Nick DeGroote
Optimal Control: Project No. 1
01 June 2021

1. Why is the study of Optimization in general and Optimal Control in particular important to Aerospace Engineering? Provide one meaningful non-linear constrained optimization example (try being original) including motivation, cost function, design variables and constraints? (at least one page and not to exceed two pages) (20 Points)

Optimization is important to aerospace engineering and engineering in general quite simply because it provides a way to find good solutions to problems. The goal in engineering is often find a solution to a problem in the most efficient way. Optimization provides a way to accomplish this. Optimal control is an extension of this concept where the goal is to minimize an objective function for a control system. An example of this could include finding the optimal gains for a PID controller which both minimize tracking error in the controller and does not allow the actuator outputs to saturate.

Optimization is an extremely important topic for my thesis, so I will it as an example. We can consider the problem of assigning a number of tasks to a fleet of unmanned aerial vehicles (UAVs). If each task is assumed to simply be a point in 2-dimensional space, then a version of the multiple depot multiple travelling salesman problem (MDMTSP) is formed. This problem is non-deterministic polynomial-time hard and places several constraints on the solution, such as visiting each task only once and having each UAV return to its starting location. This form of the MDMTSP has however been studied extensively.

A more original example can be analyzed by assuming that both the tasks and the UAVs in the fleet are heterogeneous. This means that each UAV has a certain number of capabilities such as its maximum range, maximum endurance, turning radius, flight speed, etc. The tasks also require a specific set of equipment which can be provided by a UAV. UAVs which do not have the appropriate equipment are unable to accomplish certain tasks. Assuming heterogeneity in turn enforces several more constraints in the problem.

For a situation such as disaster response, it is logical to attempt to accomplish all tasks as quickly as possible. Aid should be dispatched to any survivors in the minimum amount of time. This changes the objective function for the MDMTSP from attempting to minimize the total distance travelled by all UAVs to minimizing the distance travelled by the UAV with the longest route. A mathematical expression of this objective function can be seen in Equations 1 and 2.

$$J_k = \sum_{e_{ij} \in A} w_{ijk} c_{ij} \tag{1}$$

$$J_k^* = \min argmax_{k \in \{1,\ldots,m\}} J_k \tag{2}$$

Where:

- V is a set of vertices in a graph representing each task.
- A is a set of edges in the graph connecting all tasks.
- v_i is a vertex in V representing a task.
- e_{ij} is the edge between task v_i and task v_j in A.
- w_{ijk} is a binary variable that takes on a value of one if UAS k travels from task i to task j and zero otherwise.
- c_{ij} is the cost of travelling from task i to task j.
- J_k is the the flight time for an individual UAS k.
- m is the number of UAS in the fleet.

2. Describe the formulation of the Optimization problem. Develop a block diagram presenting a general approach to solving an optimization problem of a continuous cost function which is differentiable with an arbitrary number of equality and inequality constraints (10 Points)

An algorithmic process can be implemented for solving continuous, differentiable optimizations using Lagrange multipliers and slack variables. The process is taken from the Lesson 2 slide set. A block diagram of the process is shown below in Figure 1.

