Propulsion Theory

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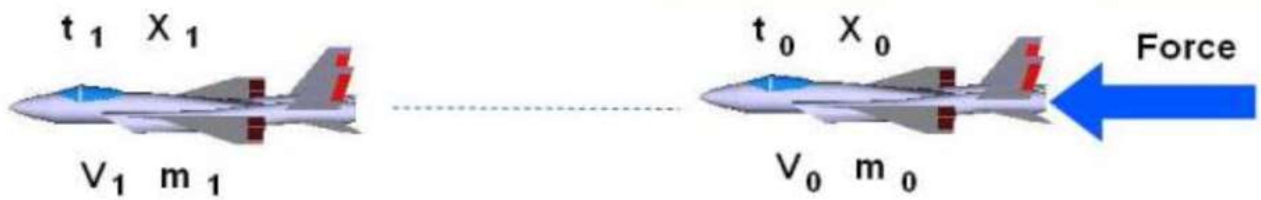
10/22/2019

An Aside on Newton's 2nd Law

National Aeronautics and Space Administration

Newton's Second Law



Force = Change of Momentum with Change of Time

Difference form: $$F = \frac{m_1 V_1 - m_0 V_0}{t_1 - t_0}$$

With constant mass: $$F = m\,\frac{V_1 - V_0}{t_1 - t_0}$$

t = time
X = location
m = mass
V = Velocity

$$F = m\,a$$

Force = mass x acceleration

This is conservation of momentum!!!

An Aside on Newton's 2nd Law

General Form: $\vec{F} = \frac{d}{dt}(m\vec{v})$

Constant Mass: $\vec{F} = m\frac{d}{dt}\vec{v} = m\vec{a}$

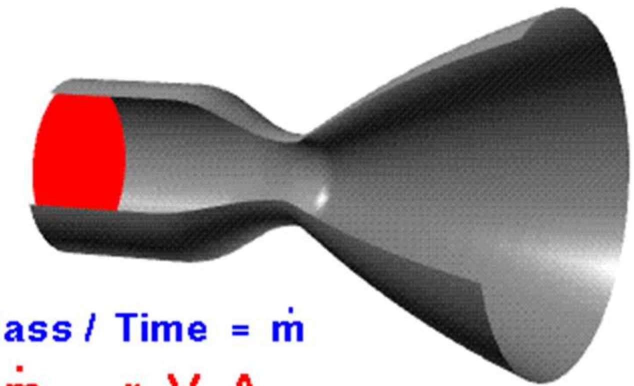
Constant Velocity: $\vec{F} = \vec{v}\frac{d}{dt}m = \dot{m}\vec{v}$

1-Dimensional Incompressible Flow

r = Density

V = Velocity

A = Area



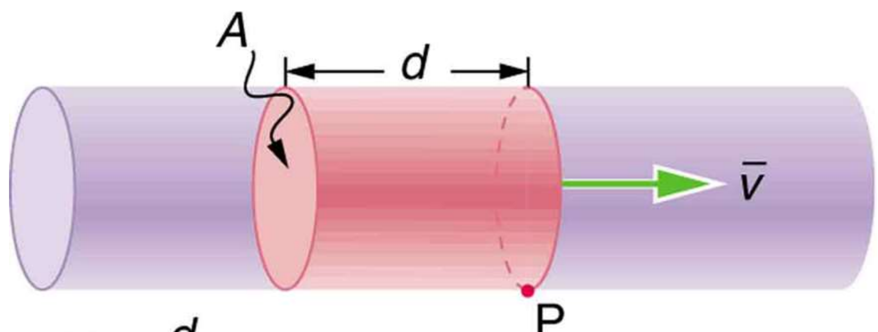
Mass Flow Rate = Mass / Time = $\dot{m}$

$\dot{m}$ = r V A

Units Check: $\frac{\text{mass}}{\text{length}^3} \; \frac{\text{length}}{\text{time}} \; \text{length}^2 = \frac{\text{mass}}{\text{time}}$

Continuity : r V A = Constant

$$\boxed{\dot{m} = \rho_1 A_1 v_1 = \rho_2 A_2 v_2}$$



$$\bar{v} = \frac{d}{t}$$

$$Q = \frac{V}{t} = \frac{Ad}{t} = A\bar{v}$$

WISC SPACE RACE

Conservation of Mass & Momentum

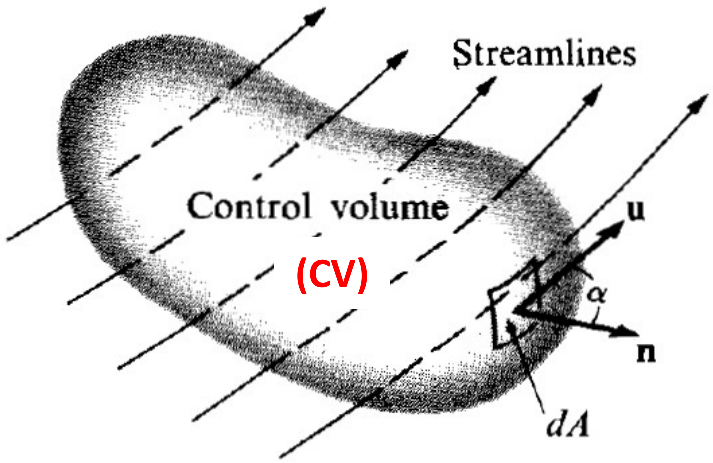


FIGURE 2.1 Fluid flow through a control volume.

$$d\dot{m} = \rho |\mathbf{u}| \cos \alpha \, dA$$

$$m_{cv} = \int_{cv} \rho \, d\mathcal{V}$$

Conservation of Mass:

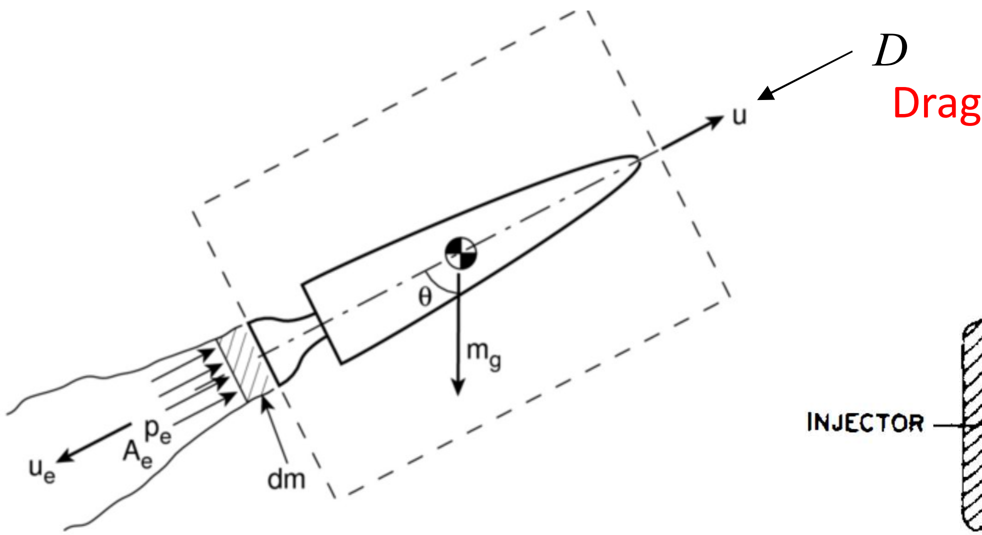
$$\overbrace{\frac{d}{dt} \int_{cv} \rho \, d\mathcal{V}}^{\text{Rate of change of mass \underline{inside} CV}} + \overbrace{\int_{cs} \rho \mathbf{u} \cdot \mathbf{n} \, dA}^{\text{Rate of change of mass \underline{leaving} CV}} = 0$$

