# 3.5

# De Kettingregel en Parametervergelijkingen

Hoe leiden we $(g \circ f)$ af , zoals bijvoorbeeld $\cos(x^2+1)$ en $(3x^2+1)^2$ ?

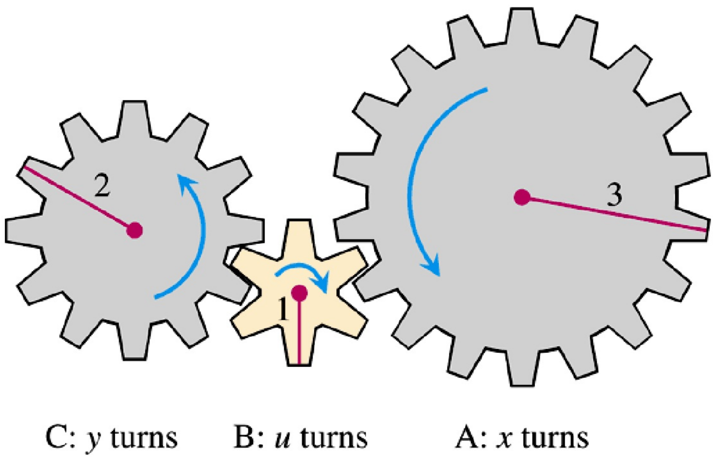



**FIGURE 3.26** When gear A makes $x$ turns, gear B makes $u$ turns and gear C makes $y$ turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

$\cos(x^2+1)$

$x \rightarrow x^2+1 = u \rightarrow \cos u$

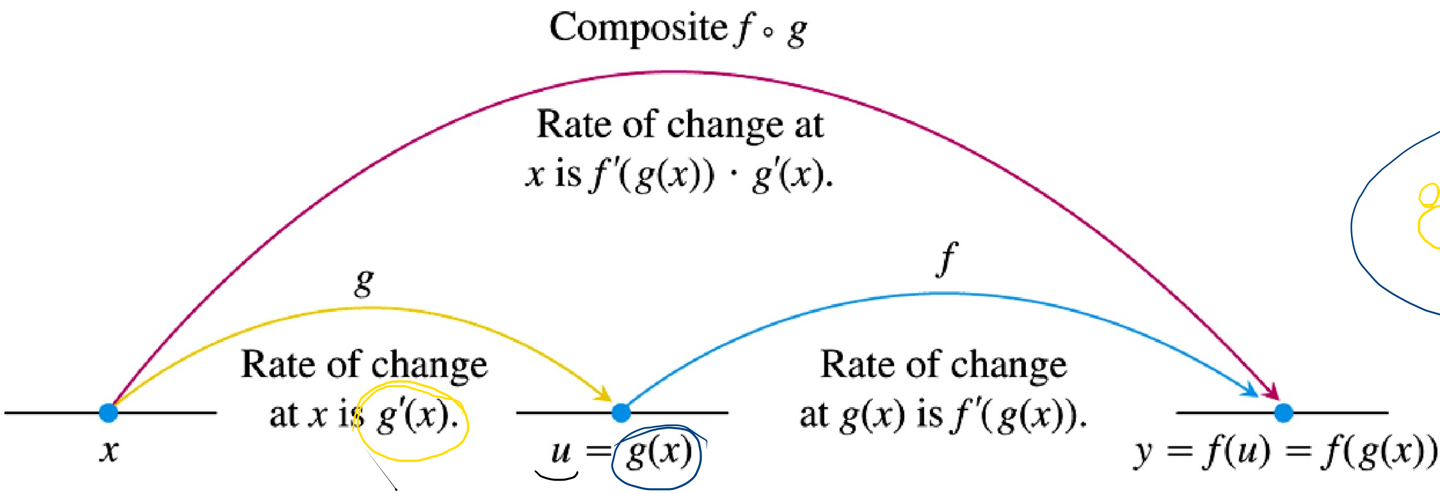
$f(x) = x^2+1$

$g(x) = \cos x$

$g \circ f(x) = \cos(x^2+1)$

$g(f(x)) = \cos(x^2+1)$

# Kettingregel




$g'(x)\, f'(g(x))$

**Stelling**

Indien g(x) differentieerbaar is in x en f(u) differentieerbaar in g(x) dan is de samengestelde functie $y(x) = (f \circ g)(x) = f(g(x))$ differentieerbaar in x:

$$y'(x) = (f \circ g)'(x) = f'(g(x))g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

**Notatie van Leibniz:** $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

**in x**   **in x**   **in u(x)**

**Voorbeelden:**

Outside-inside regel !

$$(\sin(x))^5 = \sin^5 x$$

1. Zoek de raaklijn aan $y=\sin^5 x$ voor $x = \pi/3$ ?

$$x \longrightarrow \sin(x) = u \longrightarrow u^5$$

$$\cos(x) \qquad 5u^4$$

$$(\sin^5 x)' = \cos(x)\, 5 \sin^4 x$$

2. Bereken $\dfrac{d\left(\sqrt{\sin(2x)}\right)}{dx} = 2\cos(2x)\dfrac{1}{2\sqrt{\sin(2x)}} = \dfrac{\cos(2x)}{\sqrt{\sin(2x)}}$

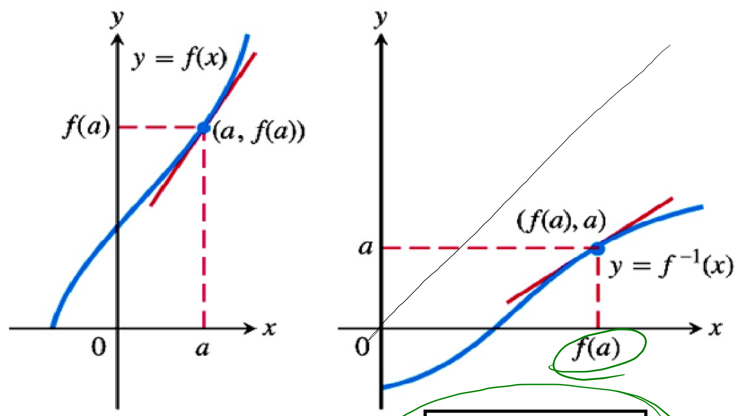
$$x \longrightarrow 2x = u \longrightarrow \sin u = v \longrightarrow \sqrt{v} = v^{1/2}$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\underbrace{2} \qquad \underbrace{\cos u} \qquad \underbrace{\frac{1}{2} v^{-1/2} = \frac{1}{2\sqrt{v}}}$$

