EE363 Review Session 2:
Invariant subspaces, Sylvester equation, PBH

This review session summarizes our treatment of invariant subspaces, the sylvester operator, and the PBH criterion. The material in this review session can be found in EE363 Lecture 6.

Announcements: None.

Invariant subspaces

Suppose A is an $n \times n$ matrix. A subspace $\mathcal{V}$ of $\mathbf{R}^n$ or $\mathbf{C}^n$ (if A is complex) is called A-invariant or an *invariant subspace* of A, if for every $v \in \mathcal{V}$ the vector Av is also in $\mathcal{V}$. A simple example is when $\mathcal{V}$ is spanned by a single eigenvector of A. More generally, $\mathcal{V}$ may be spanned by a subset of the eigenvectors of A.

Example: Suppose that $\mathcal{V}$ is A-invariant. Show that if $\mathcal{R}(B) \subseteq \mathcal{V}$, then $\mathcal{R}(\mathcal{C}) \subseteq \mathcal{V}$. Here $\mathcal{C}$ is the controllability matrix associated with A and B.

Solution: Let v, w be two vectors such that $v = \mathcal{C}w$. Then,

$$v = Bw_1 + ABw_2 + \cdots + A^{n-1}Bw_n$$

since $\mathcal{R}(B) \subseteq \mathcal{V}$ the vectors Bw_i will all lie in $\mathcal{V}$, since $\mathcal{V}$ is $A-$invariant A^kBw_{k+1} will also lie in $\mathcal{V}$. By observing that v above is a linear combination of vectors that all lie in $\mathcal{V}$, we have that $v \subseteq \mathcal{V}$. Since this holds for all w's we get $\mathcal{R}(\mathcal{C}) \subseteq \mathcal{V}$.

Example: It was shown in the notes that $\mathcal{R}(M)$ is A-invariant if and only if there is an X such that $AM = MX$. In addition, if this condition holds and $M \in \mathbf{R}^{n\times k}$ has rank k, then the eigenvalues of X are a subset of the eigenvalues of A. *Claim:* each Jordan block of X is a submatrix of a Jordan block of A. Provide a proof or counterexample.

Solution: Take any Jordan block of X. Suppose it is associated with eigenvalue λ and has size m. Let $v_1, \ldots, v_m$ be its generalized eigenvectors, and by definition, we have $Xv_1 = \lambda v_1, Xv_2 = \lambda v_2 + v_1, \cdots, Xv_m = \lambda v_m + v_{m-1}$. Using the condition $AM = MX$, we get $A(Mv_1) = \lambda(Mv_1), A(Mv_2) = \lambda(Mv_2) + (Mv_1), \cdots, A(Mv_m) = \lambda(Mv_m) + (Mv_{m-1})$. Since M is full column rank, $Mv_1, \ldots, Mv_m$ are also linear independent. Hence they are the generalized eigenvectors of A with eigenvalue λ, which implies that A has a Jordan block associated with λ of size at least m.

Sylvester operator

The sylvester equation is

$$AX + XB = C,$$

where $A, B, X, C \in \mathbf{R}^{n\times n}$. Expressing as $S(X) = C$, we refer to $S(X)$ as the sylvester operator.

Example Assuming that A and B are diagonalizable, show that the eigenvalues of the Sylvester operator are $\lambda_i + \mu_j$ for $i, j = 1, \ldots, n$, where λ_i is an eigenvalue of A and μ_j is an eigenvalue of B. What are the associated eigenvectors (matrices) X_{ij}?

Solution: Use $X = v_i w_j^T$, where $Av_i = \lambda_i v_i$ and $w_j^T B = \mu_j w_j^T$. This gives

$$S(v_i w_j^T) = A(v_i w_j^T) + (v_i w_j^T)B = (\lambda_i + \mu_j)v_i w_j^T$$

So $\lambda_i + \mu_j$ is an eigenvalue of S with eigenvector $v_i w_j^T$.

Example: Show that the set of n^2 matrices X_{ij}, $i,\ j = 1, \ldots, n$, spans $\mathbf{R}^{n\times n}$. This means that the Sylvester operator is diagonalizable (when A and B are).

Solution: To show that the matrices $v_i w_j^T$ span $\mathbf{R}^{n\times n}$, we need to show that these matrices are linear independent, *i.e.*,

$$\sum_{i,j=1}^{n} \alpha_{ij} v_i w_j^T = 0$$

-> Matrices independent?

if and only if $\alpha_{ij} = 0$ for all i, j. Let $\hat{w}_k^T$ be the kth left eigenvector of A and $\hat{v}_l$ be the lth right eigenvector of B. By the orthogonality of left and right eigenvectors,

-> But only the EV from different EW are orthogonal!

$$\hat{w}_k^T \left(\sum_{i,j=1}^{n} \alpha_{ij} v_i w_j^T \right) \hat{v}_l = \alpha_{kl} = 0$$

Since this holds for all k, l, the eigenvectors of the Sylvester operator span $\mathbf{R}^{n\times n}$.

