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RESEARCH ARTICLE

Robust cooperative output regulation for a network of parabolic PDEs with input and communication delays

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Summary

KEYWORDS:
Parabolic systems, multi-agent systems, cooperative output regulation, backstepping, time delays.

1 | INTRODUCTION

1.1 | Background and motivation

1.2 | Contributions

1.3 | Organization

1.4 | Notation

In order to increase readability, partial derivatives w. r. t. time are represented by $\dot{x}(z,t) = \partial_t x(z,t)$. Similarly, $x'(z,t) = \partial_z x(z,t)$ and $x''(z,t) = \partial_{zz} x(z,t)$ denote partial derivatives w. r. t. space. For the ordinary derivative the Leibniz's notation $\mathrm{d}_z = \mathrm{d}/\mathrm{d}z$ or Lagrange's notation $(\cdot)' = \mathrm{d}/\mathrm{d}z$ are utilized. Furthermore, it is convenient to define $1_N = \mathrm{col}(1,\dots,1) \in \mathbb{R}^N$.

2 | PROBLEM FORMULATION

Consider a heterogeneous multi-agent system (MAS) with a communication topology described by a connected graph $\mathcal{G}$ with the *Laplacian matrix*

$$L_{\mathcal{G}} = \begin{bmatrix} L_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ L_{1r} & \dots & L_{rr} \end{bmatrix}, \tag{1}$$

in which the matrices $L_{ii} \in \mathbb{R}^{N_i \times N_i}$, $i = 1,\dots,r$, $N_1 + \dots + N_r = N > 1$ are *irreducible* (see Appendix A). Note that this means no loss of generality. If the graph $\mathcal{G}$ is not strongly connected, then there always exists a similarity transformation with a permutation matrix into this form (see, e. g., [6, Ch. 2.8.1]). Otherwise $r = 1$ and $N_1 = N$ follows. As a consequence of (1), the corresponding *adjacency matrix* $A_{\mathcal{G}}$ takes the form

$$A_{\mathcal{G}} = \begin{bmatrix} A_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ A_{1r} & \dots & A_{rr} \end{bmatrix} \tag{2}$$

where $A_{ij} \in \mathbb{R}^{N_i \times N_j}$ (see Appendix A). In view of this, the heterogeneous MAS is described by r subsets of N_i homogeneous parabolic agents

$$\dot{x}_i(z,t) = \bar{\lambda}_i(z)x_i''(z,t) + \bar{a}_i(z)x_i(z,t) + G_{1,i}(z)d_i(t), \qquad (z,t) \in (0,1) \times \mathbb{R}^+ \tag{3a}$$

$$x_i'(0,t) = \bar{q}_i x_i(0,t) + G_{2,i}d_i(t), \qquad t > 0 \tag{3b}$$

$$x_i(1,t) = u_i(t - \bar{D}_i) + G_{3,i}d_i(t), \qquad t > 0 \tag{3c}$$

$$y_i(t) = \bar{C}_i[x_i(t)] + G_{4,i}d_i(t), \qquad t \geq 0 \tag{3d}$$

for $i = 1, \dots, r$. They arise from, for example, the processing and connecting time for the information packets arriving at each agent. The state of (3) is $x_i(z,t) \in \mathbb{R}^{N_i}$, $\bar{\lambda}_i = 1 + \Delta\lambda_i \in C^2[0,1]$ and $\bar{a}_i = a_i + \Delta a_i \in C[0,1]$. Note that by making use of the spatial and time transformations in [5, Ch. 2.8 & 4.8], the PDE (3a) can always be obtained for general $\bar{\lambda}_i(z)$ and an additional advection term. The input locations of the *local disturbance* $d_i(t) \in \mathbb{R}^{m_i}$ are characterized by $G_{1,i}(z) \in \mathbb{R}^{N_i \times m_i}$ with piecewise continuous elements and $G_{k,i} \in \mathbb{R}^{N_i \times m_i}$, $k = 2,3,4$, which have not to be available for the controller design. In (3b) the coefficient $\bar{q}_i = q_i + \Delta q_i \in \mathbb{R}$, specifies Robin or Neumann BCs and $u_i(t) \in \mathbb{R}$ is the input, which is subject to input delays with delay time $\bar{D}_i = D_i + \Delta D_i \in \mathbb{R}^+$.

