Robust cooperative output regulation for multi-agent systems with long distributed input and communication delays

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***Abstract*—**

***Index Terms*—Multi-agent systems, delays, cooperative output regulation, distributed-parameter systems, boundary control.**

I. INTRODUCTION

A. Motivation and Literature Review

B. Contributions

C. Organization

Notation. Throughout the paper the shorthand notation $\operatorname{diag}()N(h_i) = \operatorname{diag}(h_1, \ldots, h_N)$ is used.

II. PROBLEM FORMULATION

Consider the *multi-agent system (MAS)* consisting of the *uncertain agents* → robustness → output feedback

$$\dot{x}_i(t) = A_i x_i(t) + B_i \bar{u}_i(t) + G_{1,i} d_i(t), \qquad t > 0 \tag{1a}$$

$$y_i(t) = C_i x_i(t) + G_{2,i} d_i(t), \qquad t \geq 0 \tag{1b}$$

signal model

for $i = 1, \ldots, N$, $N > 1$, with the state $x_i(t) \in \mathbb{R}^n$, the disturbances $d_i(t) \in \mathbb{R}^{q_i}$ and the output $y_i(t) \in \mathbb{R}^p$. The inputs $\bar{u}_i = \operatorname{col}(\bar{u}_i^1, \ldots, \bar{u}_i^p)$ of the agents are subject to the uncertain distributed input delays

$$\bar{u}_i^j(t) = \int_0^{D_j^i} \beta_j^i(\tau) u_i^j(t-\tau)\mathrm{d}\tau, \; i = 1, \ldots, N, j = 1, \ldots, p \tag{2}$$

→ uncertain

with $D_j^i = D_j + \Delta D_j^i \in \mathbb{R}^+$ and

↳ if no uncertainty all agents have the same maximum delay time

$$\beta_j^i(\tau) = \beta_{0,j}^i(\tau) + \sum_{k=1}^{l_j} \beta_{k,j}^i \delta(\tau - \tau_{k,j}^i) \tag{3}$$

holds for $i = 1, \ldots, N$, $j = 1, \ldots, p$, with a piecewise continuous function $\beta_{0,j}^i(\tau) = \beta_{0,j}(\tau) + \Delta\beta_{0,j}^i(\tau) \in \mathbb{R}$ describing distributed delays as well as $\beta_{k,j}^i(\tau) = \beta_{k,j}(\tau) + \Delta\beta_{k,j}^i(\tau) \in \mathbb{R}$, $k = 1, \ldots, l_j$, and $\tau_{k,j}^i = \tau_{k,j} + \Delta\tau_{k,j}^i \in (0, D_j^i)$ specifying lumped delays. They arise from, for example, the processing and connecting time for the information packets arriving at each agent. In particular, distributed delays allow to model continuous streams of packets in packet-switch communication networks (see [8]). Consequently, the initial conditions (ICs) of (1) are $x(0) = x_0 \in \mathbb{R}^n$ and the input delay IC $\bar{u}_i^j(t) = \bar{u}_{i0}^j(t) \in \mathscr{H}_{ij}^{\bar{u}}$, $i = 1, \ldots, N$, $j = 1, \ldots, p$. Therein, $\mathscr{H}_{ij}^{\bar{u}}$ is the corresponding *initial history* depending on β_j^i, which describe the initial data stored in the delays.

The outputs $y_i = \operatorname{col}(y_i^1, \ldots, y_i^p)$, $i = 1, \ldots, N$, of the agents and the reference input $r = \operatorname{col}(r_i^1, \ldots, r_i^p)$ are transmitted with uncertain communication delays. This leads to

$$\bar{y}_i^j(t) = \int_0^{T_j^i} \gamma_j^i(\tau) y_i^j(t-\tau)\mathrm{d}\tau \tag{4a}$$

$$\bar{r}_i^j(t) = \int_0^{T_j^i} \gamma_j^i(\tau) r_i^j(t-\tau)\mathrm{d}\tau \tag{4b}$$

for $i = 1, \ldots, N$, $j = 1, \ldots, p$, representing distributed delayed measurements and reference inputs due to the network communication. In (4)

$$\gamma_j^i(\tau) = \gamma_{0,j}^i(\tau) + \sum_{k=1}^{m_j} \gamma_{k,j}^i \delta(\tau - \tau_{k,j}), \quad j = 1, \ldots, p \tag{5}$$

holds with a piecewise continuous function $\gamma_{0,j}^i(\tau) = \gamma_{0,j}(\tau) + \Delta\gamma_{0,j}^i(\tau) \in \mathbb{R}$, $\gamma_{k,j}^i = \gamma_{k,j} + \Delta\gamma_{k,j}^i \in \mathbb{R}$, $k = 1, \ldots, m_j$ and $\tau_{k,j}^i = \tau_{k,j} + \Delta\tau_{k,j}^i \in (0, T_j^i)$ as well as $T_j^i = T_j + \Delta T_j^i \in \mathbb{R}^+$. Consequently, the reference input and output delay ICs are $\bar{r}_i^j(t) \in \mathscr{H}_{ij}^{\bar{r}}$ and $\bar{y}_i^j(t) \in \mathscr{H}_{ij}^{\bar{y}}$, $i = 1, \ldots, N$, $j = 1, \ldots, p$.

Herein, *uncertain parameters* are prefixed by Δ, while the remaining parameters are the known nominal values. In particular, the uncertain matrices in (1) are $A_i = A + \Delta A_i$, $B_i = B + \Delta B_i$ and $C_i = C + \Delta C_i$, in which A, B and C are the known *nominal matrices* and ΔA_i, ΔB_i and ΔC_i represent the unknown *model uncertainties*. In the following it is assumed that (A, B) is controllable and (C, A) is observable. The disturbance input matrices $G_{j,i}$ have not be available for the design and thus have not to be known. ↳ because of the robustness?

For all agents the common reference input $r(t) \in \mathbb{R}^p$ and the local disturbances d_i, $i = 1, \ldots, N$, acting on the individual agents are described by the *signal model*

$$\dot{w}(t) = Sw(t), \qquad t > 0, \quad w(0) = w_0 \in \mathbb{R}^{n_w} \tag{6a}$$

$$r(t) = P_r w(t), \qquad t \geq 0 \tag{6b}$$

$$d_i(t) = P_i w(t), \qquad t \geq 0, \quad i = 1, \ldots, N \tag{6c}$$

with $P_r \in \mathbb{R}^{p \times n_w}$, $P_i \in \mathbb{R}^{q_i \times n_w}$ and observable pairs (P, S), (P_i, S). It is assumed that the *spectrum* $\sigma(S)$ of $S \in \mathbb{R}^{n_w \times n_w}$ has only eigenvalues on the imaginary axis, i. e., $\sigma(S) \subset \mathrm{j}\mathbb{R}$. Hence, (6) describes a wide class of signals including constant, polynomial and trigonometric functions of time as well as linear combinations thereof. In the sequel, it is assumed that only S in (6a) is known for the networked controller design.

