Nick DeGroote
Optimal Control: Homework Assignment No. 2
25 May 2021

1. Consider the problem of minimizing the performance index

$$J = x^2 + y^2$$

where the variables x and y are not independent but must satisfy the constraint

$$g(x,y) = x \cdot y - 1 = 0$$

(a) Find the value of the Lagrange Multiplier.

We begin by writing the given cost function, J and constraint, $g(x,y)$.

$$J = x^2 + y^2$$
$$g(x,y) = xy - 1 = 0$$

The general form for finding the optimal points of J with respect to g is as follows. This is termed the Lagrangian ($\mathcal{L}$).

$$\mathcal{L} = f(X) - \lambda g(X) \mid X \in \mathbb{R}^N$$

Next, we define the Lagrangian for this problem.

$$\mathcal{L} = x^2 + y^2 - \lambda\,(xy - 1)$$

Take the derivative of $\mathcal{L}$ with respect to x, y, and λ to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$J = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \lambda} \\ \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} -xy + 1 \\ -\lambda y + 2x \\ -\lambda x + 2y \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimum point.

$$\begin{bmatrix} -xy + 1 \\ -\lambda y + 2x \\ -\lambda x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms three separate equations with three different variables. The equations can be solved simultaneously for x, y, and λ. This yields 4 total solutions.

$$
\begin{aligned}
(\lambda_1, x_1, y_1) &= (-2, -i, i) \\
(\lambda_2, x_2, y_2) &= (-2, i, -i) \\
(\lambda_3, x_3, y_3) &= (2, -1, -1) \\
(\lambda_4, x_4, y_4) &= (2, 1, 1)
\end{aligned}
$$

Therefore, the value of the Lagrange multiplier has been found.

$$\boxed{\lambda = \pm 2}$$

*Note that complex values do not represent valid solutions to the problem.

(b) Validate your result using MATLAB.

Figure 1: Joint Angles

The Python script which finds the solution to this problem is attached separately to this assignment.

2. Consider the following cost function:

$$J = e^{x} \cdot (4x^2 + 2y^2 + 4xy + 2y + 1)$$

(a) Find the minimal point of performance for J. Check your result using the second differential conditions.

We begin by writing the given cost function, J. In this problem, there is no constraint.

$$J = e^x \left(4x^2 + 4xy + 2y^2 + 2y + 1\right)$$

Take the derivative of J with respect to x and y to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{L}' = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} (8x + 4y)\, e^x + \left(4x^2 + 4xy + 2y^2 + 2y + 1\right) e^x \\ (4x + 4y + 2)\, e^x \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimal point.

$$\begin{bmatrix} (8x + 4y)\, e^x + \left(4x^2 + 4xy + 2y^2 + 2y + 1\right) e^x \\ (4x + 4y + 2)\, e^x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This forms two separate equations with two different variables. The equations can be solved simultaneously for x and y. This yields 2 total solutions.

$$(x_1, y_1) = (-\frac{3}{2}, 1)$$
$$(x_2, y_2) = (\frac{1}{2}, -1)$$

The Hessian is needed to determine if the found points satisfy the necessary conditions for a minimal point. The Hessian of J is shown below.

$$\mathcal{H}(x, y) = \begin{bmatrix} 2\,(8x + 4y)\, e^x + \left(4x^2 + 4xy + 2y^2 + 2y + 1\right) e^x + 8e^x & (4x + 4y + 2)\, e^x + 4e^x \\ (4x + 4y + 2)\, e^x + 4e^x & 4e^x \end{bmatrix}$$

Substituting the two solutions for x and y yields two matrices.

$$\mathcal{H}(-3/2, 1) = \begin{bmatrix} -0.780 & 0.446 \\ 0.446 & 0.892 \end{bmatrix}$$

$$\mathcal{H}(-3/2, 1) = \begin{bmatrix} 13.189 & 6.594 \\ 6.594 & 6.594 \end{bmatrix}$$

Which has the following eigenvalues.

$$eig(\mathcal{H}(-3/2, 1)) = -0.8925, 1.0040$$
$$eig(\mathcal{H}(1/2, -1)) = 17.2656, 2.5190$$

Therefore, the point $(1/2, -1)$ is the minimum point of J. This is because both of the eigenvalues of the Hessian evaluated at $(1/2, -1)$ are positive, meaning $\mathcal{H}(1/2, -1)$ is positive definite.