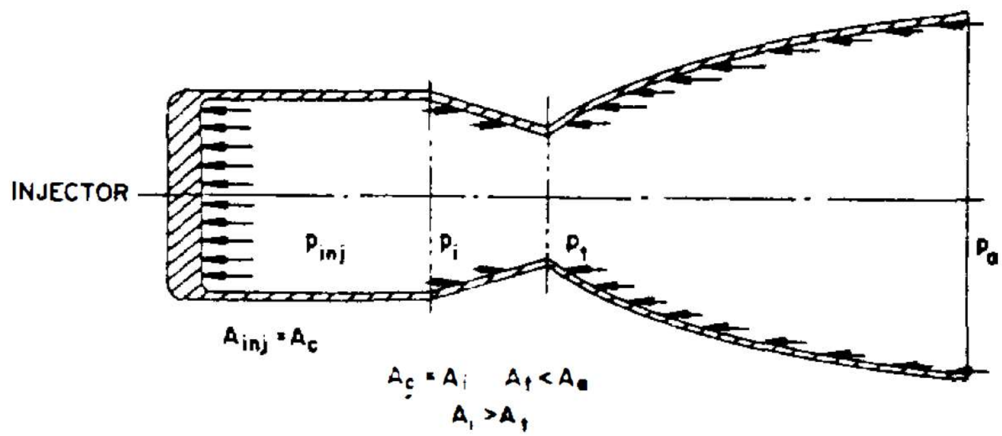
Conservation of Momentum:

$$\sum \mathbf{F} = \underbrace{\frac{d}{dt} \int_{cv} \rho \mathbf{u} \, d\mathcal{V}}_{\text{Rate of change of momentum \underline{inside} CV}} + \underbrace{\int_{cs} \rho \mathbf{u} (\mathbf{u} \cdot \mathbf{n}) \, dA}_{\text{Rate of change of momentum \underline{leaving} CV}}$$

= 0 for steady-state flow

The Free Body Diagram (FBD) of a Rocket





$$\Sigma F = 0$$

$$M\frac{du}{dt} = \overbrace{\dot{m}u_e + (p_e - p_a)A_e}^{\text{Thrust}} - Mgcos\theta + D$$

Thrust

$$\mathcal{T} = \dot{m}u_e + (p_e - p_a)A_e$$

$$u_{eq} = \frac{(p_e - p_a)A_e}{\dot{m}}$$

$$\boxed{\mathcal{T} = \dot{m}u_{eq}}$$

Specific Impulse

$$I = \int_0^{t_b} \mathcal{T} dt = \int_0^{t_b} \dot{m} u_{eq} dt$$

$if\ u_{eq} = constant:$

$$I = u_{eq} \int_0^{t_b} \dot{m} dt = u_{eq} M_p$$

$$\frac{I}{M_p} = u_{eq} = \frac{\mathcal{T}}{\dot{m}}$$

"specific impulse"

$$I_{sp} \equiv \frac{I}{M_p g_e} = \frac{u_{eq}}{g_e} = \frac{\mathcal{T}}{\dot{m} g_e}$$

Units of I_{sp}:
(both definitions are correct)

$$\frac{lbf\text{-}s}{lbm}$$

Impulse per unit weight of propellant.

$$s$$

If the thrust of this engine were theoretically equal to the total weight of propellant, how long would it be able to operate for before running out of fuel?

The Rocket Equation

g = D = 0 (simplification)

$$M\frac{du}{dt} = \underbrace{\dot{m}u_e + (p_e - p_a)A_e}_{\mathcal{T}} - M\cancel{g}cos\theta + \cancel{D}$$

$$\dot{m} = -\frac{dM}{dt} \qquad \dot{m}u_e + (p_e - p_a)A_e = \mathcal{T} = \dot{m}u_{eq}$$

$$M\frac{du}{dt} = -\frac{dM}{dt}u_{eq}$$

$$du = -u_{eq}\frac{dM}{M} \quad \rightarrow \quad \int_{u_0}^{u(t)} du = -u_{eq}\int_{M_0}^{M(t)} \frac{dM}{M} \qquad \ldots$$

The Rocket Equation

$$\int_{u_0}^{u(t)} du = -u_{eq} \int_{M_0}^{M(t)} \frac{dM}{M} \quad \rightarrow \quad \Delta u = -u_{eq} \ln\left(\frac{M(t)}{M_0}\right) = u_{eq} \ln\left(\frac{M_0}{M(t)}\right)$$

$$\boxed{\Delta u = u_{eq} \ln\left(\frac{M_0}{M(t)}\right)}$$

Notice, thrust does not show up here at all! Turns out, thrust is not nearly as important as thrust per unit propellant mass.

Remember: $$I_{sp} = \frac{u_{eq}}{g_e} = \frac{\mathcal{T}}{\dot{m} g_e}$$

$$\Delta u_{total} = u_{eq} \ln\left(\frac{M_0}{M_b}\right)$$

$$\boxed{\Re \equiv \frac{M_0}{M_b} \qquad \Delta u_{total} = \ln(\Re)}$$

“b” subscript denotes “burnout”, i.e. when the rocket runs out of fuel

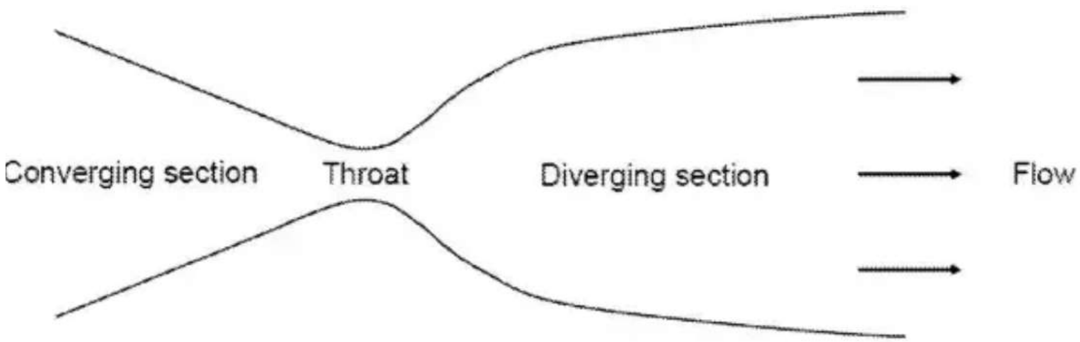
1-D Incompressible Flow

"Incompressible" means density is constant

Conservation of Mass: $\rho A u = constant$
(Continuity Equation)

Conservation of Pressure Potential Energy:
(Bernoulli's Equation)

$$P_{static} + \frac{1}{2}\rho u^2 = P_0 = constant$$



$$\boxed{\begin{array}{l} As\ A \downarrow, u \uparrow, P_{static} \downarrow \\ As\ A \uparrow, u \downarrow, P_{static} \uparrow \end{array}}$$

1-D Compressible Flow

"Compressible" means density can vary with fluid conditions

Conservation of Mass:
(Continuity Equation)

$$\rho A u = constant \rightarrow \boxed{\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0}$$

Conservation of Energy:

$$\boxed{h + \frac{u^2}{2} = h_0 = constant}$$

Newton's Second Law:

$$F = -\dot{m}u = -(\rho A u)u \rightarrow \frac{F}{A} = -(\rho u)u$$

$$\frac{F}{A} = P = -(\rho u)u \qquad \boxed{\frac{dP}{dx} = -\rho u \frac{du}{dx} \qquad du = -\frac{dp}{\rho u}}$$

1-D Compressible Flow

Ideal Gas Law:

$$pV = n\bar{R}T$$

$\bar{R} = 8.314$ [J/(mol-K)]

"Universal gas constant"

$$pV = mRT$$

R, with no bar, is a "gas constant" and is different for every ideal gas [J/kg-K]

$$p = \rho RT$$

Speed of Sound:

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p}{\rho} = \gamma RT$$

$$a = \sqrt{\gamma RT} \qquad \gamma = \frac{c_p}{c_v}$$

a is the speed of sound [m/s]

γ is the specific heat ratio

For a diatomic gas, γ is typically equal to 1.4

1-D Compressible Flow

Mach Number:

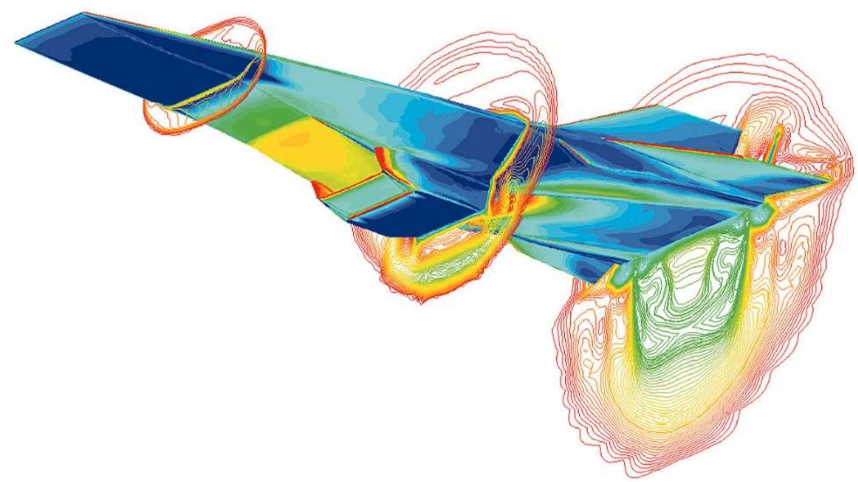
$$a = \sqrt{\gamma R T}$$

$$M = \frac{u}{a} = \frac{u}{\sqrt{\gamma R T}}$$

M < 1: subsonic
M = 1: sonic
M > 1: supersonic

Other definitions:
transonic, hypersonic, ultrasonic, reentry…

NASA Hyper X



Isentropic 1-D Compressible Flow

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0 \quad \rightarrow \quad \frac{dA}{A} = -\frac{d\rho}{\rho} - \frac{du}{u}$$