**Gevolg: De afgeleide van de inverse functie (Zie ook paragraaf 7.1 in boek)**

$\left.\frac{df}{dx}\right|_a$




The slopes are reciprocal: $\boxed{\left.\frac{df^{-1}}{dx}\right|_{f(a)} = \frac{1}{\left.\frac{df}{dx}\right|_a}}$.

$f \to f^{-1}$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

$f^{-1} \circ f (x) = x$

$(f^{-1} \circ f)' = 1$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & f(a) \arrow[r, bend left, "f^{-1}"] & a \end{tikzcd}$$

$f'(a) \cdot (f^{-1})'_{f(a)} = 1$

$$(f^{-1})'\Big|_{f(a)} = \frac{1}{f'(a)}$$

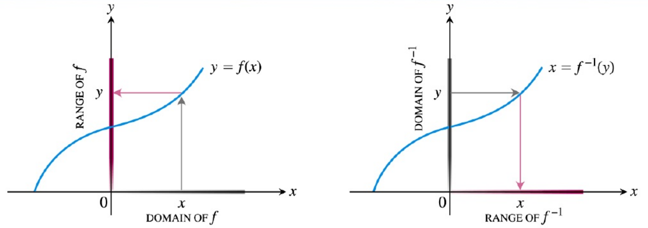
Inverse functie

↓

x en y omwisselen

↓

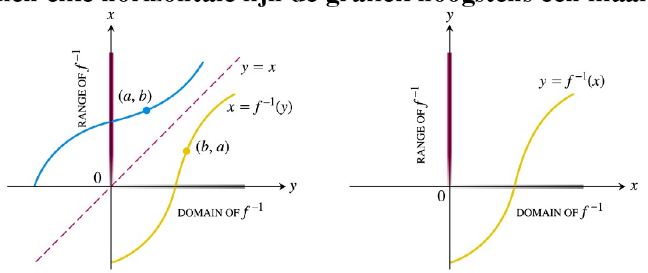
Grafiek spiegelen tov bissectrice van eerste kwadrant




(a) To find the value of $f$ at $x$, we start at $x$, go up to the curve, and then over to the $y$-axis.

(b) The graph of $f$ is already the graph of $f^{-1}$, but with $x$ and $y$ interchanged. To find the $x$ that gave $y$, we start at $y$ and go over to the curve and down to the $x$-axis. The domain of $f^{-1}$ is the range of $f$. The range of $f^{-1}$ is the domain of $f$.

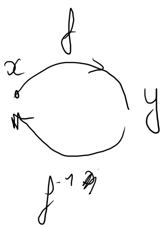
**Een inverse functie bestaat indien elke horizontale lijn de grafiek hoogstens één maal snijdt.**



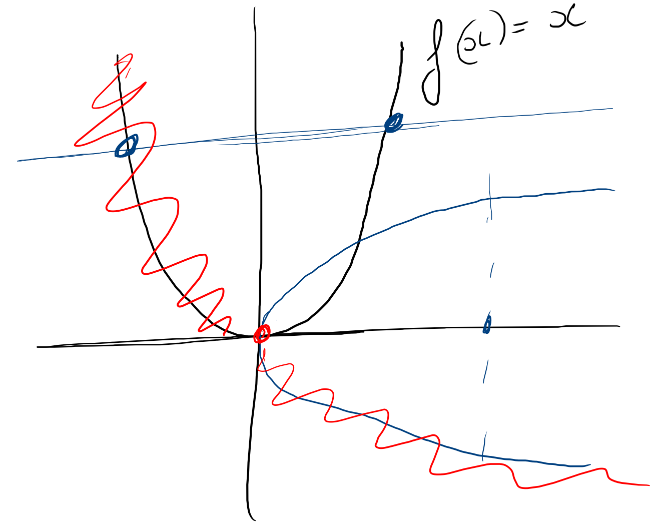

(c) To draw the graph of $f^{-1}$ in the more usual way, we reflect the system in the line $y = x$.

(d) Then we interchange the letters $x$ and $y$. We now have a normal-looking graph of $f^{-1}$ as a function of $x$.

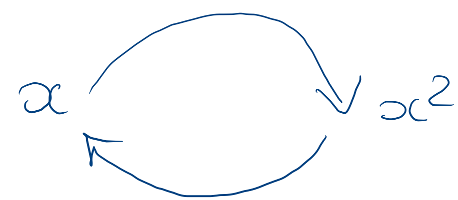
**FIGURE 7.2** Determining the graph of $y = f^{-1}(x)$ from the graph of $y = f(x)$.