Remark 1. It should be remarked that a Robin or Neumann actuation in (3c) is also possible. For this, however, the local backstepping stabilization of the agents has to be replaced by a state feedback that only assigns a finite number of eigenvalues to specify the stability margin of the closed-loop system (see [3]). $\triangleleft$

The output $y_i(t) \in \mathbb{R}^{N_i}$ to be controlled is available for the networked controller and can be defined distributed in-domain, point-wise in-domain, at the boundaries and combinations thereof. This leads to the formal *output operator*

$$\bar{C}_i[h] = \int_0^1 \bar{c}_i(\zeta)h(\zeta)\mathrm{d}\zeta + \bar{c}_{b0,i}h(0) + \bar{c}_{b1,i}h(1) \tag{4}$$

for $h(z) \in \mathbb{R}$ with $\bar{c}_{bk,i} = c_{bk} + \Delta c_{bk,i} \in \mathbb{R}$, $k = 0,1$, and

$$\bar{c}_i(z) = \bar{c}_{0,i}(z) + \sum_{k=1}^{l_i} \bar{c}_{k,i}\delta(z - z_k), \tag{5}$$

in which $\bar{c}_{0,i}(z) = c_{0,i}(z) + \Delta c_{0,i}(z) \in \mathbb{R}$ with $\bar{c}_{0,i}(z)$ piecewise continuous functions, $\bar{c}_{k,i} = c_{k,i} + \Delta c_{k,i} \in \mathbb{R}$ and $z_k \in (0,1)$, $k = 1, \dots, l_i$. The known *nominal parameters* are a_i, q_i, D_i, $c_{0,i}$, $c_{bk,i}$, $k = 0,1$ and $c_{k,i}$, z_k, $k = 1, \dots, l_i$, whereas

$$\Delta\lambda_i(z), \Delta a_i(z), \Delta q_i, \Delta D_i, \Delta c_{bk,i}(z), \Delta c_{0,i}(z) \text{ and } \Delta c_{k,i} \tag{6}$$

represent unknown *model uncertainties*.

The initial conditions (ICs) of the MAS (3) are $x_i(z,0) = x_{i,0}(z) \in \mathbb{R}^{N_i}$, $i = 1, \dots, r$, and the input delay IC $u_i(t) = u_{i0}(t) \in \mathbb{R}^{N_i}$, $-\bar{D}_i \leq t < 0$.

For all agents the common *reference input* $r(t) \in \mathbb{R}$ and the local disturbances d_i, $i = 1, \dots, r$, acting on the individual agents are described by the *signal model*

$$\dot{w}(t) = Sw(t), \qquad t > 0, \quad w(0) = w_0 \in \mathbb{R}^{n_w} \tag{7a}$$

$$r(t) = p^\top w(t), \qquad t \geq 0 \tag{7b}$$

$$d_i(t) = P_i w(t), \qquad t \geq 0 \tag{7c}$$

with $p \in \mathbb{R}^{n_w}$, $P_i \in \mathbb{R}^{m_i \times n_w}$ and the pairs $(p_r^\top, S)$, (P_i, S) observable. It is assumed that the *spectrum* $\sigma(S)$ of $S \in \mathbb{R}^{n_w \times n_w}$ has only eigenvalues on the imaginary axis, i. e., $\sigma(S) \subset \mathrm{j}\mathbb{R}$, and that S is diagonalizable. Hence, (7) describes a wide class of signals including constant and trigonometric functions of time as well as linear combinations thereof. In the sequel, it is assumed that only S in (7a) is known for the networked controller design.

The agents are divided into two groups. One group has access to the reference input r and is therefore called the *informed agents*. In contrast, the information about the reference input can only be broadcast to the remaining *uninformed agents* through a communication network. As a consequence, a cooperative state feedback regulator is required, in order to achieve output regulation for all agents. For this, introduce the *leader-follower matrix*

$$H = L_{\mathcal{G}} + L_0 = \begin{bmatrix} H_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ H_{1r} & \dots & H_{rr} \end{bmatrix} \tag{8}$$

with $H_{ii} \in \mathbb{R}^{N_i \times N_i}$, $i = 1, \dots, r$, and

$$L_0 = \operatorname{diag}(a_{10}, \dots, a_{N0}) \in \mathbb{R}^{N \times N}. \tag{9}$$

