Backstepping für parabolische MIMO-Systeme mit verteilten Totzeiten

· System

$$\dot{x} = x'' + a(z)x + \sum_{i=1}^{p} b_i(z) \int_0^{\bar{D}_i} \beta_i(\tau) u_i(t-\tau)\, d\tau$$

$$x'(0) = q_0 x(0)$$

$$x'(1) = q_1 x(1)$$

$$\eta_i = \int_0^{\bar{D}_i} \gamma_i(\tau) \int_0^1 c_i(\zeta) x(\zeta, t-\tau)\, d\zeta\, d\tau, \quad i = 1, \ldots, m$$

* Darstellung als PDE-PDE-PDE-Kaskade

Lsg der Transportgl.

$$\dot{v} = \partial_z v, \quad z \in [0, 1)$$

$$v(1) = u$$

Laplace-Trafo

$$s\check{v} = \partial_z \check{v}$$

$$\check{v}(1) = \check{u}$$

bzw.

$$\check{v}' = \frac{s}{\lambda} \check{v}$$

$$\check{v}(1) = \check{u}$$

Lsg AWP

$$\check{v} = e^{\frac{s}{\lambda} z} \check{v}(0)$$

RB

$$\check{v}(1) = e^{\frac{s}{\lambda}} \check{v}(0) \stackrel{!}{=} \check{u} \quad \Leftrightarrow \quad \check{v}(0) = e^{-\frac{s}{\lambda}} \check{u}$$

Lsg.

$$\check{v} = \underbrace{e^{\frac{s}{\lambda} z} \, e^{-\frac{s}{\lambda}}}_{e^{\frac{s}{\lambda}(z-1)}} \check{u}$$

$$\bullet\!-\!\circ$$

$$v = u\big(t + \underbrace{\tfrac{z-1}{\lambda}}_{-\tau}\big) = u\big(t + \tfrac{z}{\lambda} - \underbrace{\tfrac{1}{\lambda}}_{D}\big)$$

- Umrechnung der Integrale für Eingangsteil

$$\int_0^{D_i} b_i(\tau)\, u_i(t-\tau)\, d\tau = \int_0^1 \underbrace{\frac{b_i\big(D_i - \frac{z}{\lambda_i}\big)}{\lambda_i}}_{\tilde{b}_i(z) \,=\, D_i\, b_i(D_i(1-z))} v_i(z)\, dz$$

$$\tau = \overbrace{\tfrac{1}{\lambda}(1-z)}^{D_i - D_i z} \;\curvearrowright\; \lambda\tau = 1-z \;\curvearrowright\; z = 1-\lambda\tau$$

$$d\tau = -\tfrac{1}{\lambda}\, dz \qquad \curvearrowright\; z|_{D_i} = 1 - D_i\lambda = 0\,,\; z|_0 = 1$$

- Lsg. der Transportgl.

$$\dot{v} = \lambda v' - \tilde{\rho}_i(z) \int_0^1 c_i^T(\zeta)\, x(\zeta, t)\, d\zeta$$

$$v(1) = 0$$

$$y = v(0)$$

Laplace-Trafo

$$s\check{v} = \lambda \check{v}' - \tilde{\rho}_i(z) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta$$

$$\check{v}(1) = 0$$

$$\check{y} = \check{v}(0)$$

bzw.

$$\check{v}' = \frac{s}{\lambda} \check{v} - \frac{1}{\lambda}\tilde{\rho}_i(z) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta$$

$$\check{v}(1) = 0$$

$$\check{y} = \check{v}(0)$$

Lsg. hom. EWP

$$\check{v} = e^{\frac{s}{\lambda}(z-1)} \underbrace{\check{v}(1)}_{=0}$$

VdK

$$\check{v} = f(z)\, e^{\frac{s}{\lambda}(z-1)}$$

$$\check{v}' = f'(z)\, e^{\frac{s}{\lambda}(z-1)} + f(z)\,\frac{s}{\lambda}\, e^{\frac{s}{\lambda}(z-1)}$$

Eingehen in inhom. Dgl.