The agents are divided into two groups. One group has access to the reference input r and is therefore called the *informed agents*. In contrast, the information about the reference input can only be broadcast to the remaining *uninformed agents* through a communication network. Then, the reference model in (6) can be seen as the *leader* coinciding with agent 0, while (1) are the *followers*. It is assumed that agent 0 is the root of the digraph $\bar{\mathcal{G}}$ describing the communication network with node set $\bar{\mathcal{V}} = \{0, 1, \ldots, N\}$ and edge set $\bar{\mathcal{E}}$. Then, by removing all edges of $\bar{\mathcal{E}}$, that are incident to the root, the subgraph $\mathcal{G}$ with the node set $\mathcal{V} = \{1, \ldots, N\}$ and the edge set $\mathcal{E}$ is obtained. This graph is represented by the *Laplacian matrix* $L_{\mathcal{G}} \in \mathbb{R}^{N \times N}$ (see Appendix I for more details).

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$L_G 1_N = 0^N \Rightarrow (L_0 \otimes B_y)(1^N \otimes r) = (L_0 + L_G)(1^N \otimes r) = (H \otimes B_y)\bar{r}$

$(H\otimes B_y)\bar{y} - (L_0 \otimes B_y)(1^N \otimes r) = (H \otimes B_y)(\bar{y} - (1^N \otimes r)) = (H\otimes B_y) e$

By making use of this communication network the *cooperative state feedback regulator* → degree of freedom

$$\dot{v}_i(t) = S^* v_i(t) + B_y\Big(\underbrace{\sum_{j=1}^{N} a_{ij}(\bar{y}_i(t) - \bar{y}_j(t)) + a_{i0}(\bar{y}_i(t) - \bar{r}(t))}_{\text{should go to zero, if not the signal model gets stimulated}}\Big) \quad (7a)$$

$$u_i(t) = \mathcal{K}_i[v_i(t), x_i(t), x_j(t)], \quad j \in \mathcal{N}_i, t > 0 \quad (7b)$$

for (7a) on $t > 0$, $i = 1, \ldots, N$, with IC $v_i(0) = v_{i,0} \in \mathbb{R}^{pn_v}$ and

$$S^* = \operatorname{diag}(S_{\min}, \ldots, S_{\min}) \in \mathbb{R}^{pn_v \times pn_v} \quad (8a)$$

$$B_y = \operatorname{diag}(b_1, \ldots, b_p) \in \mathbb{R}^{pn_v \times p} \quad (8b)$$

is designed to ensure the tracking of the leader in the presence of disturbances and model uncertainty. Therein, $S_{\min}$ is obtained from S by taking only the largest Jordan block of each eigenvalue (see [2]) and the vectors $b_i \in \mathbb{R}^p$ are chosen such that the pairs $(S_{\min}, b_i)$, $i = 1, \ldots, p$, are controllable. Consequently, the controller (7) incorporates a *minimum p-copy internal model* of (6) for each agent. Furthermore, $\mathcal{N}_i$ is the neighbor set w.r.t. agent i and the node set $\mathcal{V}$. For the informed agents $a_{i0} > 0$ is satisfied, while $a_{i0} = 0$ is valid for the uninformed agents. The operator $\mathcal{K}_i$ in (7b) is a formal feedback operator, which also involves distributed delays of the states. In order to implement the cooperative state feedback regulator (7b), the *cooperative observer*

?

$$\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + B\bar{u}_i(t) + L(\bar{\eta}_i(t) - \hat{\bar{\eta}}_i(t)) \quad (9)$$

→ also transmitted from every agent?

with the *observer gain* $L \in \mathbb{R}^{n \times p}$ is designed, which only uses the distributed delayed *relative measurement*

$$\bar{\eta}_i(t) = \sum_{j=1}^{N} a_{ij}(\bar{y}_i(t) - \bar{y}_j(t)) + a_{i0}\bar{y}_i(t), \quad i = 1, \ldots, N. \quad (10)$$

Then, by combining (7) with (9) a *cooperative output feedback regulator* is obtained. → coop. observer + regulator

The *robust cooperative output regulation problem* amounts to designing a networked controller, in order to ensure stability of the resulting networked controlled MAS and the *reference tracking* → doesn't reference tracking imply stability?

$$\lim_{t\to\infty} e_{y_i}(t) = \lim_{t\to\infty}(y_i(t) - r(t)) = 0, \quad (11)$$

$i = 1, \ldots, N$, for all ICs of the MAS (1), the signal model (6) and the controller (7), (9). Furthermore, the property (11) should be robust in the sense that it holds despite of all model uncertainties, for which the nominal networked controller stabilizes the networked controlled MAS. This means that the outputs y_i, $i = 1, \ldots, N$, of all agents (1), i. e., the followers, synchronize with the output r of the leader (6) in the presence of the disturbances, the model uncertainty and the delays.

III. Cooperative State Feedback Regulator Design

→ inform!

According to the *cooperative internal model principle* (see [10]) the state feedback (7b) is designed to stabilize the nominal MAS (1) (i.e., setting all with Δ prefixed values to zero) extended by the cooperative internal models (7a). For the state feedback regulator design the state $x(t) = \operatorname{col}(x_1(t), \ldots, x_N(t)) \in \mathbb{R}^{Nn}$, the input $\bar{u}(t) = \operatorname{col}(\bar{u}_1(t), \ldots, \bar{u}_N(t)) \in \mathbb{R}^{Np}$ and the output $y(t) = \operatorname{col}(y_1(t), \ldots, y_N(t)) \in \mathbb{R}^{Np}$ are introduced. With this, the *aggregated MAS*

$$\dot{x}(t) = (I_N \otimes A)x(t) + (I_N \otimes B)\bar{u}(t) \quad (12a)$$

$$y(t) = (I_N \otimes C)x(t) \quad (12b)$$

in nominal case follows, in which the disturbance have not to be considered in the solution of the posed stabilization problem. Similarly, defining $v(t) = \operatorname{col}(v_1(t), \ldots, v_N(t)) \in \mathbb{R}^{Npn_v}$ the *aggregated cooperative internal model*

$$\dot{v}(t) = (I_N \otimes S^*)v(t) + (H \otimes B_y)\bar{y}(t) \quad (13)$$

→ why?

is obtained, where the reference input has not to be taken into account in the stabilization. The matrix H in (13) is the *leader-follower matrix*

$$H = L_{\mathcal{G}} + \operatorname{diag}(a_{01}, \ldots, a_{0N}) \quad (14)$$

(see, e.g., [6, Ch. 3.2.2]), which describes both the communication among the followers as well as between the leader and the followers.

