1. Modus Ponens relativizado:

$(\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma) \Rightarrow \rho \Rightarrow \tau$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \rho \vdash \rho \Rightarrow \sigma \Rightarrow \tau}\, ax \qquad \dfrac{}{\Gamma, \rho \vdash \rho}\, ax}{\Gamma, \rho \vdash \sigma \Rightarrow \tau}\, \Rightarrow_e \qquad \dfrac{\dfrac{}{\Gamma, \rho \vdash \rho \Rightarrow \sigma}\, ax \qquad \dfrac{}{\Gamma, \rho \vdash \rho}\, ax}{\Gamma, \rho \vdash \sigma}\, \Rightarrow_e}{\Gamma, \rho \vdash \tau}\, \Rightarrow_e}{\Gamma \vdash \rho \Rightarrow \tau}\, \Rightarrow_i}{(\rho \Rightarrow \sigma \Rightarrow \tau) \vdash (\rho \Rightarrow \sigma) \Rightarrow \rho \Rightarrow \tau}\, \Rightarrow_i}{\vdash (\rho \Rightarrow \sigma \Rightarrow \tau) \Rightarrow (\rho \Rightarrow \sigma) \Rightarrow \rho \Rightarrow \tau}\, \Rightarrow_i
$$

Llamo $\Gamma = (\rho \Rightarrow \sigma \Rightarrow \tau), (\rho \Rightarrow \sigma)$

2. Reducción al absurdo:

$(\rho \Rightarrow \bot) \Rightarrow \neg\rho$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{(\rho \Rightarrow \bot), \rho \vdash \rho \Rightarrow \bot}\, ax \qquad \dfrac{}{(\rho \Rightarrow \bot), \rho \vdash \rho}\, ax}{(\rho \Rightarrow \bot), \rho \vdash \bot}\, \Rightarrow_e}{(\rho \Rightarrow \bot) \vdash \neg\rho}\, \neg_i}{\vdash (\rho \Rightarrow \bot) \Rightarrow \neg\rho}\, \Rightarrow_i
$$

3. Introducción de la doble negación:

$\rho \Rightarrow \neg\neg\rho$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\rho, \neg\rho \vdash \rho}\, ax \qquad \dfrac{}{\rho, \neg\rho \vdash \neg\rho}\, ax}{\rho, \neg\rho \vdash \bot}\, \neg_e}{\rho \vdash \neg\neg\rho}\, \neg_i}{\vdash \rho \Rightarrow \neg\neg\rho}\, \Rightarrow_i
$$

4. Eliminación de la triple negación:

$\neg\neg\neg\rho \Rightarrow \neg\rho$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\neg\neg\rho, \rho, \neg\rho \vdash \rho}\, ax \qquad \dfrac{}{\neg\neg\neg\rho, \rho, \neg\rho \vdash \neg\rho}\, ax}{\neg\neg\neg\rho, \rho, \neg\rho \vdash \bot}\, \neg_e}{\neg\neg\neg\rho, \rho \vdash \neg\neg\rho}\, \neg_i \qquad \dfrac{}{\neg\neg\neg\rho, \rho \vdash \neg\neg\neg\rho}\, ax}{\neg\neg\neg\rho, \rho \vdash \bot}\, \neg_e}{\neg\neg\neg\rho \vdash \neg\rho}\, \neg_i}{\vdash \neg\neg\neg\rho \Rightarrow \neg\rho}\, \Rightarrow_i
$$

5. Contraposición:

$(\rho \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg\rho)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho \Rightarrow \sigma, \neg\sigma, \rho \vdash \rho \Rightarrow \sigma}\,ax \qquad \dfrac{}{\rho \Rightarrow \sigma, \neg\sigma, \rho \vdash \rho}\,ax}{\rho \Rightarrow \sigma, \neg\sigma, \rho \vdash \sigma}\,\Rightarrow_e \qquad \dfrac{}{\rho \Rightarrow \sigma, \neg\sigma, \rho \vdash \neg\sigma}\,ax}{\rho \Rightarrow \sigma, \neg\sigma, \rho \vdash \bot}\,\neg_e}{\rho \Rightarrow \sigma, \neg\sigma \vdash \neg\rho}\,\neg_i}{\rho \Rightarrow \sigma \vdash \neg\sigma \Rightarrow \neg\rho}\,\Rightarrow_i}{\vdash (\rho \Rightarrow \sigma) \Rightarrow (\neg\sigma \Rightarrow \neg\rho)}\,\Rightarrow_i$$

