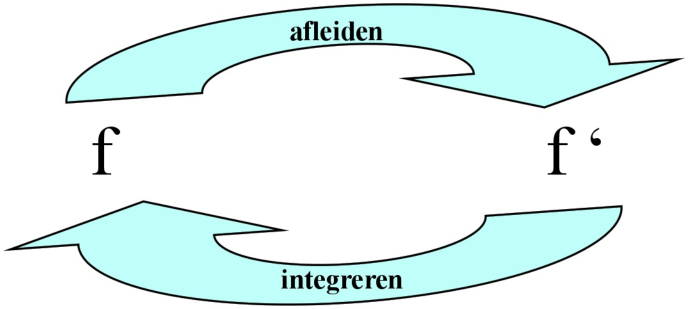
4.8

Primitieve functies en

Onbepaalde integralen



Definitie
Een functie F(x) is een **primitieve functie** van een functie f(x) als:
$$F'(x) = f(x)$$
en dit voor alle x-waarden van het domein van f

$$\begin{tikzcd} F \arrow[r, bend left, "\text{afleiden}"] & f = F' \arrow[l, bend left, "\text{integreren}"] \end{tikzcd}$$

bv: $f(x) = x$

$\Rightarrow F(x) = \frac{x^2}{2}$

of $F(x) = \frac{x^2}{2} + 35$

of . . . .

Onbepaalde integraal

Er zijn natuurlijk meerdere functies F mogelijk, die allen op een constante na verschillen: Voor elke constante C geldt dat F(x)+C ook een primitieve functie is

Definitie

De verzameling van alle primitieve functies van een functie f noemen we de **onbepaalde integraal** van f (voor x) en we noteren:

$$\int f(x)dx$$

- het uitgerekte S-symbool heet het integratieteken
- de functie f heet het integrandum
- x is de integratievariabele

$F(x)$ primitieve functie van $f(x)$

"general antiderivative"

TABLE 4.3 Antiderivative linearity rules

		Function	Onbepaalde integraal
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$,
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

k
e
n
n
e
n

TABLE 4.2 Antiderivative formulas

	Function	Onbepaalde integraal
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

Kromme ?
Helling in punt (x,y) is 3 x^2
en gaat door punt (1,-1)

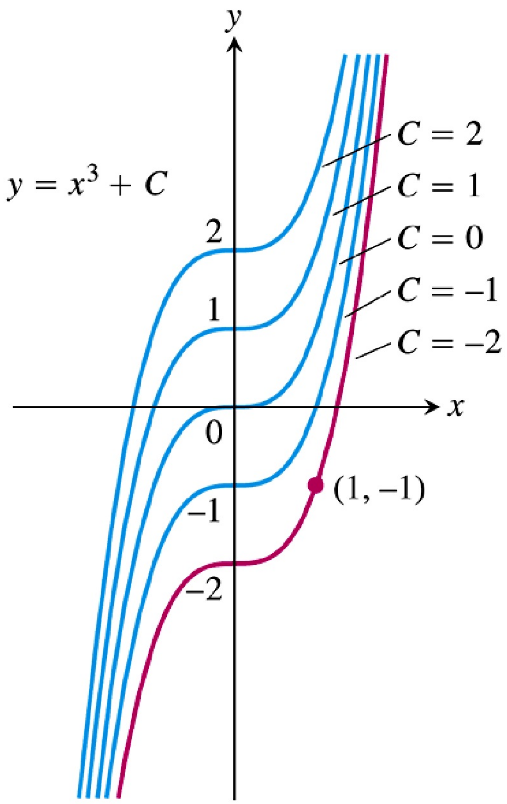


FIGURE 4.54 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 5, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

$y(x) = ?$

$$\frac{dy}{dx} = 3x^2$$

$$y(x) = \int 3x^2 \, dx$$

$$= x^3 + Cst$$

↑

$y(1) = -1$

$1 + Cst = -1$

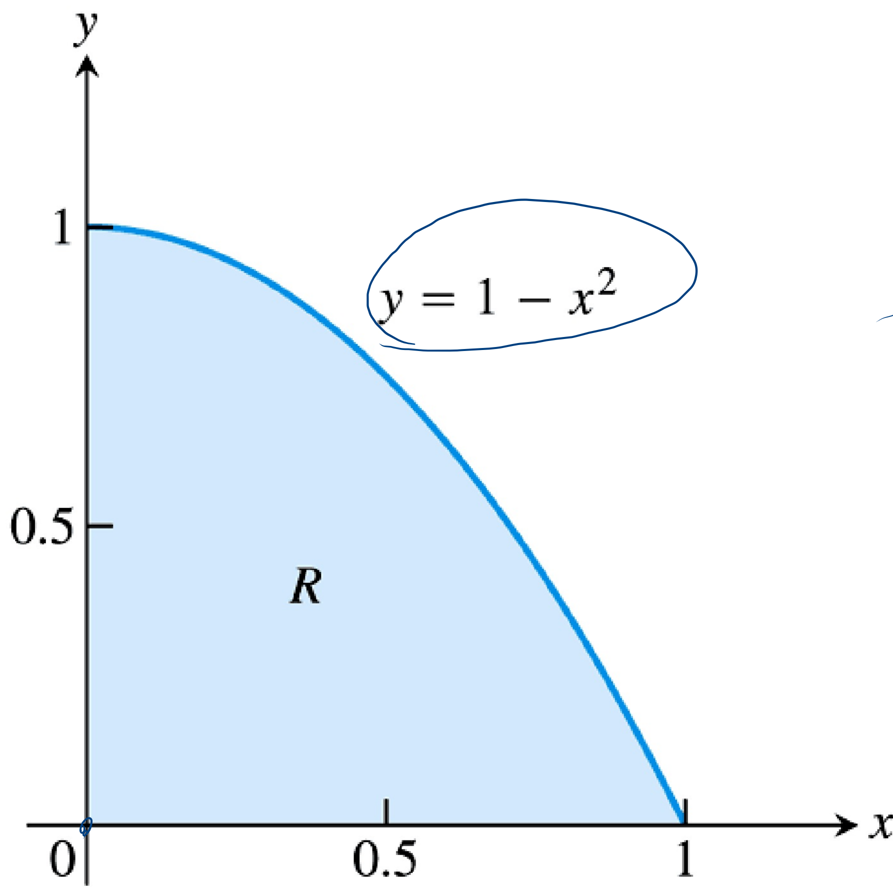
$\Rightarrow Cst = -2$

Chapter 5

Integratie

5.1

Het schatten van een oppervlakte met eindige sommen



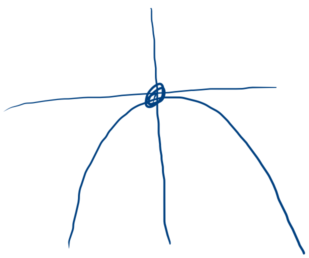
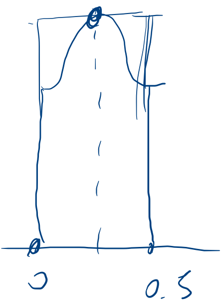
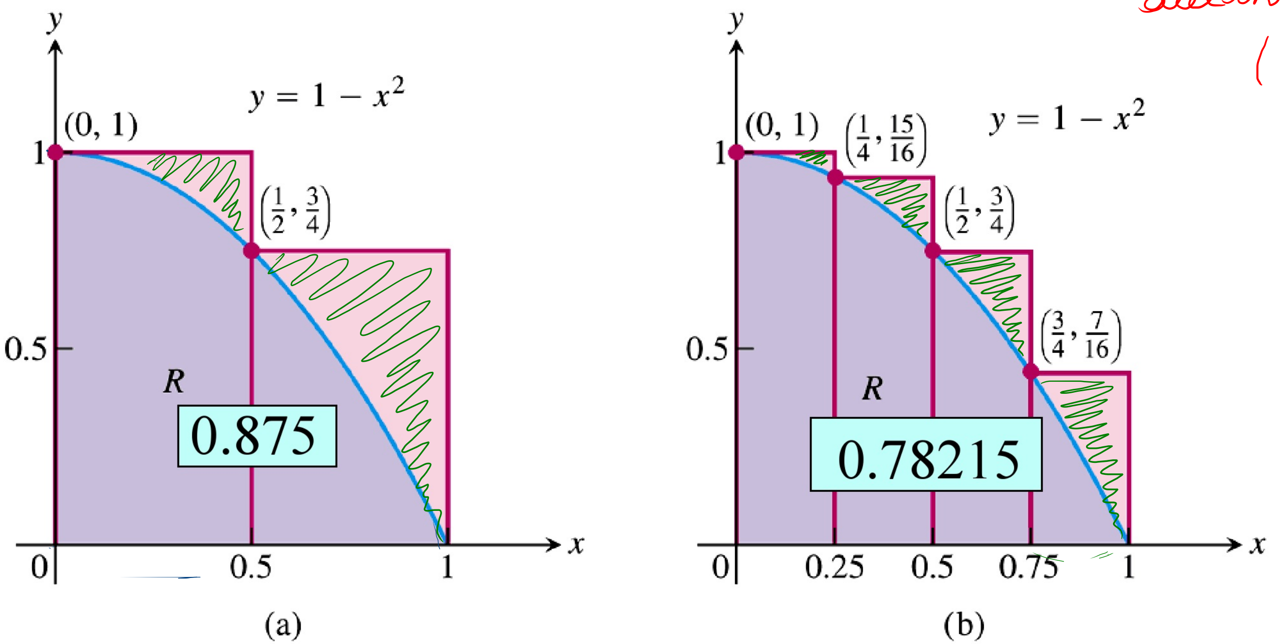


FIGURE 5.1 The area of the region *R* cannot be found by a simple geometry formula (Example 1).