$$\frac{dA}{A} = -\frac{d\rho}{\rho} - \frac{du}{u} = -\underbrace{\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s}_{d\rho} dp - \underbrace{\frac{1}{u}\left(-\frac{dp}{\rho u}\right)}_{du} \ldots$$

$$\boxed{a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s}$$

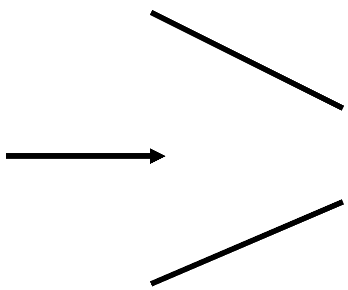
$$\ldots = -\frac{1}{\rho}\frac{1}{a^2}dp + \frac{1}{\rho u^2}dp = \frac{dp}{\rho}\left(\frac{1}{u^2} - \frac{1}{a^2}\right) \ldots$$

$$\ldots = \frac{dp}{\rho u^2}\left(1 - \frac{u^2}{a^2}\right) = \frac{dp}{\rho u^2}(1 - M^2) \qquad \boxed{\frac{dA}{A} = \frac{dp}{\rho u^2}(1 - M^2)} \quad !!!$$

Isentropic 1-D Compressible Flow

$$\boxed{\frac{dA}{A} = \frac{dp}{\rho u^2}(1 - M^2)} \qquad du = -\frac{dp}{\rho u} \qquad \boxed{\frac{du}{u} = \frac{dA}{A}\frac{1}{M^2 - 1}}$$

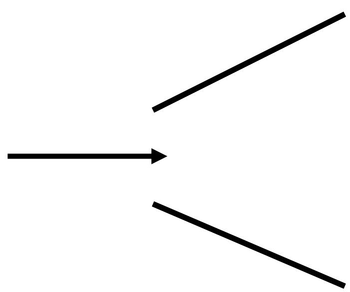
M < 1 (subsonic)



$dA < 0$

$dp < 0$ (Nozzle)

$du > 0$

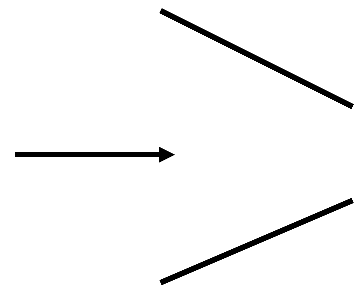


$dA > 0$

$dp > 0$ (Diffuser)

$du < 0$

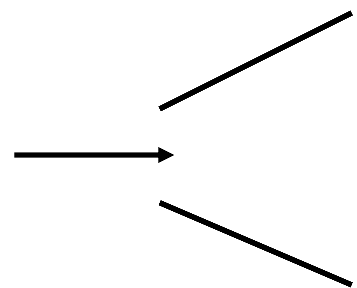
M > 1 (supersonic)



$dA < 0$

$dp > 0$ (Diffuser)

$du < 0$



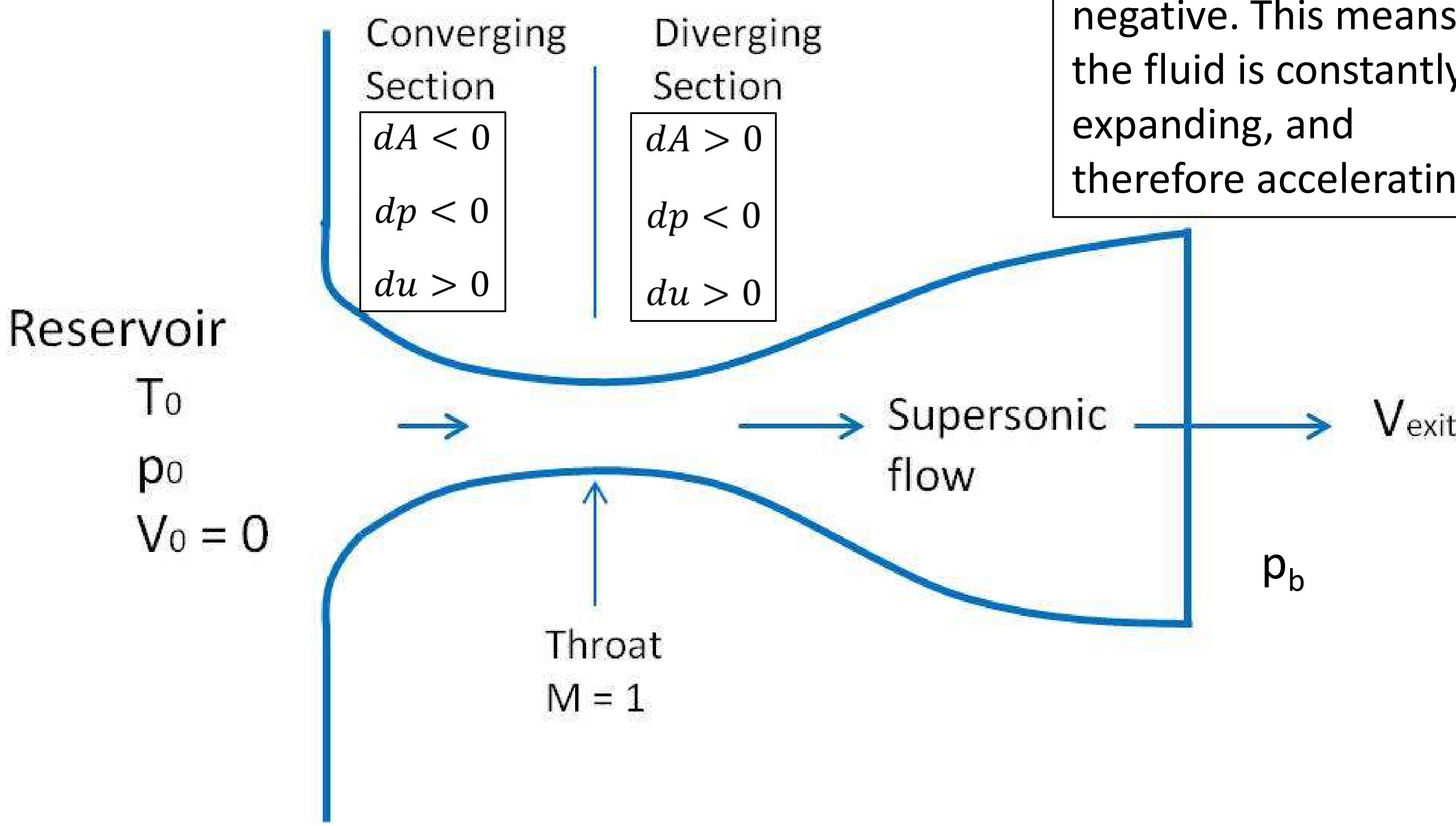
$dA > 0$

$dp < 0$ (Nozzle)

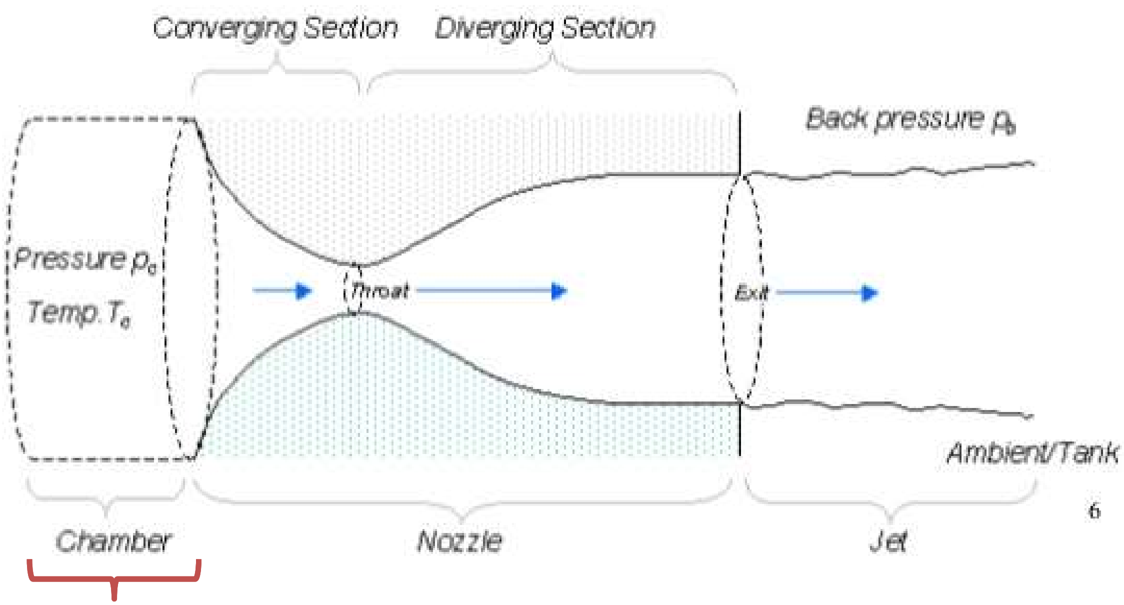
$du > 0$

Converging-Diverging Nozzle

Notice, dp is always negative. This means the fluid is constantly expanding, and therefore accelerating.