$f(x) = x^2$



$f^{-1}(x) = \sqrt{x}$



Voor de functie $y=x^2$ bestaat er enkel een inverse indien we het domein beperken tot $x \geq 0$

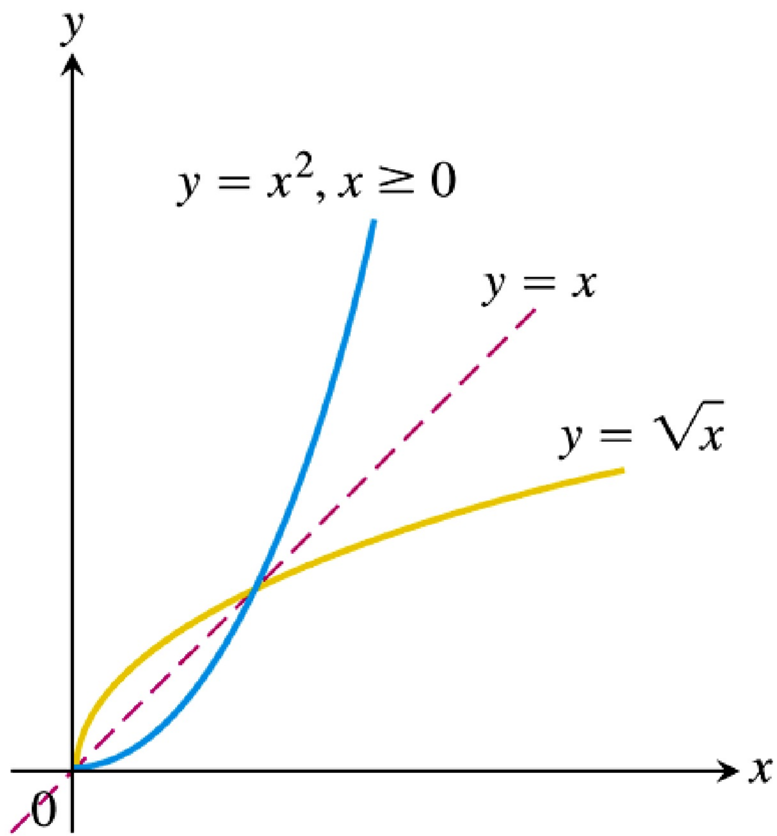



**FIGURE 7.4** The functions $y = \sqrt{x}$ and $y = x^2, x \geq 0$, are inverses of one another (Example 3).

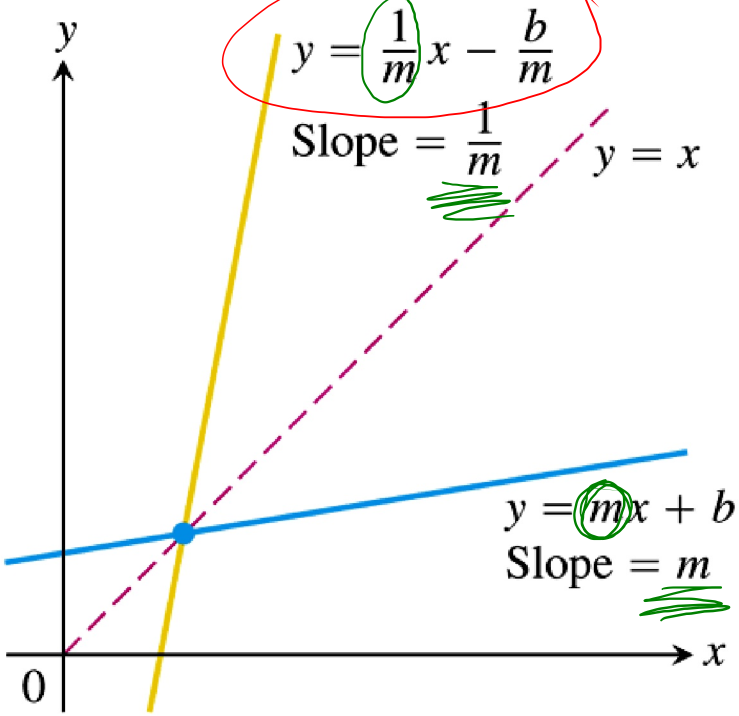
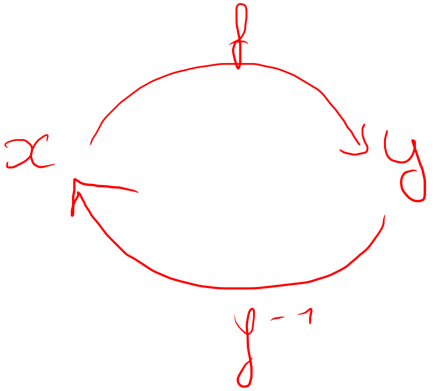



**FIGURE 7.5** The slopes of nonvertical lines reflected through the line $y = x$ are reciprocals of each other.

$y = mx + b$




$mx = y - b$

$x = \frac{1}{m}y - \frac{b}{m}$

Voorbeeld van afgeleide van een inverse functie

$y'(2) = 4$

$\left(f^{-1}\right)'\Big|_{f(2)} = \dfrac{1}{y'(2)}$

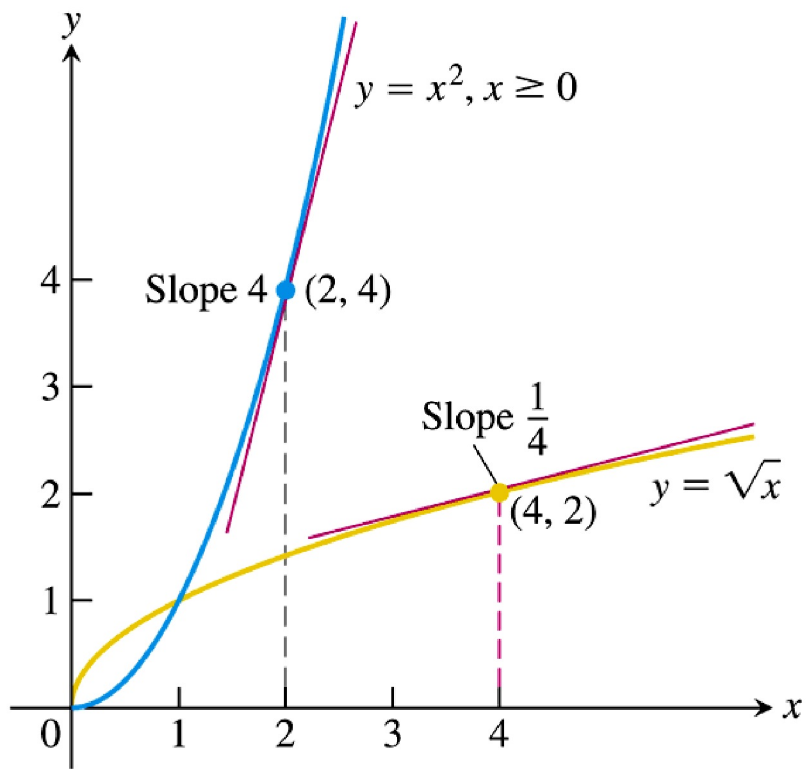



**FIGURE 7.7** The derivative of $f^{-1}(x) = \sqrt{x}$ at the point (4, 2) is the reciprocal of the derivative of $f(x) = x^2$ at (2, 4) (Example 4).

