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11/17/2013

Tetrimino Rotation Proof

Premise:

In order to succinctly describe the shape of a Tetrimino piece, Tetrocity's Shape class stores only a list of relative matrix-coordinates, where each nonzero entry indicates the presence of a Tetrimino block (the individual constituents of a whole Tetrimino piece) at that coordinate position. The matrix of a Shape is simply the smallest possible matrix that can describe the Tetrimino shape. Consider a particular orientation of an "L-piece" Tetrimino. The following shows how it is described using this scheme:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Note that this is the smallest possible matrix capable of describing that shape. The relative-matrix coordinates that correspond to this shape are: $\{(0,0),(1,0),(2,0),(2,1)\}$, and we say that the shape has a "height" of 3. I will refer to the height of a Shape as $mHeight$, as this is the variable name used in the Tetrocity source code. It is important to note that a Shape object does not store that matrix, but only the list of coordinates.

We now make two important observations that will be used in this proof: First, there is always a block located in the 0^{th} column. If there weren't, the matrix would not be as small as possible. By identical reasoning, there is always a block located in the $(mHeight-1)^{th}$ row.

Because Tetrimino rotation is a central feature of Tetrocity, it is critical that it is done as quickly as possible such that the controls remain responsive, and multiple rotations can be performed in quick succession. In order to rotate a Tetrimino, we must not only map each block to a new coordinate, but ensure that the resulting piece still adheres the minimum-size matrix principle we've discussed. In our example above, a clockwise rotation would look as follows:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Thorem:

Given a list of block matrix-coordinates $\{(r_0,c_0),(r_1,c_1)...\}$, a $90°$ clockwise Tetrimino rotation can be accomplished by applying the following transformation to each coordinate:

$$(r_i,\ c_i) \rightarrow (c_i,\ mHeight-1-r_i)$$

Proof:

First consider some shape-matrix that allows for negative indices, such that any block rotated to such a negative index is still in the matrix. Then, consider the rotation to be about some point (r^*,c^*). We will see that this specific point does not matter, but if it suits the reader, let that point be the center of mass of the Tetrimino. Also consider some block with matrix-coordinate point (r,c). For simplicity, we define the x-axis distance between these two points as $x=c^*-c$, and the y-axis distance $y = r^* - r$. These are not magnitudes. $x < 0$ implies the block is to the right of the rotational point.

If we consider the rotation as rotating the entire matrix 90° clockwise, it's self-evident to see the rotation accomplishes the following simple transformation:

$$
\begin{gathered}
r \to r^* - x \\
c \to c^* + y
\end{gathered}
$$

However, this transformation does not guarantee that the desired matrix properties are preserved. First, the coordinates may have negative values, which is not allowed. Secondly, the matrix may not be as small as possible. In order to fix this, we must find the index of the smallest row, and subtract that from each coordinate's row. Then, we must find the index of the smallest column, and subtract that from each coordinate's column. Then, each coordinate will be representative of a smallest-possible matrix. The full transformation is:

$$
\begin{gathered}
r \to r^* - x - (\text{Smallest row after rotation}) \\
c \to c^* + y - (\text{Smallest column after rotation})
\end{gathered}
$$

<u>Lemma:</u>

After the transformation applied above, the following is true in all cases:

$$
\begin{gathered}
\text{Topmost row} = r^* - c^* \\
\text{Leftmost column} = c^* - (mHeight - 1 - r^*)
\end{gathered}
$$

Proof:

To see this, we will now use the two observations made earlier in this paper. Namely, that there is always a block in column 0 and another in row $mHeight - 1$. It's clear that before the rotation, the leftmost block will become the topmost after the rotation. More specifically, the horizontal distance between the leftmost block and the rotational point before the rotation is equal to the vertical distance between the topmost block and rotational point after the rotation. However, since there is always a block in the 0 column, the largest initial horizontal distance and hence largest final vertical distance will always be $c^* - 0 = c^*$. The final row coordinate of this block will then be $r^* - c^*$.

To prove the second relationship, we apply identical logic, noting that the bottom-most block, which has row $mHeight - 1$, before the rotation will always become the leftmost block after the rotation. The initial vertical distance between that block and rotational coordinate is $(mHeight - 1 - r^*)$. Thus, the column coordinate of that block after the rotation is $= c^* - (mHeight - 1 - r^*)$. □

We can now derive our final result. Plugging in the results of the lemma:

$$
\begin{gathered}
r \to r^* - x - (r^* - c^*) = r^* - (c^* - c) - (r^* - c^*) = c \\
c \to c^* + y - (c^* - (mHeight - 1 - r^*)) = c^* + (r^* - r) - (c^* - (mHeight - 1 - r^*)) = mHeight - 1 - r
\end{gathered}
$$

We consider it self evident that the full rotation can be performed by applying this transformation to each block coordinate individually. □

Thorem:

Given a list of block matrix-coordinates $\{(r_0, c_0), (r_1, c_1)...\}$, a 90° counter-clockwise Tetrimino rotation can be accomplished by applying the following transformation to each coordinate:

$$(r_i,\ c_i) \to (mHeight - 1 - c_i,\ r_i)$$

Proof:

We simply reverse the clockwise transformation:

$$r_i \leftarrow c_i \Rightarrow c_i \to r_i$$
$$c_i \leftarrow mHeight - 1 - r_i \Rightarrow r_i \to mHeight - 1 - c_i \ \square$$