$$f'(z)\, e^{\frac{s}{\lambda}(z-1)} + \cancel{f(z)\cdot\frac{s}{\lambda}\, e^{\frac{s}{\lambda}(z-1)}} = \cancel{\frac{s}{\lambda}\, f(z)\, e^{\frac{s}{\lambda}(z-1)}}$$

$$- \tilde{\gamma}_i(z) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta$$

Dgl. für f

$$f' = - \tilde{\gamma}_i(z) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta\, e^{\frac{s}{\lambda}(1-z)}$$

mit EB $f(1) = 0$

$$\int_1^z f'(\bar{z})\, d\bar{z} = -\int_1^z \tilde{\gamma}_i(\bar{z}) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta\, e^{\frac{s}{\lambda}(1-\bar{z})}\, d\bar{z}$$

$$= - \int_1^z e^{\frac{s}{\lambda}(1-\bar{z})}\, \tilde{\gamma}_i(\bar{z}) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta\, d\bar{z}$$

$$= \int_z^1 e^{\frac{s}{\lambda}(1-\bar{z})}\, \tilde{\gamma}_i(\bar{z}) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta\, d\bar{z}$$

Lsg.

$$\check{v} = \int_z^1 \underbrace{e^{\frac{s}{\lambda}(1-\bar{z})} \, e^{\frac{s}{\lambda}(z-1)}}_{e^{\frac{s}{\lambda}(z-\bar{z})}} \tilde{\gamma}_i(\bar{z}) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta \, d\bar{z}$$

Ausgangsgl.

$$\check{y} = \check{v}(0) = \int_0^1 e^{-\frac{s}{\lambda}\bar{z}} \tilde{\gamma}_i(\bar{z}) \int_0^1 c_i(\zeta)\, \check{x}(\zeta)\, d\zeta\, d\bar{z}$$

$$y = \int_0^1 \tilde{\gamma}_i(\bar{z}) \int_0^1 c_i(\zeta)\, x(\zeta, t - \underbrace{\frac{\bar{z}}{\lambda}}_{D\bar{z}})\, d\zeta\, d\bar{z}$$

$$= \int_0^D \underbrace{\cancel{\frac{\tilde{\gamma}_i(\frac{\tau}{D})}{D}}}_{\gamma_i(\tau)} \int_0^1 c_i(\zeta)\, x(\zeta, t-\tau)\, d\zeta\, d\tau$$

$\tau = \frac{\bar{z}}{\lambda} \wedge \tau = D\tau'$

$\wedge \quad \tilde{\gamma}_i(\frac{\tau}{D}) = D\gamma_i(\tau) \Rightarrow \tilde{\gamma}_i(\tau) = D\gamma_i(D\tau)$

Also PDE-PDE-ODE-Kaskade mit Randeingang und -ausgang

$$\dot{x} = x'' + a(z)x + \underbrace{\underbrace{\sum_{i=1}^{p} b_i(z) \int_0^1 \beta_i(\zeta)\, v_i(\zeta)\, d\zeta}_{\int b^T(z,\zeta)\, v(\zeta)\, d\zeta \quad \text{mit } b^T(z,\zeta) = [b_1(z)\beta_1(\zeta) \ldots b_p \ldots]}}_{\int_0^1 [b_1 \ldots b_p] \begin{bmatrix} \beta_1 & & \\ & \ddots & \\ & & \beta_p \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_p \end{bmatrix} d\zeta}$$

$x'(0) = q_0 x(0)$

$x'(1) = q_1 x(1)$

$\dot{v} = \Lambda_v v', \quad z \in [0,1), \quad \Lambda_v = \operatorname{diag}(\frac{1}{D_i})$

$v(1) = u$

$$\dot{w} = \Lambda_w w' + \underbrace{\begin{bmatrix} \gamma_1(z) \int_0^1 c_1(\zeta)\, x(\zeta,t)\, d\zeta \\ \vdots \\ \gamma_m(z) \int_0^1 c_m(\zeta)\, x(\zeta,t)\, d\zeta \end{bmatrix}}_{\int C(z,\zeta)\, x(\zeta,t)\, d\zeta \quad \text{mit } C(z,\zeta) = \underbrace{\begin{bmatrix} \gamma_1(z) c_1(\zeta) \\ \vdots \\ \gamma_m(z) c_m(\zeta) \end{bmatrix}}_{\begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_m \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}}}, \quad z \in [0,1), \quad \Lambda_w = \operatorname{diag}(\frac{1}{\tau}$$