Bovensom

maximum in elk deelinterval
(≠ linker punt)





$1 \times \frac{1}{2} + \frac{3}{4} \times \frac{1}{2}$

$= \frac{7}{8} = 0{,}875$

FIGURE 5.2 (a) We get an upper estimate of the area of R by using two rectangles containing R. (b) Four rectangles give a better upper estimate. Both estimates overshoot the true value for the area.

Ondersom | Middelpuntsregel

minimum in elk deel interval

midden van interval

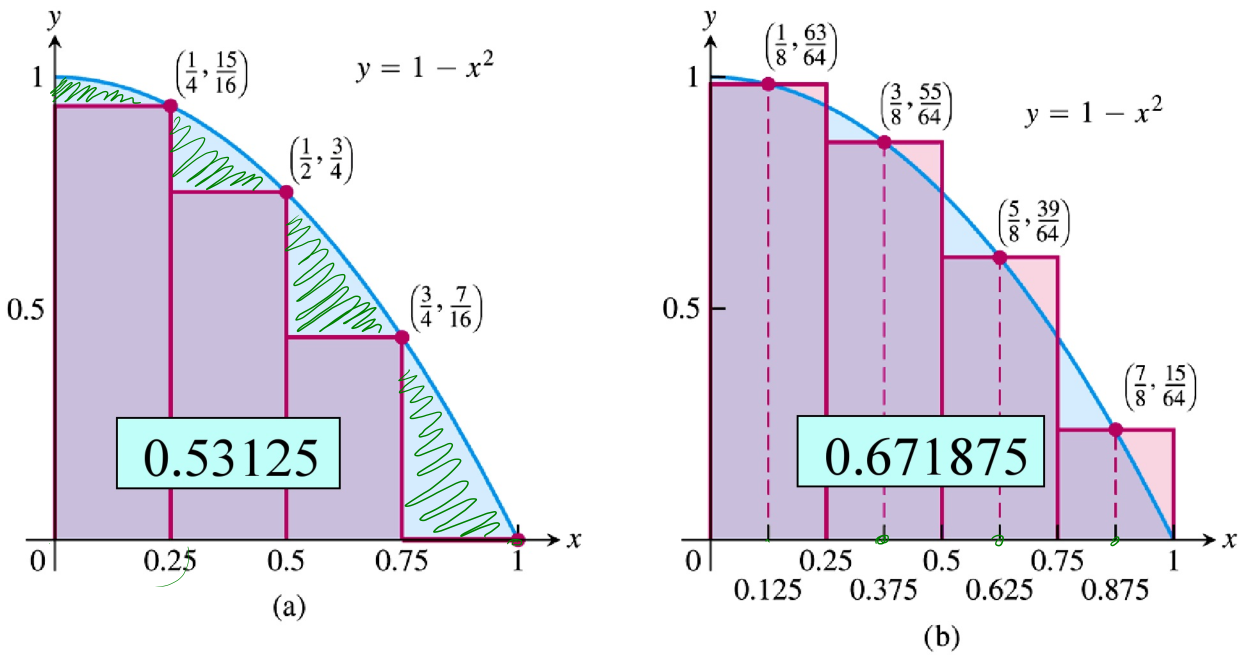
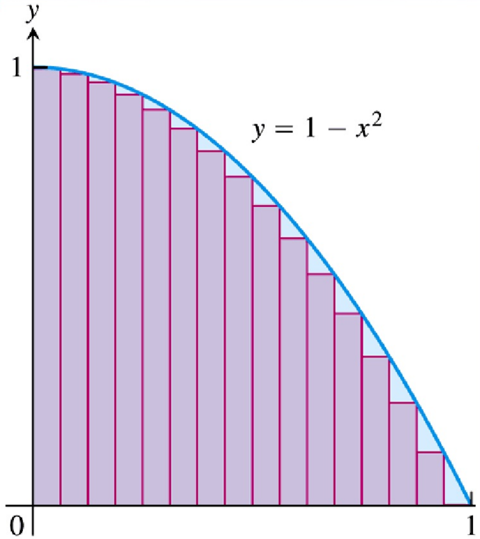


FIGURE 5.3 (a) Rectangles contained in R give an estimate for the area that undershoots the true value. (b) The midpoint rule uses rectangles whose height is the value of $y = f(x)$ at the midpoints of their bases.



(a)

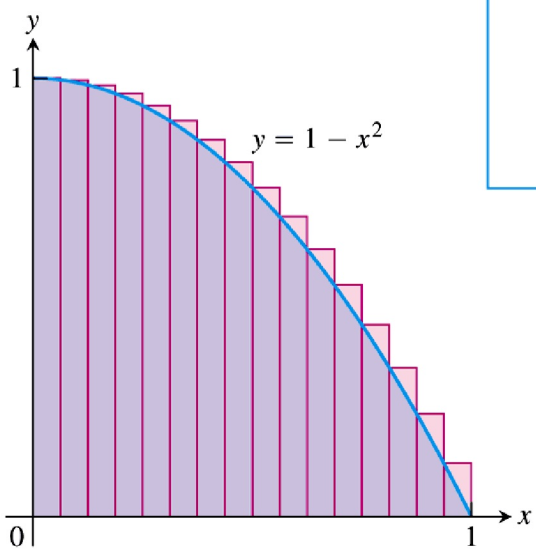


FIGURE 5.4 (a) A lower sum using 16 rectangles of equal width $\Delta x = 1/16$. (b) An upper sum using 16 rectangles.

TABLE 5.1 Finite approximations for the area of R

Number of subintervals	ondersom Lower sum	middelpuntsregel Midpoint rule	bovensom Upper sum
2	.375	.6875	.875
4	.53125	.671875	.78125
16	.634765625	.6669921875	.697265625
50	.6566	.6667	.6766
100	.66165	.666675	.67165
1000	.6661665	.66666675	.6671665

Hoe meer intervallen hoe dichter de sommen bij elkaar liggen en de oppervlakte benaderen. De sommen zijn gelijk aan de oppervlakte in de limiet voor het aantal intervallen gaande naar oneindig?

5.2

Sigma Notatie en Sommaties

The index k ends at $k = n$.

The summation symbol (Greek letter sigma)

$$\sum_{k=1}^{n} a_k$$

a_k is a formula for the kth term.

The index k starts at $k = 1$.

The sum in sigma notation	**The sum written out, one term for each value of k**	**The value of the sum**
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

Algebra Rules for Finite Sums

1. *Sum Rule*: $$\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

2. *Difference Rule*: $$\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

3. *Constant Multiple Rule*: $$\sum_{k=1}^{n} c a_k = c \cdot \sum_{k=1}^{n} a_k$$ (Any number c)

4. *Constant Value Rule*: $$\sum_{k=1}^{n} c = n \cdot c$$ (c is any constant value.)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

$$1 + 2 + \dots + (n-1) + n \leftarrow$$
$$n + (n-1) + \dots + 2 + 1 \leftarrow$$
$$n(n+1) = 2\sum_{k=1}^{n} k$$

The first n squares: $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

The first n cubes: $$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$$

bewijs door inductie

$$1^2 = \frac{1 \times 2 \times 3}{6} = 1$$

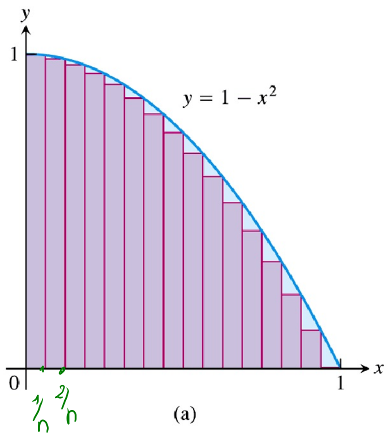
stap 1. OK voor $k = 1$? ✓

stap 2. als je veronderstelt dat OK voor n bewijs dan dat ook OK voor $n+1$

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)}{6}\left(2n^2 + n + 6n + 6\right) = \frac{n+1}{6}\left(2n^2 + 7n + 6\right)$$

$$= \frac{(n+1)}{6}(n+2)(2n+3) \rightarrow \text{OK}$$



$\frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \dots$

n equidistante deelintervallen

$\downarrow$

Ondersom

$$\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left(1-\left(\frac{k}{n}\right)^2\right)\left(\frac{1}{n}\right) = \sum_{k=1}^{n}\left(\frac{1}{n} - \frac{k^2}{n^3}\right)$$

$$= 1 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

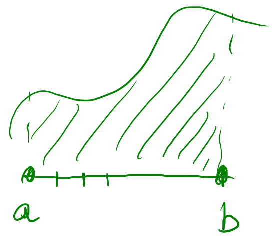
$$= 1 - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$=1-\frac{2n^3+3n^2+n}{6n^3}$$

