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Consensus reaching in multi-agent packet-switched networks with non-linear coupling

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An important task for multi-agent systems (MAS) is to reach a consensus, e.g. to align their velocity vectors. Recent results propose appropriate consensus protocols to achieve such tasks, but most of them do not consider the effect of communication constraints such as the presence of time-delays in the exchange of information between the agents. In this article, we provide conditions for a non-linear, locally passive MAS of arbitrary size to reach a consensus, when the agents communicate over a packet-switched network that is characterised by a given topology. Both the cases of constant and switching topologies are considered. The nature of the communication channel imposes constraints that are modelled using stochastic delays of arbitrary distribution. We first embed this model in another, distributed but deterministic delay model and provide conditions for the error introduced by this simplification. In our main result, we provide conditions for the locally passive MAS with distributed delays to reach a consensus. In the case of a fixed topology, the underlying directed graph has to contain a spanning tree. In the case of a switching topology, only the union graph of all graphs that persist over time is required to contain a spanning tree. These conditions are independent of the distribution and the size of the packet delays. To show attractivity of the consensus set, we use an invariance principle for systems described by functional differential equations based on an appropriate Lyapunov–Razumikhin function. This methodological approach is the main contribution of this work and can also be applied to other consensus problems with delays. We illustrate our results by numerical simulations showing synchronisation of non-linear Kuramoto oscillators over a digital network.

Keywords: multi-agent systems; non-linear consensus protocols; packet-switched networks; switched systems; stochastic delays; distributed delays; kuramoto oscillators

1. Introduction

In recent years, there has been an increasing interest in understanding the properties of networked systems. This term refers to systems of subunits that exchange information in an effort to achieve a collective task in a decentralised way. Such systems are often called *multi-agent systems* (MAS). It is important to understand the way these subsystems manage to accomplish a collective behaviour, as such phenomena are observed in nature. Examples are schools of fishes, flashing fireflies, synchronisation of neurons and behaviour of crowds in panic; see, for example, Vicsek, Czirók, Ben-Jacob, Cohen, and Shochet (1995); Vicsek (2001); Strogatz (2003); Couzin, Krause, Franks, and Levin (2005). At the same time, these collective tasks are also key objectives for typical engineering problems, including formation control (Fax and Murray 2004; Marshall, Broucke, and Francis 2004; Lin, Francis, and Maggiore 2005; Ren 2006), flocking (Jadbabaie, Lin, and Morse 2003; Cortés, Martínez, and Bullo 2006; Olfati-Saber 2006; Tanner, Jadbabaie, and Pappas 2007; Wieland, Kim, Scheu and Allgöwer 2008), uninhabited autonomous vehicles (UAV) and robotic networks (Finke, Passino, and Sparks 2006; Liu and Passino 2006; Stubbs et al. 2006; Martínez, Bullo, Cortés, and Frazzoli 2007a,b), collision avoidance (Dimarogonas, Loizou, Kyriakopoulos, and Zavlanos 2006; Ferrari-Trecate, Buffa, and Gati 2006; Wieland and Allgöwer 2007; Wieland, Ebenbauer, and Allgöwer 2007), cooperative transportation (Yang, Watanabe, Izumi, and Kiguchi 2004), synchronisation (Strogatz 2000; Jadbabaie, Motee, and Barahona 2004; Wu 2006), agreement (Blondel, Hendrickx, Olshevsky, and Tsitsiklis 2005; Hatano and Mesbahi 2005; Wang and Slotine 2006), and load balancing (Kashyap, Başar, and Srikant 2006a,b; Xiao, Boyd, and Kim 2007). For more details, the interested reader is refered to a recent special issue on communicating-agent networks (Roy, Saberi, and Stoorvogel 2007) and recent reviews on consensus and cooperation are given in Olfati-Saber, Fax, and Murray (2007) and Ren, Beard, and Atkins (2007).

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Typically, we are interested in the *robust* functionality of the network. Robust means that the objectives are met for a range of uncertainties. For example, in the case of multi-agent consensus, the requirement is that all agents should eventually agree irrespective of their initial condition or the size and structure of the topology, as long as the network is somehow 'connected'. Other modelling uncertainties can also be a subject of analysis: Can a consensus be reached even if the topology may be changing with time (Jadbabaie et al. 2003; Fang and Antsaklis 2005; Moreau 2005; Ren and Beard 2005)? What happens if we include communication constraints like delays in the communication channels (Olfati-Saber and Murray 2004; Bliman and Ferrari-Trecate 2008; Lee and Spong 2006)?