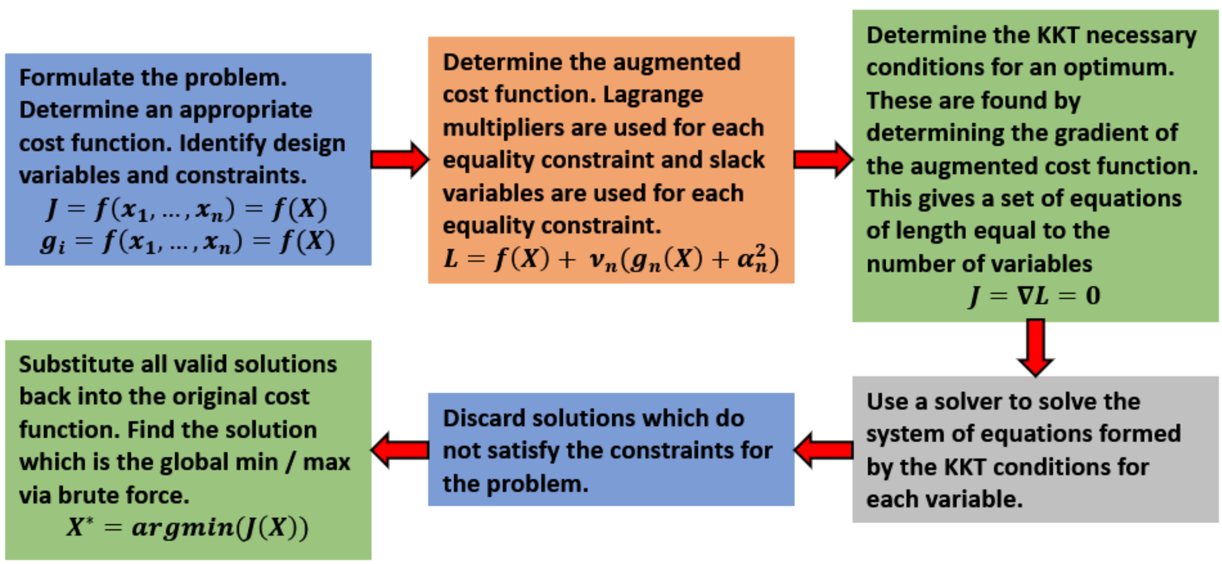

Figure 1: A block diagram showing the general process for solving constrained optimization problems using Lagrange multipliers and slack variables.

3. Develop the KKT (Karush-Kuhn-Tucker) necessary conditions for the problem. Show all the steps in a neat and organized manner. Include all necessary explanations (15 Points).

We begin by writing the given cost function, J and constraints, $\theta_i(x, y, z)$.

$$J = -5x - 4y - 6z$$

$$\theta_1 = x - y + z - 20 \leq 0$$

$$\theta_2 = 3x + 2y + 4z - 42 \leq 0$$

$$\theta_3 = 3x - 2y - 30 \leq 0$$

$$\theta_4 = -x \leq 0$$

$$\theta_5 = -y \leq 0$$

$$\theta_6 = -z \leq 0$$

The KKT necessary conditions involve a first derivative test. To meet the KKT necessary condition for a local optimum, all components of the gradient must be equal to zero. For a constrained optimization problem, the solution can also lie on the boundary of one of the constraints. To account for this, Lagrange multipliers and slack variables are introduced.

The Lagrange multipliers and slack variables are introduced as shown below. The augmented cost function is denoted by $\mathcal{J}'$.

$$\mathcal{J} = \lambda_0 \left(\alpha_0^2 + x - y + z - 20\right) + \lambda_1 \left(\alpha_1^2 + 3x + 2y + 4z - 42\right) + \lambda_2 \left(\alpha_2^2 + 3x + 2y - 30\right)$$
$$+ \lambda_3 \left(\alpha_3^2 - x\right) + \lambda_4 \left(\alpha_4^2 - y\right) + \lambda_5 \left(\alpha_5^2 - z\right) - 5x - 4y - 6z$$

Take the derivative of $\mathcal{J}$ with respect to x, y, and λ to find the gradient. We use the Jacobian as a matrix representation for convenience. Setting each element in the matrix equal to zero forms the KKT conditions for the problem. This forms a total of 15 equations with 15 different independent variables. These equations can be solved simultaneously for candidates for optimal points.

$$\mathcal{J}' = \nabla J = \begin{bmatrix} 2\alpha_0\lambda_0 \\ 2\alpha_1\lambda_1 \\ 2\alpha_2\lambda_2 \\ 2\alpha_3\lambda_3 \\ 2\alpha_4\lambda_4 \\ 2\alpha_5\lambda_5 \\ \alpha_0^2 + x - y + z - 20 \\ \alpha_1^2 + 3x + 2y + 4z - 42 \\ \alpha_2^2 + 3x + 2y - 30 \\ \alpha_3^2 - x \\ \alpha_4^2 - y \\ \alpha_5^2 - z \\ \lambda_0 + 3\lambda_1 + 3\lambda_2 - \lambda_3 - 5 \\ -\lambda_0 + 2\lambda_1 + 2\lambda_2 - \lambda_4 - 4 \\ \lambda_0 + 4\lambda_1 - \lambda_5 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4. Solve the optimum design problem described above using MATLAB (only allowed to use SOLVE and not the Optimization Toolbox). (35 Points)