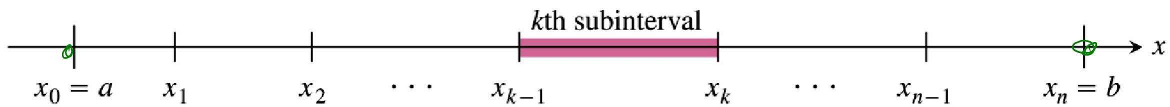
De Ondersom convergeert naar $\lim_{n\to\infty}\left(1-\frac{2n^3+3n^2+n}{6n^3}\right)=1-\frac{2}{6}=\frac{2}{3}$

$0{,}6666\ldots$

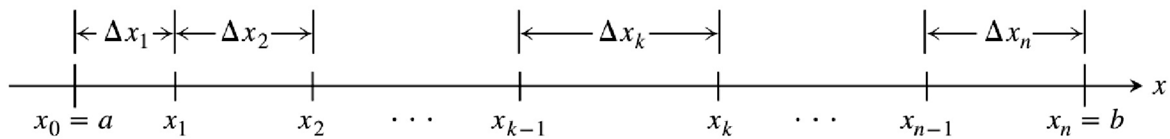
Een **partitie** van het interval [a,b] is een verzameling punten P = {a, x1, x2,…,xn-1, b} met a < x1 < x2 < ... < xn-1 < b (notatie a = xo en b = xn)

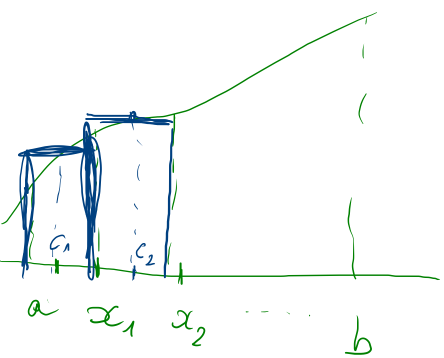


The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the k**th subinterval of** P is $[x_{k-1}, x_k]$, for k an integer between 1 and n.

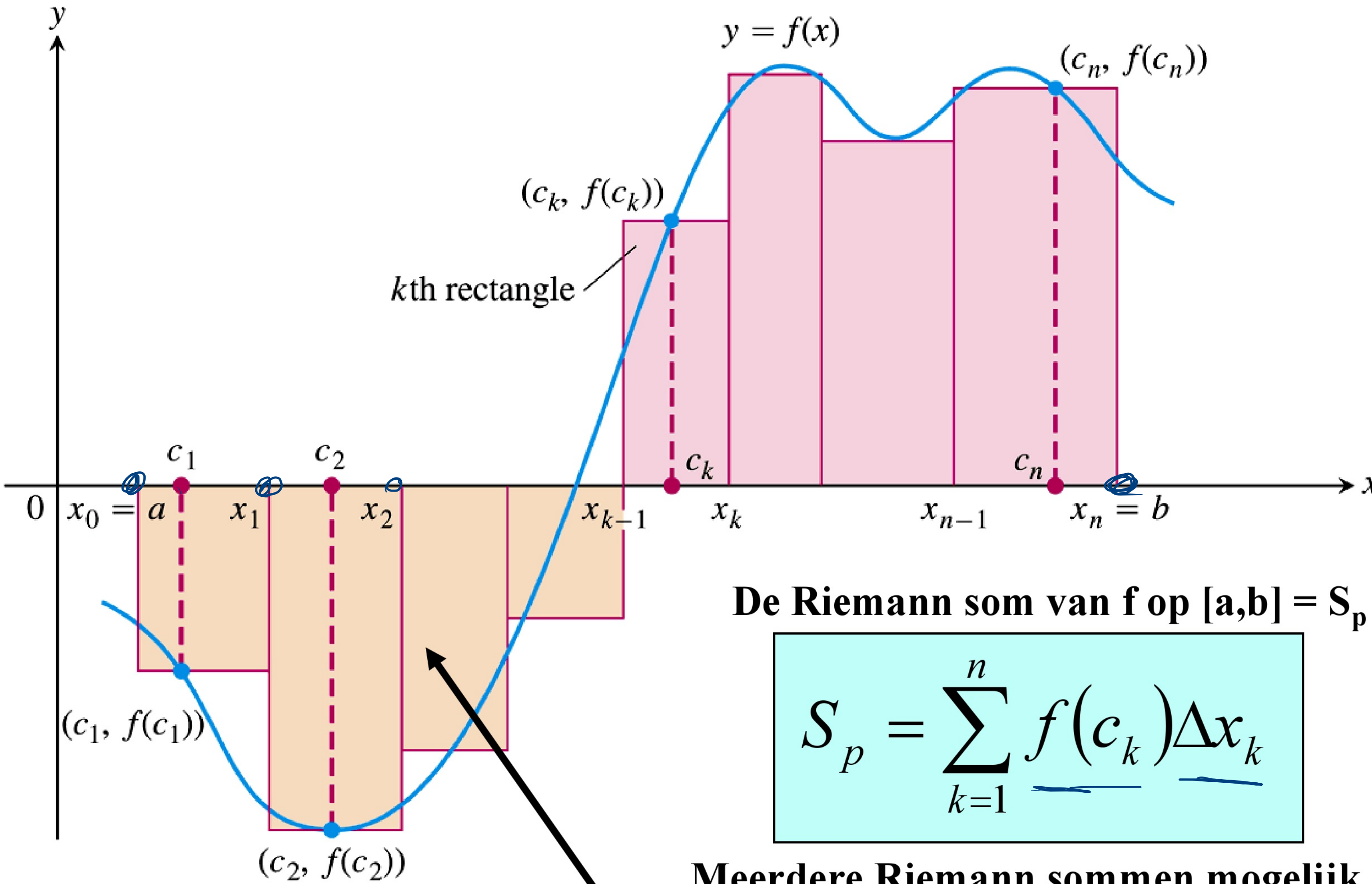


The width of the first subinterval $[x_0, x_1]$ is denoted Δx_1, the width of the second $[x_1, x_2]$ is denoted Δx_2, and the width of the kth subinterval is $\Delta x_k = x_k - x_{k-1}$. If all n subintervals have equal width, then the common width Δx is equal to $(b - a)/n$.





Keuze van een punt c_k in elk k-de subinterval

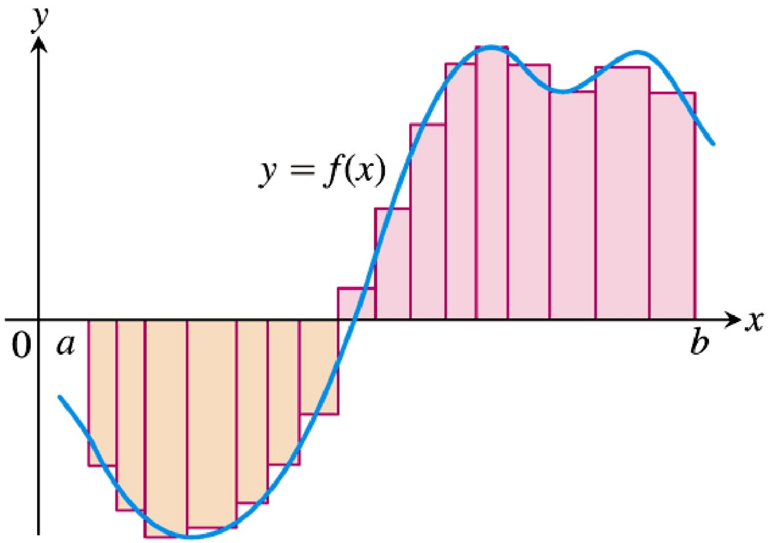


De Riemann som van f op [a,b] = S_p met

$$S_p = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

Meerdere Riemann sommen mogelijk (P en c_k)

FIGURE 5.9 The rectangles approximate the region between the graph of the function $y = f(x)$ and the x-axis.

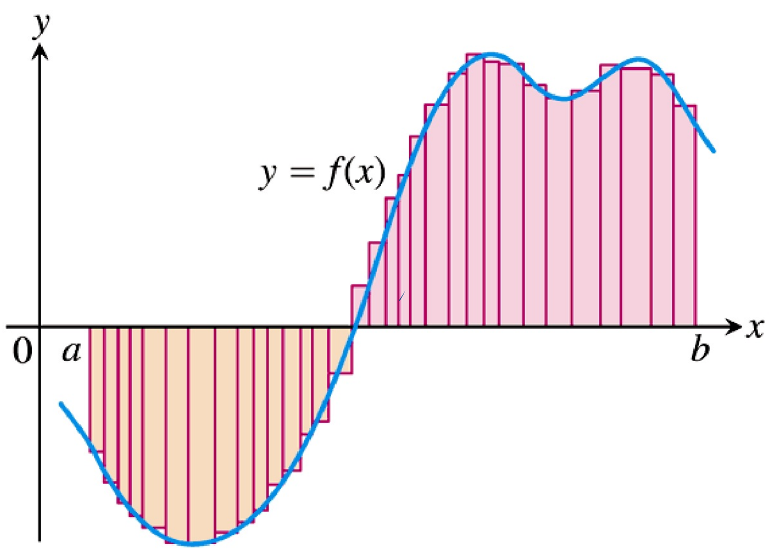


(a)

Definitie

De **norm** van een partitie P is de breedte van het grootste subinterval van die partitie = ||P||

||P||



(b)

FIGURE 5.10 The curve of Figure 5.9 with rectangles from finer partitions of $[a, b]$. Finer partitions create collections of rectangles with thinner bases that approximate the region between the graph of f and the x-axis with increasing accuracy.

