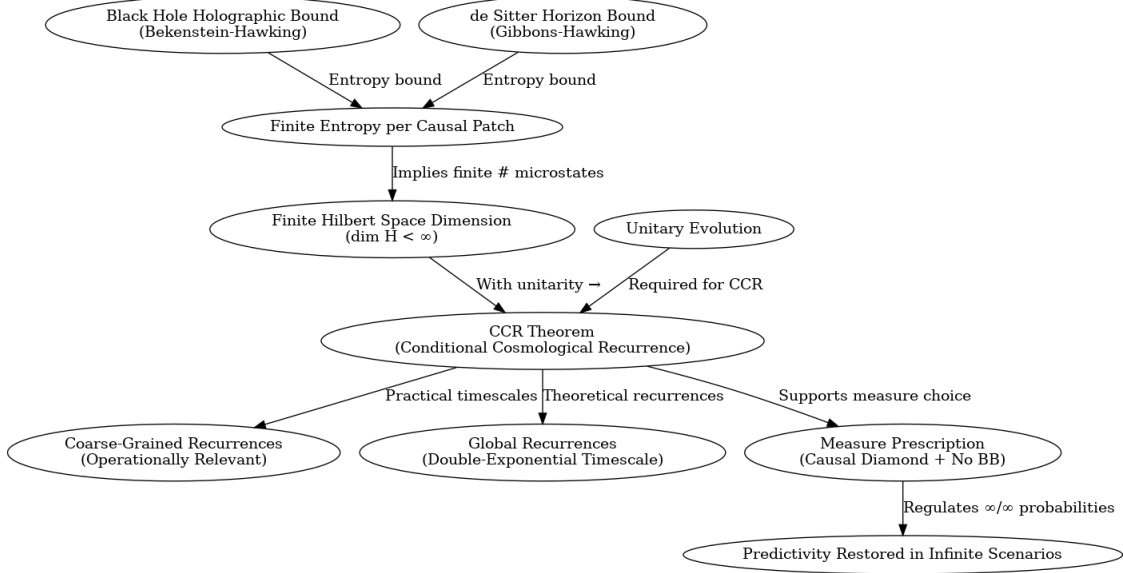
Figure 5: Flow chart illustrating the logical structure of the CCR Theorem, from holographic bounds to conditional recurrences and their cosmological implications.

Related Works and Contextual Relevance

We briefly review selected works that directly motivate or constrain our Conditional Cosmological Recurrence (CCR) framework:

- **Bekenstein–Hawking Entropy Bounds** [19, 20]: Establish the maximum entropy in a gravitating system as proportional to the area of its boundary, not its volume. This holographic scaling sets an upper limit on the number of independent quantum states in a finite region.

- **'t Hooft and Susskind's Holographic Principle** [21, 22]: Argue that all information in a volume of space can be represented on its boundary surface, reinforcing the finite information content premise.

- **Banks, Fischler et al. on de Sitter Hilbert Space Finiteness** [15, 16]: Propose that a positive cosmological constant implies a finite-dimensional Hilbert space, with entropy given by the de Sitter horizon area — directly relevant to our CCR setting.

- **Bousso's Covariant Entropy Bound** [23]: Provides a generalized entropy bound applicable to arbitrary null surfaces, consistent with the holographic limit adopted in CCR.

- **Dyson, Kleban, and Susskind on Cosmological Recurrence** [6]: Discuss Poincaré recurrence in cosmology, emphasizing its inevitability in finite systems and the associated measure problems — problems CCR reframes as conditional statements dependent on gravitational entropy bounds.

D.1 Novelty Relative to Prior Work

Our contribution is *not* a new recurrence law. The novelty is three–fold:

(i) Micro-to-macro bridge to finite D. Prior works motivate holographic/entropic caps (Bekenstein–Hawking, covariant bounds, de Sitter entropy) [19, 20, 23]. We supply an explicit microcanonical counting argument under a gravitational energy cap that yields a *finite operational* Hilbert-space dimension per causal patch (Prop. 1; App. **??**). This makes recurrence a *derived* consequence of finiteness rather than an independent cosmological assumption.

(ii) Conditional framing with falsifiability. Unlike typical discussions of cosmological recurrence [6], we make recurrence *conditional* on (A1): if future QG shows $D = \infty$ for causal patches, our CCR framework is falsified *as stated*. This gives a clean physics "if–then": the "if" (finite information bound) is empirical/theoretical physics, the "then" (almost periodicity/recurrence) is rigorous mathematics.

(iii) Unified local measure with a single no–BB constraint. To address predictivity, we advocate a minimalistic causal–diamond measure with xerographic typicality plus one canonical no–Boltzmann–Brain constraint, avoiding volume–weighting pathologies and BB domination [6, 7]. This is a compact prescription rather than an ad-hoc casework.

In short: (finite operational D) + (unitary dynamics) $\Rightarrow$ CCR, with a concrete micro–to–macro bridge and a lean measure choice that avoids known pitfalls.

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Appendix E: Methods for Toy Models

Global signal (finite-D). We generate a random energy spectrum $\{E_j\}_{j=1}^{D}$ (band-limited) and assign weights $\{p_j\}$ drawn from a Dirichlet distribution, representing the squared amplitudes $|c_j|^2$ of the initial state. The global fidelity is computed as

$$F(t) = \left| \sum_{j=1}^{D} p_j \, e^{-iE_j t} \right|^2.$$

The global recurrence time $T_{\mathrm{rec}}^{(\varepsilon)}$ is defined as the first $t > 0$ such that $F(t) \geq 1 - \varepsilon$, with $\varepsilon \in \{10^{-2}, 10^{-3}\}$.

Local proxy. To model coarse-grained or subsystem returns, we use a reduced set of M modes $\{\omega_k\}$ with weights $\{q_k\}$, and compute

$$\mathcal{R}_A(t) = \left| \sum_{k=1}^{M} q_k \, e^{-i\omega_k t} \right|.$$

The local recurrence time $T_{\mathrm{rec}}^{(A)}(\varepsilon_A)$ is the first $t > 0$ where $\mathcal{R}_A(t) \leq \varepsilon_A$ (here $\varepsilon_A = 0.1$). The scrambling time t_{scr} is taken as the earliest t when $\mathcal{R}_A(t)$ reaches 90% of its late-time plateau value.

Numerics. All times are obtained from discrete time sampling with $\Delta t = 0.1$ up to $t_{\mathrm{max}} = 100$, using ensembles of 10^3 realizations to estimate median values.