The latter matrix describes the communication between the agents and the leader represented by the global reference model contained in (7). In particular, $a_{i,0} > 0$ holds for the informed agents, while $a_{i,0} = 0$ is valid for the uninformed agents. In the following it is assumed that the leader coinciding with agent 0 is the root of the digraph $\bar{\mathcal{G}}$ describing the communication network with node set $\bar{\mathcal{V}} = \{0, 1, \dots, N\}$ and edge set $\bar{\mathcal{E}}$. Then, by removing all edges of $\bar{\mathcal{E}}$, that are incident to the root, the subgraph $\mathcal{G}$ with node set $\mathcal{V} = \{1, \dots, N\}$ and the edge set $\mathcal{E}$ is obtained. With this, the *cooperative state feedback regulator* respecting the communication topology described by (8) has the following form

$$\dot{v}_i(t) = (I_{N_i} \otimes S)v_i(t) + \sum_{j=1}^{i} \big((L_{ij} \otimes b_{y_j})\bar{y}_j(t) + a_{i0}(\bar{y}_i(t) - \bar{r}(t))\big), \qquad t > 0 \tag{10a}$$

$$u_i(t) = \mathcal{K}_i[v(t), x(t)], \qquad t \geq 0 \tag{10b}$$

for $i = 1, \dots, r$ (see also [2]). In (10a) the pairs (S, b_{y_j}), $j = 1, \dots, i$, are controllable and

$$\bar{y}_j(t) = \int_0^{\bar{T}_j} \bar{\gamma}_j(\tau) y_j(t - \tau)\mathrm{d}\tau \tag{11a}$$

$$\bar{r}(t) = \int_0^{\bar{T}_i} \bar{\gamma}_i(\tau) r(t - \tau)\mathrm{d}\tau \tag{11b}$$

represent distributed delayed measurements and reference inputs due to the network communication. Therein,

$$\bar{\gamma}_j(\tau) = \bar{\gamma}_{0,j}(\tau) + \sum_{k=1}^{\bar{l}_j} \bar{\gamma}_{k,j}\delta(\tau - \tau_k) \tag{12}$$

holds with $\bar{\gamma}_{0,j} = \gamma_{0,j} + \Delta\gamma_{0,j} \in \mathbb{R}$ piecewise continuous functions, $\bar{\gamma}_{k,j} = \gamma_{k,j} + \Delta\gamma_{k,j} \in \mathbb{R}$ and $\bar{T}_j = T_j + \Delta T_j \in \mathbb{R}^+$. In (10b) $\mathcal{K}_i$, $i = 1, \dots, r$, is a formal *feedback operator* acting on the states $x = \operatorname{col}(x_1, \dots, x_r)$ and $v = \operatorname{col}(v_1, \dots, v_r)$. It should be remarked that the state feedback in (10b) has to respect the communication topology. This leads to constraints in the networked regulator design. It is assumed that the nominal parameters $\gamma_{k,j}$, $k = 0, 1, \dots, \bar{l}_j$, and T_j are known. As a consequence of (10) and (11a), these system parameters have to be the same for all N_i agents described by (3), but need not coincide for different i in (3). The unknown model uncertainties for (11) are

$$\Delta\gamma_{k,j} \quad \text{and} \quad \Delta T_j. \tag{13}$$

Remark 2. If the graph $\mathcal{G}$ is *cycle free* (see Appendix A), then the Laplacian matrix $L_{\mathcal{G}}$ can be mapped with a similarity transformation using a permutation matrix into a lower triangular matrix (see [8]). Hence, $N_i = 1$, $i = 1, \dots, r$, and $r = N$ holds in (1). Consequently, all agents and delays can be different. ◁

The IC of the networked controller (10) are the state IC $v_i(0) = v_{i,0} \in \mathbb{R}^{N_i n_w}$ and the communication delay IC $\bar{y}_j(t) = y_{j0}(t) \in \mathbb{R}^{N_j}$, $-\bar{T}_j \leq t < 0$.