5.3

De Bepaalde Integral

Definitie
Zij f een functie gedefinieerd op een interval [a,b] en zij P een willekeurige partitie van [a,b]. Zij c_k willekeurige punten in de subintervallen $[x_{k-1}, x_k]$.
Indien de volgende limiet bestaat :

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

en onafhankelijk van de gekozen partitie en van de gekozen punten, dan zeggen we dat **f integreerbaar** is op [a,b] en we noemen I de **bepaalde integraal** van f over [a,b].

We zeggen dat de Riemann som convergeert naar de bepaalde integraal en dat f integreerbaar is over [a,b]

$\lim_{\|P\|\to 0}$ betekent dat de lengte van alle subintervallen of de lengte van het grootste subinterval (de norm van de partitie), naar nul gaat.

We kunnen ook zeggen $\forall \varepsilon > 0, \exists \delta > 0$ zodat voor elke partitie P van [a,b] met $\|P\| < \delta$ en voor elke keuze van c_k in de subintervallen $[x_{k-1}, x_k]$ geldt dat

$$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - I \right| < \varepsilon$$

Notatie van de bepaalde integraal van f over [a,b]

Integrandum
The function is the integrand.

bovengrens
Upper limit of integration

x is the variable of integration.
integratievariabele

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

ondergrens
Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

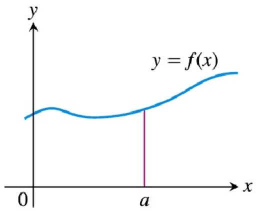
Stelling
Alle continue functies zijn integreerbaar.

NB: om niet integreerbaar te zijn moet f “serieus discontinu” zijn !

Indien f integreerbaar is bestaan alle nevenstaande integralen. Er geldt tevens:

TABLE 5.3 Rules satisfied by definite integrals

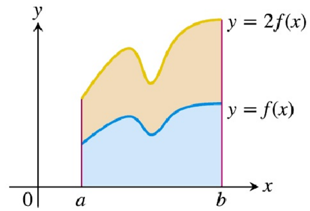
1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	Also a Definition
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any Number k
		$\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$	$k = -1$
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

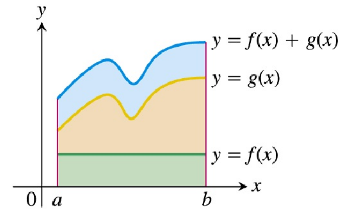
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

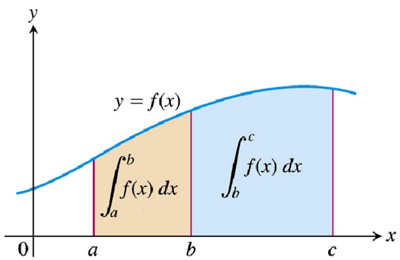
(Shown for $k = 2$.)



(c) *Sum:*

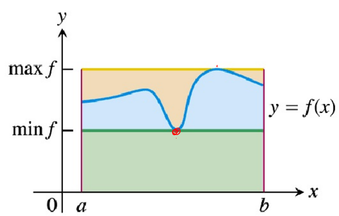
$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)



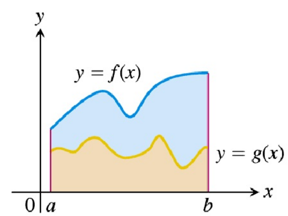
(d) *Additivity for definite integrals:*

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



(e) *Max-Min Inequality:*

$$\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a)$$

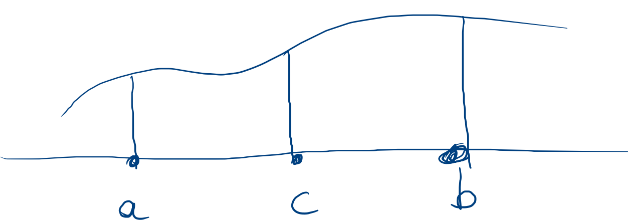


(f) *Domination:*

$f(x) \ge g(x)$ on $[a, b]$

$$\Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

FIGURE 5.11



$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_{\cancel{c}\;b}^{\cancel{b}\;c} f(x)\,dx$$

Oppervlakte onder de grafiek van een positieve functie

Zij y=f(x) een positieve integreerbare functie over een gesloten interval [a,b], dan wordt de oppervlakte A tussen de curve y=f(x) en de X-as, tussen a en b gegeven door:

$$A = \int_a^b f(x)dx$$

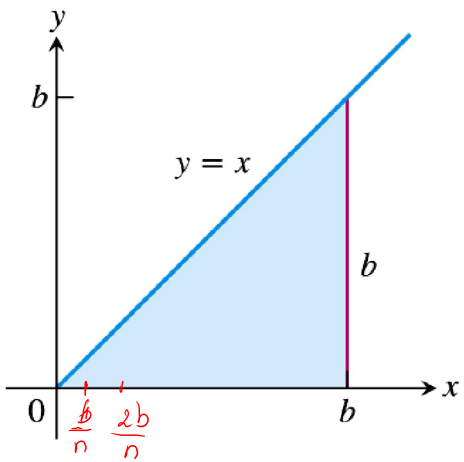


FIGURE 5.12 The region in Example 4 is a triangle.

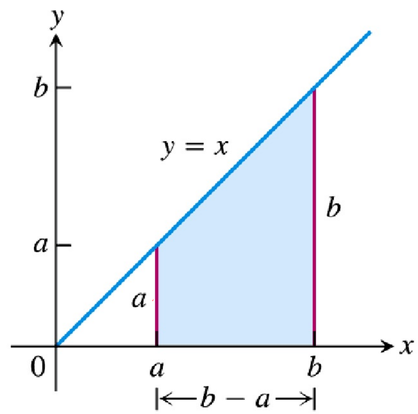


FIGURE 5.13 The area of this trapezoidal region is $A = (b^2 - a^2)/2$.

$$\frac{b^2}{2} - \frac{a^2}{2}$$

$$P = \left\{0, \frac{b}{n}, \frac{2b}{n}, \ldots, b\right\}$$

$$c_k = \frac{kb}{n}$$

$$I = \sum_{k=1}^{n} \frac{b}{n} \cdot f\left(\frac{kb}{n}\right) = \sum_{k=1}^{n} \frac{b}{n}\frac{kb}{n} = \frac{b^2}{n^2}\sum_{k=1}^{n} k = \frac{b^2}{n^2}\frac{n(n+1)}{2} = \frac{b^2(n^2+n)}{2n^2}$$

$$\lim_{n\to\infty} I = \lim_{n\to\infty} \frac{b^2}{2}\left(\frac{n^2+n}{n^2}\right) = \frac{b^2}{2}$$

$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b \tag{1}$$

$$\int_a^b c\,dx = c(b - a), \qquad c \text{ any constant} \tag{2}$$

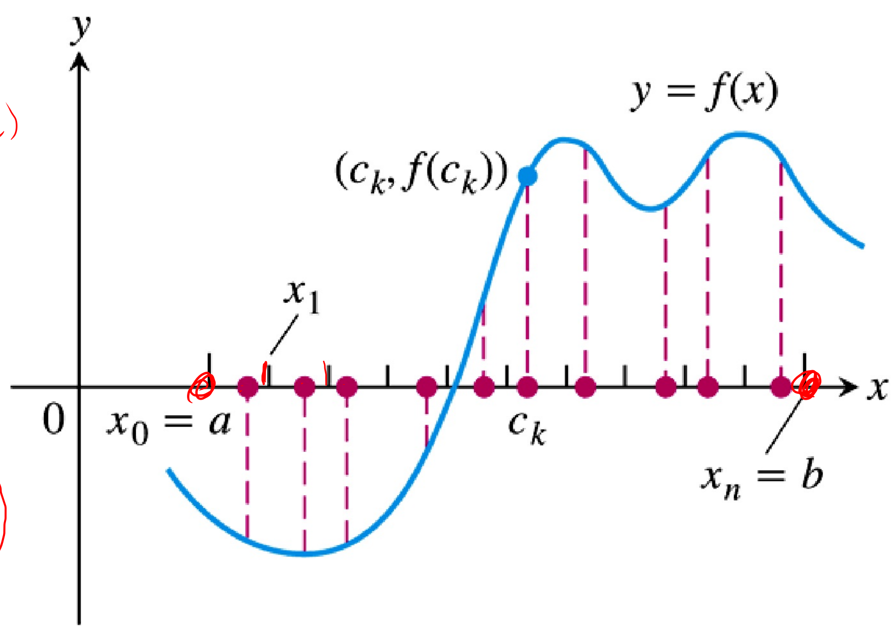
$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b \tag{3}$$

Gemiddelde waarde van een arbitraire continue functie

met $\Delta x = \dfrac{b-a}{n}$ $\rightarrow \dfrac{1}{n} = \dfrac{\Delta x}{(b-a)}$

$$\frac{f(c_1)+f(c_2)+\ldots f(c_n)}{n} =$$

$$\frac{1}{n}\sum_{k=1}^{n} f(c_k) = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\Delta x$$



$\Delta x_k = \dfrac{b-a}{n}$

FIGURE 5.14 A sample of values of a function on an interval $[a, b]$.

