Last Time:

- MEKF warm up

Today:

- Gyro model
- Real MEKF

* M+C Ch. 6.2

MEKF Recap:

- Essential Idea: Run an EKF using a local 3-parameter (axis-angle) error vector, but keep track of global state using a quaternion or rotation matrix to avoid singularities.

- Since we use a 3-parameter error vector, the filter covariance can be represented by a 3×3 non-singular matrix

- We can also eliminate rigid body dynamics from the filter prediction by assuming a near-perfect gyro.

Real Gyro Errors:

- In addition to white noise, gyros also exhibit bias (their errors do not have zero mean)

- Since we assumed zero mean Gaussian noise in the EKF, we have to do something about this.

- The bias is slowly time varying and can be modeled as a random walk:

$$\omega_k = \omega_k^{true} + \beta_k + v_\omega \quad , \quad v_\omega \sim N(0, V_\omega)$$

$$\beta_{k+1} = \beta_k + v_\beta \quad , \quad v_\beta \sim N(0, V_\beta)$$

- A random walk is the integral (or sum in discrete time) of a white-noise process. It is also called a "drunkard's walk" and is related to Brownian Motion.

- If starting from the origin at t_0, the expected $\|\beta_k\|$ is proportional to $\sqrt{k}$.

- For a good gyro V_ω and V_β will be small, but β can be large.

- This corrupts our predictions in the EKF

* Solution: Add β to the filter state and estimate it online

Dynamics with Gyro Bias:

* State Vector: $x_k = \begin{bmatrix} q_k \\ \beta_k \end{bmatrix}$

* State Propagation:

$$x_{k+1} = f(x_k, \omega_k) = \left[\begin{array}{c} q_k \begin{bmatrix} r \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} \\ \hline \beta_k \end{array}\right], \quad \theta = \|\omega_k - \beta_k\| \delta t$$

$$r = \frac{\omega_k - \beta_k}{\|\omega_k - \beta_k\|}$$

* Linearization:

$$\delta x_{k+1} = A_k \delta x_k + \underbrace{B_k \delta \omega_k}_{\text{this term doesn't affect the covariance so we don't need it}}, \quad \delta x_k = \begin{bmatrix} \phi_k \\ \delta \beta_k \end{bmatrix} \in \mathbb{R}^6$$

axis-angle error vector $\in \mathbb{R}^3$

bias error $\in \mathbb{R}^3$

$$A_k = \left[\begin{array}{c|c} \frac{\partial \phi_{k+1}}{\partial \phi_k} & \frac{\partial \phi_{k+1}}{\partial \beta_k} \\ \hline \frac{\partial \beta_{k+1}}{\partial \phi_k} & \frac{\partial \beta_{k+1}}{\partial \beta_k} \end{array}\right]$$

* $\frac{\partial \beta_{k+1}}{\partial \phi_k} = 0$

* $\frac{\partial \beta_{k+1}}{\partial \beta_k} = I$

$\dfrac{\partial \phi_{k+1}}{\partial \phi_k}$:

$$q_{k+1}\begin{bmatrix}\frac{1}{2}\phi_{k+1}\\ 1\end{bmatrix} = q_k\begin{bmatrix}\frac{1}{2}\phi_k\\ 1\end{bmatrix}\underbrace{\begin{bmatrix} r\sin(\theta/2)\\ \cos(\theta/2)\end{bmatrix}}_{p_k = q_k^{+} q_{k+1}}$$

-expand:

$$q_{k+1} + q_{k+1}\begin{bmatrix}\frac{1}{2}\phi_{k+1}\\ 0\end{bmatrix} = \left(q_k + q_k\begin{bmatrix}\frac{1}{2}\phi_k\\ 0\end{bmatrix}\right) p_k$$

$$\Rightarrow \quad q_{k+1}\begin{bmatrix}\frac{1}{2}\phi_{k+1}\\ 0\end{bmatrix} = q_k\begin{bmatrix}\frac{1}{2}\phi_k\\ 0\end{bmatrix} p_k$$

$$\Rightarrow \quad \begin{bmatrix}\frac{1}{2}\phi_{k+1}\\ 0\end{bmatrix} = q_{k+1}^{+} q_k\begin{bmatrix}\frac{1}{2}\phi_k\\ 0\end{bmatrix} p_k = \underbrace{p_k^{+}\begin{bmatrix}\frac{1}{2}\phi_k\\ 0\end{bmatrix} p_k}_{\text{this is just rotating } \phi_k \text{ by the quaternion } p_k}$$

$$\Rightarrow \quad \phi_{k+1} = {}^{B_{k+1}}R^{B_k}\,\phi_k$$

$$\boxed{\frac{\partial \phi_{k+1}}{\partial \phi_k} = {}^{B_{k+1}}R^{B_k} = e^{-\widehat{(\omega_k - \beta_k)}\,\delta t}}$$

$\dfrac{\partial \phi_{k+1}}{\partial \beta_k}$:

* Assuming small angles:

$$q_{k+1} + q_{k+1}\begin{bmatrix}\frac{1}{2}\phi_{k+1}\\ 0\end{bmatrix} = q_k\begin{bmatrix}\frac{1}{2}(\omega_k - \beta_k)\delta t - \frac{1}{2}\delta\beta_k\delta t\\ 1\end{bmatrix}$$

$$\approx q_{k+1} - q_{k+1}\begin{bmatrix}\frac{1}{2}\delta\beta_k\delta t\\ 0\end{bmatrix}$$

$$\Rightarrow \quad \boxed{\frac{\partial \phi_{k+1}}{\partial \beta_k} \approx -\delta t\, I}$$

$$\Rightarrow \quad \boxed{A_k = \begin{bmatrix} {}^{B_{k+1}}R^{B_k} & -\delta t I \\ 0 & I \end{bmatrix}}$$

* Covariance Prediction:

$$P_{k+1|k} = A_k P_{k|k} A_k^T + \underbrace{W}$$

related to V_ω and V_β for gyro

Measurement Model With Gyro Bias:

- We still have the same measurements:

$$y_k = \begin{bmatrix} {}^B r_1 \\ {}^B r_2 \\ \vdots \\ {}^B r_N \end{bmatrix} = \begin{bmatrix} {}^B Q^N_k & & & 0 \\ & {}^B Q^N_k & & \\ & & \ddots & \\ 0 & & & {}^B Q^N_k \end{bmatrix} \begin{bmatrix} {}^N r_1 \\ {}^N r_2 \\ \vdots \\ {}^N r_N \end{bmatrix}$$

- We need $C_k = \dfrac{\partial y_k}{\partial x_k}$, $\delta x_k = \begin{bmatrix} \phi_k \\ \beta_k \end{bmatrix}$

$$\frac{\partial {}^B r}{\partial \phi} : \left({}^B r + \delta {}^B r\right) = \left({}^N Q^B e^{\hat{\phi}}\right)^T {}^N r \approx (I - \hat{\phi})\, {}^B Q^N\, {}^N r$$

$$= \underbrace{{}^B Q^N\, {}^N r}_{{}^B r} - \hat{\phi}\, \underbrace{{}^B Q^N\, {}^N r}_{{}^B r}$$

$$\Rightarrow \quad \delta {}^B r = -\hat{\phi}\, {}^B r = {}^B \hat{r}\, \phi \quad \Rightarrow \quad \boxed{\frac{\partial {}^B r}{\partial \phi} = {}^B \hat{r}}$$

$$C_k = \left[\frac{\partial y_k}{\partial \phi_k} \;\vdots\; \frac{\partial y_k}{\partial \beta_k} \right] = \begin{bmatrix} {}^B \hat{r}_1 & \vdots & 0 \\ {}^B \hat{r}_2 & \vdots & 0 \\ \vdots & & \vdots \\ {}^B \hat{r}_N & \vdots & 0 \end{bmatrix}$$

"Mission Mode" MEKF Algorithm:

1) Initialize with $\bar{X}_{0|0} = \begin{bmatrix} \bar{q}_{0|0} \\ \bar{\beta}_{0|0} \end{bmatrix}$, $P_{0|0}$

2) Predict:

$$\bar{X}_{k+1|k} = f(x_k, \omega_k), \quad P_{k+1|k} = A_k P_{k|k} A_k^T + W$$

3) Innovation:

$$Z_{k+1} = \begin{bmatrix} {}^B r_1 \\ \vdots \\ {}^B r_N \end{bmatrix} - \begin{bmatrix} {}^B Q^N_{k+1|k} & & 0 \\ & \ddots & \\ 0 & & {}^B Q^N_{k+1|k} \end{bmatrix} \begin{bmatrix} {}^N r_1 \\ \vdots \\ {}^N r_N \end{bmatrix}$$

$$C_{k+1} = \left[\begin{array}{c|c} {}^B\hat{r}_1 & 0 \\ \vdots & 0 \\ \vdots & \\ {}^B\hat{r}_N & 0 \end{array}\right]$$

$$S_{k+1} = C_{k+1} P_{k+1|k} C_{k+1}^T + V$$

4) Kalman Gain:

$$L_{k+1} = P_{k+1|k} C_{k+1}^T S_{k+1}^{-1}$$

5) Update:

$$\delta\bar{X}_{k+1|k+1} = L_{k+1} Z_{k+1}$$

$$\bar{q}_{k+1|k+1} = \bar{q}_{k+1|k} \begin{bmatrix} r \sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix}, \quad \theta = \|\bar{\phi}_{k+1|k+1}\|, \quad r = \frac{\bar{\phi}_{k+1|k+1}}{\theta}$$

$$\beta_{k+1} = \beta_k + \delta\beta$$

$$P_{k+1|k+1} = (I - L_{k+1} C_{k+1}) P_{k+1|k} (I - L_{k+1} C_{k+1})^T + L_{k+1} V L_{k+1}^T$$

6) Go to 2