De raaklijn van de functie $f(x)=x^2$ in het punt (2,4) met rico 4 wordt na spiegeling de raaklijn van de inverse functie in het punt (4,2) met rico ¼.

Dit bevestigt de regel:

$$\boxed{\left.\frac{df^{-1}}{dx}\right|_{f(a)} = \frac{1}{\left.\frac{df}{dx}\right|_{a}}.}$$

waarbij a=2 en f(a)=4

Vb:

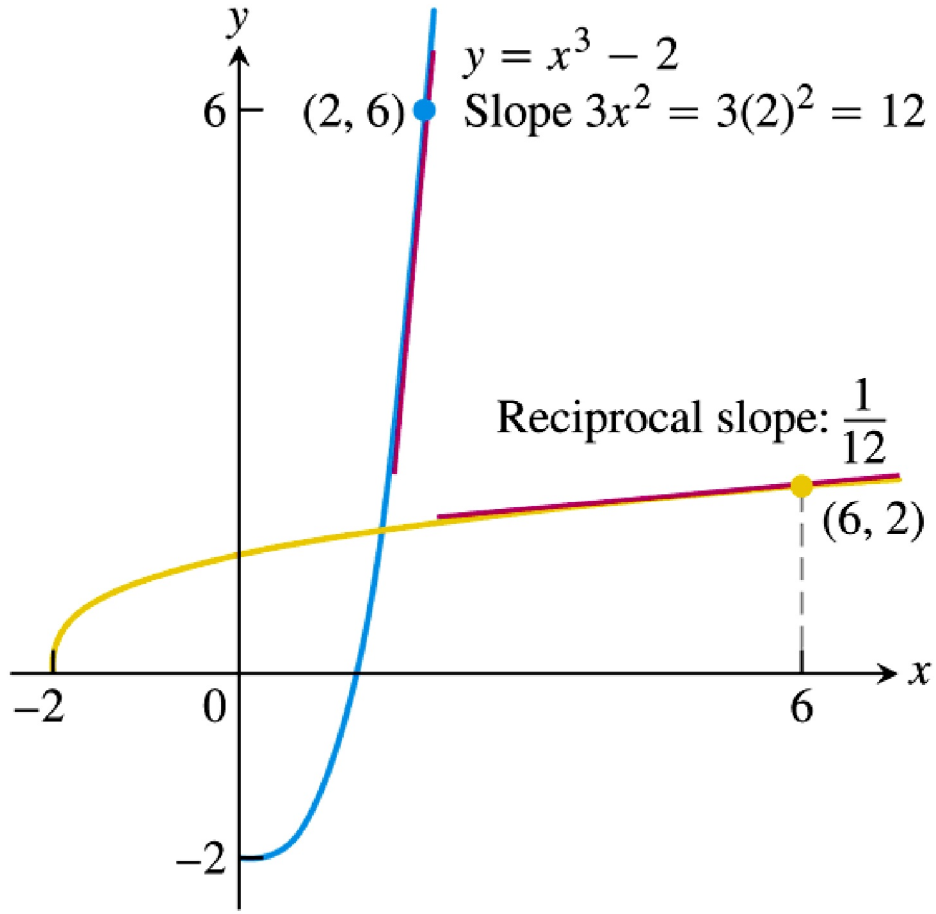



**FIGURE 7.8** The derivative of $f(x) = x^3 - 2$ at $x = 2$ tells us the derivative of $f^{-1}$ at $x = 6$ (Example 5).

**!!! De trigonometrische formules voor afgeleiden gelden enkel indien we in radialen werken !!!!**

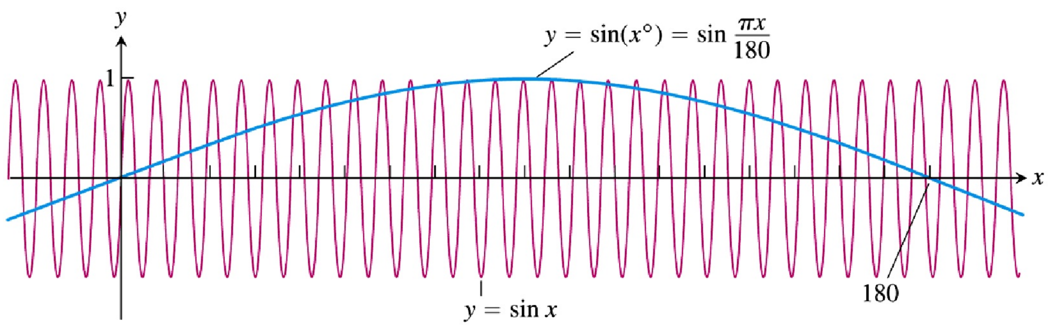



**FIGURE 3.28** Sin $(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 8).

Helling van grafiek
Veel minder steil

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°)$$





$$180° \rightarrow \pi \text{ rad}$$
$$1° \rightarrow \frac{\pi}{180} \text{ rad}$$
$$x° \rightarrow \frac{\pi x}{180} \text{ rad}$$

$$x \rightarrow \frac{\pi x}{180} = u \rightarrow \sin u$$
$$\downarrow \qquad\qquad\qquad \downarrow$$
$$\frac{\pi}{180} \qquad\qquad\qquad \cos u$$

**Parametervoorstelling van een 2-dimensionale kromme:**

**x en y zijn functies van een parameter t:**

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases} \quad \text{voor} \quad a \le t \le b$$