Modelling the delayed input $\bar{u}$ and output $\bar{y}$ in terms of the solution of transport equations (see, e.g., [1, Ch. 3]) leads to the system consisting of the *PDE-ODE cascade*

→ Elemente sind $\frac{1}{D_i}$

→ should yield (2)

$$\dot{\varphi}(z,t) = (I_N \otimes \Lambda)\varphi'(z,t) \quad (15a)$$

$$\varphi(1,t) = u(t) \quad (15b)$$

$$\dot{x}(t) = (I_N \otimes A)x(t) + \int_0^1 (I_N \otimes B\Phi(\zeta))\varphi(\zeta,t)\mathrm{d}\zeta \quad (15c)$$

describing the agents with input delays and the *PDE-ODE cascade*

$$\dot{\psi}(z,t) = (I_N \otimes \Theta)\psi'(z,t) + (I_N \otimes \Gamma(z)C)x(t) \quad (16a)$$

$$\psi(1,t) = 0 \quad (16b)$$

$$\dot{v}(t) = (I_N \otimes S^*)v(t) + (H \otimes B_y)\boxed{\psi(0,t)} \quad (16c)$$

→ should be (4a)?

representing the cooperative internal models with nominal communication delays. The ODE states $x(t)$ and $v(t)$ evolve on $t \in \mathbb{R}_0^+$ as well as the PDE states $\varphi(z,t) \in \mathbb{R}^{Np}$ and $\psi(z,t) \in \mathbb{R}^{Np}$ are defined on the domain $(z,t) \in [0,1] \times \mathbb{R}_0^+$. Furthermore, $\Lambda = \operatorname{diag}(1/D_1, \ldots, 1/D_p)$, $\Phi(\zeta) = \operatorname{diag}(D_1\beta_1(D_1(1-\zeta)), \ldots, D_p\beta_p(D_p(1-\zeta)))$, $\Theta = \operatorname{diag}(1/T_1, \ldots, 1/T_p)$ and $\Gamma(z) = \operatorname{diag}(\gamma_1(T_1 z), \ldots, \gamma_p(T_p z))$ hold. ??

A. Local Stabilization of the MAS

In the first step of determining the networked controller (7) the agents (15) are locally stabilized. This requires to determine the *local decoupling transformation* → in ODE no more PDE-state!

$$e_x(t) = x(t) - \int_0^1 (I_N \otimes Q(\zeta))\varphi(\zeta,t)\mathrm{d}\zeta \quad (17)$$

with $Q(z) \in \mathbb{R}^{n \times p}$ and

→ after good stabilization this will go to zero. Woher kommt das?

$$u(t) = (I_N \otimes K_1)e_x(t) + \bar{u}(t) \quad (18)$$

with $K_1 \in \mathbb{R}^{p \times n}$ to map (15) into the *target system*

$$\dot{e}_x(t) = (I_N \otimes (A - B_1 K_1))e_x(t) - (I_N \otimes B)\bar{u}(t) \quad (19a)$$

$$\dot{\varphi}(z,t) = (I_N \otimes \Lambda)\varphi'(z,t) \quad (19b)$$

$$\varphi(1,t) = (I_N \otimes K_1)e_x(t) + \bar{u}(t) \quad (19c)$$

$$\dot{\psi}(z,t) = (I_N \otimes \Theta)\psi'(z,t) + (I_N \otimes \Gamma(z)C)e_x(t) + \int_0^1 (I_N \otimes \Gamma(\zeta)CQ(\zeta))\varphi(\zeta,t)\mathrm{d}\zeta \quad (19d)$$

$$\psi(1,t) = 0 \quad (19e)$$

$$\dot{v}(t) = (I_N \otimes S^*)v(t) + (H \otimes B_y)\psi(0,t) \quad (19f)$$

where

$$B_1 = Q(1)\Lambda. \quad (20)$$

In order to determine $Q(z)$, differentiate (17) as well as insert (15) and (19). This directly yields after an integration by parts the *local decoupling equations*

$$Q'(z)\Lambda = -AQ(z) - B\Phi(z), \quad z \in (0,1] \quad (21a)$$

$$Q(0) = 0. \quad (21b)$$

The next lemma asserts the solvability of (21) by providing an explicit solution.

Lemma 1 (Solvability of the local decoupling equations): The *local decoupling equations* (21) have a continuous solution $Q(z) \in \mathbb{R}^{n\times p}$.

Proof: Assume that the matrix $A \in \mathbb{R}^{n\times n}$ has the *Jordan blocks* J_i, $i = 1,\dots,\rho$, to which the *Jordan chains*

possible for every nxn m trix

$$\omega_{i(1)}^\top A = \mu_i \omega_{i(1)}^\top \tag{22a}$$
$$\omega_{i(k)}^\top A = \mu_i \omega_{i(k)}^\top + \omega_{i(k-1)}^\top, \quad k = 2,\dots,l_i, \tag{22b}$$

are associated and $l_1 + \ldots + l_\rho = n$. In (22) the vectors $\omega_{i(k)} \in \mathbb{C}^n$ are the *generalized left eigenvectors* of A w.r.t. the eigenvalue μ_i, $i = 1,2,\dots,\rho$, in which ρ is the number of eigenvalues of A with linearly independent eigenvectors (see, e. g., [5, Ch. 6] for details on the related *Jordan canonical form*). The generalized eigenvectors $\omega_{i(k)}$, $i = 1,\dots,\rho$, $k = 1,\dots,l_i$, are determined such that they form a basis for $\mathbb{C}^n$, which is always possible. Premultiply (21) by the generalized left eigenvectors $\omega_{i(k)}^\top$, $i = 1,\dots,\rho$, $k = 1,\dots,l_i$, as well as define $q_{i(k)}^\top = \omega_{i(k)}^\top Q$ and $\theta_{i(k)}^\top = \omega_{i(k)}^\top B\Phi$. Then, (21) results in the initial value problem (IVP)

from (22b)

$$\mathrm{d}_z q_{i(k)}^\top(z) = -\big(\mu_i q_{i(k)}^\top(z) + \theta_{i(k)}^\top(z) + q_{i(k-1)}^\top(z)\big)\Lambda^{-1} \tag{23a}$$
$$q_{i(k)}^\top(0) = 0^\top \tag{23b}$$

on $z \in (0,1]$ for $i = 1,\dots,\rho$, $k = 1,\dots,l_i$ and setting $q_{i(0)}^\top = 0^\top$. The IVP (23) is solved by

$$q_{i(k)}^\top(z) = -\int_0^z (\theta_{i(k)}^\top(\zeta) + q_{i(k-1)}^\top(\zeta))\Lambda^{-1}\mathrm{e}^{-\mu_i\Lambda^{-1}(z-\zeta)}\mathrm{d}\zeta. \tag{24}$$