6. Adjunción:

$((\rho \wedge \sigma) \Rightarrow \tau) \Leftrightarrow (\rho \Rightarrow \sigma \Rightarrow \tau)$
Direccion: $\Rightarrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{(\rho \wedge \sigma) \Rightarrow \tau, \rho, \sigma \vdash (\rho \wedge \sigma) \Rightarrow \tau}\,ax \qquad \dfrac{\dfrac{}{(\rho \wedge \sigma) \Rightarrow \tau, \rho, \sigma \vdash \rho}\,ax \qquad \dfrac{}{(\rho \wedge \sigma) \Rightarrow \tau, \rho, \sigma \vdash \sigma}\,ax}{(\rho \wedge \sigma) \Rightarrow \tau, \rho, \sigma \vdash (\rho \wedge \sigma)}\,\wedge_i}{(\rho \wedge \sigma) \Rightarrow \tau, \rho, \sigma \vdash \tau}\,\Rightarrow_i}{(\rho \wedge \sigma) \Rightarrow \tau, \rho \vdash \sigma \Rightarrow \tau}\,\Rightarrow_i}{(\rho \wedge \sigma) \Rightarrow \tau \vdash \rho \Rightarrow \sigma \Rightarrow \tau}\,\Rightarrow_i}{\vdash ((\rho \wedge \sigma) \Rightarrow \tau) \Rightarrow \rho \Rightarrow \sigma \Rightarrow \tau}\,\Rightarrow_i$$

Direccion: $\Leftarrow$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho, \sigma, \tau, (\rho \wedge \sigma) \vdash \tau}\,ax}{\rho, \sigma, \tau \vdash (\rho \wedge \sigma) \Rightarrow \tau}\,\Rightarrow_i}{\rho, \sigma \vdash \tau \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)}\,\Rightarrow_i}{\rho \vdash \sigma \Rightarrow \tau \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)}\,\Rightarrow_i}{\vdash \rho \Rightarrow \sigma \Rightarrow \tau \Rightarrow ((\rho \wedge \sigma) \Rightarrow \tau)}\,\Rightarrow_i$$

7. de Morgan (I):

$\neg(\rho \vee \sigma) \Leftrightarrow (\neg\rho \wedge \neg\sigma)$
Direccion $\Rightarrow$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho}\,ax}{\Gamma \vdash (\rho \vee \sigma)}\,\vee_i 1 \qquad \dfrac{}{\Gamma \vdash \neg(\rho \vee \sigma)}\,ax}{\Gamma \vdash \bot}\,\neg_e}{\neg(\rho \vee \sigma) \vdash \neg\rho}\,\neg_i \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Delta \vdash \sigma}\,ax}{\Delta \vdash (\rho \vee \sigma)}\,\vee_i 2 \qquad \dfrac{}{\Delta \vdash \neg(\rho \vee \sigma)}\,ax}{\Delta \vdash \bot}\,\neg_e}{\neg(\rho \vee \sigma) \vdash \neg\sigma}\,\neg_i}{\dfrac{\neg(\rho \vee \sigma) \vdash \neg\rho \wedge \neg\sigma}{\vdash \neg(\rho \vee \sigma) \Rightarrow (\neg\rho \wedge \neg\sigma)}\,\Rightarrow_i}\,\wedge_i
$$

$\Gamma = \{\neg(\rho \vee \sigma), \rho\}$ y $\Delta = \{\neg(\rho \vee \sigma), \sigma\}$