**De punten (x,y)=(f(t),g(t)) vormen de kromme.**

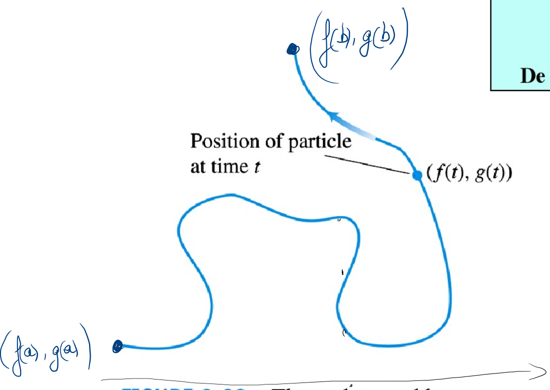
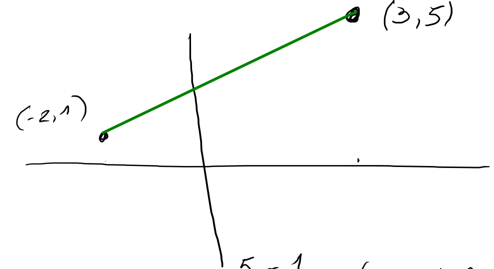



**FIGURE 3.29** The path traced by a particle moving in the $xy$-plane is not always the graph of a function of $x$ or a function of $y$.

(f(a),g(a)) is het startpunt en
(f(b),g(b)) is het eindpunt
van de kromme

Met een parametervergelijking kun je krommes beschrijven die geen functies zijn

Voorbeeld: Zoek het lijnsegment dat de punten (-2,1) en (3,5) verbindt




$$y - 1 = \frac{5-1}{3-(-2)}(x+2)$$

$$y - 1 = \frac{4}{5}(x+2)$$

(1) $\dfrac{y-1}{4} = \dfrac{x+2}{5} = t$

$$\begin{cases} x = 5t - 2 \\ y = 4t + 1 \end{cases} \quad 0 \le t \le 1$$

(2) $y = \dfrac{4}{5}x + \dfrac{13}{5}$

$$\begin{cases} x = t \\ y = \dfrac{4}{5}t + \dfrac{13}{5} \end{cases} \quad -2 \le t \le 3$$

**Parametervoorstelling van een cirkel met straal 1**

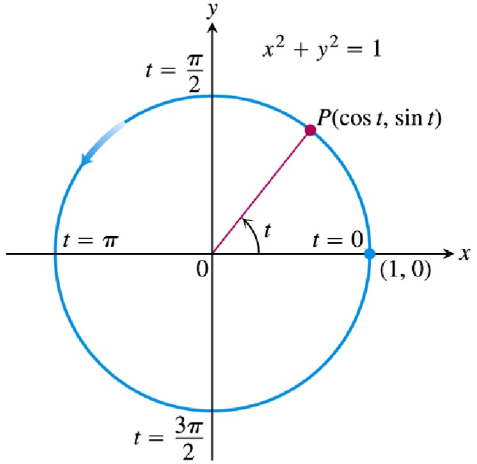



**FIGURE 3.30** The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing $t$ (Example 9).



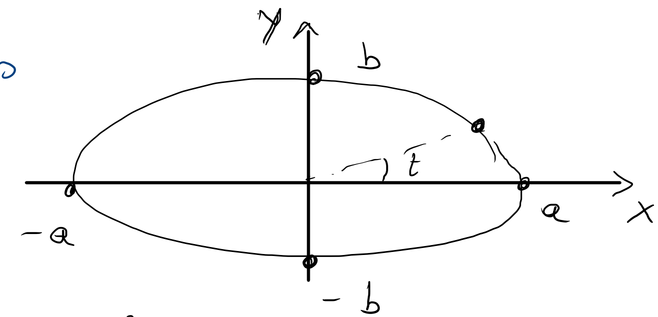


cirkel met straal a

$$\begin{cases} x(t) = a\cos(t) \\ y(t) = a\sin(t) \end{cases}$$

$$0 \leq t < 2\pi$$

ellips




$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\begin{cases} x(t) = a\cos t \\ y(t) = b\sin t \end{cases} \qquad 0 \leq t < 2\pi$$

## Parametervoorstelling van de parabool $y=x^2$ voor positieve x-waarden

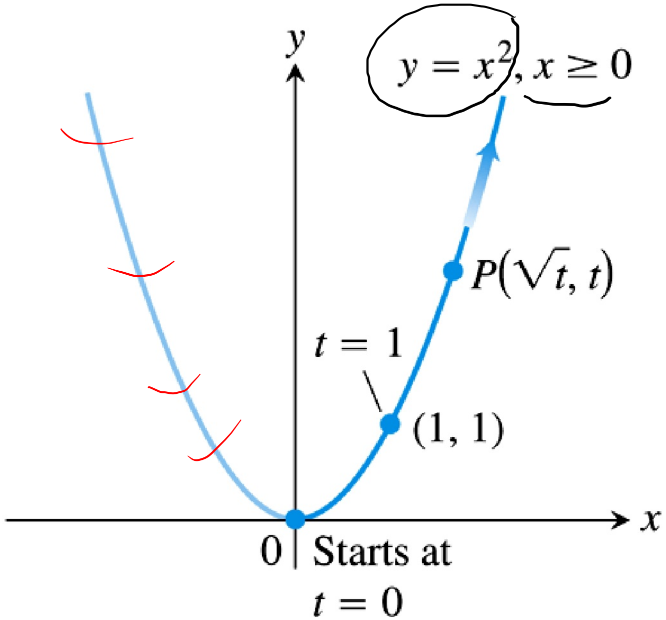



**FIGURE 3.31** The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 10).

$$\begin{cases} x(t) = \sqrt{t} \\ y(t) = t \end{cases} \quad t \geq 0$$

of

$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \quad t \geq 0$$

**! Interval van t bij 2 verschillende parametervoorstellingen van dezelfde kromme kan verschillen !** → Vb van lijnstuk

## Helling van de raaklijn aan een parametrische kromme

**Parametric Formula for $dy/dx$**

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (2)$$

~~Bewijs: Kettingregel~~ $\cancel{\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}}$