The *robust cooperative output regulation* amounts to designing the networked controller (10), in order to ensure stability of the resulting networked controlled MAS and the *reference tracking*

$$\lim_{t\to\infty} e_{y_i}(t) = \lim_{t\to\infty}(y_i(t) - 1_N r(t)) = 0, \tag{14}$$

$i = 1, \dots, r$, for all ICs of the MAS (3), the signal model (7) and the controller (10). Furthermore, the property (14) should be robust in the sense that it holds despite of all model uncertainties (6) and (13), for which the nominal networked controller stabilizes the networked controlled MAS.

3 | COOPERATIVE STATE FEEDBACK REGULATOR DESIGN

According to the *cooperative internal model principle* (see [12] for lumped-parameter and [2] for distributed-parameter systems) the regulator (10) is designed to stabilize the nominal MAS (3), that results from setting the model uncertainties (6) and (13) to zero. For this, the lumped and distributed delays appearing in (10) are represented by the solution of transport equations (see

[1, Ch. 3]. This leads to the *hyperbolic-parabolic ODE-PDE cascade*

$$\dot{\varphi}(z,t) = \Lambda\varphi'(z,t) \qquad (z,t) \in (0,1)\times\mathbb{R}^+ \tag{15a}$$
$$\varphi(1,t) = u(t), \qquad t>0 \tag{15b}$$
$$\dot{x}(z,t) = x''(z,t) + A(z)x(z,t), \qquad (z,t) \in (0,1)\times\mathbb{R}^+ \tag{15c}$$
$$x'(0,t) = Q_0 x(0,t), \qquad t>0 \tag{15d}$$
$$x(1,t) = \varphi(0,t), \qquad t>0 \tag{15e}$$
$$\dot{\psi}(z,t) = \bar{\Lambda}\psi'(z,t) + B(z)\mathcal{C}[x(t)], \qquad (z,t) \in (0,1)\times\mathbb{R}^+ \tag{15f}$$
$$\psi(1,t) = 0, \qquad t>0 \tag{15g}$$
$$\dot{v}(t) = (I_N \otimes S)v(t) + (H \boxtimes b_y)\psi(0,t), \qquad t>0 \tag{15h}$$

with $\varphi(z,t) \in \mathbb{R}^N$, $\psi(z,t) \in \mathbb{R}^N$ and

$$\Lambda = \operatorname{diag}(\lambda_1 I_{N_1},\dots,\lambda_r I_{N_r}) \tag{16a}$$
$$A(z) = \operatorname{diag}(a_1(z) I_{N_1},\dots,a_r(z) I_{N_r}) \tag{16b}$$
$$Q_0 = \operatorname{diag}(q_1 I_{N_1},\dots,q_r I_{N_r}) \tag{16c}$$
$$\bar{\Lambda} = \operatorname{diag}(\bar{\lambda}_1 I_{N_1},\dots,\bar{\lambda}_r I_{N_r}) \tag{16d}$$
$$B(z) = \operatorname{diag}(\kappa_1(z) I_{N_1},\dots,\kappa_r(z) I_{N_r}) \tag{16e}$$
$$\mathcal{C}[x] = \operatorname{col}(\mathcal{C}_1[x_1],\dots,\mathcal{C}_r[x_r]) \tag{16f}$$
$$b_y = \operatorname{col}(b_{y_1},\dots,b_{y_r}), \tag{16g}$$

in which $\lambda_i = \frac{1}{D_i}$, $\bar{\lambda}_i = \frac{1}{T_i}$, $\kappa_i(z) = \gamma_i(T_i z)$, $i = 1,\dots,r$ and $\mathcal{C}$ are the output operators (4) in the nominal case. The *extended Kronecker product* $\boxtimes$ in (15h) is defined in Appendix A.

3.1 | Local stabilization of the MAS

In the first step of determining the stabilizing networked controller (10) the agents are locally stabilized. This requires to determine the *local backstepping transformation*

$$\tilde{x}(z,t) = x(z,t) - \int_0^z K(z,\zeta)x(\zeta,t)\mathrm{d}\zeta = \mathcal{T}[x(t)](z) \tag{17a}$$
$$\tilde{\varphi}(z,t) = \varphi(z,t) - \int_0^z P(z,\zeta)\varphi(\zeta,t)\mathrm{d}\zeta - \int_0^1 L(z,\zeta)x(\zeta,t)\mathrm{d}\zeta \tag{17b}$$