In this article, we investigate the consensus properties of MAS of arbitrary size that exchange state information over a packet-switched digital communication network, e.g. an IP-network. Such networks use a shared media, see Tanenbaum (2002). Hence, data is buffered before it is sent over the communication channel, and this introduces a deterministic time-delay in the transmission. As data is sent in packets over the network, the queuing of packets at the servers and routers introduces additional stochastic delays, called *jitter*. So far, the effect of jitter on MAS consensus has not been investigated. Since the behaviour of the packets is related to all other packets in their environment, an accurate modelling of communication channels is very complicated and is still an open issue; see Richard (2003).

We first develop a modelling approach for digital network channels with stochastic packet delays, using a distributed delay description. The transformation from stochastic delays to distributed delays introduces an error, for which we provide an analytic upper bound. We give a condition for the channel input that guarantees the accuracy of this simplification. This transformation allows us to analyse the resulting model more easily, while we can explicitly use information about the stochastic distribution of the delay in the subsequent analysis. This avoids the pitfall of assuming that the delay is fixed which could wrongly predict instability, while the distributed delay model is stable and vice versa, cf Michiels, Assche, and Niculescu (2005). The use of a distributed delay framework to model packet-switched networks (PSN) has been suggested in the past by Roesch, Roth, and Niculescu (2005); Morărescu, Niculescu, and Gu (2007). However, no conditions for the accuracy of the transformation have been given so far.

Having obtained this transformation, we present a continuous-time consensus protocol for a non-linear, locally passive MAS that guarantees a consensus over PSN with arbitrary delay distributions both for fixed and switching network topologies, i.e. a consensus is reached independent of delay. The only requirement for the fixed topology case is that the underlying graph contains a spanning tree. For the switching topology case, only the union graph of all subgraphs that persist over time has to contain a spanning tree. Moreover, we consider both the leaderless and the leader-driven consensus problem. For our proofs, we use an invariance principle for functional differential equations based on Lyapunov–Razumikhin functions from Haddock and Terjéki (1983). In order to solve the switching topology case, we combine this invariance principle with recent results for switching systems from Hespanha, Liberzon, Angeli, and Sontag (2005). This methodology is the main contribution of this work because it can also be applied to consensus problems with constant or time-varying delays; see Papachristodoulou and Jadbabaie (2006); Münz, Papachristodoulou, and Allgöwer (2008). Simulations illustrate our results, using synchronisation of coupled Kuramoto oscillators (Kuramoto 1984) on a digital network as an example.

Previously, Sipahi, Atay, and Niculescu (2007); and Michiels, Morărescu, and Niculescu (2007) discussed MAS with distributed delays but only for gamma and uniform distributions and specific topologies, like ring and linear topology. Olfati-Saber and Murray (2004) and Bliman and Ferrari-Trecate (2008) presented results for MAS with delayed communications, either with switching topologies and identical, discrete delays or with fixed undirected graphs and time-varying delays. In the considered models, both the neighbour's state and the agent's own state suffer identical delays, which makes a fundamental difference to the work at hand. Liu and Passino (2006) and Xiao and Wang (2006) deal with MAS with switching topologies and time-varying delays but in discrete time. From Tanner and Christodoulakis (2005), we know that under specific assumptions the communication delay has no effect on consensus reaching for discrete-time MAS. In Ghabcheloo, Aguiar, Pascoal, and Silvestre (2007) and Papachristodoulou and Jadbabaie (2006, 2005), a synchronisation problem with switching topology has been considered but for constant delays. Non-linear consensus problems without delay have been previously studied in Bauso, Giarré, and Pesenti (2006); Lin, Francis, and Maggiore (2007); Qu, Chunyu, and Wang (2007); Cortés (2008). Some preliminary results of the work presented in this article have been published in Münz, Papachristodoulou, and Allgöwer (2007).

This article is structured as follows: Some background information on time-delay systems and algebraic graph theory is given in §2. The problem is posed in § 3. Modelling of communication channels with stochastic packet delays is shown in § 4. In § 5, we present conditions for consensus over PSN for a fixed topology. In § 6, these results are extended to switching topologies. Simulation results are presented in § 7. This article is concluded in § 8.

2. Preliminaries

In this section, we review briefly some stability and invariance results for functional differential equations using Lyapunov–Razumikhin functions as well as some tools and notation from Algebraic Graph Theory.

2.1 *Stability of functional differential equations*

This section gives a brief summary of stability results for functional differential equations. The interested reader is referred to Hale and Lunel (1993) and Haddock and Terjéki (1983) for details.