With this, the solution

$$Q(z) = \mathrm{col}(\omega_{1(1)}^\top,\dots,\omega_{1(l_1)}^\top,\dots,\omega_{\rho(l_\rho)}^\top)^{-1} \cdot \mathrm{col}(q_{1(1)}^\top(z),\dots,q_{1(l_1)}^\top(z),\dots,q_{\rho(l_\rho)}^\top(z)) \tag{25}$$

of (21) can be obtained. Therein, the continuity of $Q(z)$ is inherited from the assumed piecewise continuity of the functions appearing in (24). ■

This result shows that the stabilization of the delayed agent (15) can be traced back to the classical state feedback controller design for (A, B_1) (see (19a)). In particular, assume that the latter matrix pair is controllable, then there exists a feedback gain K_1 ensuring exponential stability of (19a). With this, the delayed agents (19a)–(19c) are stabilized, as (19b)–(19c) are a finite-time stable cascade of transport equations.

The next lemma shows that the controllability of (A, B_1) and thus the stabilizability of the nominal agents depends on the transfer behaviour of the transport equations modelling the distributed input delays.

Lemma 2 (Controllability of (A, B_1)): Consider the transfer matrix

$$F_i(s) = \int_0^1 \Phi(\zeta)\mathrm{e}^{s\Lambda^{-1}(\zeta-1)}\mathrm{d}\zeta \in \mathbb{C}^{p\times p} \tag{26}$$

for $i = 1,\dots,N$ of the subsystems

$$\dot\varphi_i(z,t) = \Lambda\varphi_i'(z,t), \qquad (z,t) \in [0,1)\times\mathbb{R}^+ \tag{27a}$$
$$\varphi_i(1,t) = u_i(t), \quad \text{input} \qquad t > 0 \tag{27b}$$
$$\text{outp t} \quad \bar u_i(t) = \textstyle\int_0^1 \Phi(\zeta)\varphi_i(\zeta,t)\mathrm{d}\zeta, \qquad t \geq 0 \tag{27c}$$

from u_i to $\bar u_i$ in (15) modelling the distributed input delays. The pair (A, B_1) is controllable if (A, B) is controllable and $\det F_i(\lambda) \neq 0$, $\forall\lambda \in \sigma(A)$.

Why is that? → look in proof!

Proof: The transfer matrix $F_i(s)$ of (59) in the sense of [11] is readily obtained by a formal Laplace transform of (59) assuming vanishing IC and solving the resulting IVP in the frequency domain.

PBH oder ?

By the *PHB eigenvector test* for controllability, $\omega_{i(1)}^\top B_1 \neq 0^\top$ has to hold for all linearly independent left eigenvectors $\omega_{i(1)}^\top$, $i = 1,\dots,\rho$, $\rho \leq n$, of A w.r.t. the eigenvalue μ_i so that (A, B_1) is controllable (see [4, Th. 6.2-5]). In view of (20) and (24) it follows

$$\omega_{i(1)}^\top B_1 = q_{i(1)}^\top(1)\Lambda = -\omega_{i(1)}^\top BF(\mu_i). \tag{28}$$

For this, $\omega_{i(1)}^\top B \neq 0^\top$ is necessary, which requires the controllability of the pair (A, B). Then, $\omega_{i(1)}^\top B_1 \neq 0^\top$ holds for $\det F_i(\mu_i) \neq 0$ verifying the conditions of the lemma. ■

So far local stabilization ⟺ $e_{x,i}$ is stable

B. Simultaneous Stabilization of the Internal Model

In order to stabilize the MAS, the *cooperative decoupling transformation*

$$e_v(t) = v(t) - \int_0^1 (H\otimes P_1(\zeta))\psi(\zeta,t)\mathrm{d}\zeta - \int_0^1 (H\otimes P_2(\zeta))\varphi(\zeta,t)\mathrm{d}\zeta - (H\otimes P_3)e_x(t) \tag{29}$$

with $P_1(z) \in \mathbb{R}^{pn_v\times p}$, $P_2(z) \in \mathbb{R}^{pn_v\times p}$ and $P_3 \in \mathbb{R}^{pn_v\times n}$ as well as

$$\bar u(t) = (I_N \otimes K_2)e_v(t) \tag{30}$$

with $\in \mathbb{R}^{p\times pn_v}$ are used to map (19) into the *final target system*

$$\begin{bmatrix}\dot e_x(t)\\ \dot e_v(t)\end{bmatrix} = \underbrace{\begin{bmatrix} I_N\otimes(A - B_1K_1) & I_N\otimes(-B_1K_2)\\ 0 & I_N\otimes S^* - H\otimes B_2K_2\end{bmatrix}}_{A_{\mathrm{cl}}}\underbrace{\begin{bmatrix}e_x(t)\\ e_v(t)\end{bmatrix}}_{e(t)} \tag{31a}$$

$$\dot\varphi(z,t) = (I_N\otimes\Lambda)\varphi'(z,t) \tag{31b}$$
$$\varphi(1,t) = (I_N\otimes K_1)e_x(t) + (I_N\otimes K_2)e_v(t) \tag{31c}$$
$$\dot\psi(z,t) = (I_N\otimes\Theta)\psi'(z,t) + (I_N\otimes\Gamma(z)C)e_x(t) + \textstyle\int_0^1 (I_N\otimes\Gamma(\zeta)CQ(\zeta))\varphi(\zeta,t)\mathrm{d}\zeta \tag{31d}$$
$$\psi(1,t) = 0 \tag{31e}$$

with

$$B_2 = P_2(1)\Lambda - P_3B_1. \tag{32}$$

Different from (17) the cooperative decoupling transformation (29) uses the information from neighbouring agents and the leader to decouple the ODE subsystem (19f). This follows from the leader-follower matrix H appearing in (29). With this, the ODE-PDE cascade (31) is obtained. Therein, the stabilization of the cooperative internal model (7) is traced back to a *simultaneous stabilization* of the e_v-subsystem in (31a) by determining the feedback gain K_2.

Differentiating (29), inserting (19) and (29) as well as using an integration by parts results in the *cooperative decoupling equations*

$$P_1'(z)\Theta = -S^*P_1(z), \quad z \in (0,1] \tag{33a}$$
$$P_1(0)\Theta = -B_y \tag{33b}$$
$$P_2'(z)\Lambda = -S^*P_2(z) + \int_0^1 P_1(\zeta)\mathrm{d}\zeta\Gamma(z)CQ(\cancel{\zeta}), z \in (0,1] \tag{33c}$$
$$P_2(0) = 0 \tag{33d}$$
$$S^*P_3 - P_3(A - B_1K_1) = \int_0^1 P_1(\zeta)\Gamma(\zeta)\mathrm{d}\zeta C + P_2(1)\Lambda K_1 \tag{33e}$$

to be solved for determining (29).

