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Optimal Control: Homework Assignment No. 5
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$$J(x(\cdot)) = \int_a^b L(t, x(t), \dot{x}(t))\mathrm{d}t$$

1. $L(t, x(t), \dot{x}(t)) = [1 - (\dot{x}(t))^2]^2$
2. $L(t, x(t), \dot{x}(t)) = [1 - (\dot{x}(t))^2]$
3. $L(t, x(t), \dot{x}(t)) = [(\dot{x}^2 - 2x\dot{x} - x^2]$
4. $L(t, x(t), \dot{x}(t)) = [\dot{x}^2 + t^2]$

IMPORTANT: Check the conditions of Legendre and Weierstrass for all 4 problems

Problem 1: $L(t, x(t), \dot{x}(t)) = [1 - (\dot{x}(t))^2]^2$

We begin with the Euler-Lagrange Equation.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

Where

$$L = [1 - (\dot{x}(t))^2]^2$$
$$q = x(t)$$
$$\dot{q} = \dot{x}(t)$$

Applying the Euler-Lagrange equation:

$$\left(4\dot{x}^2 - 4\right)\ddot{x} + 8\dot{x}^2\ddot{x} = 0$$
$$4\left(3\dot{x}^2 - 1\right)\ddot{x} = 0$$

Therefore, for the first order functional given, we find the following general solution from Equation 3.22 in the Ben Asher textbook.

$$
\begin{aligned}
&4\left(3\dot{x}^*(t)^2-1\right)\ddot{x}^*(t)=0\\
&\dot{x}^*(t)=c\\
&x^*(t)=ct+b
\end{aligned}
$$

Next we look for the Legendre Necessary Condition via the second derivative test. We can see that if a minimum exists, then $x^*(t)$ is the optimal solution because there is a positive slope.

$$\frac{\partial^2 L}{\partial \dot{q}^2}=4\left(3\dot{x}^*(t)^2-1\right)\geq 0$$

Finally, the Weierstrass Necessary Condition is analyzed. The general form for the Weierstrass Necessary Condition is as follows.

$$E(x,\dot{x},t,\dot{z})=f(x,\dot{z},t)-f(x,\dot{x},t)-(\dot{z}-\dot{x})f_x(x,\dot{x},t)\geq 0$$

Applying the equation to this problem, we see that the Weierstrass Necessary condition can be greater than or equal to zero for values of $\dot{x}$. Therefore the condition is satisfied.

$$
\begin{aligned}
E(x,\dot{x},t,\dot{z})&=\left(1-\dot{z}^2\right)^2-\left(1-\dot{x}^2\right)^2+4\left(1-\dot{x}^2\right)(\dot{z}-\dot{x})\,\dot{x}\\
&=\left(-\dot{z}+\dot{x}\right)^2\left(\dot{z}^2+2\dot{z}\dot{x}+3\dot{x}^2-2\right)\geq 0
\end{aligned}
$$

Problem 2: $L(t,x(t),\dot{x}(t))=[1-(\dot{x}(t))^2]$

We begin with the Euler-Lagrange Equation.

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=0$$

Where

$$
\begin{aligned}
&L=[1-(\dot{x}(t))^2]\\
&q=x(t)\\
&\dot{q}=\dot{x}(t)
\end{aligned}
$$

Applying the Euler-Lagrange equation:

$$-2\ddot{x} = 0$$

Therefore, for the first order functional given, we find the following general solution from Equation 3.22 in the Ben Asher textbook.

$$\begin{aligned} & -2\ddot{x}^*(t) = 0 \\ & \dot{x}^*(t) = c \\ & x^*(t) = ct + b \end{aligned}$$

Next we look for the Legendre Necessary Condition via the second derivative test. We can see that the calculated value is less than zero, meaning the condition is not satisfied.

$$\frac{\partial^2 L}{\partial \dot{q}^2} = -2 \leq 0$$

Finally, the Weierstrass Necessary Condition is analyzed. The general form for the Weierstrass Necessary Condition is as follows.

$$E(x, \dot{x}, t, \dot{z}) = f(x, \dot{z}, t) - f(x, \dot{x}, t) - (\dot{z} - \dot{x}) f_x(x, \dot{x}, t) \geq 0$$