As stated in Part 4 of this project, the KKT conditions form a system of 15 equations. Solving a system of 15 equations analyically presents its own challenge. Solving this type of problem by hand would be tedious and time consuming. Fortunately there are software packages available which can easily solve systems of equations with many variables. In this assignment, Python is used in conjunction with the Sympy library for symbolic mathematics to provide this functionality.

Using the Jacobian matrix simplifies the problem by allowing all equations to be packaged together in a convenient form. After setting the Jacobian equal to zero, Sympy's *solve* command can be used to find all possible solutions to the system. We find that there are a total of 144 possible solutions to the problem after solving the system.

Not every one of the 144 potential solutions is valid. To represent a valid solution, all variables in the solution must be real. We can easily test if this criteria is met for each of the solutions. Finally, the global optimum can be found by substituting all of the valid solutions back into the original cost function. This is a brute-force search, but the operation is not computationally expensive and can be performed quickly. After accounting for valid solutions, the final answer to this optimization is shown below.

$$x = 0$$

$$y = 15$$

$$z = 3$$

$$\alpha_1 = -4\sqrt{2}$$

$$\alpha_2 = 0$$

$$\alpha_3 = 0$$

$$\alpha_4 = 0$$

$$\alpha_5 = -\sqrt{15}$$

$$\alpha_6 = -\sqrt{3}$$

$$\nu_1 = 0$$

$$\nu_2 = 3/2$$

$$\nu_3 = 1/2$$

$$\nu_4 = 1$$

$$\nu_5 = 0$$

$$\nu_6 = 0$$

Substituting the point $(0, 15, 3)$ into the original cost function.

$$J(0, 3, 15) = -78$$

5. What are the lessons learned when you reflect upon the above optimization process? You need to be specific in relation to this project as well describing what you take away from this in general (at least one page and not to exceed two pages) (20 Points)

This project allows for several insights into the optimization process. We will first consider the takeaways for this project in particular. The first thing to note is how algorithmic the process is for solving constrained optimization problems with Lagrange multipliers and slack variables. As shown in the flow chart in Problem 2, there are a number of steps which can be followed to find the answer for this type of problem every time. With a few *for* loops, it was relatively easy to develop a function that solves the general form of the problem.

The other big takeaway that I have from this project is the concern of scalability for large problems. In the previous homework assignment, the KKT conditions could be solved easily in a fraction of a second with one to two constraints. After increasing the number of constraints to 6, the time to solve the problem increases to 17.8 seconds. While it is impossible to determine the exact time complexity without taking a deeper look into the solver, at first glance it appears to be either exponential or high order polynomial.

The final takeaway I have from this project is some of the limitations of this type of solution method. As stated previously, when a large number of constraints are enforced the computation time required to solve the problem increases quickly. Depending on the application, this may be acceptable; however, if there is any sort of on-line use case that needs to execute in real time, then other solution methods may be needed.

In general, there is a trade-off between execution time and the quality of the solution for optimization problems. There are numerical solvers which can provide solutions to constrained optimization problems much more quickly than an analytical solution. There is however a chance that these types of solutions could become stuck in a local minimum, missing the global optimum for the problem because many numerical solvers are gradient based.

In a way, I feel that the optimization process is in it self a sort of optimization problem. Based on the type of problem presented, there needs to be design considerations to find the solution as quickly and as accurately as possible. We have to adhere to constraints such as the available computing power and the maximum amount of time allowed for computations. After that, an appropriate solution method needs to be implemented to actually solve the problem.

I find the optimization process very satisfying because it provides a way to quantify which solutions are good and which are bad. If the objective function is specified correctly, then optimization can be used to find the best course of action. Obviously there are more complex optimization problems than the one in this project, but rationale behind solving the problme stays largely the same.