The next lemma provides an explicit solution of (33) verifying the existence of the cooperative decoupling transformation (29).

Lemma 3 (Solvability of the cooperative decoupling equations): The *cooperative decoupling equations* (33a)–(33b) and (33c)–(33d) have a solution $P_1(z) \in \mathbb{R}^{pn_v \times p}$ and $P_2(z) \in \mathbb{R}^{pn_v \times p}$, in which the elements of $P_1(z)$ and $P_2(z)$ are continuous functions. If $A - B_1K_1$ is a Hurwitz matrix, then (33e) has a solution $P_3 \in \mathbb{R}^{pn_v \times n}$ → then (33e) is a Sylvester equ. but why Hurwitz necessary?

Proof: Denote the *Jordan blocks* of the matrix $S \in \mathbb{R}^{n_v \times n_v}$ by J_i, $i = 1, \ldots, \rho_v$, with the corresponding *Jordan chains*

$$\nu_{i(1)}^\top S^* = \sigma_i \nu_{i(1)}^\top \qquad V_{i(k)}^\top \tag{34a}$$

$$\nu_{i(k)}^\top S^* = \sigma_i \cancel{\omega_{i(k)}} + \nu_{i(k-1)}, \quad k = 2, \ldots, l_i, \tag{34b}$$

with $l_1 + \ldots + l_{\rho_v} = n_v$. Therein, $\nu_{i(k)} \in \mathbb{C}^{n_v}$ are the *generalized left eigenvectors* of S^* w.r.t. the eigenvalue σ_i, $i = 1, 2, \ldots, \rho_v$, with ρ_v denoting the number of eigenvalues of S with linearly independent eigenvectors (see, e. g., [5, Ch. 6] for details on the related *Jordan canonical form*). Without loss of generality it is assumed that the generalized eigenvectors $\nu_{i(k)}$, $i = 1, \ldots, \rho$, $k = 1, \ldots, l_i$, form a basis for $\mathbb{C}^{n_v}$. Introduce the abbreviations $p_{1,i(k)}^\top(z) = \nu_{i(k)}^\top P_1(z)$ and $b_{i(k)}^\top = \nu_{i(k)}^\top B_y$, $i = 1, \ldots, \rho_v$, $k = 1, \ldots, l_i$. Then, by premultiplying (33a)–(33b) with $\nu_{i(k)}^\top$ and solving the resulting IVP (see also proof of Lemma 1), the solution

$$p_{1,i(k)}^\top(z) = \big(b_{i(k)}^\top \mathrm{e}^{-\sigma_i \Theta^{-1} z} \overset{2?\ \text{minus}}{-} \int_0^z p_{1,i(k-1)}^\top(\zeta) \mathrm{e}^{-\sigma_i \Theta^{-1}(z-\zeta)} \mathrm{d}\zeta\big)\Theta^{-1} \tag{35}$$

for $i = 1, \ldots, \rho_v$, $k = 1, \ldots, l_i$. With this, the continuous solution of (33a)–(33b) is

$$P_1(z) = \Upsilon^{-1} \operatorname{col}(p_{1,1(1)}^\top(z), \ldots, p_{1,1(l_1)}^\top(z), \ldots, p_{1,\rho_v(l_{\rho_v})}^\top(z)), \tag{36}$$

where

$$\Upsilon = \operatorname{col}(\nu_{1(1)}^\top, \ldots, \nu_{1(l_1)}^\top, \ldots, \nu_{\rho(l_{\rho_v})}^\top). \tag{37}$$

Similarly, premultiplying (33c)–(33d) by $\nu_{i(k)}^\top$ and using $p_{2,i(k)}^\top = \nu_{i(k)}^\top P_2$ gives

$$p_{2,i(k)}^\top(z) = -\Big(\int_0^z p_{2,i(k-1)}^\top(\zeta) \mathrm{e}^{-\sigma_i \Lambda^{-1}(z-\zeta)} \mathrm{d}\zeta + \int_0^z \int_0^1 p_{1,i(k)}^\top(\bar\zeta) \mathrm{d}\bar\zeta\, \Gamma(\zeta) C Q(\zeta) \mathrm{e}^{-\sigma_i \Lambda^{-1}(z-\zeta)} \mathrm{d}\zeta \Big)\Lambda^{-1} \tag{38}$$

for $i = 1, \ldots, \rho_v$, $k = 1, \ldots, l_i$, after solving the resulting IVP. Then,

$$P_2(z) = \Upsilon^{-1} \operatorname{col}(p_{2,1(1)}^\top(z), \ldots, p_{2,1(l_1)}^\top(z), \ldots, p_{2,\rho_v(l_{\rho_v})}^\top(z)) \tag{39}$$

is the continuous solution of (33c)–(33d). Finally, by premultiplying (33e) with $\nu_{i(k)}^\top$ and defining $p_{3,i(k)}^\top = \nu_{i(k)}^\top P_3$ leads to

$$p_{3,i(k)}^\top = \big(-p_{3,i(k-1)}^\top + \textstyle\int_0^1 p_{1,i(k)}^\top(\zeta)\Gamma(\zeta) C \mathrm{d}\zeta + p_{2,i(k)}^\top(1)\Lambda K_1\big) \cdot (\sigma_i I - A + B_1K_1)^{-1} \tag{40}$$

for $i = 1, \ldots, \rho_v$, $k = 1, \ldots, l_i$. This result exists if $\sigma(S^*) \cap \sigma(A - B_1K_1) = \emptyset$. Since $\sigma(S^*) \in \mathrm{j}\mathbb{R}$ by assumption the latter condition is fulfilled if $A - B_1K_1$ is Hurwitz. Hence, the solution of (33e) is

$$P_3 = \Upsilon^{-1} \operatorname{col}(p_{3,1(1)}^\top(z), \ldots, p_{3,1(l_1)}^\top, \ldots, p_{3,\rho_v(l_{\rho_v})}^\top) \tag{41}$$

if $A - B_1K_1$ is a Hurwitz matrix. ∎

A prerequisite for the simultaneous stabilization of the e_v-system in (31a) is the controllability of (S^*, B_2). The next lemma shows that this property depends on the transfer behaviour of the nominal agents (1). This result uses the matrix fraction description (MFD) of a transfer matrix in terms of polynomial matrices, for which details can be found in, e.g., [3, Ch. 1].