(b) Validate your result using MATLAB.

The Python script which finds the solution to this problem is attached separately to this assignment.

3. Consider the cost function:

$$J = x^2 + y^2$$

(a) Find the optimal solution for the following constraint: $g = x \cdot y - 1 \geq 0$

(b) Find the optimal solution for the following constraint: $g = x \cdot y - 1 \leq 0$

(c) Validate results in (a) and (b) using MATLAB.

Part a.

We begin by writing the given cost function, J and constraint, $g(x, y)$.

$$J = x^2 + y^2$$
$$g(x, y) = xy - 1 >= 0$$

The constraint must be set to be less than or equal to zero. Inverting the inequality sign finds the new constraint.

$$g(x, y) = -xy + 1 <= 0$$

We can write the general form of the augmented performance index, $\mathcal{J}'$, as follows.

$$\mathcal{J}' = f(X) + \nu(g(X) + \alpha^2) \mid X \in \mathbb{R}^N$$

Next, we define $\mathcal{J}$ for this problem.

$$\mathcal{J}' = x^2 + y^2 + \nu\left(-xy + 1 + \alpha^2\right)$$

Take the derivative of $\mathcal{J}$ with respect to x, y, ν, and α to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{J}' = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 2\alpha\lambda \\ \alpha^2 - xy + 1 \\ -\lambda y + 2x \\ -\lambda x + 2y \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimum point.

$$\begin{bmatrix} 2\alpha\lambda \\ \alpha^2 - xy + 1 \\ -\lambda y + 2x \\ -\lambda x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms four separate equations with four different variables. The equations can be solved simultaneously for x, y, and λ. This yields 6 total solutions.

$$
\begin{aligned}
(\alpha_1, \nu_1, x_1, y_1) &= (0, -2, -i, i) \\
(\alpha_2, \nu_2, x_2, y_2) &= (0, -2, i, -i) \\
(\alpha_3, \nu_3, x_3, y_3) &= (0, 2, -1, -1) \\
(\alpha_4, \nu_4, x_4, y_4) &= (0, 2, 1, 1) \\
(\alpha_5, \nu_5, x_5, y_5) &= (-i, 0, 0, 0) \\
(\alpha_6, \nu_6, x_6, y_6) &= (i, 0, 0, 0)
\end{aligned}
$$

The Hessian is needed to determine if the found points satisfy the necessary conditions for a minimal point. The Hessian of J is shown below. Note that we only need to be concerned with the solutions that have real values.

$$\mathcal{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\mathcal{H}$ has the following eigenvalues.

$$eig(H) = 2, 2$$

Therefore, the points $(1, 1)$ and $(-1, -1)$ are both minimums of J. This is because both of the eigenvalues of the Hessian are positive, meaning $\mathcal{H}$ is positive definite.

Next, we substitute all three points into J to determine the global minimum.

$$
\begin{aligned}
J(1, 1) &= 2 \\
J(-1, -1) &= 2
\end{aligned}
$$

The point $(\pm 1, \pm 1)$ optimizes the performance index.

Part b.

We begin by writing the given cost function, J and constraint, $g(x, y)$.

$$
\begin{aligned}
J &= x^2 + y^2 \\
g(x, y) &= xy - 1 <= 0
\end{aligned}
$$

The constraint is already less than or equal to zero, so no changes are necessary.

We can write the general form of the augmented performance index, $\mathcal{J}'$, as follows.

$$\mathcal{J}' = f(X) + \nu(g(X) + \alpha^2) \mid X \in \mathbb{R}^N$$

Next, we define $\mathcal{J}$ for this problem.

$$\mathcal{J}' = x^2 + y^2 + \nu \left(xy - 1 + \alpha^2 \right)$$

Take the derivative of $\mathcal{J}$ with respect to x, y, ν, and α to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{J}' = \begin{bmatrix} 2\alpha\lambda \\ \alpha^2 + xy - 1 \\ \lambda y + 2x \\ \lambda x + 2y \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimum point.

$$\begin{bmatrix} 2\alpha\lambda \\ \alpha^2 + xy - 1 \\ \lambda y + 2x \\ \lambda x + 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms four separate equations with four different variables. The equations can be solved simultaneously for x, y, and λ. This yields 6 total solutions.