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with the standard norm $|\cdot|$. Let $\mathcal{C}([a,b],\mathbb{R}^n)$ denote the Banach space of continuous functions mapping the interval $[a,b]\subset\mathbb{R}$ into $\mathbb{R}^n$ with the topology of uniform convergence. For easier notation, we drop the argument of $\mathcal{C}$ if $a=-\mathcal{T}$ and $b=0$ for a given $\mathcal{T}>0$, i.e. $\mathcal{C}=\mathcal{C}([-\mathcal{T},0],\mathbb{R}^n)$. The norm on $\mathcal{C}$ is defined as $\|\varphi\|=\sup_{-\mathcal{T}\le s\le 0}|\varphi(s)|$. Let $\rho\ge 0$ and $x\in\mathcal{C}([-\mathcal{T},\rho],\mathbb{R}^n)$, then for any $t\in[0,\rho]$, define a segment $x_t\in\mathcal{C}$ of x such that $x_t(s)=x(t+s)$, $s\in[-\mathcal{T},0]$.

Let Ω be a subset of $\mathcal{C}$, $f:\Omega\to\mathbb{R}^n$ a given function, and '˙' represent the right-hand Dini derivative. Then, we call

$$\dot{x}(t)=f(x_t) \tag{1}$$

an autonomous Retarded Functional Differential Equation (RFDE) on Ω. Given $\varphi\in\Omega$ and $\rho>0$, a function $x(\varphi)\in\mathcal{C}([-\mathcal{T},\rho],\mathbb{R}^n)$ is said to be a solution to (1) with initial condition φ, if $x_t(\varphi)\in\Omega$, $x(\varphi)(t)$ satisfies (1) for $t\in[0,\rho)$, and $x_0(\varphi)=\varphi$. Such a solution exists and is unique if f is continuous and $f(\varphi)$ is Lipschitzian in each compact set in Ω. Note that $x(\varphi)(t)\in\mathbb{R}$, whereas $x_t(\varphi)\in\mathcal{C}$. We denote the value of the segment $x_t(\varphi)$ at time $s, s\in[-\mathcal{T},0]$, as $x_t(\varphi)(s)=x(\varphi)(t+s)$. For easier notation, we often drop the initial condition φ of x and x_t.

An element $\phi\in\mathcal{C}$ is called a steady-state or equilibrium of (1) if $x_t(\phi)=\phi$ for all $t\ge 0$. Without loss of generality we assume that $\phi=0$ is an equilibrium of (1). The stability of (1) around such a steady-state is defined in a way similar to the stability of non-linear ordinary differential equations (ODE) using an ϵ-δ argument; see Hale and Lunel (1993).

There are two types of Lyapunov theorems for stability of equilibria of RFDEs, namely, Lyapunov–Krasovskii and Lyapunov–Razumikhin. Lyapunov–Krasovskii is the natural extension of Lyapunov's theorem from ODEs to RFDEs. It is based on non-increasing Lyapunov–Krasovskii functionals. In this work, we will be applying Lyapunov–Razumikhin-type theorems to prove consensus, which use functions instead of functionals.

Let $D\subseteq\mathbb{R}^n$. By a Lyapunov–Razumikhin function $V=V(x)$, we mean a continuous function $V:D\to\mathbb{R}$. The upper right-hand Dini derivative of V with respect to (1) is defined by

$$\dot{V}(\varphi)=\limsup_{h\to 0^+}\frac{1}{h}\big(V(\varphi(0)+hf(\varphi))-V(\varphi(0))\big).$$

With this definition, we have the following Lyapunov–Razumikhin theorem:

Theorem 2.1: *Suppose $f:\Omega\to\mathbb{R}^n$ maps bounded subsets of Ω into bounded subsets of $\mathbb{R}^n$ and consider* (1). *Suppose v, $w:\mathbb{R}^+\to\mathbb{R}^+$ are continuous, non-decreasing functions, $v(s)$ positive for $s>0$, $v(0)=0$. If there is a Lyapunov-Razumikhin function $V:D\to\mathbb{R}$ such that*:

(i) $V(x)\ge v(|x|)$ *for* $x\in D$,
(ii) $\dot{V}(\varphi(0))\le -w(|\varphi(0)|)$
if $V(\varphi(0))=\max_{-\mathcal{T}\le s\le 0} V(\varphi(s))$,

then the equilibrium $x=0$ of (1) *is stable.*

Note that the function V in Razumikhin's theorem need not be non-increasing along the system trajectories, but may indeed increase within a delay interval. However, it is required that V decreases 'on average' in the sense described in the theorem. The proof of Razumikhin's theorem is based on the fact that

$$\overline{V}(\varphi)=\max_{-\mathcal{T}\le s\le 0} V(\varphi(s)) \tag{2}$$

is a Lyapunov–Krasovskii functional that is non-increasing along the system trajectories. This is an important fact that we will be using in our proofs.