Definitie
Als f integreerbaar is op [a,b] dan wordt de gemiddelde waarde van f op het interval [a,b] gedefinieerd als:

$$gem(f) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

De gemiddelde waarde van een positieve functie (f(x)≥0 voor alle x)

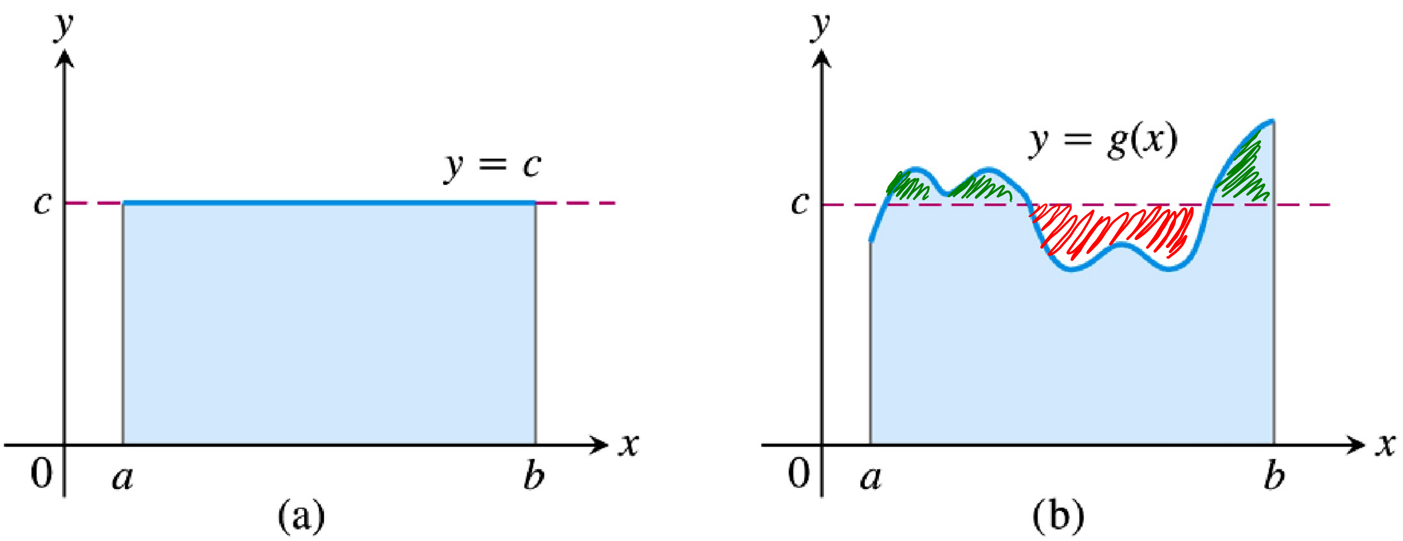


FIGURE 5.5 (a) The average value of $f(x) = c$ on $[a, b]$ is the area of the rectangle divided by $b - a$. (b) The average value of $g(x)$ on $[a, b]$ is the area beneath its graph divided by $b - a$.

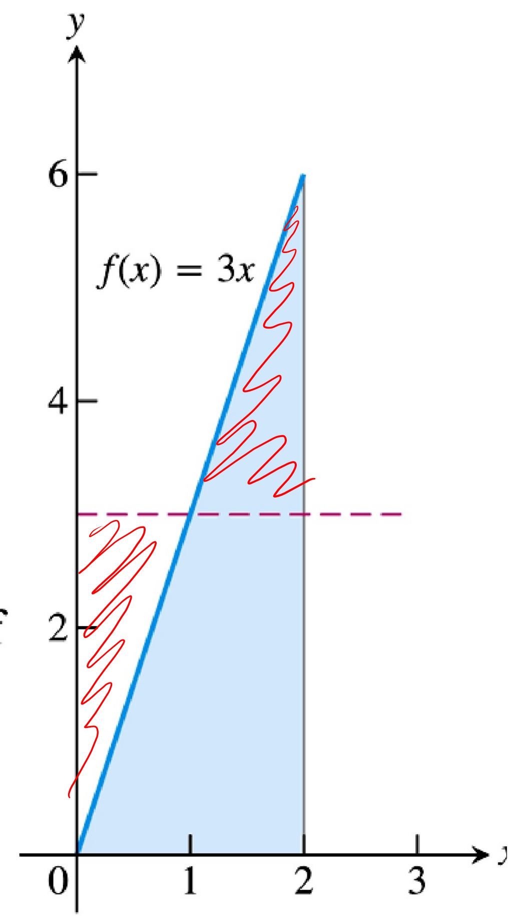


FIGURE 5.6 The average value of $f(x) = 3x$ over $[0, 2]$ is 3 (Example 3).

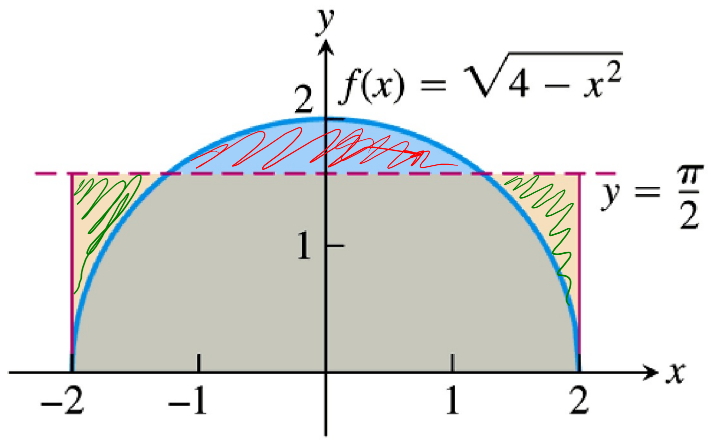


FIGURE 5.15 The average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is $\pi/2$ (Example 5).

$$\frac{1}{4} \int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

Opp v. cirkel $\rightarrow \pi R^2$

blauwe opp $\rightarrow \dfrac{\pi (2)^2}{2}$

$$\text{gem}(f) = \frac{1}{4} \; \frac{4\pi}{2} = \frac{\pi}{2}$$

5.4

Het Fundamentele Theorema van de Calculus

Het middelwaarde theorema voor bepaalde integralen

Indien f continu is op het interval [a,b] dan bestaat er een punt c in dit interval [a,b] zodat:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

Bewijs NIET

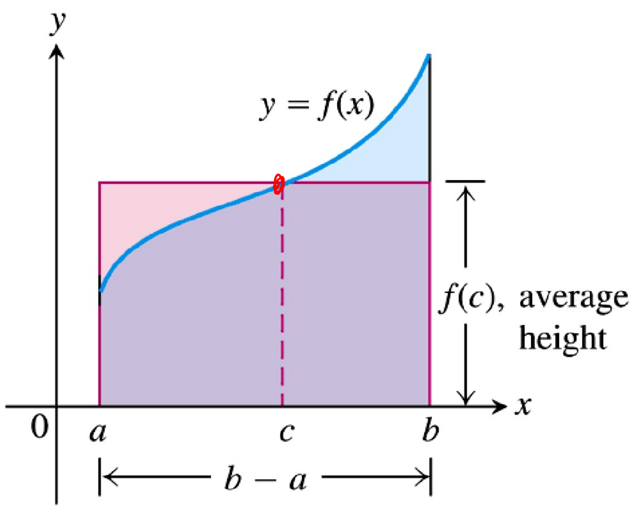


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

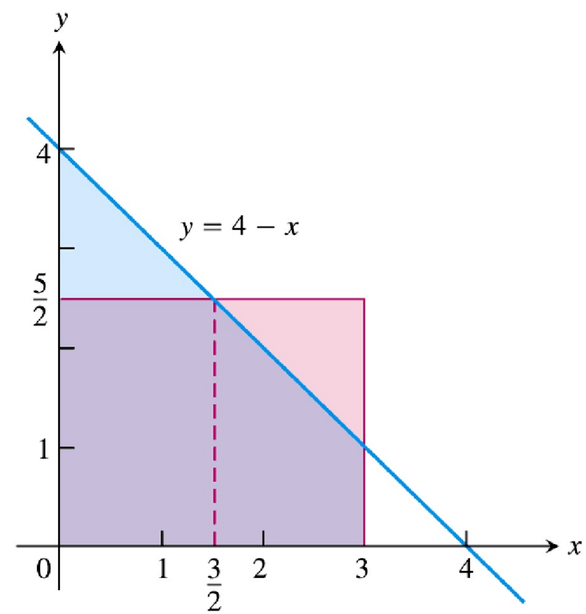


FIGURE 5.18 The area of the rectangle with base [0, 3] and height 5/2 (the average value of the function $f(x) = 4 - x$) is equal to the area between the graph of f and the x-axis from 0 to 3 (Example 1).