$$\begin{aligned}
(\alpha_1, \nu_1, x_1, y_1) &= (-1, 0, 0, 0) \\
(\alpha_2, \nu_2, x_2, y_2) &= (0, -2, -1, -1) \\
(\alpha_3, \nu_3, x_3, y_3) &= (0, -2, 1, 1) \\
(\alpha_4, \nu_4, x_4, y_4) &= (0, 2, -i, i) \\
(\alpha_5, \nu_5, x_5, y_5) &= (0, 2, i, -i) \\
(\alpha_6, \nu_6, x_6, y_6) &= (1, 0, 0, 0)
\end{aligned}$$

The Hessian is needed to determine if the found points satisfy the necessary conditions for a minimal point. The Hessian of J is shown below. Note that we only need to be concerned with the solutions that have real values.

$$\mathcal{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$\mathcal{H}$ has the following eigenvalues.

$$eig(H) = 2, 2$$

Therefore, the points $(0, 0)$, $(1, 1)$, and $(-1, -1)$ are minimums of J. This is because both of the eigenvalues of the Hessian are positive, meaning $\mathcal{H}$ is positive definite.

Next, we substitute all three points into J to determine the global minimum.

$$\begin{aligned}
J(0, 0) &= 0 \\
J(1, 1) &= 2 \\
J(-1, -1) &= 2
\end{aligned}$$

The point $(0, 0)$ optimizes the performance index.

Part c.

The Python script which finds the solution to this problem is attached separately to this assignment.

4.

Given the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

a) Find the rectangle of maximum perimeter that can be inscribed in the ellipse.
b) Find the rectangle of maximum area that can be inscribe in the ellipse.

Part a.

We begin by writing the given cost function, J and constraint, $g(x, y)$. We can write the perimeter as a function of x and y.

$$J = 4x + 4y$$
$$g(x, y) = \frac{y^2}{b^2} + \frac{x^2}{a^2} - 1 <= 0$$

We begin by writing the Lagrangian, $\mathcal{L}$ for this problem.

$$\mathcal{L} = -\lambda \left(-1 + \frac{y^2}{b^2} + \frac{x^2}{a^2}\right) + 4x + 4y$$

Take the derivative of J with respect to x and y to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{L}' = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} \\ 4 - \frac{2\lambda x}{a^2} \\ 4 - \frac{2\lambda y}{b^2} \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimal point.

$$\begin{bmatrix} 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} \\ 4 - \frac{2\lambda x}{a^2} \\ 4 - \frac{2\lambda y}{b^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms three separate equations with three different variables. The equations can be solved simultaneously for x, y, and λ. This yields 2 total solutions.

$$(\lambda_1, x_1, y_1) = \left(-2\left(a^2 + b^2\right)\sqrt{\frac{1}{a^2 + b^2}},\ -a^2\sqrt{\frac{1}{a^2 + b^2}},\ -b^2\sqrt{\frac{1}{a^2 + b^2}}\right)$$
$$(\lambda_2, x_2, y_2) = \left(2\left(a^2 + b^2\right)\sqrt{\frac{1}{a^2 + b^2}},\ a^2\sqrt{\frac{1}{a^2 + b^2}},\ b^2\sqrt{\frac{1}{a^2 + b^2}}\right)$$

We know that only positive values of a and b are valid, so (λ_2, x_2, y_2) must be the maximum point.

Substituting (λ_2, x_2, y_2) back into the objective function finds the following maximized perimeter in terms of a and b.

$$4\left(a^2+b^2\right)\sqrt{\frac{1}{a^2+b^2}} = 4\sqrt{a^2+b^2}$$

We then have the following for a final answer.

$$x = a^2\sqrt{\frac{1}{a^2+b^2}}$$
$$y = b^2\sqrt{\frac{1}{a^2+b^2}}$$
$$J = 4\sqrt{a^2+b^2}$$

Part b.