Direccion $\Leftarrow$

$$
\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \rho \vee \sigma}\,ax \qquad \dfrac{}{\Gamma, \rho \vdash \rho}\,ax \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \sigma \vdash \sigma}\,ax \qquad \dfrac{\dfrac{}{\Gamma, \sigma \vdash \neg\rho \wedge \neg\sigma}\,ax}{\Gamma, \sigma \vdash \neg\sigma}\,\wedge_e 2}{\Gamma, \sigma \vdash \bot}\,\neg_e}{\Gamma, \sigma \vdash \rho}\,\bot_e}{\Gamma \vdash \rho}\,\vee_e \qquad \dfrac{\dfrac{}{\Gamma \vdash \neg\rho \wedge \neg\sigma}\,ax}{\Gamma \vdash \neg\rho}\,\wedge_e 1}{\dfrac{\dfrac{\Gamma \vdash \bot}{\neg\rho \wedge \neg\sigma \vdash \neg(\rho \vee \sigma)}\,\neg_i}{\vdash (\neg\rho \wedge \neg\sigma) \Rightarrow \neg(\rho \vee \sigma)}\,\Rightarrow_i}\,\neg_e
$$

$\Gamma = \{(\neg\rho \wedge \neg\sigma), \neg(\rho \vee \sigma)\}$

8. De Morgan II:

$\neg(\rho \wedge \sigma) \Leftrightarrow \neg\rho \vee \neg\sigma$
Direccion $\Rightarrow$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_0 \vdash \neg\rho}\,ax}{\Gamma_0 \vdash \Delta_0}\,\vee_{i_1} \qquad \dfrac{}{\Gamma_0 \vdash \Delta}\,ax}{\Gamma_0 \vdash \bot}\,\neg_e}{\Gamma \vdash \rho}\,{\scriptstyle pbc} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_1 \vdash \rho}\,ax \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma_2 \vdash \neg\sigma}\,ax}{\Gamma_2 \vdash \Delta_0}\,\vee_{i_2} \qquad \dfrac{}{\Gamma_1 \vdash \Delta}\,ax}{\Gamma_2 \vdash \bot}\,\neg_e}{\Gamma_1 \vdash \sigma}\,{\scriptstyle pbc}}{\Gamma_1 \vdash \rho \wedge \sigma}\,\wedge_i \qquad \dfrac{}{\Gamma_1 \vdash \neg(\rho \wedge \sigma)}\,ax}{\Gamma_1 \vdash \bot}\,\neg_e}{\Gamma \vdash \neg\rho}\,{\scriptstyle pbc}}{\dfrac{\Gamma \vdash \bot}{\vdash \neg(\rho \wedge \sigma) \Rightarrow \neg\rho \vee \neg\sigma}\,{\scriptstyle pbc}}\,\neg_e
$$

$\Gamma = \{\neg(\rho \wedge \sigma), \neg(\neg\rho \vee \neg\sigma)\}$
$\Gamma_0 = \{\neg(\rho \wedge \sigma), \neg(\neg\rho \vee \neg\sigma), \neg\rho\}$
$\Gamma_1 = \{\neg(\rho \wedge \sigma), \neg(\neg\rho \vee \neg\sigma), \rho\}$
$\Gamma_2 = \{\neg(\rho \wedge \sigma), \neg(\neg\rho \vee \neg\sigma), \rho, \neg\sigma\}$
$\Delta = \neg(\neg\rho \vee \neg\sigma)$

$\Delta_0 = \neg\rho \vee \neg\sigma$

Direccion $\Leftarrow$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{i_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \Delta$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_1 \vdash \neg\rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_2, \rho \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_2$}
\UnaryInfC{$\Gamma_2, \rho \vdash \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma_2, \rho \vdash \neg\sigma$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma_2, \rho \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\Gamma_2 \vdash \neg\rho$}
\RightLabel{$\vee_e$}
\TrinaryInfC{$\Gamma \vdash \neg\rho$}
\RightLabel{$\neg_e$}
\BinaryInfC{$\Gamma \vdash \bot$}
\RightLabel{$\neg_i$}
\UnaryInfC{$\vdash \neg\rho \vee \neg\sigma \Rightarrow \neg(\rho \wedge \sigma)$}
\end{prooftree}
$$