**Voorbeeld: parabool**

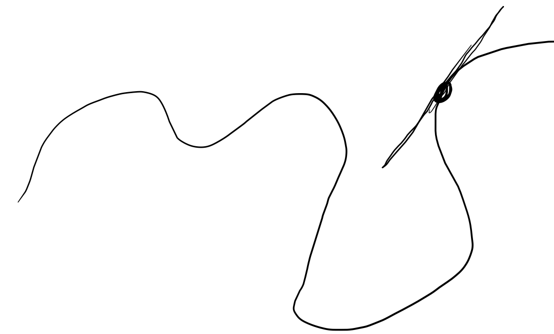
$$\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases} \longrightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{1} = 2t$$

**Parametric Formula for $d^2y/dx^2$**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

**NB:** y' uitrekenen in functie van t mbv 1ste formule

**Vb: parabool** $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{1} = 2t \longrightarrow \frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{2}{1} = 2$



$\frac{dy}{dx}$

$\begin{cases} x(t) \\ y(t) \end{cases}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

**Parametric Formula for $d^2y/dx^2$**

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{\dfrac{dy'}{dt}}{\dfrac{dx}{dt}}$$

**NB:** y' uitrekenen in functie van t mbv 1ste formule

**Vb: parabool**

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{2t}{1} = 2t \longrightarrow \frac{d^2y}{dx^2} = \frac{\dfrac{dy'}{dt}}{\dfrac{dx}{dt}} = \frac{2}{1} = 2$$

1) **Waar landt de cargo ?**
2) **Geef een cartesische vergelijking van de baan van de cargo.**
3) **Hoe snel daalt de hoogte met x bij landing ?**

$$\begin{cases} x = 120t \\ y = -16t^2 + 500 \end{cases}$$

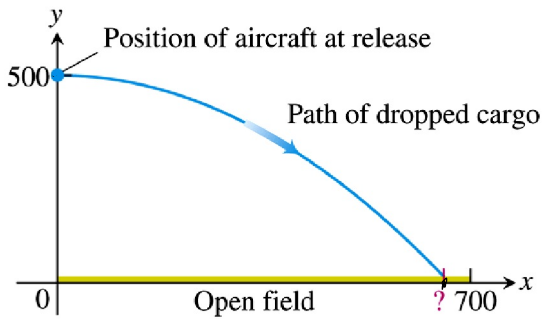



**FIGURE 3.32** The path of the dropped cargo of supplies in Example 15.

NB : in "feet"





1) $y(t_e) = 0$

$-16 t_e^2 + 500 = 0$

$t_e^2 = \frac{500}{16} = \frac{125}{4}$

$t_e = \sqrt{\frac{125}{4}} = \frac{\sqrt{125}}{2}$

$$\boxed{x(t_e) = 120 \times \frac{\sqrt{125}}{2}}$$

2) $t = \frac{x}{120}$

$y = -16 \frac{x^2}{120^2} + 500$

3) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-32t}{120}$

$\left.\frac{dy}{dx}\right|_{t_e} = -\frac{32}{120} t_e$

## Standard Parametrizations and Derivative Rules

CIRCLE $x^2 + y^2 = a^2$:

$$x = a\cos t$$
$$y = a\sin t$$
$$0 \le t \le 2\pi$$

ELLIPSE $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$:

$$x = a\cos t$$
$$y = b\sin t$$
$$0 \le t \le 2\pi$$

FUNCTION $y = f(x)$:

$$\begin{cases} x = t \\ y = f(t) \end{cases}$$

DERIVATIVES

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

# 3.6

## Impliciete Afgeleiden

**Tot nu toe: afgeleide van functie gedefinieerd indien functie gegeven wordt als**

- **$y = f(x)$**
- **Een parametervergelijking** $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

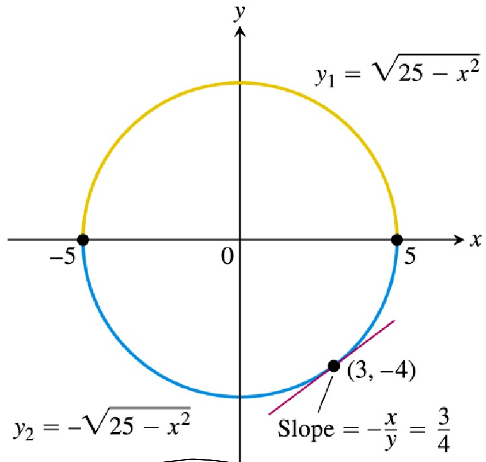
↕

**Nu : functie impliciet gedefinieerd**
**bv: $x^2 + y^2 = 25$**

**Afgeleide van zo'n functie in een punt ?**

↓

-Differentieer linker- en rechterlid naar x waarbij je y als een differentieerbare functie naar x beschouwt
- Los de bekomen uitdrukking op naar dy/dx