with the kernels $K(z,\zeta) \in \mathbb{R}^{N\times N}$, $P(z,\zeta) \in \mathbb{R}^{N\times N}$ and $L(z,\zeta) \in \mathbb{R}^{N\times N}$, that is independently applied to each agent. For this, the kernels need to have a particular structure. More specifically, they are of the form

$$K(z,\zeta) = \operatorname{diag}(k_1(z,\zeta) I_{N_1},\dots,k_r(z,\zeta) I_{N_r}) \tag{18a}$$
$$P(z,\zeta) = \operatorname{diag}(p_1(z,\zeta) I_{N_1},\dots,p_r(z,\zeta) I_{N_r}) \tag{18b}$$
$$L(z,\zeta) = \operatorname{diag}(l_1(z,\zeta) I_{N_1},\dots,l_r(z,\zeta) I_{N_r}), \tag{18c}$$

in which $k_i(z,\zeta) \in \mathbb{R}$, $p_i(z,\zeta) \in \mathbb{R}$ and $l_i(z,\zeta) \in \mathbb{R}$ for $i = 1,\dots,r$. The latter kernels are determined such that (15) is mapped with the feedback

$$u(t) = \int_0^1 P(1,\zeta)\varphi(\zeta,t)\mathrm{d}\zeta + \int_0^1 L(1,\zeta)x(\zeta,t)\mathrm{d}\zeta + \bar{u}(t) \tag{19}$$

and the new input $\bar{u}(t) \in \mathbb{R}^N$ into the *intermediate target system*

$$\dot{\tilde{\varphi}}(z,t) = \Lambda \tilde{\varphi}'(z,t) \tag{20a}$$
$$\tilde{\varphi}(1,t) = \bar{u}(t) \tag{20b}$$
$$\dot{\tilde{x}}(z,t) = \tilde{x}''(z,t) + \tilde{A}\tilde{x}(z,t) \tag{20c}$$
$$\tilde{x}'(0,t) = 0 \tag{20d}$$
$$\tilde{x}(1,t) = \tilde{\varphi}(0,t) \tag{20e}$$
$$\dot{\psi}(z,t) = \bar{\Lambda}\psi'(z,t) + B(z)C\mathcal{T}^{-1}[\tilde{x}(t)] \tag{20f}$$
$$\psi(1,t) = 0 \tag{20g}$$
$$\dot{v}(t) = (I_N \otimes S)v(t) + (H \boxtimes b_y)\psi(0,t) \tag{20h}$$

with

$$\tilde{A} = \operatorname{diag}(-\mu_1 I_{N_1}, \dots, -\mu_r I_{N_r}). \tag{21}$$

Therein, the PDE subsystem (20c)–(20e) describing the agent target dynamics is exponentially stable for suitable $\mu_i \in \mathbb{R}$, $i = 1, \dots, r$. For this, the backstepping transformation (17a) is utilized to map (15c)–(15e) into the *target system*

$$\dot{\tilde{x}}(z,t) = \tilde{x}''(z,t) + \tilde{A}\tilde{x}(z,t) \tag{22a}$$
$$\tilde{x}'(0,t) = 0 \tag{22b}$$
$$\tilde{x}(1,t) = \varphi(0,t) - \int_0^1 K(1,\zeta)x(\zeta,t)\mathrm{d}\zeta. \tag{22c}$$

By making use of the calculation in, e. g., [4] and taking the block structure of $A(z)$, Q_0, $\tilde{A}$ in (16b), (16c), (21) into account, the *kernel equations*

$$k_{i,zz}(z,\zeta) - k_{i,\zeta\zeta}(z,\zeta) = (\mu_i + a_i(\zeta))k_i(z,\zeta), \quad 0 < \zeta < z < 1 \tag{23a}$$
$$k_{i,\zeta}(z,0) = q_i k_i(z,0) \tag{23b}$$
$$k_i(z,z) = q_i - \tfrac{1}{2}\int_0^z (\mu_i + a_i(\zeta))\mathrm{d}\zeta \tag{23c}$$

for $i = 1, \dots, r$ result. It is verified in [10] that (23) has a unique C^2-solution. Furthermore, the inverse transformation exists and is given by

$$x(z,t) = \tilde{x}(z,t) + \int_0^z K_I(z,\zeta)\tilde{x}(\zeta,t)\mathrm{d}\zeta = \mathcal{T}^{-1}[\tilde{x}(t)](z) \tag{24}$$

with

$$K_I(z,\zeta) = \operatorname{diag}(k_{1,I}(z,\zeta)I_{N_1}, \dots, k_{r,I}(z,\zeta)I_{N_r}). \tag{25}$$