Stagnation State



Everything in the chamber is known as the “stagnation state”. This refers to the flow being stagnant, i.e. having no velocity. While the velocity of the flow is nonzero in the chamber, it can be approximated as such, given that its velocity is very low relative to the rest of the nozzle. Assuming isentropic, 1-dimensional flow, the stagnation state will not change throughout the nozzle. In other words, at any place in the flow, if you were to bring a moving particle to a halt isentropically, it would be at the stagnation state (P_0, T_0)

$$P_{static} + \frac{1}{2}\rho \cancelto{\approx 0}{u^2} = P_0 = constant \qquad h + \frac{\cancelto{\approx 0}{u^2}}{2} = h_0 = constant \qquad h_0 = c_p T_0$$

Temperature/Mach Number Relation

Here we begin to derive the governing equations for a converging-diverging nozzle.

$$h + \frac{u^2}{2} = h_0 = constant$$

$$h = c_p T$$

$$c_p T + \frac{u^2}{2} = c_p T_0$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2 c_p T}$$

$$c_p = c_v + R \qquad \gamma = \frac{c_p}{c_v}$$

$$c_p = \frac{\gamma}{\gamma - 1} R$$

$$a^2 = \gamma R T \;\rightarrow\; T = \frac{a^2}{\gamma R} \qquad M = \frac{u}{a}$$

$$\frac{T_0}{T} = 1 + \frac{u^2}{2 \frac{\gamma}{\gamma - 1} R \frac{a^2}{\gamma R}} \rightarrow \boxed{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2}$$

Pressure and Density/Mach Number Relation

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

Recall from thermodynamics:
For an isentropic process…

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1}$$

$$\boxed{\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}}$$

$$\boxed{\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}}}$$

Area Ratio and Mach Number

$$Mass\ Flux \equiv \frac{\dot{m}}{A} = \rho u$$

$$\left.\begin{aligned} u &= Ma \\ a &= \sqrt{\gamma R T} \\ T &= T_0\left(1 + \frac{\gamma - 1}{2} M^2\right)^{-1} \end{aligned}\right\} \quad \underline{u = M\sqrt{\frac{\gamma R T_0}{1 + \frac{\gamma - 1}{2} M^2}}}$$

Area Ratio and Mach Number

$$\rho = \rho_0 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{1-\gamma}}$$

$$Ideal\ Gas: \rho_0 = \frac{p_0}{RT_0}$$

$$\rho = \frac{p_0}{RT_0} \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{1-\gamma}}$$

$$Mass\ Flux \equiv \frac{\dot{m}}{A} = \rho u \longrightarrow \frac{\dot{m}}{A} = p_0 \sqrt{\frac{\gamma}{RT}} M \left(\frac{1}{1 + \frac{\gamma - 1}{2} M^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

Area Ratio and Mach Number

$$\frac{\dot{m}}{A} = p_0 \sqrt{\frac{\gamma}{RT_0}} M \left(\frac{1}{1 + \frac{\gamma - 1}{2} M^2} \right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{d\left(\frac{\dot{m}}{A}\right)}{dM} = 0 \rightarrow Maximum\ mass\ flux\ when\ M = 1$$

$$\left(\frac{\dot{m}}{A}\right)_{max} = \left.\frac{\dot{m}}{A}\right|_{M=1} = p_0 \sqrt{\frac{\gamma}{RT_0}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

Area Ratio and Mach Number

$Define\ conditions\ of\ M = 1\ with\ a\ ^*$

$$A|_{M=1} = A^* \qquad T|_{M=1} = T^*$$

$$p|_{M=1} = p^* \qquad \rho|_{M=1} = \rho^*$$

$For\ a\ fixed\ \dot{m}, \dfrac{\dot{m}}{A}\ is\ maximized\ when\ A\ is\ minimum\ (at\ throat)$

$When\ flow\ is\ choked, A^* = A_{throat} \equiv A_t$

Area Ratio and Mach Number

$$Divide \left(\frac{\dot{m}}{A}\right)_{max} \; by \; \frac{\dot{m}}{A}$$

$$\left(\frac{\dot{m}}{A}\right)_{max} = \frac{\dot{m}}{A^*} = p_0 \sqrt{\frac{\gamma}{RT_0}} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{\dot{m}}{A} = p_0 \sqrt{\frac{\gamma}{RT_0}} M \left(\frac{1}{1 + \frac{\gamma - 1}{2} M^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{\cancel{\dot{m}}}{A^*}\frac{A}{\cancel{\dot{m}}} = \boxed{\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

Nozzle geometry alone determines the Mach number!!!

Isentropic Flow Equations Summary

(standard form of equations)

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{T_0}{T} = 1+\frac{\gamma-1}{2}M^2$$

$$\frac{p_0}{p} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}$$

$$\frac{\rho_0}{\rho} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}}$$

Isentropic Flow Equations Summary

(standard form of equations)

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{T}{T_0} = \left(1+\frac{\gamma-1}{2}M^2\right)^{-1}$$

$$\frac{p}{p_0} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$\frac{\rho}{\rho_0} = \left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{1-\gamma}}$$

This form tends to be a bit more useful in my opinion, since stagnation conditions are usually known. Note though that these equations are identical to those on the previous page.

e.g. To solve for pressure along the nozzle:

$$p = p_0\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{1-\gamma}}$$

Note that these equations are identical to those on the previous page.

2 Solutions to the Area/Mach Equation

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

This is a quadratic equation! Therefore, it has two roots, i.e. two solutions of M for a given area ratio. One happens to be less than 1, and the other is greater than 1. Therefore, **for a given area ratio, there is a subsonic (M_{sub}) and supersonic (M_{sup}) solution!**

Example

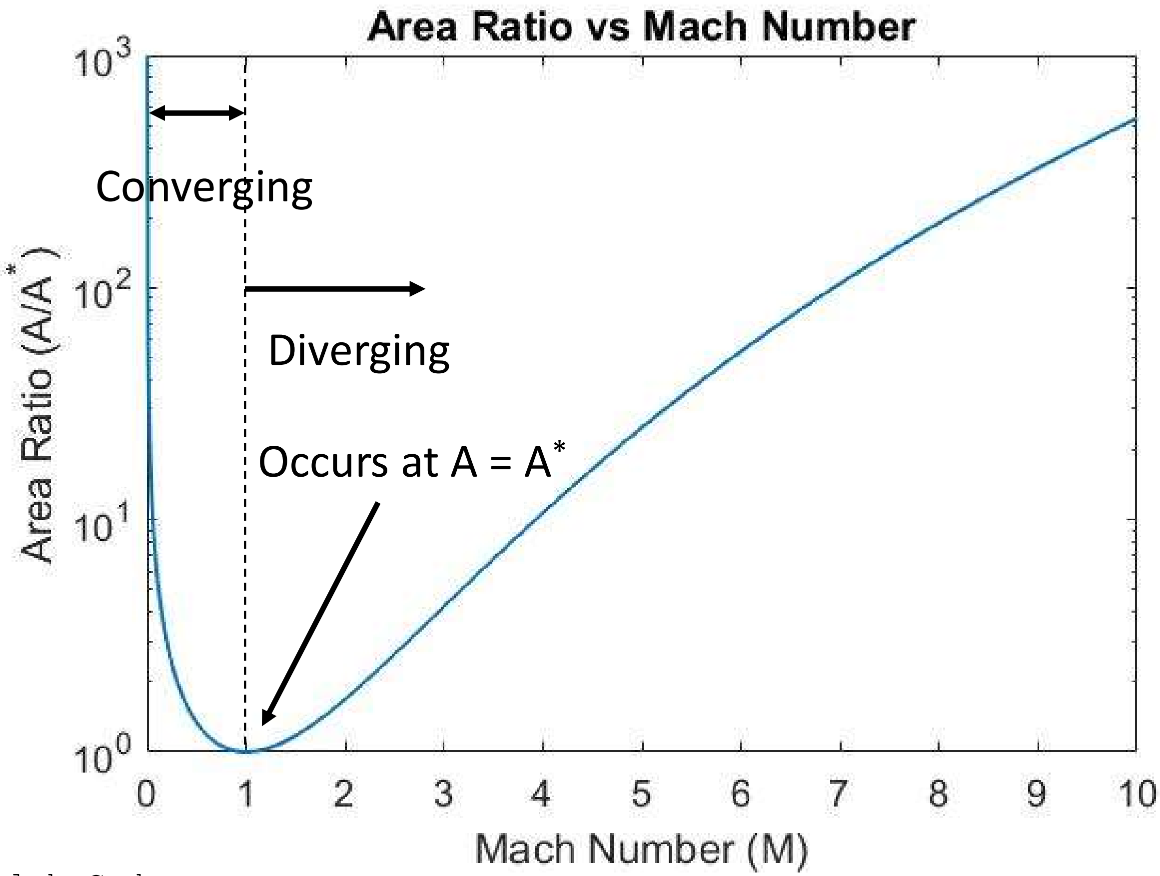
A nozzle has a throat area of 0.1 m^2, and an exit area of 1.0 m^2. What are the two isentropic solutions of the Mach number at the exit of the nozzle?