$\Gamma = \{\neg\rho \vee \neg\sigma, \rho \wedge \sigma\}$
$\Gamma_1 = \{\neg\rho \vee \neg\sigma, \rho \wedge \sigma, \neg\rho\}$
$\Gamma_2 = \{\neg\rho \vee \neg\sigma, \rho \wedge \sigma, \neg\sigma\}$
$\Delta = \{\neg\rho \vee \neg\sigma\}$

9. Conmutatividad ($\wedge$):

$(\rho \wedge \sigma) \Rightarrow (\sigma \wedge \rho)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho \wedge \sigma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_2}$}
\UnaryInfC{$\rho \wedge \sigma \vdash \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\rho \wedge \sigma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{e_1}$}
\UnaryInfC{$\rho \wedge \sigma \vdash \rho$}
\RightLabel{$\wedge_i$}
\BinaryInfC{$\rho \wedge \sigma \vdash \sigma \wedge \rho$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash \rho \wedge \sigma \Rightarrow \sigma \wedge \rho$}
\end{prooftree}
$$

10. Asociatividad ($\wedge$):

$((\rho \wedge \sigma) \wedge \tau) \Leftrightarrow (\rho \wedge (\sigma \wedge \tau))$
Direccion $\Rightarrow$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \wedge \tau$}
\RightLabel{$\wedge_{i_1}$}
\UnaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{i_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \wedge \tau$}
\RightLabel{$\wedge_{i_1}$}
\UnaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\RightLabel{$\wedge_{i_2}$}
\UnaryInfC{$\Gamma \vdash \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \wedge \tau$}
\RightLabel{$\wedge_{i_2}$}
\UnaryInfC{$\Gamma \vdash \tau$}
\RightLabel{$\wedge_e$}
\BinaryInfC{$\Gamma \vdash \sigma \wedge \tau$}
\RightLabel{$\wedge_e$}
\BinaryInfC{$\Gamma \vdash \rho \wedge (\sigma \wedge \tau)$}
\RightLabel{$\Rightarrow_i$}
\UnaryInfC{$\vdash (\rho \wedge \sigma) \wedge \tau \Rightarrow \rho \wedge (\sigma \wedge \tau)$}
\end{prooftree}
$$

$\Gamma = (\rho \wedge \sigma) \wedge \tau$

Direccion $\Leftarrow$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \wedge (\sigma \wedge \tau)$}
\RightLabel{$\wedge_{i_1}$}
\UnaryInfC{$\Gamma \vdash \rho$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \wedge (\sigma \wedge \tau)$}
\RightLabel{$\wedge_{i_2}$}
\UnaryInfC{$\Gamma \vdash \sigma \wedge \tau$}
\RightLabel{$\wedge_{i_1}$}
\UnaryInfC{$\Gamma \vdash \sigma$}
\RightLabel{$\wedge_e$}
\BinaryInfC{$\Gamma \vdash \rho \wedge \sigma$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$\Gamma \vdash \rho \wedge (\sigma \wedge \tau)$}
\RightLabel{$\wedge_{i_2}$}
\UnaryInfC{$\Gamma \vdash \sigma \wedge \tau$}
\RightLabel{$\wedge_{i_2}$}
\UnaryInfC{$\Gamma \vdash \tau$}
\RightLabel{$\wedge_{i_2}$}
\BinaryInfC{$\Gamma \vdash (\rho \wedge \sigma) \wedge \tau$}
\UnaryInfC{$\vdash \rho \wedge (\sigma \wedge \tau) \Rightarrow (\rho \wedge \sigma) \wedge \tau$}
\end{prooftree}
$$

$\Gamma = \rho \wedge (\sigma \wedge \tau)$