Therein, the kernel $K_I(z,\zeta) \in \mathbb{R}$ follows from similar kernel equations. With this, the operator $C\mathcal{T}^{-1}$ in (20f) takes the form

$$C\mathcal{T}^{-1}[\tilde{x}(t)] = \int_0^1 \tilde{C}(\zeta)\tilde{x}(\zeta,t)\mathrm{d}\zeta + C_{b,0}\tilde{x}(0,t) + C_{b,1}\tilde{x}(1,t) \tag{26}$$

with

$$\tilde{C}(z) = \operatorname{diag}(\tilde{c}_1(z)I_{N_1}, \dots, \tilde{c}_r(z)I_{N_r}) \tag{27a}$$
$$C_{b,0} = \operatorname{diag}(c_{b0,1}I_{N_1}, \dots, c_{b0,r}I_{N_r}) \tag{27b}$$
$$C_{b,1} = \operatorname{diag}(c_{b1,1}I_{N_1}, \dots, c_{b1,r}I_{N_r}) \tag{27c}$$

and

$$\tilde{c}_i(z) = c_{b1,i}k_{I,i}(1,z) + c_i(z) + \int_z^1 c_i(\zeta)k_{I,1}(\zeta,z)\mathrm{d}\zeta \tag{28}$$

by changing the order of integration and utilizing simple calculations.

In order to map (15a), (15b) and (22) into (20a)–(20e) apply (17b) to (15a), (15b). With the same calculations as in [3] and taking (16a)–(16c) and (18a) into account the *kernel equations*

$$p_{i,z}(z,\zeta) + p_{i,\zeta}(z,\zeta) = 0, \quad 0 < \zeta < z < 1 \tag{29a}$$
$$p_i(z,0)\lambda_i = -l_{i,\zeta}(z,1) \tag{29b}$$

and

$$\lambda_i l_{i,z}(z,\zeta) = l_{i,\zeta\zeta}(z,\zeta), \quad (z,\zeta) \in (0,1)^2 \tag{30a}$$
$$l_{i,\zeta}(z,0)\lambda_i = q_i l_i(z,0) \tag{30b}$$
$$l_i(z,1) = 0 \tag{30c}$$
$$l_i(0,\zeta) = k_i(1,\zeta) \tag{30d}$$

are obtained for $i = 1, \dots, r$. By making use of the method of characteristics, the solution of (29) can be determined explicitly. In addition, it is easily verified that (30) has a unique mild solution in $L_2(0,1)$. This follows from the fact that (30) can be seen as an initial boundary value problem, in which ζ is the spatial and z is the temporal variable (see also [3]).

3.2 | Cooperative decoupling of the cooperative internal model

The *final target system* is the ODE-PDE-ODE cascade

$$\dot{e}_v(t) = (I_N \otimes S - H \boxtimes \mathrm{col}_r(\lambda_i \tilde{s}_i(1)k_{v_i}^\top))e_v(t) \tag{31a}$$
$$\dot{\tilde{\varphi}}(z,t) = \Lambda\tilde{\varphi}'(z,t) \tag{31b}$$
$$\tilde{\varphi}(1,t) = (I_N \boxtimes K_v)e_v(t) \tag{31c}$$
$$\dot{\tilde{x}}(z,t) = \tilde{x}''(z,t) + \tilde{A}\tilde{x}(z,t) \tag{31d}$$
$$\tilde{x}'(0,t) = 0 \tag{31e}$$
$$\tilde{x}(1,t) = \tilde{\varphi}(0,t) \tag{31f}$$
$$\dot{\psi}(z,t) = \bar{\Lambda}\psi'(z,t) + B(z)C\mathcal{T}^{-1}[\tilde{x}(t)] \tag{31g}$$
$$\psi(1,t) = 0, \tag{31h}$$

in which

$$\mathrm{col}_r(\lambda_i \tilde{s}_i(1)k_{v_i}^\top) = \mathrm{col}(\lambda_1 \tilde{s}_1(1)k_{v_1}^\top, \dots, \lambda_r \tilde{s}_r(1)k_{v_r}^\top). \tag{32}$$

This allows a systematic specification of the dynamics of the networked controlled MAS by independently assigning the dynamics of each subsystem.