Answer:

$M_{sub} = 0.0580$

$M_{sup} = 3.923$

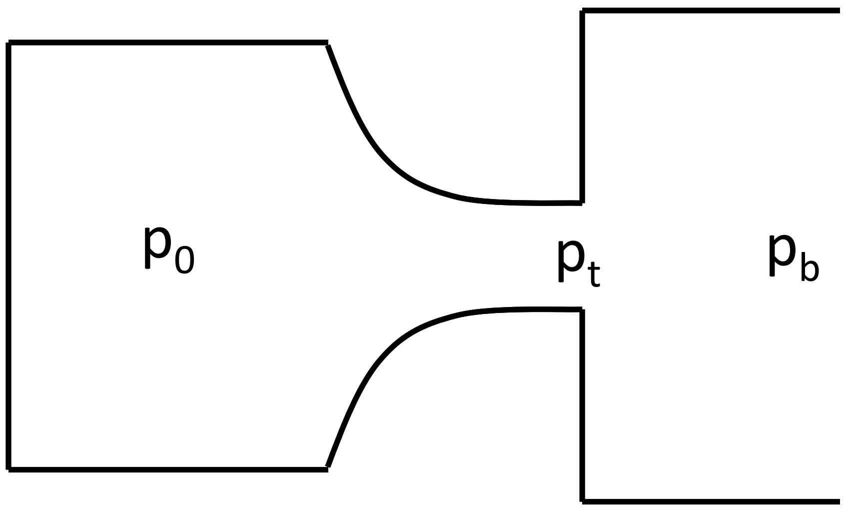
2 Solutions to the Area/Mach Equation



Looks like a nozzle! To achieve supersonic flow, need a **converging and diverging** section of nozzle.

```
%% Matlab Code
gamma = 1.4
M = 0:1e-5:10;
Ae_Astar = 1./M.*(2/(gamma+1)*(1 + (gamma - 1)/2*M.^2)).^((gamma+1)/(2*(gamma-1)));
semilogy(M, Ae_Astar)
```

Converging Nozzle



$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{1-\gamma}}$$

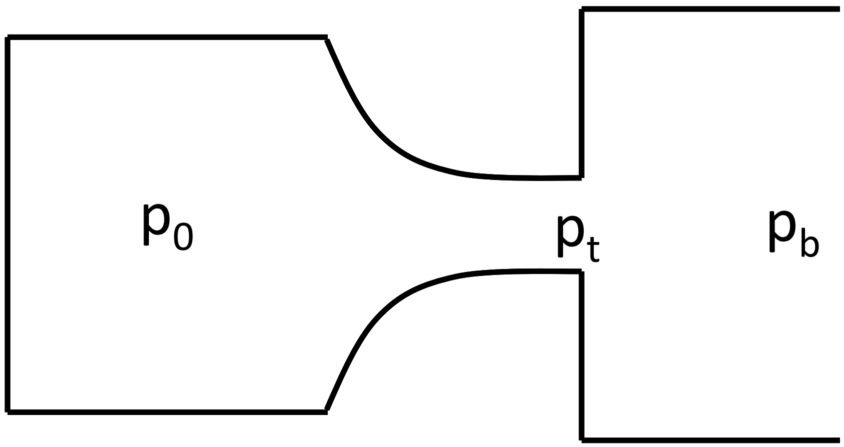
Flow is choked if:

$$p_t = p^* = p_0\left(1 + \frac{\gamma - 1}{2}\right)^{\frac{\gamma}{1-\gamma}}$$

$$\frac{p^*}{p_0} = 0.528\ (for\ \gamma = 1.4)$$

"Choked" flow occurs when M = 1. This refers to the fact that when the flow is choked, the volumetric flow rate at the throat is no longer influenced by the back pressure ratio p_b / p_0. The only way to increase mass flow rate, then, is to increase the density of the flow. Having a higher p_0 will accomplish this, but having a lower p_b will not.

Converging Nozzle



$$\frac{p}{p_0} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{1-\gamma}}$$

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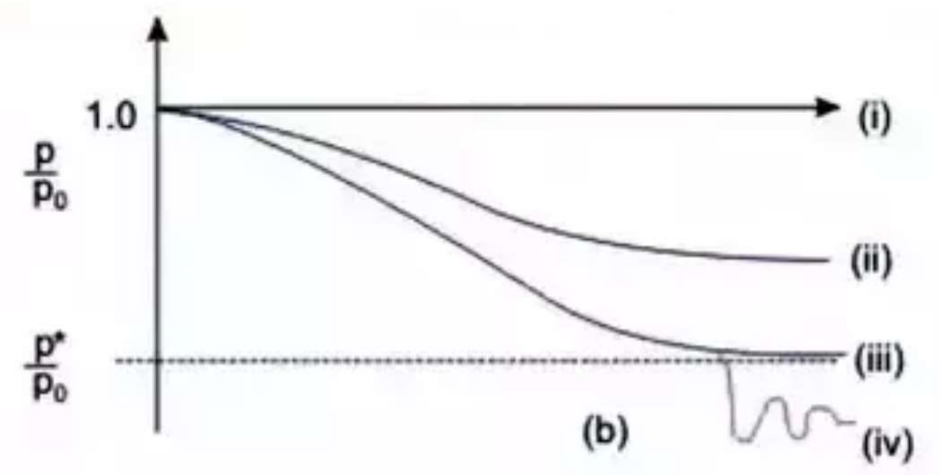


$(i)\ p_b = p_0$

$(ii)\ p^* < p_b < p_0$

$(iii)\ p_b = p^*$

$(iv)\ p_b < p^*$

Converging Nozzle

What's the mass flow rate?