Applying the equation to this problem, we see that the Weierstrass Necessary condition is not greater than or equal to zero. Therefore the condition is not satisfied. It is worth noting that a failure for the Legendre Necessary Condition will also imply a failure of the Weierstrass Necessary Condition.

$$\begin{aligned} E(x, \dot{x}, t, \dot{z}) &= -\dot{z}^2 + 2\left(\dot{z} - \dot{x}\right)\dot{x} + \dot{x}^2 \\ &= -\left(-\dot{z} + \dot{x}\right)^2 \leq 0 \end{aligned}$$

Problem 3: $L(t, x(t), \dot{x}(t)) = [\dot{x}^2 - 2x\dot{x} - x^2]$

We begin with the Euler-Lagrange Equation.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

Where

$$
\begin{aligned}
L &= [\dot{x}^2 - 2x\dot{x} - x^2] \\
q &= x \\
\dot{q} &= \dot{x}
\end{aligned}
$$

Applying the Euler-Lagrange equation:

$$
\begin{aligned}
2x + 2\ddot{x} &= 0 \\
2\left(x + \ddot{x}\right) &= 0
\end{aligned}
$$

Therefore, for the first order functional given, we find the following general solution from Equation 3.24 in the Ben Asher textbook. Note that this equation is regular, so the general solution is different from the previous two problems.

$$
\begin{aligned}
&2\left(x^*(t) + \ddot{x}^*(t)\right) = 0 \\
&\dot{x}^*(t) = a\sin(t) + b\cos(t)
\end{aligned}
$$

Next we look for the Legendre Necessary Condition via the second derivative test. We can see that the calculated value is greater than zero, therefore the condition is satisfied.

$$
\frac{\partial^2 L}{\partial \dot{q}^2} = 2 \geq 0
$$

Finally, the Weierstrass Necessary Condition is analyzed. The general form for the Weierstrass Necessary Condition is as follows.

$$
E(x, \dot{x}, t, \dot{z}) = f(x, \dot{z}, t) - f(x, \dot{x}, t) - (\dot{z} - \dot{x}) f_x(x, \dot{x}, t) \geq 0
$$

Applying the equation to this problem, we see that the Weierstrass Necessary condition can be greater than or equal to zero for values of $\dot{x}$. Therefore the condition is satisfied.

$$
\begin{aligned}
E(x, \dot{x}, t, \dot{z}) &= \dot{z}^2 - 2\dot{z}x - (\dot{z} - \dot{x})\left(-2x + 2\dot{x}\right) + 2x\dot{x} - \dot{x}^2 \\
&= \left(-\dot{z} + \dot{x}\right)^2 \geq 0
\end{aligned}
$$

Problem 3: $L(t, x(t), \dot{x}(t)) = [\dot{x}^2 + t^2]$

We begin with the Euler-Lagrange Equation.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

Where

$$\begin{aligned} L &= [\dot{x}^2 + t^2] \\ q &= x \\ \dot{q} &= \dot{x} \end{aligned}$$

Applying the Euler-Lagrange equation:

$$2\ddot{x} = 0$$

Therefore, for the first order functional given, we find the following general solution from Equation 3.22 in the Ben Asher textbook.

$$\begin{aligned} 2\ddot{x}^*(t) &= 0 \\ \dot{x}^*(t) &= c \\ x^*(t) &= ct + b \end{aligned}$$

Next we look for the Legendre Necessary Condition via the second derivative test. We can see that the calculated value is greater than zero, therefore the condition is satisfied.

$$\frac{\partial^2 L}{\partial \dot{q}^2} = 2 \geq 0$$

Finally, the Weierstrass Necessary Condition is analyzed. The general form for the Weierstrass Necessary Condition is as follows.

$$E(x, \dot{x}, t, \dot{z}) = f(x, \dot{z}, t) - f(x, \dot{x}, t) - (\dot{z} - \dot{x}) f_x(x, \dot{x}, t) \geq 0$$

Applying the equation to this problem, we see that the Weierstrass Necessary condition can be greater than or equal to zero for values of $\dot{x}$. Therefore the condition is satisfied.

$$\begin{aligned} E(x, \dot{x}, t, \dot{z}) &= \dot{z}^2 - 2\left(\dot{z} - \dot{x}\right)\dot{x} - \dot{x}^2 \\ &= \left(-\dot{z} + \dot{x}\right)^2 \geq 0 \end{aligned}$$