In order to map (20) into (31) the *cooperative decoupling transformation*

$$e_v(t) = v(t) - \int_0^1 (H \boxtimes \tilde{s}(\zeta))\tilde{\varphi}(\zeta,t)\mathrm{d}\zeta - \int_0^1 (H \boxtimes \tilde{q}(\zeta))\tilde{x}(\zeta,t)\mathrm{d}\zeta - \int_0^1 (H \boxtimes r(\zeta))\psi(\zeta,t)\mathrm{d}\zeta \tag{33}$$

with $r(\zeta) \in \mathbb{R}^{n_v}$, $\tilde{q}(\zeta) \in \mathbb{R}^{n_v}$ and $\tilde{s}(\zeta) \in \mathbb{R}^{n_v}$ is introduced. Note that different from the previous transformations (17) the decoupling transformation (33) respects the communication topology characterized by leader-follower matrix H (see (8)). This introduces constraints in the design, which lead to simultaneous stabilization problems for the resulting ODE subsystem (31a).

Differentiating (33) w.r.t. time and inserting (20) leads to

$$\begin{aligned}\dot{e}_v(t) = {} & \big(I_N \otimes S - H \boxtimes \mathrm{col}(\lambda_i \tilde{s}_i(1)k_{v_i}^\top)\big)e_v(t) \\ & + \big(H \boxtimes (b_y + \mathrm{col}(\bar{\lambda}_i r_i(0))\big)\psi(0,t) \\ & + \int_0^1 \big(H \boxtimes \mathrm{col}(Sr_i(\zeta) - \bar{\lambda}_i r_i'(\zeta))\big)\psi(\zeta,t)\mathrm{d}\zeta \\ & + \int_0^1 \Big(H \boxtimes \mathrm{col}(S\tilde{q}_i(\zeta) - \tilde{c}_i(\zeta)\int_0^1 \gamma_i(\zeta)r_i(\zeta)\mathrm{d}\zeta - \tilde{q}_i''(\zeta) + \mu_i\tilde{q}_i(\zeta))\Big)\tilde{x}(\zeta,t)\mathrm{d}\zeta \\ & + H \boxtimes \big(\mathrm{col}(\tilde{q}_i'(1) - c_{b0,i}\int_0^1 \gamma_i(\zeta)r_i(\zeta)\mathrm{d}\zeta + \lambda_i\tilde{s}_i(0))\big) - \big(H \boxtimes \tilde{q}(1)\big)\tilde{x}'(1,t).\end{aligned} \tag{34}$$

Consequently, the target system (31) is obtained if $r_i(\zeta)$, $\tilde{q}_i(\zeta)$ and $\tilde{s}_i(\zeta)$ are the solution of the *decoupling equations*

$$\bar{\lambda}_i r_i'(\zeta) + Sr_i(\zeta) = 0, \quad \zeta \in (0,1] \tag{35a}$$
$$\bar{\lambda}_i r_i(0) = -b_{y_i}, \tag{35b}$$

$$\tilde{q}_i''(\zeta) - \mu_i q_i(\zeta) = -\tilde{c}_i(\zeta)\int_0^1 \gamma_i(\zeta)r_i(\zeta)\mathrm{d}\zeta, \quad \zeta \in (0,1] \tag{36a}$$
$$\tilde{q}_i(0) = -c_{b0,i}\int_0^1 \gamma_i(\zeta)r_i(\zeta)\mathrm{d}\zeta \tag{36b}$$
$$\tilde{q}_i(1) = 0 \tag{36c}$$

and

$$\lambda_i\tilde{s}_i'(\zeta) + S\tilde{s}_i(\zeta) = 0, \quad \zeta \in (0,1] \tag{37a}$$
$$\lambda_i\tilde{s}_i(0) = c_{b1,i}\int_0^1 \gamma_i(\zeta)r_i(\zeta)\mathrm{d}\zeta - \tilde{q}_i'(1), \tag{37b}$$

for $i = 1, \dots, r$.