We begin by writing the given cost function, J and constraint, $g(x,y)$. We can write the area as a function of x and y.

$$J = (2x)(2y)$$
$$g(x,y) = \frac{y^2}{b^2} + \frac{x^2}{a^2} - 1 <= 0$$

We begin by writing the Lagrangian, $\mathcal{L}$ for this problem.

$$\mathcal{L} = -\lambda\left(-1 + \frac{y^2}{b^2} + \frac{x^2}{a^2}\right) + 4xy$$

Take the derivative of $\mathcal{L}$ with respect to x and y to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{L}' = \begin{bmatrix} 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} \\ 4y - \frac{2\lambda x}{a^2} \\ 4x - \frac{2\lambda y}{b^2} \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimal point.

$$\begin{bmatrix} 1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} \\ 4y - \frac{2\lambda x}{a^2} \\ 4x - \frac{2\lambda y}{b^2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms three separate equations with three different variables. The equations can be solved simultaneously for x, y, and λ. This yields 4 total solutions.

$$(x_1, y_1) = (-\frac{3}{2}, 1)$$
$$(x_2, y_2) = (\frac{1}{2}, -1)$$

$$(\lambda_1, x_1, y_1) = \left(-2ab,\ -\frac{\sqrt{2}a}{2},\ \frac{\sqrt{2}b}{2}\right)$$
$$(\lambda_2, x_2, y_2) = \left(-2ab,\ \frac{\sqrt{2}a}{2},\ -\frac{\sqrt{2}b}{2}\right)$$
$$(\lambda_3, x_3, y_3) = \left(2ab,\ -\frac{\sqrt{2}a}{2},\ -\frac{\sqrt{2}b}{2}\right)$$
$$(\lambda_4, x_4, y_4) = \left(2ab,\ \frac{\sqrt{2}a}{2},\ \frac{\sqrt{2}b}{2}\right)$$

We know that only positive values of a and b are valid, and we only consider values in the first quadrant, so (λ_4, x_4, y_4) must be the maximum point.

Substituting (λ_4, x_4, y_4) back into the objective function finds the following maximized area in terms of a and b.

$$2ab$$

We then have the following for a final answer.

$$x = \frac{\sqrt{2}a}{2}$$
$$y = \frac{\sqrt{2}b}{2}$$
$$J = 2ab$$

5.

A tin can manufacturer wants to find the dimensions of a cylindrical can (closed top and bottom) such that, for a given amount of tin, the volume of the can is a maximum. If the thickness of the tine stock is constant, a given amount of tin implies a given surface area of the can. Use height and radius as variables, and use a Lagrange multiplier. Show that the optimal height is twice the radius.

We begin by writing the given cost function, J and constraint, $g(x, y)$. We can write the volume and surface area as a function of r and h. Note c is a constant representing the constant amount of material.

$$\begin{aligned} J &= \pi r^2 h \\ g(x, y) &= 2\pi h r + 2\pi r^2 - c = 0 \end{aligned}$$

We begin by writing the Lagrangian, $\mathcal{L}$ for this problem.

$$\mathcal{L} = \pi h r^2 - \lambda\left(-c + 2\pi h r + 2\pi r^2\right)$$

Take the derivative of $\mathcal{L}$ with respect to x, y, and λ to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{L}' = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} -2\pi\lambda r + \pi r^2 \\ c - 2\pi h r - 2\pi r^2 \\ 2\pi h r - \lambda\left(2\pi h + 4\pi r\right) \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimal point.

$$\begin{bmatrix} -2\pi\lambda r + \pi r^2 \\ c - 2\pi h r - 2\pi r^2 \\ 2\pi h r - \lambda\left(2\pi h + 4\pi r\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms three separate equations with three different variables. The equations can be solved simultaneously for h and λ. This yields 3 total solutions.

$$\begin{aligned} (\lambda_1, h_1) &= (0, 0) \\ (\lambda_2, h_2) &= (0, 0) \\ (\lambda_3, h_3) &= (r/2, 2r) \end{aligned}$$

We know that only positive values of r and h are valid, so (λ_3, h_3) must be the maximum point.

Substituting λ_3, h_3 back into the objective function finds the following maximized volume and surface area.

$$V = 2\pi r^3$$
$$c = 6\pi r^2$$

We can therefore see that the optimal design does in fact have a height value equal to twice the radius.

6.

Determine the point x_1, x_2 at which the function

$$J = x_1 + x_2$$

is a minimum subject to the constraint

$$x_1^2 + x_1 x_2 + x_2^2 = 1 \ .$$

Show that $x_1 = x_2 = -1/\sqrt{3}$.

Part a.