$w(1) = 0$

$y = w(0)$

* Entwurf der Zustandsrückführung

$$u = -\int_0^1 k_x(z)\, x(z)\, dz - \int_0^1 K_v(z)\, v(z)\, dz$$

· Entkopplung x-System

$$e_x = x - \int_0^1 q^T(z,\zeta)\, v(\zeta)\, d\zeta \quad \text{bzw.} \quad x = e_x + \int_0^1 q^T(z,\zeta)\, v(\zeta)\, d\zeta$$

· System in neuen Koord.

$$\dot e_x = \dot x - \int_0^1 q^T(z,\zeta)\, \dot v(\zeta)\, d\zeta$$

$$= \cancel{x''} + \cancel{a(z)x} + \int_0^1 b^T(z,\zeta)\, v(\zeta)\, d\zeta - \int_0^1 q^T(z,\zeta)\, \Lambda_v\, v'(\zeta)\, d\zeta$$

$$+ e_x'' + a(z) e_x - \tilde b^T(z) \int_0^1 k_x(z) e_x(z)\, dz$$

$$- \cancel{x''} - \int_0^1 q^T_{zz}(z,\zeta)\, v(\zeta)\, d\zeta - \cancel{a(z)x} + \int_0^1 a(z)\, q^T(z,\zeta)\, v(\zeta)\, d\zeta$$

$$+ \tilde b^T(z) \int_0^1 k_x(z)\, e_x(z)\, dz$$

Mit

$$-\int_0^1 q^T(z,\zeta)\, \Lambda_v\, v'(\zeta)\, d\zeta = -q^T(z,1)\, \Lambda_v \underbrace{v(1)}_{u} + q^T(z,0)\, \Lambda_v\, v(0)$$

$$+ \int_0^1 q^T_\zeta(z,\zeta)\, \Lambda_v\, v(\zeta)\, d\zeta$$

Koeff.vgl.

$$\int_0^1 \left(b^T(z,\zeta) + q^T_{zz}(z,\zeta) + a(z)\, q^T(z,\zeta) - q^T_\zeta(z,\zeta)\, \Lambda_v \right) v(\zeta)\, d\zeta$$

$$+ \tilde b^T(z) \int_0^1 k_x(z)\, e_x(z)\, dz - \underbrace{q^T(z,1)\, \Lambda_v}_{\tilde b^T(z)} \underbrace{\int_0^1 k_x(z)\, e_x(z)\, dz}_{= u}$$

$$+ q^T(z,0)\, \Lambda_v\, v(0)$$

$$\overset{!}{=} 0$$

RB

$$e_x' = x' - \int_0^1 q_z^T(z,\zeta)\, v(\zeta)\, d\zeta$$

Dann ist

$$e_x'(0) = x'(0) - \int_0^1 q_z^T(0,\zeta)\, v(\zeta)\, d\zeta$$

$$= q_0 x(0) - \int_0^1 q_z^T(0,\zeta)\, v(\zeta)\, d\zeta$$

$$= q_0 e_x(0) + \int_0^1 \underbrace{(q_0 q^T(0,\zeta) - q_z^T(0,\zeta))}_{\doteq 0^T} v(\zeta)\, d\zeta$$

$$e_x'(1) = q_1 e_x(1) + \int_0^1 \underbrace{(q_1 q^T(1,\zeta) - q_z^T(1,\zeta))}_{\doteq 0^T} v(\zeta)\, d\zeta$$

Also

$$\dot e_x = e_x'' + a(z) e_x - \tilde b^T(z) \int_0^1 k_x(z) e_x(z)\, dz$$

$$e_x'(0) = q_0 e_x(0)$$

$$e_x'(1) = q_1 e_x(1)$$

$$\dot v = \Lambda_v v'$$

$$v(1) = \int_0^1 k_x(z) e_x(z)\, dz$$

BEM: Entwurf von $k_x(z)$ auch von Tonelen

d.h. wird kein Entkopplungs-Trafo benötigt?