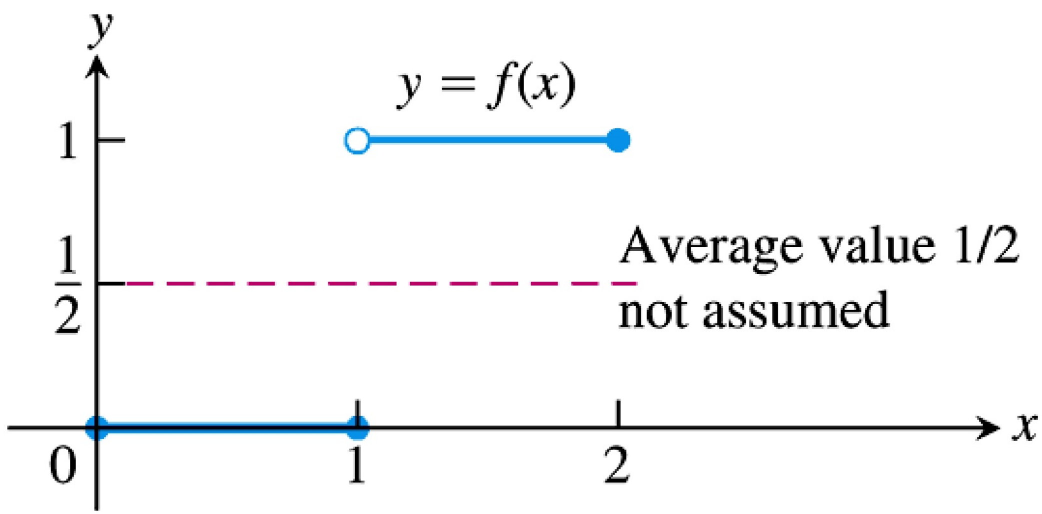


FIGURE 5.17 A discontinuous function need not assume its average value.

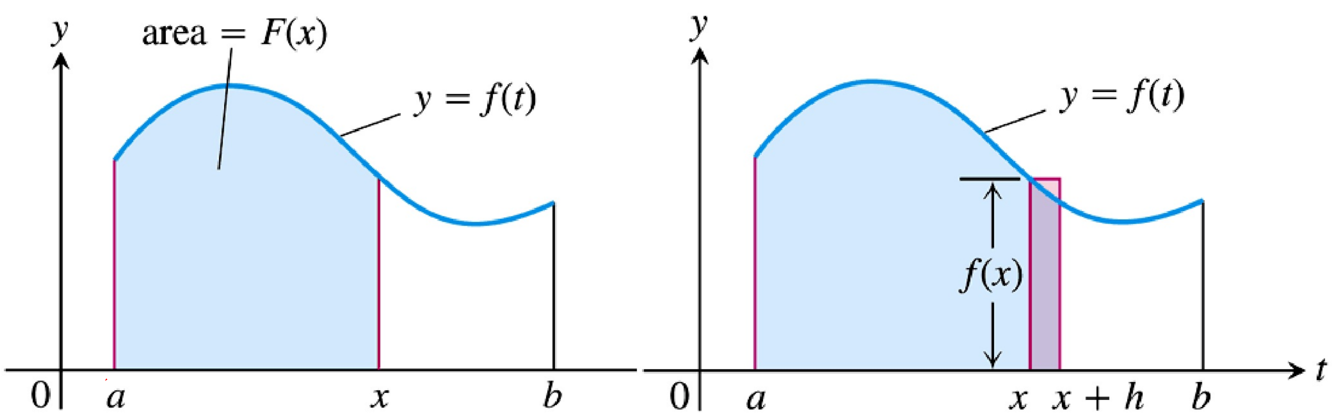
Fundamentele stelling van de calculus (deel 1)
Indien f continu is op [a,b] dan is de functie F(x) gedefinieerd als:

$$F(x) = \int_a^x f(t)\,dt$$

continu en heeft tevens een afgeleide in elk punt van het interval]a,b[gegeven door:

$$\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Bewijs NIET



Fundamentele stelling van de calculus (deel 2)
Zij f continu op [a,b] en zij F een primitieve functie van f op [a,b] dan geldt :

$$\int_a^b f(x)\,dx = F(b) - F(a) = \left[F(x)\right]_a^b = F(x)\Big|_a^b$$

~~Bewijs NIET~~

1 Zoek y' als $y(x)=\int_{1}^{x^2}\cos(t)dt$ Kettingregel !

2 Zoek y' als $y(x)=\int_{x}^{5}3t\sin(t)dt$

2

$$-y(x)=\int_{5}^{x}\underbrace{3t\sin(t)}dt$$

$$\left(-y(x)\right)'=3x\sin x \implies y'(x)=-3x\sin x$$

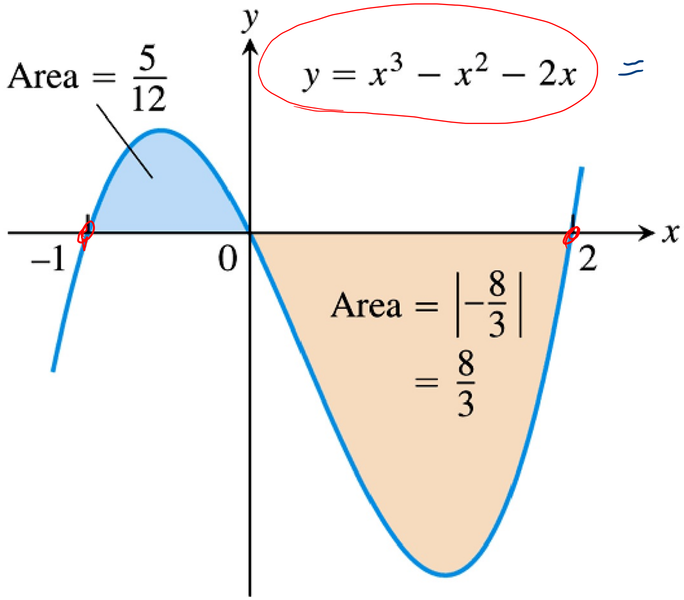
1

$$x \xrightarrow[2x]{} x^2=u \xrightarrow[\cos(u)]{} \int_{1}^{u}\cos(t)\,dt$$

$$y'(x)=2x\cos(x^2)$$

Oppervlaktes berekenen tussen curve van f en X-as over een interval [a,b]:

- zoek nulpunten van f in interval
- verdeel het interval bij de nulpunten
- integreer f over de deelintervallen
- sommeer de absolute waardes van de deelintegralen



$= x(x^2 - x - 2) = x(x-2)(x+1)$

$\int_{-1}^{0} f(x)\,dx$

$\int_{0}^{2} f(x)\,dx$

FIGURE 5.23 The region between the curve $y = x^3 - x^2 - 2x$ and the x-axis (Example 8).

Integratieformules gekend verondersteld (oefeningensessies)!

Partiële integratie (Integration by parts zie ook §8.2)

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx$$

Integratie door substitutie (inverse van kettingregel)

$$\int_a^b f(g(x))g'(x)dx = \int_\alpha^\beta f(u)du \quad \text{met} \quad \begin{cases} \alpha = g(a) \\ \beta = g(b) \end{cases}$$

$$\int_a^b (fg)' dx = \int_a^b f' g\, dx + \int_a^b g' f\, dx$$

$$[fg]_a^b =$$

Vb. $\int_0^{\frac{\pi}{2}} x \cos x \, dx$

$f(x) = x \qquad g'(x) = \cos x$

$f'(x) = 1 \qquad g(x) = \sin x$

$$= x \sin x \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$$

Vb. $\int_0^{\frac{\pi}{2}} x \cos(x^2) dx = \frac{1}{2} \int_0^{\frac{\pi}{4}^2} \cos u \, du = \frac{1}{2} \sin u \Big|_0^{\frac{\pi}{4}^2}$

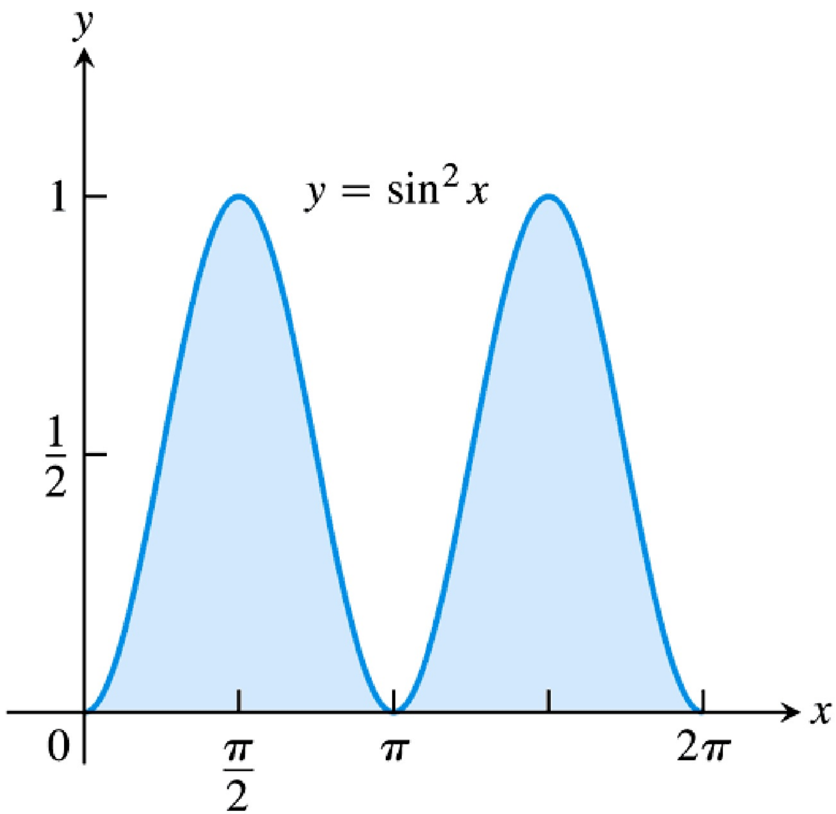
$x^2 = u$

$2x = \frac{du}{dx}$

$2x\,dx = du$

$x\,dx = \frac{1}{2}\,du$

$$= \frac{1}{2} \sin\left(\frac{\pi}{4}^2\right) - 0$$



$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

FIGURE 5.24 The area beneath the curve $y = \sin^2 x$ over $[0, 2\pi]$ equals π square units (Example 8).