Lemma 4 (Controllability of (S^*, B_2)): Consider the right coprime MFD $G_i(s) = C(sI - A)^{-1}B = N(s)D^{-1}(s)$ of the transfer behaviour of the nominal agents (1) from u_i to y_i, $i = 1, \ldots, N$, with the numerator matrix $N(s) \in \mathbb{C}^{p \times p}$ and the column reduced ? denominator matrix $D(s) \in \mathbb{C}^{p \times p}$. Then, the pair (S^*, B_2) is controllable if (S^*, b_i), $i = 1, \ldots, p$, are controllable and

$$\det(M_1(\lambda)N(\lambda) - M_2(\lambda)D(\lambda)) \neq 0, \quad \forall \lambda \in \sigma(S^*), \tag{42}$$

where

$$M_1(\lambda) = -\int_0^1 \mathrm{e}^{-\lambda\Theta^{-1}\zeta}\Gamma(\zeta)\mathrm{d}\zeta \qquad \text{fehlt hier nicht } B_2\text{?} \tag{43a}$$

$$M_2(\lambda) = \tfrac{\Theta}{\lambda}(I - \mathrm{e}^{-\lambda\Theta^{-1}})\int_0^1 \Gamma(\zeta) C Q(\zeta) \mathrm{e}^{-\lambda\Lambda^{-1}(1-\zeta)}\mathrm{d}\zeta. \tag{43b}$$

Proof: The pair (S^*, B_2) is controllable iff $\nu_{i(1)}^\top B_2 = \nu_{i(1)}^\top P_2(1)\Lambda - \nu_{i(1)}^\top P_3 = p_{2,i(1)}^\top(1)\Lambda - p_{3,i(1)}^\top \neq 0^\top$ in view of (32) and the proof of Lemma 3 for all linearly independent left eigenvectors $\nu_{i(1)}^\top$, $i = 1, \ldots, \rho_v$, $\rho \leq n_v$, of S^* w.r.t. the eigenvalue σ_i (see [4, Th. 6.2-5]). Inserting the solution $p_{j,i(1)}^\top$, $j = 2, 3$, determined in the proof of Lemma 3 yields → wurde oben also nur vergessen

$$\nu_{i(1)}^\top B_2 = b_{i(1)}^\top \Theta^{-1}\big(M_1(\sigma_i)F_1(\sigma_i) + M_2(\sigma_i)F_2(\sigma_i)\big) \tag{44}$$

with $F_1(s) = C(sI - A + B_1K_1)^{-1}B_1$ and $F_2(s) = K_1(sI - A + B_1K_1)^{-1}B_1 - I$. Consider the right coprime MFD

$$N_x(s)\tilde{D}^{-1}(s) = (sI - A + B_1K_1)^{-1}B_1, \tag{45}$$

⇒ $F_1(s) = CN_x(s)\tilde{D}^{-1}(s)$

in which the denominator matrix $\tilde{D}(s)$ is defined by ⇒ $F_1(s) = \frac{CN_x(s)}{D(s) + K_1N_x(s)}$

$$\tilde{D}(s) = D(s) + K_1N_x(s) \tag{46}$$

with the right coprime MFD $N_x(s)D^{-1}(s) = (sI - A)^{-1}B_1$ (see [3, Ch. 2]). Consequently,

$$F_1(s) = C(sI - A + B_1K_1)^{-1}B_1 = N(s)\tilde{D}^{-1}(s) \tag{47}$$

with $N(s) = CN_x(s)$ is obtained. The transfer matrix $F_2(s)$ has the MFD

$$\begin{aligned} F_2(s) &= K_1(sI - A + B_1K_1)^{-1}B_1 - I \\ &= (K_1N_x(s) - \tilde{D}(s))\tilde{D}^{-1}(s) = -D(s)\tilde{D}^{-1}(s) \end{aligned} \tag{48}$$

in view of (45) and (46). As a consequence, the result

$$\nu_{i(1)}^\top B_2 = b_{i(1)}^\top \Theta^{-1}\big(M_1(\sigma_i)N(\sigma_i) - M_2(\sigma_i)D(\sigma_i)\big)\tilde{D}^{-1}(\sigma_i) \tag{49}$$

follows from (44), (47) and (48). If (S^*, b_i), $i = 1, \ldots, p$, are controllable, then $b_{i(1)}^\top = \nu_{i(1)}^\top B_y \neq 0^\top$ by [4, Th. 6.2-5]. Since $\sigma(S^*) \in \mathrm{j}\mathbb{R}$ and $\det\tilde{D}(s) = \det(sI - A + B_1K_1)$ is a Hurwitz polynomial by Lemma 3 (see [3, Ch. 2]) the inverse $\tilde{D}^{-1}(\sigma_i)$ exists. Then, $\nu_{i(1)}^\top B_2 \neq 0^\top$ holds for (42). ∎

If the conditions of Lemma 4 are satisfied, then a systematic solution for the simultaneous stabilization of the e_v-subsystem in (31a) using a Riccati approach can be found in [6, Th 3.1]. The next theorem asserts the stability of the nominal networked controlled MAS.

Theorem 1 (Stability of the networked controlled MAS): Assume that $A - B_1K_2$ and $I_N \otimes S^* - H \otimes B_2K_2$ are Hurwitz matrices. Then, the nominal networked controlled MAS consisting of the plant (1) and the controller (7) specified by (18) and (30) is asymptotically stable pointwise in space for any ICs $x_i(0) \in \mathbb{R}^n$, $v_i(0) \in \mathbb{R}^{pn_v}$, $i = 1, \ldots, N$ and piecewise continuous ICs of the corresponding transport equations in (15) and (16) modelling the input and output delays.

Proof: The matrix in (31a) is a Hurwitz matrix, which follows from its upper block triangular form and the fact that $A - B_1K_2$ and $I_N \otimes S^* - H \otimes B_2K_2$ are Hurwitz. Consequently, the ODE subsystem (31a) is exponentially stable. Since the PDE subsystem (31b)–(31d) is a cascade of transport PDEs, also the corresponding state asymptotically vanish pointwise in space for the considered regularity of the ICs. Hence, the target system is asymptotically stable pointwise in space. With this, asymptotic stability in the original coordinates follows from going through the chain of the bounded inverse decoupling transformations. ■

IV. Cooperative observer design

In order to design the cooperative observer (9), introduce $\bar{\eta} = \mathrm{col}(\bar{\eta}_1, \ldots, \bar{\eta}_N)$ and consider its aggregation

$$\dot{\hat{x}}(t) = (I_N \otimes A)\hat{x}(t) + (I_N \otimes B)\bar{u}(t) + (H \otimes L)(\bar{\eta}(t) - \hat{\bar{\eta}}(t)). \quad (50)$$

