a) Want to show core tensor C_j can be estimated as

$$\hat{C}_j = R_j \times_1 (Z_j^T Z_j)^{-1} Z^T \times_2 V_1^T \times_3 V_2^T \dots \times_{d+1} V_d^T$$

$\operatorname{argmin}_c \| R_{j(1)} - X_{j(1)} B_j \|_F^2$ — use $vec(X)$ — $Z_j = X_{j(1)}(U_{jl} \otimes U_{jl-1} \otimes \dots \otimes U_{j1})$

$= \operatorname{argmin} \| vec(R_{j(1)}) - vec(X_{j(1)} B_j) \|_2^2$ — $R_j = Y - \sum_{i \neq j} B_j * X_j$

plug in B_j — $B_j = C_j \times_1 U_{j1} \times_2 \dots \times_{l_j} U_{jl} \times_{l_{j+1}} \dots V_1 \times_{l_j+2} \dots \times_{l_j+d} V_d$

$= \operatorname{argmin} \| vec(R_{j(1)}) - vec(X_{j(1)}(U_l \otimes U_{l-1} \otimes \dots \otimes U_{jl_j}) C_j (V_d \otimes \dots \otimes V_1)^T) \|_2^2$

use def of Z_j — swap + transpose

$= \operatorname{argmin} \| vec(R_{j(1)}) - vec(Z_j C_j (V_d \otimes \dots \otimes V_1)^T) \|_2^2$

lastly, use $vec(ABC^T) = (C \otimes A) vec(B)$

$= \operatorname{argmin} \| vec(R_{j(1)}) - (V_d \otimes \dots \otimes V_1 \otimes Z_j) vec(C_j) \|_2^2$ ✓

Now, optimal soln when

$$vec(R_{j(1)}) = (V_d \otimes \dots \otimes V_1 \otimes Z_j) vec(C_j)$$

$$Vec(C_j) = (V_d \otimes \dots \otimes V_1 \otimes Z_j)^{-1} vec(R_{j(1)})$$

use 1. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

2. $AB^{-1} = B^{-1} A^{-1}$

3. $V_i^T V_i = I_{\hat{Q}_i}$

$Vec(C_j) =$

$$V_d^T \otimes V_{d-1}^T \dots \otimes V_1^T (Z_j^T Z_j)^{-1} Z_j^T vec(R_{j(1)})$$

Now invoke $(A \otimes B)(C \otimes D) = (A * C) \otimes (B * D)$

$$C_j = R_j \times_1 (Z^T Z_j)^{-1} Z_j \times_2 V_1^T \times_3 \dots$$ ✓

b) Tucker decomp

$$\mathcal{X} = \mathcal{G} \times_1 A \times_2 B \times_3 C = \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{r=1}^{R} g_{pqr}\, a_p \circ b_q \circ c_r = [[\mathcal{G}; A, B, C]]$$

from online proof we can say $\mathcal{X}$ is 3 way tensor $\in \mathbb{R}^{I \times J \times K}$

so $$= \sum^{P} \sum^{Q} \sum^{R} g_{pqr}\, a_{ip}\, b_{jq}\, c_{kr}$$ pull out P summation

$$= \sum^{P} \left(\sum_{q \neq p}^{Q} \sum^{R} g_{pqr}\, a_{ip}\, b_{jq}\, c_{kr} + \sum^{R} g_{ppr}\, a_{ip}\, b_{jp}\, c_{kr} \right)$$

G is superdiagonal

$$= \sum^{P} \sum^{R} g_{ppr}\, a_{ip}\, b_{jp}\, c_{kr}$$ do same for R

$$= \sum^{P} \left(\sum_{r \neq p}^{R} g_{ppr}\, a_{ip}\, b_{jp}\, c_{kr} + g_{ppp}\, a_{ip}\, b_{jp}\, c_{kp} \right)$$

G is superdiag

$$= \sum_{p=1}^{P} a_{ip}\, b_{jp}\, c_{kp}$$ so if $P = Q = R$ + G is super diag ✓