Theorem 7

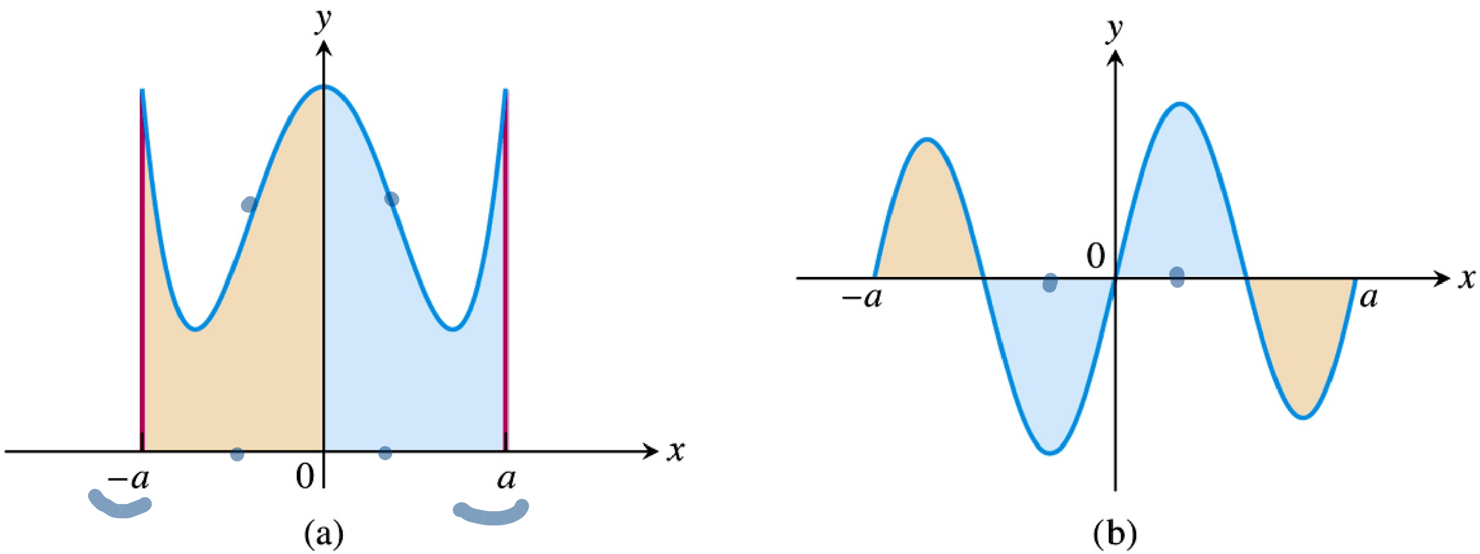
Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

f even $\Longleftrightarrow f(x) = f(-x)$

f oneven $\Longleftrightarrow f(x) = -f(-x)$



Co **FIGURE 5.26** (a) f even, $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ (b) f odd, $\int_{-a}^{a} f(x)\,dx = 0$

5.6

Oppervlakte tussen krommes

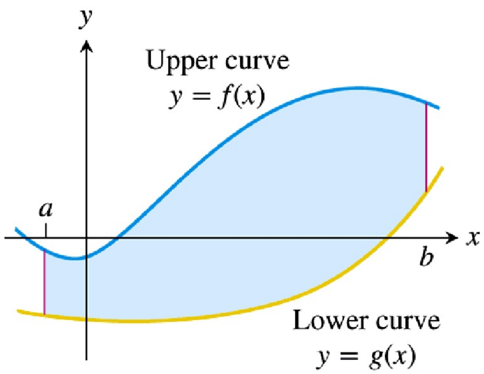


FIGURE 5.27 The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

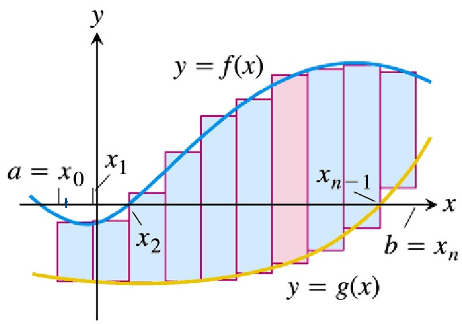
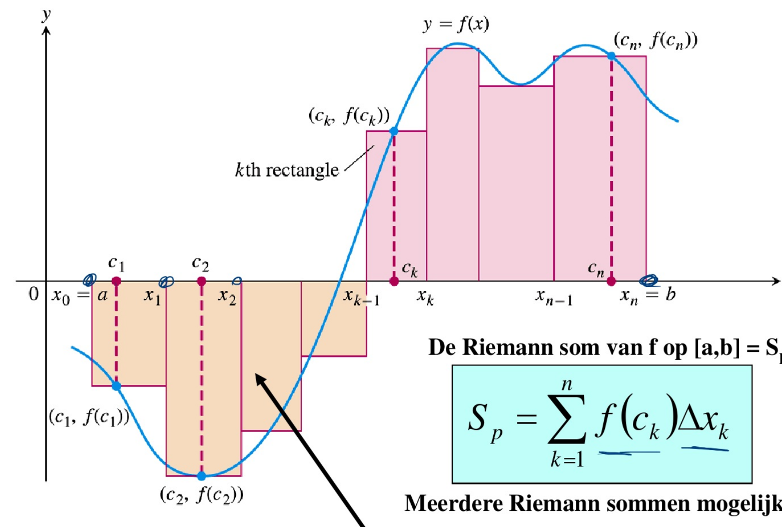


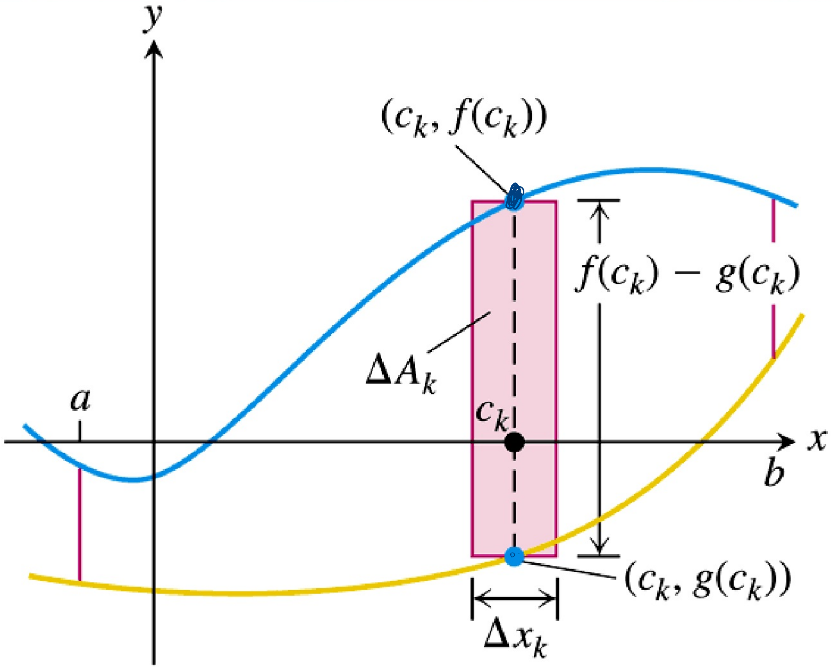
FIGURE 5.28 We approximate the region with rectangles perpendicular to the x-axis.