$$x^2 + y^2 = 25$$

$$(y(x))^2$$

$$x \rightarrow y(x) = u \rightarrow u^2$$

$$y' \qquad 2u$$

1. Samenstelling van 2 functies $y_1$ en $y_2$

$$y_1' = \frac{-\cancel{2}x}{\cancel{2}\sqrt{25-x^2}} = \frac{-x}{y_1}$$

$$y_2' = \frac{\cancel{2}x}{\cancel{2}\sqrt{25-x^2}} = \frac{x}{-y_2}$$

2. parametervergelijking

$$\begin{cases} x(t) = 5\cos t \\ y(t) = 5\sin t \end{cases}$$

$$\frac{dy}{dx} = \frac{5\cos t}{-5\sin t} = \frac{x}{-y}$$

3. impliciet afleiden

$$\cancel{2}x + \cancel{2}y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

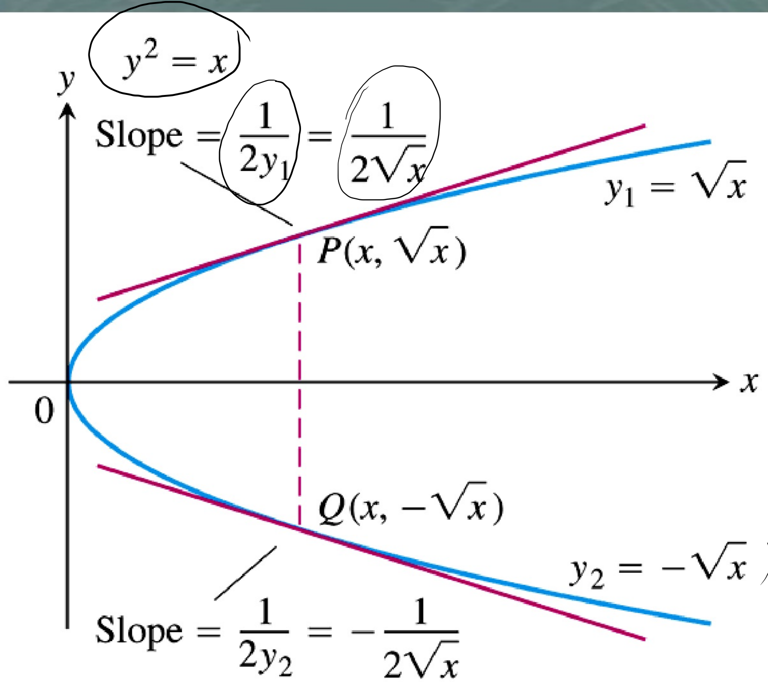



**FIGURE 3.37** The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of $x$ on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for $y$.

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

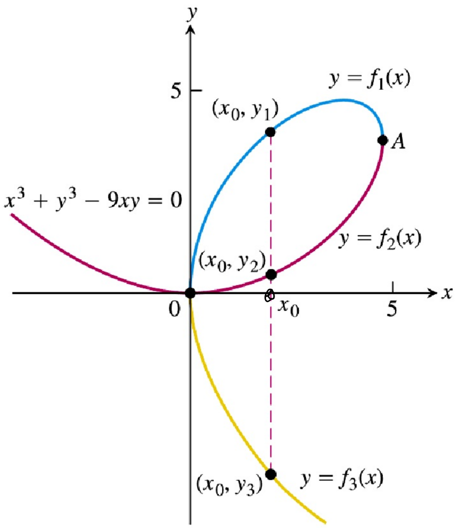



**FIGURE 3.38** The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of $x$. The curve can, however, be divided into separate arcs that *are* the graphs of functions of $x$. This particular curve, called a *folium*, dates to Descartes in 1638.

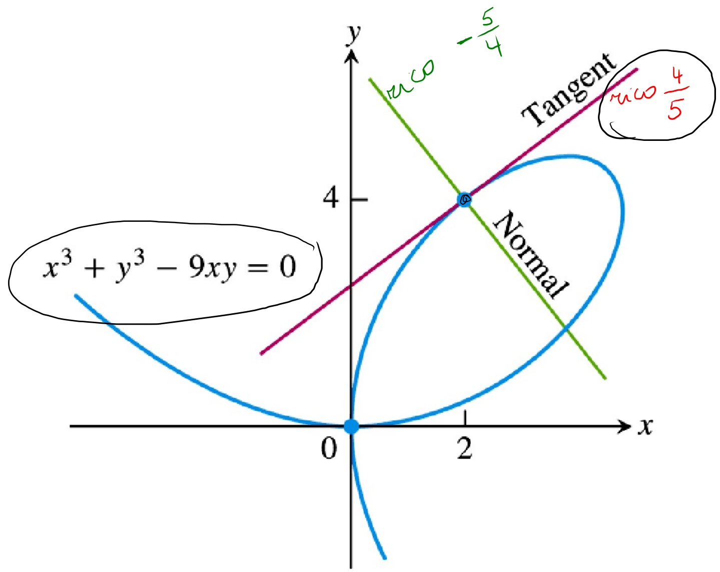



**FIGURE 3.41** Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

$$x^3 + y^3 - 9xy = 0$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(y + x\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx}\left(3y^2 - 9x\right) = -3x^2 + 9y$$

$$\frac{dy}{dx} = \frac{\overset{3}{\cancel{9}}y - \cancel{3}x^2}{\cancel{3}y^2 - \underset{3}{\cancel{9}}x}$$

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{12-4}{16-6} = \frac{8}{10} = \frac{4}{5}$$

$$-\frac{1}{\frac{4}{5}} = -\frac{5}{4}$$

**Bereken de helling van de onderstaande grafiek in elk punt (x,y)**

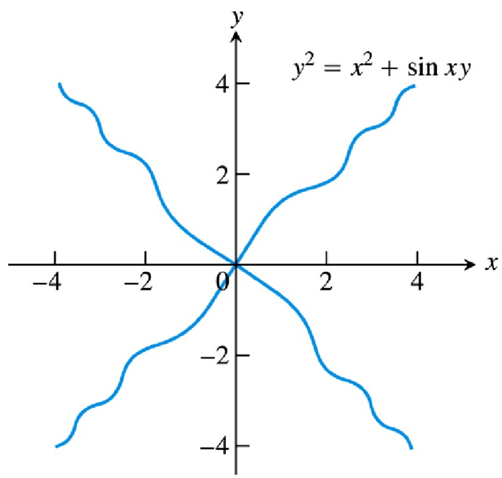



**FIGURE 3.39** The graph of $y^2 = x^2 + \sin xy$ in Example 3. The example shows how to find slopes on this implicitly defined curve.