falls $q^T(z,\zeta)$ Lsg der Entkopplungsglen

$$q_\zeta^T(z,\zeta)\Lambda_v = -q_{zz}^T(z,\zeta) - a(z) q^T(z,\zeta) - b^T(z,\zeta)$$

$$q_z^T(0,\zeta) = q_0 q^T(0,\zeta)$$

$$q_z^T(1,\zeta) = q_1 q^T(1,\zeta)$$

$$q^T(z,0) = 0^T$$

mit $\tilde b^T = q^T(z,1)\Lambda_v$

* Lsg. der Entkopplungsglgn

· Modalkoord. von Λ_v

$$\underbrace{q_\zeta^T(z,\zeta)\Lambda_v e_i}_{\lambda_{v,i}\, q_{i,\zeta}(z,\zeta)} = -\underbrace{q_{zz}^T(z,\zeta)e_i}_{q_{i,zz}(z,\zeta)} - a(z)\underbrace{q^T(z,\zeta)e_i}_{q_i(z,\zeta)} - \underbrace{b^T(z,\zeta)e_i}_{b_i(z)\tilde{\beta}_i(\zeta)}$$

$$\underbrace{q_z^T(0,\zeta)e_i}_{q_{i,z}(0,\zeta)} = q_0\,\underbrace{q^T(0,\zeta)e_i}_{q_i(0,\zeta)}$$

$$\underbrace{q_z^T(1,\zeta)e_i}_{q_{i,z}(1,\zeta)} = q_1\,\underbrace{q^T(1,\zeta)e_i}_{q_i(1,\zeta)}$$

$$\underbrace{q^T(z,0)e_i}_{q_i(z,0)} = 0$$

bzw.

$$q_{i,\zeta}(z,\zeta) = -\frac{1}{\lambda_{v,i}}\left(q_{i,zz}(z,\zeta) - a(z)q_i(z,\zeta) - b_i(z)\tilde{\beta}_i(\zeta)\right), \quad (z,\zeta)\in(0,1)^2$$

$$q_{i,z}(0,\zeta) = q_0\, q_i(0,\zeta)$$

$$q_{i,z}(1,\zeta) = q_1\, q_i(1,\zeta)$$

$$q_i(z,0) = 0$$

. Modalkoord. von $d_z^2 + a(z)$

$$q_i(z,\zeta) = \sum_{j=1}^{\infty} q_{ij}^*(\zeta)\, \phi_j(z)$$

Einsetzen in PDE mit $b_i(z) = \sum_{j=1}^{\infty} b_{ij}^* \phi_j(z)$

$$\sum_{j=1}^{\infty} \dot q_{ij}^*(\zeta)\, \phi_j(z) = -\frac{1}{\lambda_{\kappa i}} \sum_{j=1}^{\infty} q_{ij}^*(\zeta) \underbrace{\left(\phi_j''(z) + a(z)\phi_j(z)\right)}_{\lambda_j \phi_j(z)}$$

$$+ \sum_{j=1}^{\infty} b_{ij}^* \tilde\beta_i(\zeta)\, \phi_j(z)$$

Mit AB $\quad q_i(z,0) = \sum_{j=1}^{\infty} \underbrace{q_{ij}^*(0)}_{=0} \phi_j(z) = 0 \quad$ folgt

$$\dot q_{ij}^*(\zeta) = -\frac{\lambda_j}{\lambda_{\kappa i}} q_{ij}^*(\zeta) + b_{ij}^* \tilde\beta_i(\zeta)$$

$$q_{ij}^*(0) = 0$$

Lsg

$$q_{ij}^*(\zeta) = \int_0^{\zeta} e^{-\frac{\lambda_j}{\lambda_{\kappa i}}(\zeta - \bar\zeta)}\, b_{ij}^* \tilde\beta_i(\bar\zeta)\, d\bar\zeta$$

$\rightarrow$ Konvergenz in L_2 folgt aus HG-Theorie

BEM: Bei konzentrierten Tokeinen ist $\tilde\beta_i(\zeta)$ eine δ-Fkt, was wg Integralbildung erlaubt ist

Modale Steuerbarkeit

Betrachten

$$\tilde{b}^T = q^T(z,1)\,\Lambda_v = \sum_{j=1}^{r} \underbrace{\left[q_{1j}^*(1) \;\dots\; q_{rj}^*(1) \right] \Lambda_v}_{\neq 0^T} \phi_j(z)$$

Also

$$q_{ij}^*(1) = \int_0^1 e^{-\frac{\lambda_j}{x_1}(1-\bar{\tau})}\, b_{ij}^*\, \tilde{\beta}_i(\bar{\tau})\, d\bar{\tau} \neq 0$$

für mind. ein $i \in \{1,\dots,p\}$ $\Rightarrow$ λ_j modal steuerbar