After modelling the delayed measurements with transport equations (cf. (16)), the observer

$$\begin{aligned}\dot{\hat{x}}(t) &= (I_N \otimes A)\hat{x}(t) + (I_N \otimes B)\bar{u}(t)\\ &\quad + (H \otimes L)(\psi(0,t) - \hat{\psi}(0,t)) \quad (51a)\\ \dot{\hat{\psi}}(z,t) &= (I_N \otimes \Theta)\hat{\psi}'(z,t) + (I_N \otimes \Gamma(z)C)\hat{x}(t)\\ &\quad + (H \otimes M(z))(\psi(0,t) - \hat{\psi}(0,t)) \quad (51b)\\ \hat{\psi}(1,t) &= 0 \quad (51c)\end{aligned}$$

with the additional *observer gain* $M(z) \in \mathbb{R}^{p\times p}$ has to be considered. By defining $\tilde{x} = x - \hat{x}$ and $\tilde{\psi} = \psi - \hat{\psi}$, the *observer error dynamics*

$$\begin{aligned}\dot{\tilde{x}}(t) &= (I_N \otimes A)\tilde{x}(t) - (H \otimes L)\tilde{\psi}(0,t) \quad (52a)\\ \dot{\tilde{\psi}}(z,t) &= (I_N \otimes \Theta)\tilde{\psi}'(z,t) + (I_N \otimes \Gamma(z)C)\tilde{x}(t)\\ &\quad - (H \otimes M(z))\tilde{\psi}(0,t) \quad (52b)\\ \tilde{\psi}(1,t) &= 0 \quad (52c)\end{aligned}$$

is readily obtained for (52), when considering the nominal case. For the purpose of stabilizing (52), the *local decoupling transformation*

$$\varepsilon(z,t) = \tilde{\psi}(z,t) - (I_N \otimes P(z))\tilde{x}(t) \quad (53)$$

with $P(z) \in \mathbb{R}^{p\times n}$ and

$$M(z) = P(z)L \quad (54)$$

are utilized mapping (52) into the *target system*

$$\begin{aligned}\dot{\tilde{x}}(t) &= (I_N \otimes A - H \otimes LC_1)\tilde{x}(t) - (H \otimes LC_1)\varepsilon(0,t) \quad (55a)\\ \dot{\varepsilon}(z,t) &= (I_N \otimes \Theta)\varepsilon'(z,t) \quad (55b)\\ \varepsilon(1,t) &= 0 \quad (55c)\end{aligned}$$

with

$$C_1 = P(0). \quad (56)$$

This is a *PDE-ODE cascade*, in which the $\tilde{x}$-system can be simultaneously stabilized and the ε-system is finite-time stable. Differentiating (53) w.r.t. time and inserting (52) and (55) yields the *local decoupling equations*

$$\begin{aligned}\Theta P'(z) &= P(z)A - \Gamma(z)C, \quad z \in [0,1) \quad (57a)\\ P(1) &= 0. \quad (57b)\end{aligned}$$

The next lemma clarifies the solvability of (57).

Lemma 5 (Solvability of the local observer decoupling equations): The *local decoupling equations* (57) have a continuous solution $P(z) \in \mathbb{R}^{p\times n}$.

By transposing (57a) the ODE (21a) is obtained. Hence, Lemma 5 directly follows from the same reasoning as in the proof Lemma 1 by taking the slightly different IC (57b) into account.

The next lemma presents a condition for the observability of (C_1, A), which is a prerequisite for the simultaneous stabilization of (55a).

Lemma 6 (Observability of (C_1, A)): Consider the transfer matrix

$$G_i(s) = \int_0^1 \mathrm{e}^{-s\Theta^{-1}\zeta}\Gamma(\zeta)\mathrm{d}\zeta \in \mathbb{C}^{p\times p} \quad (58)$$

for $i = 1, \ldots, N$ of the subsystems

$$\begin{aligned}\dot{\psi}_i(z,t) &= \Theta\psi_i'(z,t) + \Gamma(z)y_i(t), \quad (z,t) \in [0,1) \times \mathbb{R}^+ \quad (59a)\\ \psi_i(1,t) &= 0, \quad t > 0 \quad (59b)\\ \bar{y}_i(t) &= \psi_i(0,t), \quad t \geq 0 \quad (59c)\end{aligned}$$

from y_i to $\bar{y}_i$ in (16) modelling the distributed output delays. The pair (C_1, A) is observable if (C, A) is observable and $\det G_i(\lambda) \neq 0$, $\forall \lambda \in \sigma(A)$.

Proof: The transfer matrix $G_i(s)$ of (59) in the sense of [11] is readily obtained by a formal Laplace transform of (59) assuming vanishing IC and solving the resulting IVP in the frequency domain. By the *PHB eigenvector test* for the observability of (C_1, A) (see [4, Th. 6.2-5]), $C_1\nu_i \neq 0$ has to hold for all linearly independent right eigenvectors $\nu_i^\top$, $i = 1, \ldots, \bar{\rho}$, $\bar{\rho} \leq n$, of A w.r.t. the eigenvalue μ_i. Following the same lines as in the proof of Lemma 6 it can be shown that

$$C_1\nu_i = P(0)\nu_i = G_i(\mu_i)C\nu_i. \quad (60)$$

By the observability of (C, A) it follows that $C\nu_i \neq 0$. With this, $C_1\nu_i \neq 0$ is satisfied for $\det G_i(\mu_i) \neq 0$, which verifies the conditions of the lemma. ■

If the conditions of this lemma are satisfied, then the $\tilde{x}$-system (55a) can be simultaneously stabilized by making use of the systematic Riccati approach presented in [6, Th. 3.2]. The next theorem states the stability of the observer error dynamics (52).

Theorem 2 (Stability of the observer error dynamics): Assume that $I_N \otimes A - H \otimes LC_1$ is a Hurwitz matrix. Then, the observer error dynamics (52) are asymptotically stable pointwise in space for any ICs $x_i(0) \in \mathbb{R}^n$, $\hat{x}_i(0) \in \mathbb{R}^n$, $i = 1, \ldots, N$, and piecewise continuous ICs of the transport equations in (52).