Remark. This shows that, when A and B are diagonalizable, the Sylvester operator is nonsingular if and only if no eigenvalue of A is the negative of an eigenvalue of B. In fact, this holds even when A and B are not diagonalizable.

PBH

The PBH controllability and observability tests give us alternative ways to test for controllabilty and observability.

PBH Controllability Test: (A, B) is controllable, if and only if there exists no left eigenvector of A orthogonal to the columns of B.

Remarks: There are two implications to prove. In both cases, we will prove the contrapositive statement.

1. If there exists a left eigenvector of A orthogonal to the columns of B, then (A, B) is uncontrollable:

   Proof: Suppose that $w \in \mathbf{C}^n, w \neq 0$ is a left eigenvector of A, that is orthogonal to the columns of B. Then

   $$w^T A = \lambda w^T, \quad w^T B = 0,$$ → Diese Eigenmode kann nicht kontrolliert werden

   so

   $$w^T [\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array}] = 0,$$

   and therefore

   $$\textbf{Rank}\left([\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array}]\right) < n,$$

   which means the controllability matrix has linearly dependent rows, and is not controllable.

   Example: Let $\dot{x}_t = Ax_t + Bu_t$, where

   $$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}.$$

   Is (A, B) controllable?

   Solution: The eigenvalues of A are found to be $\lambda_1 = -1, \lambda_2 = -2, \lambda_3 = -3$, associated with left eigenvectors

   $$w_1 = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}, w_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

   We find that $w_3^T B = 0$. In addition,

   $$w_3^T \left[\begin{array}{ccc} B & AB & A^2B \end{array}\right] = \left[\begin{array}{ccc} 0 & 0 & 0 \end{array}\right].$$

   Hence (A, B) is *not* controllable.

2. If (A, B) is not controllable, there exists a left eigenvector of A orthogonal to the columns of B.

   Proof: Suppose

   $$\textbf{Rank}\left([\begin{array}{cccc} B & AB & \dots & A^{n-1}B \end{array}]\right) < n,$$

   we can change coordinates as is shown in EE363 Lecture 6,

   $$\tilde{A} = T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = T^{-1}B = \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix}.$$

Let λ be an eigenvalue of $\tilde{A}_{22}$, and w_{22} be an associated left eigenvector. Define

$$w = T^{-T} \begin{bmatrix} 0 \\ w_{22} \end{bmatrix} \neq 0,$$

and so

$$\begin{aligned}
w^T A &= \begin{bmatrix} 0 \\ w_{22} \end{bmatrix}^T T^{-1} \left(T \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} T^{-1} \right) \\
&= \begin{bmatrix} 0 & w_{22}^T \tilde{A}_{22} \end{bmatrix} T^{-1} \\
&= \begin{bmatrix} 0 & \lambda w_{22}^T \end{bmatrix} T^{-1} \\
&= \lambda \begin{bmatrix} 0 & w_{22}^T \end{bmatrix} T^{-1} \\
&= \lambda w^T
\end{aligned}$$

similarly,

$$\begin{aligned}
w^T B &= \begin{bmatrix} 0 \\ w_{22} \end{bmatrix}^T T^{-1} \left(T \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} 0 & w_{22}^T \end{bmatrix} \begin{bmatrix} \tilde{B}_{11} \\ 0 \end{bmatrix} \\
&= 0. \qquad \square
\end{aligned}$$

Example: Using (A, B) from the previous example, construct a left eigenvalue of A that is orthogonal to the columns of B

Solution: First, we form the controllabilty matrix

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -3 & 7 \\ -3 & 7 & -15 \end{bmatrix}$$

from where we can construct T,

$$T = \begin{bmatrix} 1 & -3 & 1 \\ -3 & 7 & 0 \\ -7 & -15 & 0 \end{bmatrix},$$

where we have concatenated e_1 to the last two columns of $\mathcal{C}$. $\lambda = \tilde{A}_{22} = -3$, and choosing $w_{22} = 1$, we get

$$w = T^{-T} \begin{bmatrix} 0 \\ 0 \\ w_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix}.$$

Check:

$$w^T A = \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix}^T \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix} = \lambda w^T,$$

$$w^T B = \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} = 0.$$

Thus, we have found a left eigenvector of A, orthogonal to the columns of B.

PBH Observability Test: PBH observability test says (C, A) is observable if and only if there exists no right eigenvector of A orthogonal to the columns of C.

Proof: similar to proof of PBH observability criterion.

Example: Given $A = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$, find necessary and sufficient conditions for (C, A) to be observable.

Solution: Suppose $v \in \mathbf{C}^n$ is an eigenvector of A. Using the PBH observability criterion, (C, A) is observable if and only if

$$Cv \neq 0,$$

i.e., every eigenspace of A does not intersect with the $\mathcal{N}(C)$, except at the origin. Since A is diagonal, this implies that the columns of C corresponding to the same diagonal elements of A should be linearly independent.