We begin by writing the given cost function, J and constraint, $g(x, y)$. We can write the volume and surface area as a function of r and h. Note c is a constant representing the constant amount of material.

$$J = x_1 + x_2$$
$$g(x, y) = x_1^2 + x_1 x_2 + x_2^2 - 1$$

We begin by writing the Lagrangian, $\mathcal{L}$ for this problem.

$$\mathcal{L} = -\lambda \left(x_1^2 + x_1 x_2 + x_2^2 - 1\right) + x_1 + x_2$$

Take the derivative of $\mathcal{L}$ with respect to x, y, and λ to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{L}' = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} -x_1^2 - x_1 x_2 - x_2^2 + 1 \\ -\lambda \left(2x_1 + x_2\right) + 1 \\ -\lambda \left(x_1 + 2x_2\right) + 1 \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimal point.

$$\begin{bmatrix} -x_1^2 - x_1 x_2 - x_2^2 + 1 \\ -\lambda \left(2x_1 + x_2\right) + 1 \\ -\lambda \left(x_1 + 2x_2\right) + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms three separate equations with three different variables. The equations can be solved simultaneously for x_1, x_2, and λ. This yields 2 total solutions.

$$(\lambda_1, x_{11}, x_{21}) = (-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$$
$$(\lambda_2, x_{12}, x_{22}) = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

The Hessian of J is zero, so it cannot be used to determine the minimum point. Instead, we know that the minimum must lie on the boundary. We substitute both solutions back into the original objective.

$$J(\lambda_1, x_{11}, x_{21}) = -\frac{2}{\sqrt{3}}$$
$$J(\lambda_2, x_{12}, x_{22}) = \frac{2}{\sqrt{3}}$$

We can therefore see that the optimal point is at $(\lambda_1, x_{11}, x_{21})$.

Part b.

The Python script which finds the solution to this problem is attached separately to this assignment.

7.

Minimize the performance index

$$J = \frac{1}{2}(x^2 + y^2 + z^2)$$

subject to the constraints

$$x + 2y - z = 3$$

$$x - y + 2z = 12.$$

Show that

$$x = 5, \quad y = 1, \quad z = 4, \quad \nu_1 = -2, \quad \nu_2 = -3.$$

Part a.

We begin by writing the given cost function, J and constraints, $g_1(x, y)$ and $g_2(x, y)$. We can write the volume and surface area as a function of r and h. Note c is a constant representing the constant amount of material.

$$\begin{aligned} J &= 0.5x^2 + 0.5y^2 + 0.5z^2 \\ g_1(x, y) &= x + 2y - z - 3 \\ g_2(x, y) &= x - y + 2z - 12 \end{aligned}$$

We begin by writing the Lagrangian, $\mathcal{L}$ for this problem.

$$\mathcal{L} = \nu_1 (x + 2y - z - 3) + \nu_2 (x - y + 2z - 12) + 0.5x^2 + 0.5y^2 + 0.5z^2$$

Take the derivative of $\mathcal{L}$ with respect to x, y, and λ to find the gradient. We use the Jacobian as a matrix representation for convenience.

$$\mathcal{L}' = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial x_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} x + 2y - z - 3 \\ x - y + 2z - 12 \\ \nu_1 + \nu_2 + 1.0x \\ 2\nu_1 - \nu_2 + 1.0y \\ -\nu_1 + 2\nu_2 + 1.0z \end{bmatrix}$$

Setting the Jacobian equal to zero gives the following matrix equation. The satisfies the sufficient conditions for being an optimal point.

$$\begin{bmatrix} x + 2y - z - 3 \\ x - y + 2z - 12 \\ \nu_1 + \nu_2 + 1.0x \\ 2\nu_1 - \nu_2 + 1.0y \\ -\nu_1 + 2\nu_2 + 1.0z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This forms five separate equations with five different variables. The equations can be solved simultaneously for x, y, z, ν_1, and ν_2. This yields 1 total solution.

$$\begin{aligned} x &= 5 \\ y &= 1 \\ z &= 4 \\ \nu_1 &= -2 \\ \nu_2 &= -3 \end{aligned}$$

The Hessian, $\mathcal{H}$, is calculated below.

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which has eigenvalues of 1 and 3. Therefore the points above satisfy the necessary conditions for a minimum.

Part b.

The Python script which finds the solution to this problem is attached separately to this assignment.