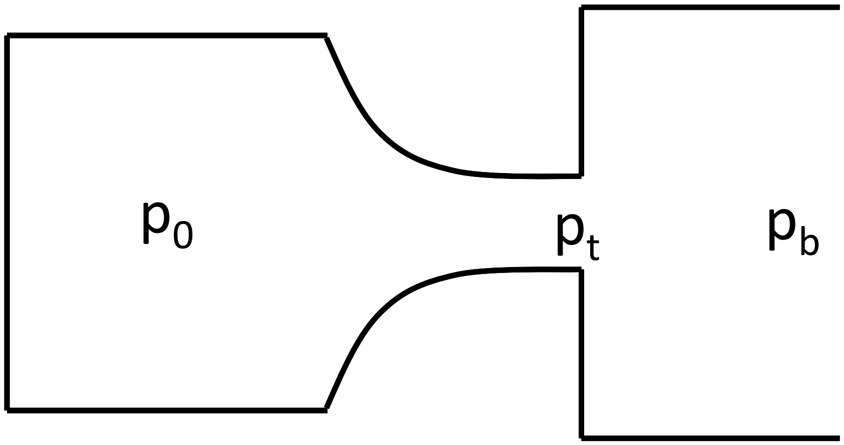
$$\dot{m} = \rho_t A_t u_t$$

$$u_t = M_t\sqrt{\gamma R T_t}$$

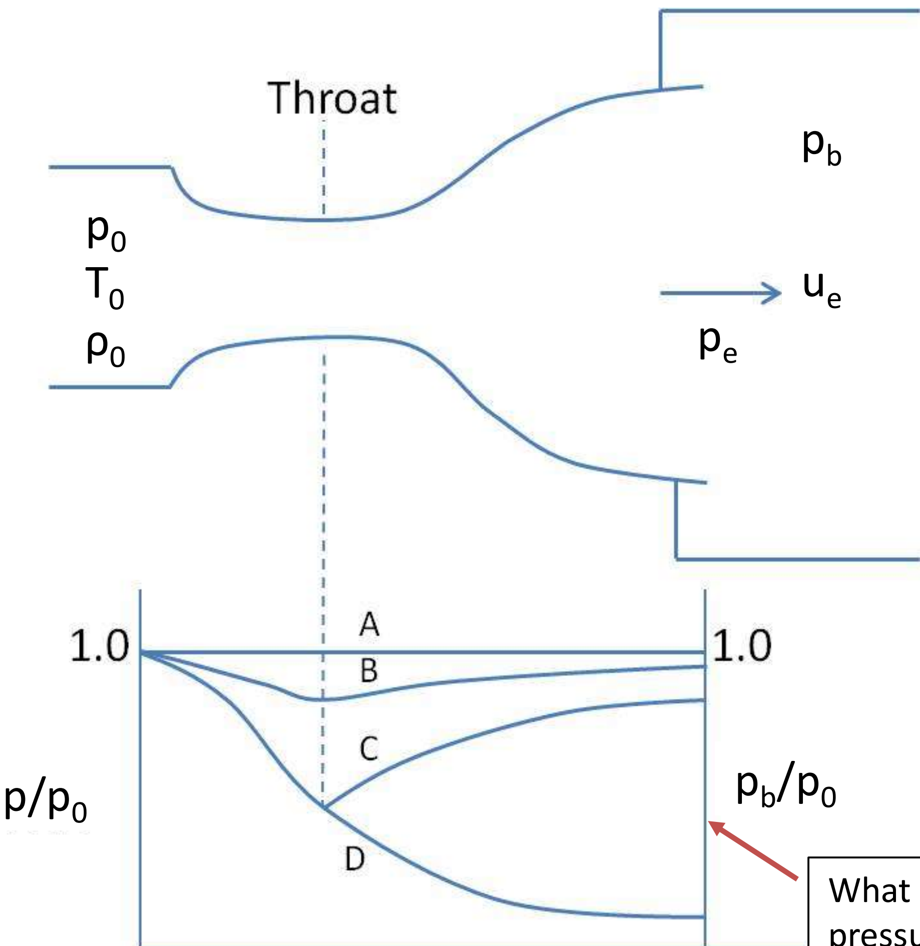
$$T_t = T_0\left(1 + \frac{\gamma - 1}{2}M_t^2\right)^{-1}$$

$$\rho_t = \frac{p_t}{RT_t}$$

$$M_t = 1\ if\ flow\ is\ choked$$



Converging-Diverging Nozzle



What about a pressure ratio in the middle of p_{sub} and p_{sup}

$$\frac{A_e}{A^*} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\,M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\rightarrow M_{sup}\ \&\ M_{sub}$$

$$p_{sup} = p_0\left(1+\frac{\gamma-1}{2}M_{sup}{}^2\right)^{\frac{\gamma}{1-\gamma}}$$

$$p_{sub} = p_0\left(1+\frac{\gamma-1}{2}M_{sub}{}^2\right)^{\frac{\gamma}{1-\gamma}}$$