Proof: Since $I_N \otimes A - H \otimes LC_1$ is a Hurwitz matrix, the ODE subsystem in (55a) is exponentially stable. For the assumed regularity of the IC, the transport equation (55b)–(55c) has a unique continuous solution so that the boundary input in (55a) is well-defined. Hence, the exponentially stable ODE (55a) is driven by a bounded input vanishing in finite time. As a consequence, its solution decays showing asymptotic stability of (55). This implies asymptotic stability of (52), because the decoupling transformation (53) is boundedly invertible. ■

V. Observer-based compensator

The *cooperative output feedback regulator* results from using the estimates of (51) in (18) and (30) yielding

$$u(t) = (I_N \otimes K_1)\hat{e}_x(t) + (I_N \otimes K_2)\hat{e}_v(t) \quad (61)$$

with $\hat{e}_x(t) = \hat{x}(t) - \int_0^1 (I_N \otimes Q(\zeta))\hat{\varphi}(\zeta,t)\mathrm{d}\zeta$ and $\hat{e}_v(t) = v(t) - \int_0^1 (H \otimes P_1(\zeta))\hat{\psi}(\zeta,t)\mathrm{d}\zeta - \int_0^1 (H \otimes P_2(\zeta))\hat{\varphi}(\zeta,t)\mathrm{d}\zeta - (H \otimes P_3)\hat{e}_x(t)$. Therein, the state estimate $\hat{\varphi}$ can be determined from the *trivial observer*

$$\begin{aligned}\dot{\hat{\varphi}}(z,t) &= (I_N \otimes \Lambda)\hat{\varphi}'(z,t), \quad (z,t) \in (0,1) \times \mathbb{R}^+ \quad (62a)\\ \hat{\varphi}(1,t) &= u(t), \quad t > 0 \quad (62b)\end{aligned}$$

with the IC $\hat{\varphi}(z,0) = \hat{\varphi}_0(z) \in \mathbb{R}^{Np}$, $z \in [0,1]$. Since the input delay IC results from applying the known input, the IC $\varphi(z,0)$ is known so that $\hat{\varphi}(z,0) = \varphi(z,0)$ can be assumed implying $\hat{\varphi}(z,t) = \varphi(z,t)$, $t \geq 0$. The next theorem clarifies the stability of the resulting closed-loop system.

Theorem 3 (Stability of the observer-based controlled MAS): Let the conditions of Theorems 1 and 2 be satisfied. Then, the nominal networked controlled MAS consisting of the augmented plant (1), (7a) and the observer-based compensator (51) with (61) is asymptotically stable pointwise in space for any ICs $x_i(0) \in \mathbb{R}^n$, $\hat{x}_i(0) \in \mathbb{R}^n$, $v_i(0) \in \mathbb{R}^{pn_v}$, $i = 1,\ldots,N$ and piecewise continuous ICs of the corresponding transport equations in (15), (16) and (51) modelling the input and output delays.

Proof: Using (61) and applying the transformations (17) and (29) to (15), (16) leads to

$$\dot{e}(t) = A_{\text{cl}}e(t) - B_e\Delta(\tilde{x}(t), \tilde{\psi}(t)) \tag{63}$$

with

$$B_e = \begin{bmatrix} I_N \otimes B_1 \\ H \otimes B_2 \end{bmatrix} \tag{64}$$

and $\Delta(\tilde{x}(t), \tilde{\psi}(t)) = (H \otimes P_2 - I_N \otimes K_1)\tilde{x}(t) + \int_0^1 (H \otimes P_1(\zeta))\tilde{\psi}(\zeta,t)\mathrm{d}\zeta$ (cf. (31)). Thereby, the PDE subsystem (19d), (19e) is left unchanged, while for the PDE subsystem (31b), (31c) only the BC $\varphi(1,t) = (I_N \otimes K_1)e_x(t) + (I_N \otimes K_2)e_v(t) + \Delta(\tilde{x}(t), \tilde{\psi}(t))$ is different. For $0 \leq t < T_{\max}$, $T_{\max} = \max_{i=1,\ldots,N,\, j=1,\ldots,p} T_i^j$ (see (4)), the state ε is bounded and so is $\tilde{x}$ in (55), since (55a) is assumed to be exponentially stable. Consequently, all closed-loop states remain bounded, because A_{cl} is Hurwitz by assumption. As $\varepsilon(z,t) = 0$ pointwise in space for $t > T_{\max}$, the result $\tilde{\psi}(z,t) = (I_N \otimes P(z))\tilde{x}(t)$ follows from (53). With this, the resulting closed-loop system is a cascade of exponentially stable ODEs for e and $\tilde{x}$ and finite-time stable transport PDEs for φ and ψ (cf. (31)) so that it is asymptotically stable pointwise in space. Then, the bounded invertibility of the utilized transformations imply the same in the original coordinates. ■

VI. Robust cooperative output regulation

VII. Example

VIII. Concluding remarks

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Appendix I
Elements from graph theory and definitions

The communication topology between the agents is described by a *digraph* $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A_{\mathcal{G}}\}$, in which $\mathcal{V}$ is a set of N *nodes* $\mathcal{V} = \{v_1,\ldots,v_N\}$, one for each agent and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is a set of *edges* that models the information flow from the node v_j to v_i by $(v_i, v_j) \in \mathcal{E}$. This flow is weighted by $a_{ij} \geq 0$, which are the element of the *adjacency matrix* $A_{\mathcal{G}} = [a_{ij}] \in \mathbb{R}^{N\times N}$ with $a_{ii} = 0$, $i = 1,\ldots,N$. From this, the *Laplacian matrix* $L_{\mathcal{G}} \in \mathbb{R}^{N\times N}$ of the graph $\mathcal{G}$ can be derived by $L_{\mathcal{G}} = D_{\mathcal{G}} - A_{\mathcal{G}}$, where $D_{\mathcal{G}} = \operatorname{diag}(d_1,\ldots,d_N)$ with $d_i = \sum_{k=1}^N a_{ik}$, $i = 1,\ldots,N$. A *path* from the node v_j to the node v_i is a sequence of $r \geq 2$ distinct nodes $\{v_{l_1},\ldots,v_{l_r}\}$ with $v_{l_1} = v_j$ and $v_{l_r} = v_i$ such that $(v_k, v_{k+1}) \in \mathcal{E}$. A graph $\mathcal{G}$ is said to be *connected* if there is a node v, called the *root*, such that, for any node $v_i \in \mathcal{V}\setminus\{v\}$, there is a path from v to v_i. For further details on graph theory see, e. g., [7, Ch. 2]. In order to model the dynamics of MAS, it is necessary to introduce the *Kronecker product* $A \otimes B = [a_{ij}B] \in \mathbb{C}^{n_1n_2\times m_1m_2}$ of two matrices $A = [a_{ij}] \in \mathbb{C}^{n_1\times m_1}$ and $B \in \mathbb{C}^{n_2\times m_2}$ (see, e. g., [9]). In what follows the property

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{65}$$

with $A \in \mathbb{R}^{n_1\times m_1}$, $B \in \mathbb{R}^{n_2\times m_2}$, $C \in \mathbb{R}^{m_1\times m_3}$ and $D \in \mathbb{R}^{m_2\times m_4}$ of the Kronecker product is needed (see [9, Ch 1.3]).