3.3 | Stability of the networked controlled MAS

4 | ROBUST COOPERATIVE OUTPUT REGULATION

5 | EXAMPLE

6 | CONCLUDING REMARKS

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AUTHOR BIOGRAPHY

APPENDIX

A ELEMENTS FROM GRAPH THEORY

The communication topology between the agents is described by a *time invariant (weighted) directed graph* $\mathcal{G}$, which is called a *digraph*. This is a triple $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A_{\mathcal{G}}\}$, in which $\mathcal{V}$ is a set of N *nodes* $\mathcal{V} = \{v_1, \dots, v_N\}$, one for each agent and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of *edges* that models the information flow from the node v_j to v_i with $(v_i, v_j) \in \mathcal{E}$. This flow is weighted by $a_{ij} \geq 0$, which are the element of the *adjacency matrix* $A_{\mathcal{G}} = [a_{ij}] \in \mathbb{R}^{N \times N}$. From this, the *Laplacian matrix* $L_{\mathcal{G}} \in \mathbb{R}^{N \times N}$ of the graph $\mathcal{G}$ can be derived by $L_{\mathcal{G}} = D_{\mathcal{G}} - A_{\mathcal{G}}$, where $D_{\mathcal{G}} = \operatorname{diag}(d_1, \dots, d_N)$ with $d_i = \sum_{k=1}^{N} a_{ik}$, $i = 1, \dots, N$, is the *degree matrix* of $\mathcal{G}$. It is assumed that there are no *self-loops*, i. e., $a_{ii} = 0$, $i = 1, \dots, N$, holds for $A_{\mathcal{G}}$. Hence, the elements l_{ij} of $L_{\mathcal{G}}$ are

$$l_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j \\ -a_{ij}, & i \neq j. \end{cases} \tag{A1}$$

A *path* from the node v_j to the node v_i is a sequence of $r \geq 2$ distinct nodes $\{v_{l_1}, \dots, v_{l_r}\}$ with $v_{l_1} = v_j$ and $v_{l_r} = v_i$ such that $(v_k, v_{k+1}) \in \mathcal{E}$. A graph $\mathcal{G}$ is said to be *connected* if there is a node v, called the *root*, such that, for any node $v_i \in \mathcal{V} \setminus \{v\}$, there is a path from v to v_i. A graph is said to be *strongly connected* if there exists a directed path between any two nodes in the graph. A necessary and sufficient conditions for this an *irreducible* Laplacian matrix L_G (see, e. g., [9, Ch. 4.2.1]). For a matrix $M \in \mathbb{R}^{N \times N}$ this means that there exist no *permutation matrix* P such that M can be mapped with P into a lower blockdiagonal matrix with irreducible matrices on the block main diagonal by a similarity transformation (see, e. g., [9, Lem. 4.2]). A graph is *cycle free* if there is no path beginning and ending at the same node. For further details on graph theory see, e. g., [7, Ch. 2]. In order to model the dynamics of MAS, it is necessary to introduce the *Kronecker product* $A \otimes B = [a_{ij}B] \in \mathbb{C}^{n_1 n_2 \times m_1 m_2}$ of two matrices $A = [a_{ij}] \in \mathbb{C}^{n_1 \times m_1}$ and $B \in \mathbb{C}^{n_2 \times m_2}$ (see, e. g., [11]). In what follows the property

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{A2}$$

with $A \in \mathbb{R}^{n_1 \times m_1}$, $B \in \mathbb{R}^{n_2 \times m_2}$, $C \in \mathbb{R}^{m_1 \times m_3}$ and $D \in \mathbb{R}^{m_2 \times m_4}$ of the Kronecker product is needed (see [11, Ch 1.3]). With this, one can define an extended form of the Kronecker product

$$A \boxtimes b = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nm} \end{bmatrix} \boxtimes \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} A_{11} \otimes b_1 & \dots & A_{1n} \otimes b_n \\ \vdots & \ddots & \vdots \\ A_{n1} \otimes b_1 & \dots & A_{nm} \otimes b_n \end{bmatrix}$$

of a block matrix $A = [A_{ij}] \in \mathbb{R}^{n \times m}$ and a block vector $b = [b_i] \in \mathbb{R}^{p \times 1}$.