De Riemann som van f op [a,b] = S_p met

$$S_p = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

Meerdere Riemann sommen mogelijk (P en c_k)



$$\sum_{k=1}^{n} \Delta x_k \left(f(c_k) - g(c_k)\right)$$

$\downarrow \; ||P|| \Rightarrow 0$

integraal

$$\int_a^b \left(f(x) - g(x)\right) dx$$

Oppervlakte tussen krommes

Indien f en g continu zijn met $f(x) \geq g(x)$ op het interval [a,b] dan wordt de oppervlakte tussen beide krommes y=f(x) en y=g(x) tussen a en b gegeven door de integraal van f-g tussen a en b :

$$A = \int_a^b [f(x) - g(x)]dx$$

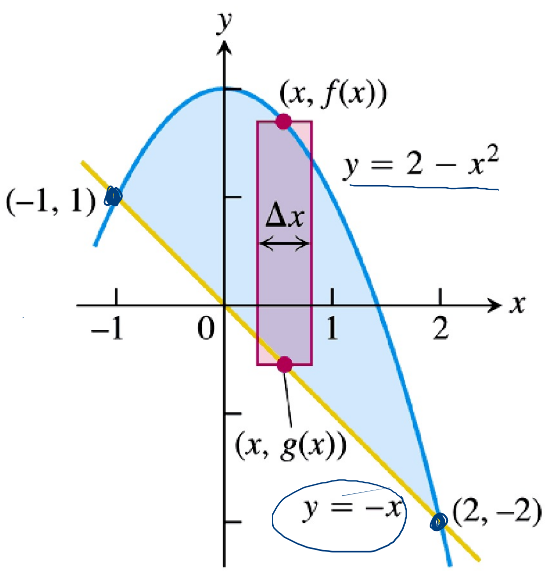


FIGURE 5.30 The region in Example 4 with a typical approximating rectangle.

$Opp = \frac{9}{2}$

1. snypunt

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x_1 = -1 \longrightarrow y_1 = 1$

$x_2 = 2 \longrightarrow y_2 = -2$

2. opp

$$\int_{-1}^{2} \left((2 - x^2) - (-x) \right) dx$$

$$= \int_{-1}^{2} (2 - x^2 + x)\, dx$$

$$= 2x - \frac{x^3}{3} + \frac{x^2}{2} \Big|_{-1}^{2}$$

$$= 4 - \frac{8}{3} + 2 - \left(-2 + 1/3 + \frac{1}{2}\right)$$

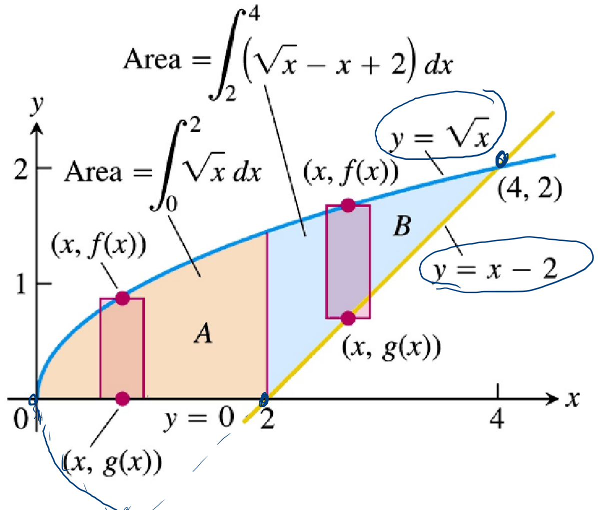


FIGURE 5.31 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 5.

snijpunten

$(0,0)$ en $(0,2)$

$$\sqrt{x} = x - 2$$
$$x = (x-2)^2$$
$$0 = x^2 - 5x + 4$$
$$0 = (x-1)(x-4)$$
$$\cancel{x_1 = 1} \qquad x_2 = 4 \rightarrow y_2 = 2$$

Opp

$$① \int_0^2 \sqrt{x}\, dx = \left.\frac{x^{3/2}}{3/2}\right|_0^2 = \frac{2}{3}\sqrt{8}$$

$$② \int_2^4 (\sqrt{x} - x + 2)\, dx$$
$$= \left.\frac{x^{3/2}}{3/2} - \frac{x^2}{2} + 2x\right|_2^4$$
$$= \frac{16}{3} \cancel{- 8} \cancel{+ 8} - \left(\frac{2}{3}\sqrt{8} - 2 + 4\right)$$

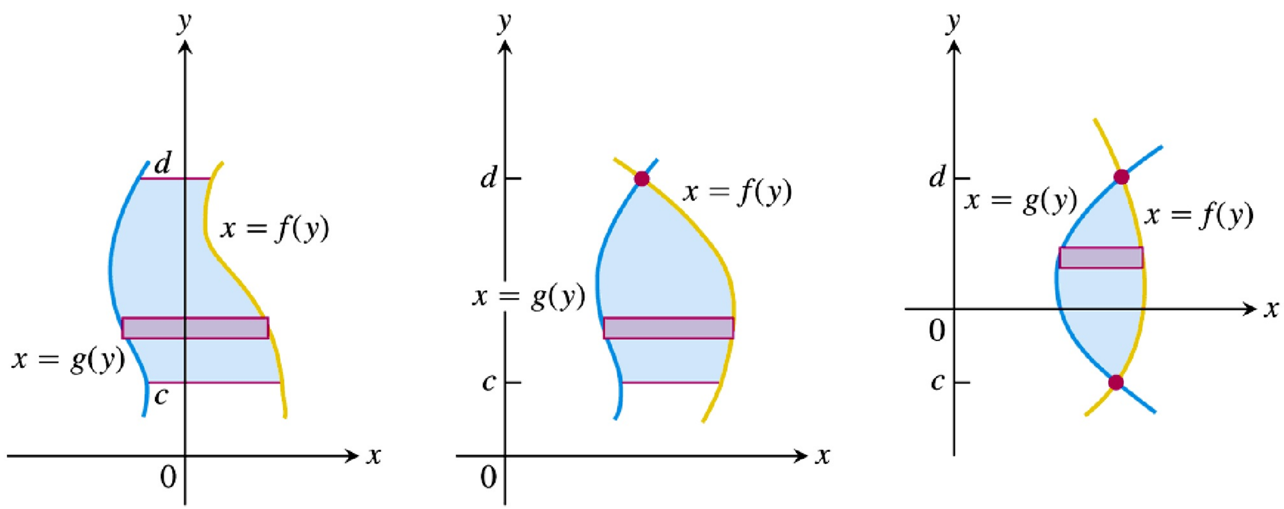
$$\text{Opp} = \frac{10}{3}$$

$$\text{Totale opp} = \frac{16}{3} - 2 = \frac{10}{3}$$

Integration with Respect to y

If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these



use the formula

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

In this equation f always denotes the right-hand curve and g the left-hand curve, so $f(y) - g(y)$ is nonnegative.

$y = \sqrt{x} \rightarrow x = y^2$

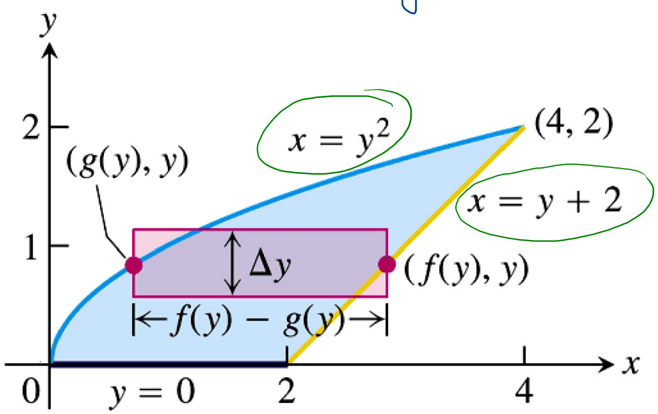


FIGURE 5.32 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 6).

$y = x - 2$
$x = y + 2$

$$\text{Opp} = \int_0^2 \left((y+2) - y^2\right) dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3} \Big|_0^2$$

$$= 2 + 4 - \frac{8}{3} - (0 + 0 + 0)$$

$$= \frac{10}{3}$$

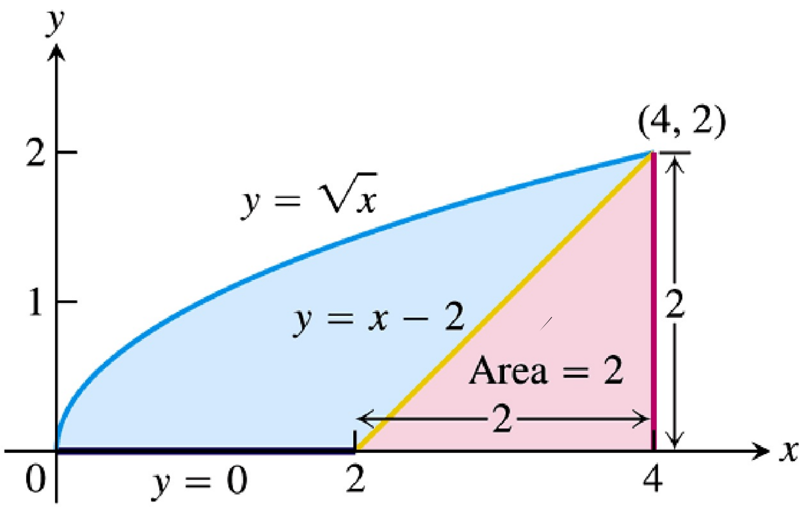


FIGURE 5.33 The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle (Example 7).

$$\int_0^4 \sqrt{x}\,dx - 2$$

$$= \left.\frac{x^{3/2}}{3/2}\right|_0^4 - 2$$

$$= \frac{16}{3} - 2 = \frac{10}{3}$$