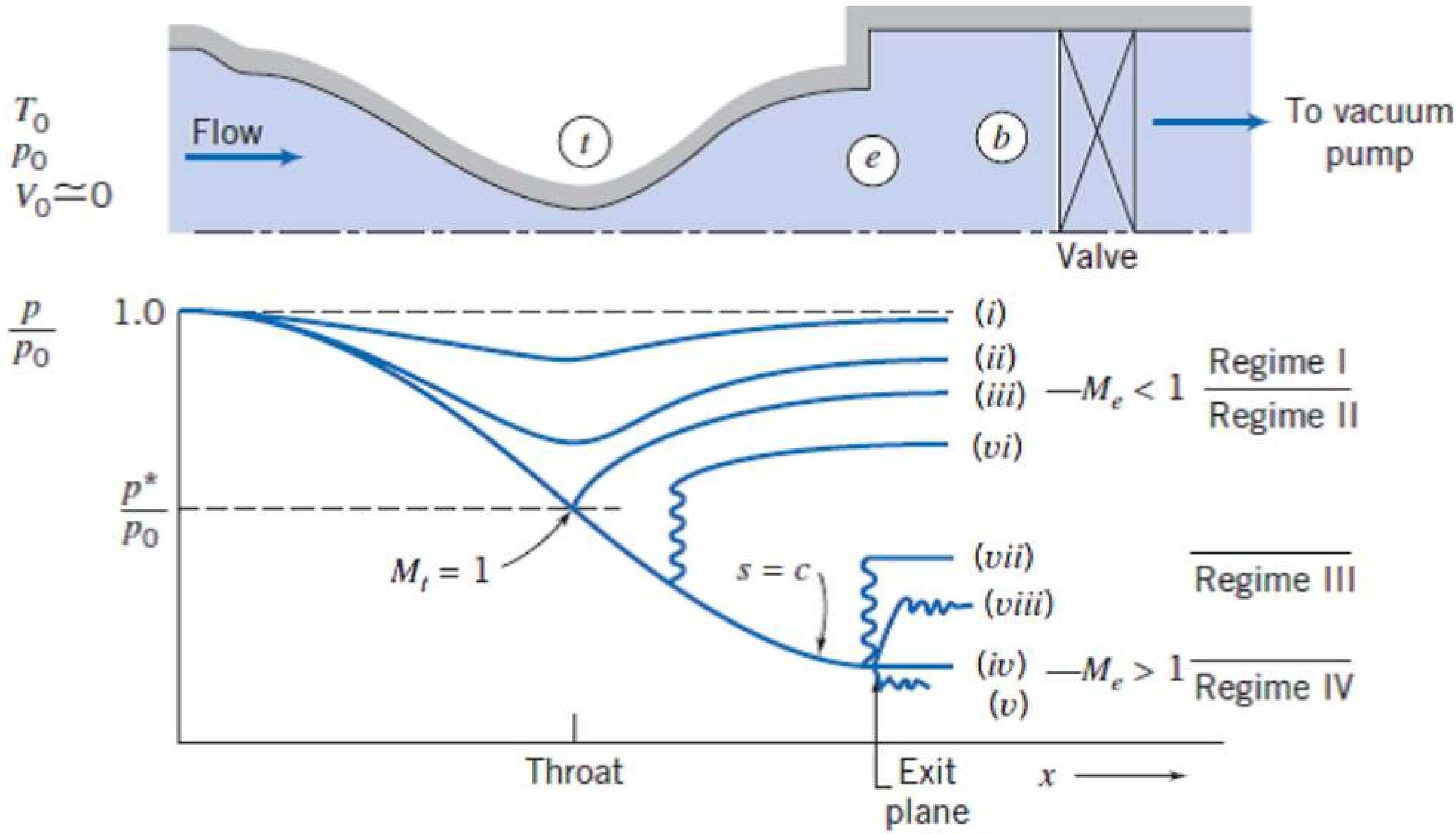
$(A)\ p_b = p_0$

$(B)\ p^* < p_b < p_0$

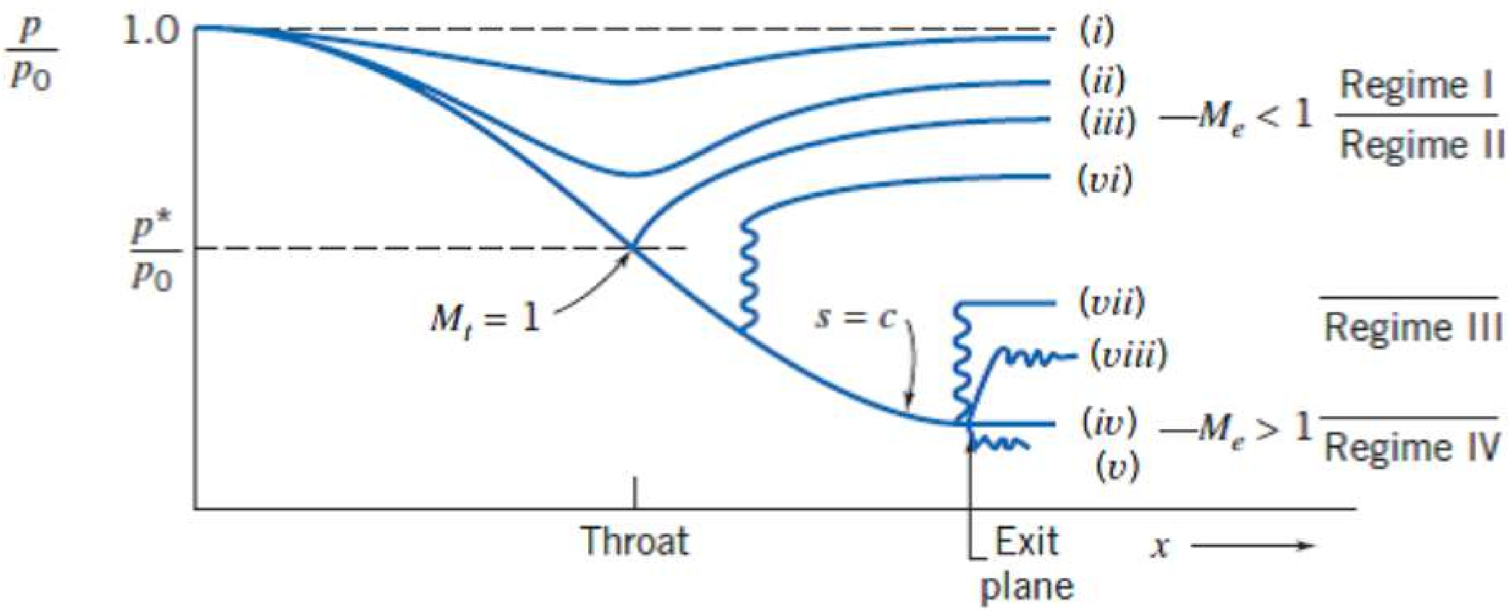
$(C)\ p_b = p_{sub}\ (A.K.A.\, p_{critical})$

$(D)\ p_b = p_{sup}\ (A.K.A.\, p_{design})$

Converging-Diverging Nozzle



Converging-Diverging Nozzle



$(i)\ p_b = p_0$

$(ii)\ p^* < p_b < p_0$

$(iii)\ p_b = p_{sub}\ (A.K.A.\,p_{critical})$

$(iv)\ p_b = p_{sup}\ (A.K.A.\,p_{design})$

$(v)\ p_b < p_{sup}\ (external\ expantion\ waves, underexpanded)$

$(vi)\ p_{shex} < p_b < p_{sub}\ (normal\ shock\ wave\ inside\ nozzle)$

$(vii)\ p_b = p_{shex}\ (normal\ shock\ wave\ at\ nozzle\ exit)$

$(viii)\ p_{sup} < p_b < p_{shex}\ (oblique\ shock\ waves, overexpanded)$

p_{shex} = back pressure resulting in a normal shock wave exactly at the nozzle exit

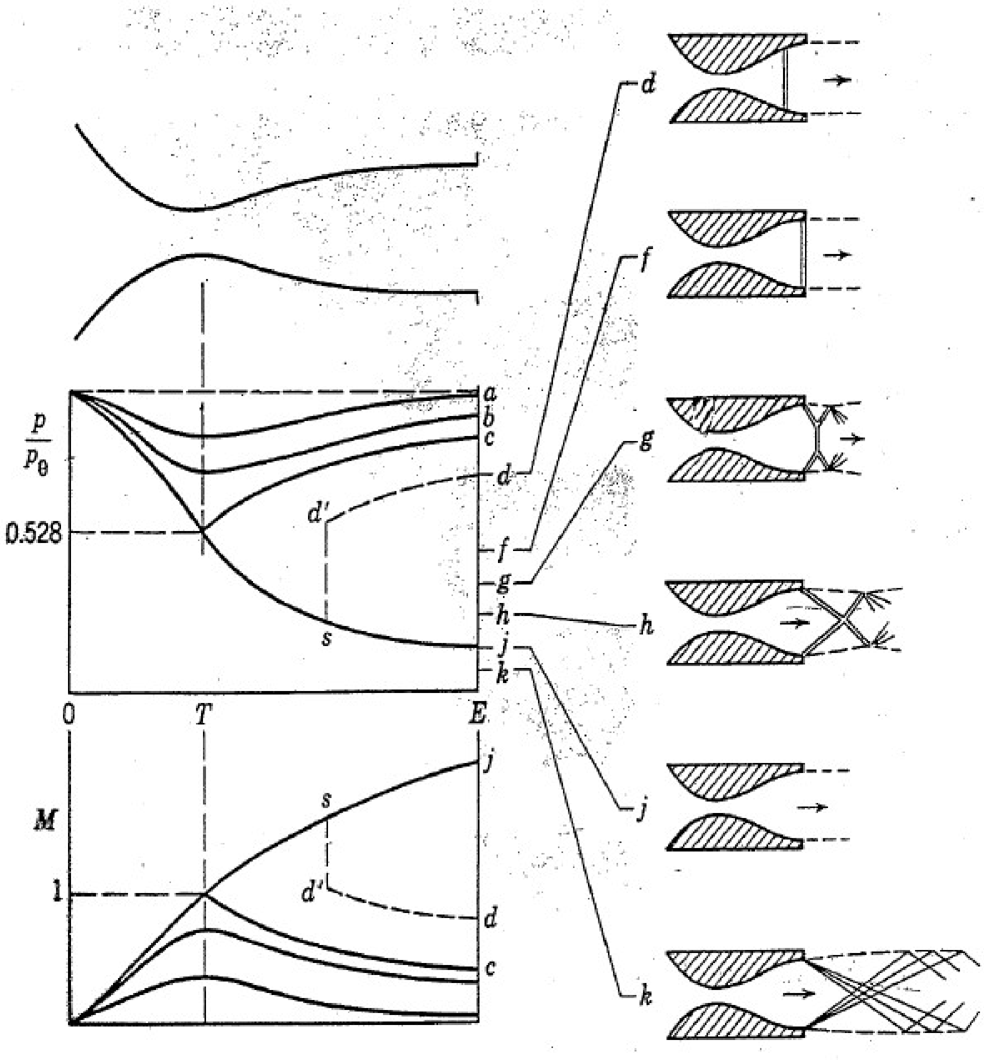


FIG. 5-3 Effect of pressure ratio on flow in a Laval nozzle.