$$y^2 = x^2 + \sin(xy)$$

$$2y\frac{dy}{dx} = 2x + \cos(xy)\left(y + x\frac{dy}{dx}\right)$$

$$\left(2y - \cos(xy)\,x\right)\frac{dy}{dx} = 2x + y\cos(xy)$$

$$\frac{dy}{dx} =$$

**Analoog: Hogere orde afgeleide**

Vb. $2x^3 - 3y^2 = 8 \longrightarrow y' = \frac{x^2}{y}$ *en* $y'' = \frac{2x}{y} - \frac{x^4}{y^3}$

Via impliciet afleiden kan men de onderstaande uitbreiding van de reeds gekende formule bewijzen

**THEOREM 4 Power Rule for Rational Powers**

If $p/q$ is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

Bewijs in de les





$$2x^3 - 3y^2 = 8$$

$$6x^2 - 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x^2}{y}$$

$$\frac{d^2y}{dx^2} = \frac{2xy - x^2\frac{dy}{dx}}{y^2} = \frac{2xy - x^2\frac{x^2}{y}}{y^2} = \frac{2x}{y} - \frac{x^4}{y^3}$$

$$(y)^q = \left(x^{p/q}\right)^q = x^p$$

$$y^q = x^p$$

$$q y^{q-1}\frac{dy}{dx} = p x^{p-1}$$

$$\frac{dy}{dx} = \frac{p}{q}x^{p-1}y^{1-q}$$

$$= \frac{p}{q}x^{p-1}\left(x^{\frac{p}{q}}\right)^{1-q}$$

$$= \frac{p}{q}x^{\frac{(p-1)q + p(1-q)}{q}}$$

$$= \frac{p}{q}x^{\frac{p-q}{q}}$$

$$= \frac{p}{q}x^{\frac{p}{q}-1}$$

# 3.7

## Vraagstukken met snelheden

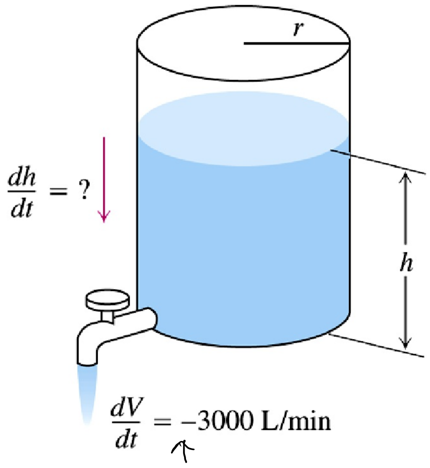



**FIGURE 3.42** The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

$m^2$ $\quad$ $m$

$$V = \pi R^2 h \qquad 1000 \frac{l}{m^3}$$

$$-3000 \frac{l}{min} = \frac{dV}{dt} = \pi R^2 \frac{dh}{dt} \, 1000 \frac{l}{m^3}$$

$$\frac{dh}{dt} = \frac{-3000 \frac{l}{min}}{\pi R^2 \, 1000 \frac{l}{m^3}}$$

$$= \frac{-3}{\pi R^2} \frac{m^3}{min}$$

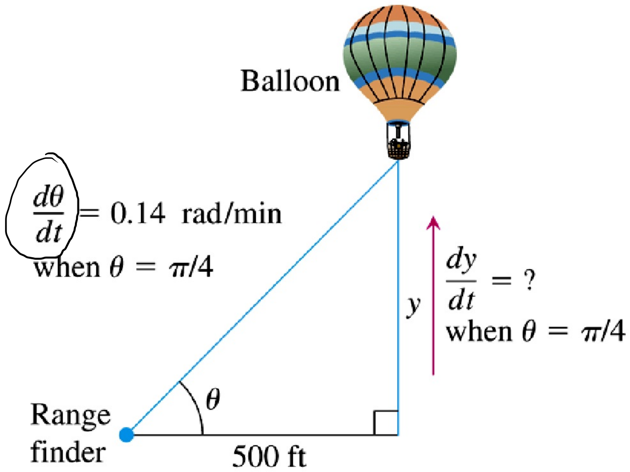



**FIGURE 3.43** The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

$$\operatorname{tg}\theta = \frac{y}{500}$$

$$\frac{\frac{d\theta}{dt}}{\cos^2\theta} = \frac{1}{500}\frac{dy}{dt}$$

$$0.14\ \frac{\text{rad}}{\text{min}}$$

$$\left.\frac{dy}{dt}\right|_{\theta=\frac{\pi}{4}} = \frac{500\,\text{ft}}{\cos^2\left(\frac{\pi}{4}\right)}$$

$$= 1000\ \text{ft} \times 0.14\ /\text{min}$$

$$= 140\ \frac{\text{ft}}{\text{min}}$$

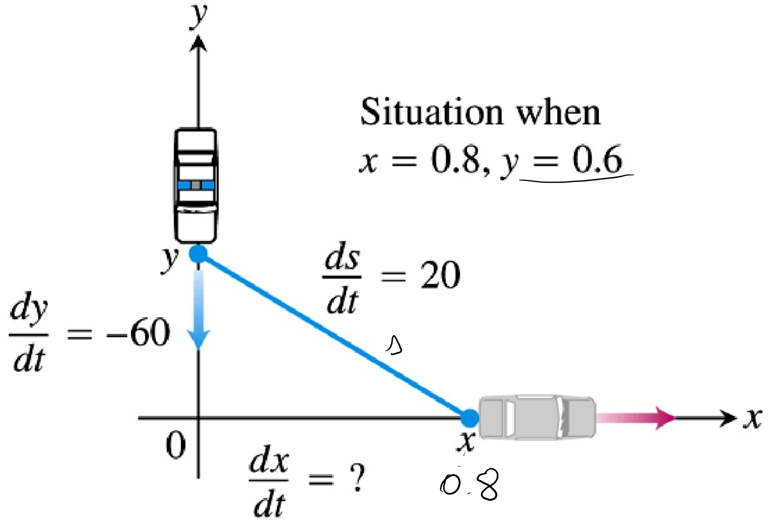



**FIGURE 3.44** The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

$$x^2 + y^2 = s^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$$

**NB: 3.8: Differentialen NIET !!!**