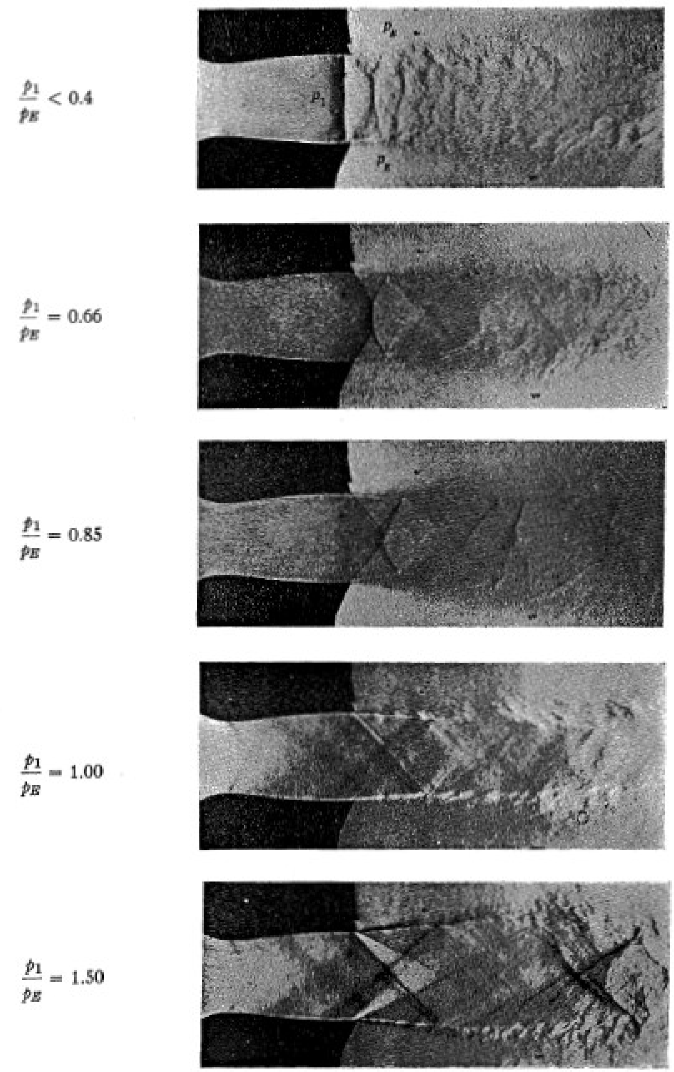
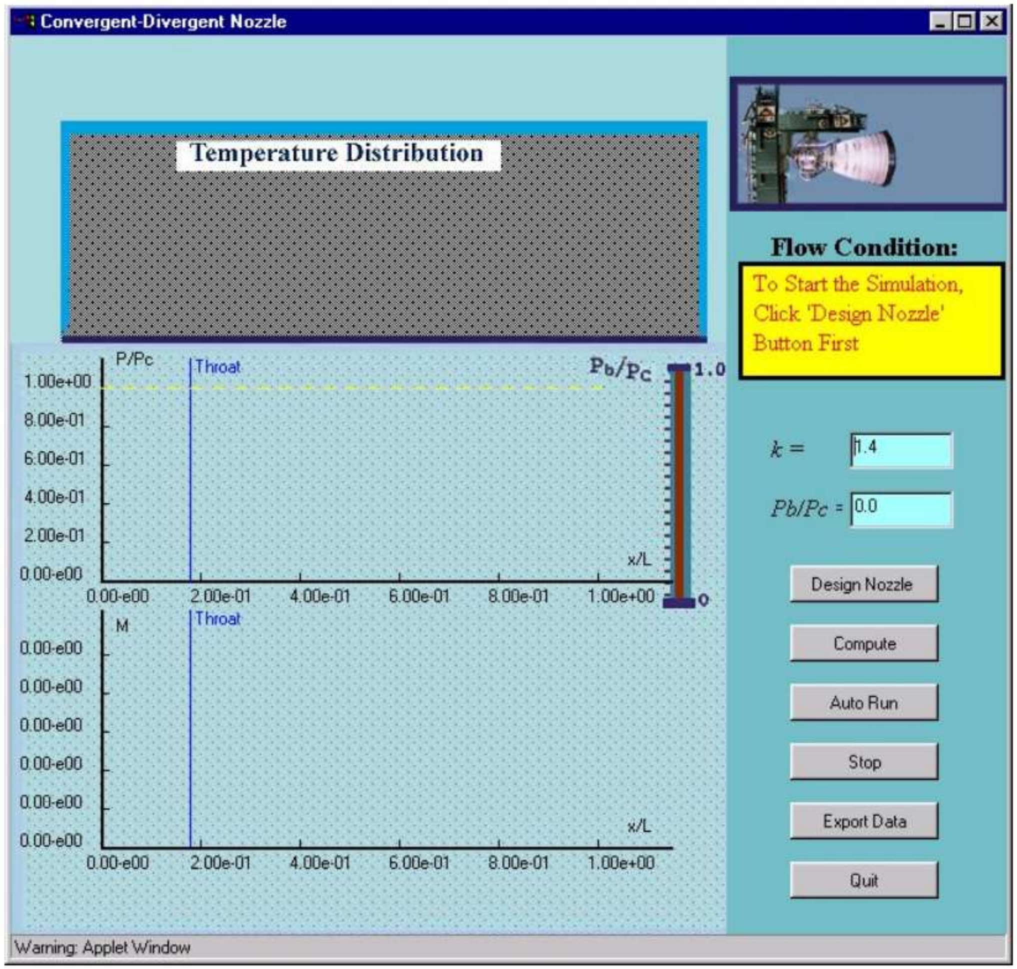


FIG. 5-4 Schlieren photographs of flow from a supersonic nozzle at different back pressures. The photographs, from top to bottom, may be compared with Fig. 5-3, sketches *d, g, h, j, k*, respectively. Reproduced from: L. Howarth (ed.), *Modern Developments in Fluid Dynamics, High Speed Flow*, Oxford, 1953.

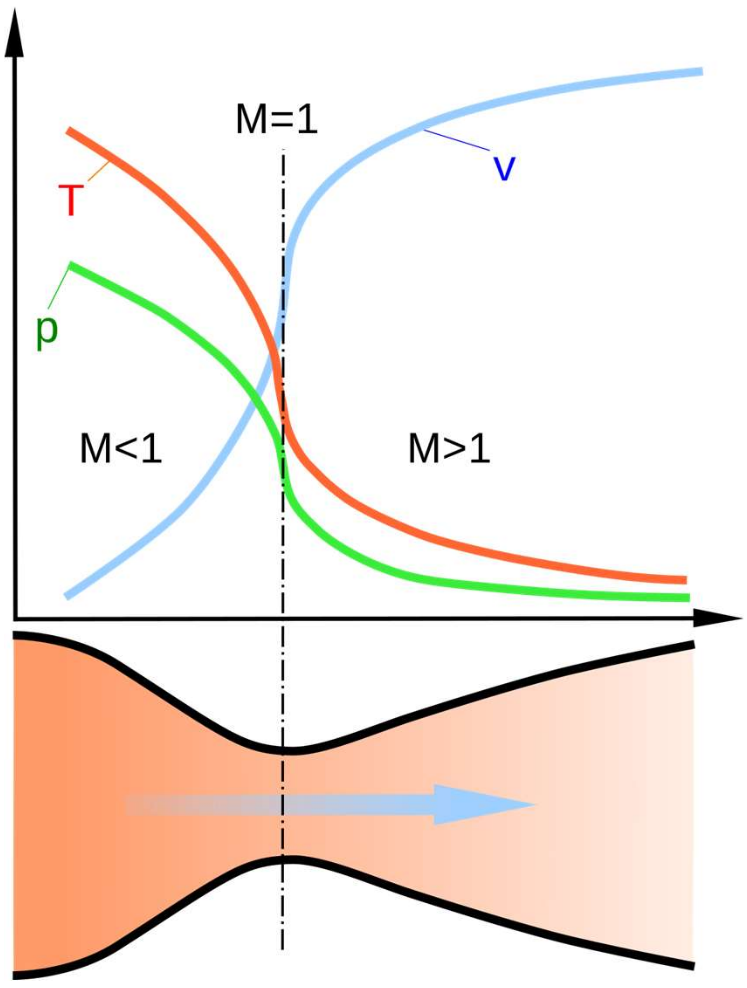
C-D Nozzle Animation



http://www.engapplets.vt.edu/fluids/CDnozzle/cdinfo.html

http://www.engapplets.vt.edu/fluids/CDnozzle/index.html

Converging-Diverging Nozzle



Thrust

$$\mathcal{T} = \dot{m}u_e + (p_e - p_a)A_e$$

$$\mathcal{T} = \dot{m}u_{eq}$$

$$u_{eq} = \frac{(p_e - p_a)A_e}{\dot{m}}$$

$$u_e = \sqrt{2c_pT_0\left(1 - \frac{T_e}{T_0}\right)}$$

$$\dot{m} = \frac{p_0A^*}{\sqrt{RT_0}}\sqrt{\gamma\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

$$u_e = \sqrt{2c_pT_0\left[1 - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

Thrust

$$\mathcal{T} = p_0 A^* \left[\sqrt{\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(\frac{p_e}{p_0} - \frac{p_a}{p_0}\right)\frac{A_e}{A^*} \right]$$

$$c_{\mathcal{T}} = \frac{\mathcal{T}}{p_0 A^*}$$

"thrust coefficient"

$$c^* \equiv \frac{p_0 A^*}{\dot{m}}$$

"characteristic velocity"
A.K.A. "cee-star"

$$\mathcal{T} = \dot{m} c^* c_{\mathcal{T}}$$

Ideal Thrust

$$(c_{\mathcal{T}})_{ideal} = \sqrt{\frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1-\left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(\frac{p_e}{p_0}-\frac{p_a}{p_0}\right)\frac{A_e}{A^*}$$

$$(c^*)_{ideal} = \sqrt{\frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}} RT_0}$$

$$\mathcal{T}_{ideal} = \dot{m}(c^*)_{ideal}\,(c_{\mathcal{T}})_{ideal}$$

Comparing to the ideal values allow you to determined how effective your engine is!

$c_{\mathcal{T}}$: Function of nozzle geometry only

c^*: Function of fuel-oxidizer composition & mixing

Final Remarks

$$\Delta u = u_{eq} \ln\left(\frac{M_0}{M(t)}\right) \qquad I_{sp} \equiv \frac{I}{M_p g_e} = \frac{u_{eq}}{g_e} = \frac{\mathcal{T}}{\dot{m} g_e}$$

$$\mathcal{T} = p_0 A^* \left[\sqrt{\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(\frac{p_e}{p_0} - \frac{p_a}{p_0}\right)\frac{A_e}{A^*} \right]$$

Thrust is not the most important quantity! We care much more about I_{sp}, which is directly related to u_e. **Thrust <u>per unit propellant mass</u>** is the real parameter of interest.

Future Seminars

- Shocks in converging-diverging nozzles
- Chemical composition in converging-diverging nozzles, and how to model it (NASA CEAM)
- Flow measurements using nozzles, orifices, and venturis