In this article, we have to prove the attractivity of a subspace of $\mathbb{R}^n$. Therefore, we will make repeated use of an invariance principle for RFDEs. For this, we need to define ω-limit sets of solutions and provide LaSalle-type theorems for RFDEs.

Definition 2.2: A set $M\subseteq\Omega$ is said to be *positively invariant* with respect to (1) if, for any $\varphi\in M$, there is a solution $x(\varphi)$ of (1) that is defined on $[-\mathcal{T},\infty)$ such that $x_t(\varphi)\in M$ for all $t\ge 0$ and $x_0=\varphi$.

Definition 2.3: Let $\varphi\in\Omega$. An element ψ of Ω is in $\omega(\varphi)$, the ω-limit set of φ, if $x(\varphi)$ is defined on $[-\mathcal{T},\infty)$

and there is a sequence $\{t_n\}$ of non-negative real numbers satisfying $t_n \to \infty$ and $\|x_{t_n}(\varphi) - \psi\| \to 0$ as $n \to \infty$.

If $x(\varphi)$ is a solution of (1) that is defined and bounded on $[-\mathcal{T}, \infty)$, then the orbit through φ, i.e. the set $\{x_t(\varphi) : t \geq 0\}$ is precompact, $\omega(\varphi)$ is non-empty, compact, connected and invariant, and $x_t(\varphi) \to \omega(\varphi)$ as $t \to \infty$.

For a given set $\Omega \subset \mathcal{C}$, define

$$E_V = \left\{ \varphi \in \Omega : \max_{s \in [-\mathcal{T},0]} V(x_t(\varphi)(s)) = \max_{s \in [-\mathcal{T},0]} V(\varphi(s)), \quad \forall t \geq 0 \right\} \tag{3}$$

$$M_V = \text{largest set in } E_V \text{ that is invariant with respect to Equation (1).} \tag{4}$$

Here, E_V is the set of functions $\varphi \in \Omega$ which can serve as initial conditions for (1) so that $x_t(\varphi)$ satisfies

$$\max_{s \in [-\mathcal{T},0]} V(x_t(\varphi)(s)) = \max_{s \in [-\mathcal{T},0]} V(\varphi(s))$$

for all $t \geq 0$. Note that the above condition guarantees that $\overline{V}$ defined in (2) satisfies $\dot{\overline{V}}(\varphi) = 0$. In particular, for a Lyapunov–Razumikhin function V and for any $\varphi \in E_V$, we have $\dot{V}(x_t(\varphi)) = 0$ for any $t > 0$ such that $\max_{s \in [-\mathcal{T},0]} V(x_t(\varphi)(s)) = V(x_t(\varphi)(0))$.

We then have the following theorem from Haddock and Terjéki (1983):

Theorem 2.4: *Suppose there exists a Lyapunov–Razumikhin function $V = V(x)$ and a closed set Ω that is positively invariant with respect to* (1) *such that*

$$\dot{V}(\varphi) \leq 0 \quad \textit{for all } \varphi \in \Omega \textit{ s.t. } V(\varphi(0)) = \max_{s \in [-\mathcal{T},0]} V(\varphi(s)). \tag{5}$$

Then, for any $\varphi \in \Omega$ such that $x(\varphi)$ is defined and bounded on $[-\mathcal{T}, \infty)$, $\omega(\varphi) \subseteq M_V \subseteq E_V$, and we have

$$x_t(\varphi) \to M_V \textit{ as } t \to \infty.$$

Theorem 2.4 will be used extensively in our work. It proves the attractivity of invariant subsets M_V of Ω for the solutions of RFDE (1).

2.2 *Algebraic graph theory*

The topology of the communication network between the agents is represented by a graph. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a set of vertices (nodes) $\mathcal{V} = \{v_i\}$, $i \in \mathcal{I} = \{1, \ldots, N\}$, which represent the agents, and a set of edges (links) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, which represent the communication channels between the agents. If $v_i, v_j \in \mathcal{V}$ and $e_{ij} = (v_i, v_j) \in \mathcal{E}$, then there is an edge (a directed arrow) from node v_i to node v_j, i.e. agent j can receive data from agent i. In this article, we assume that the graph $\mathcal{G}$ is *directed*, i.e. $e_{ij} \in \mathcal{E}$ does not necessarily imply that $e_{ji} \in \mathcal{E}$. We also assume that the network topology does not contain self-loops, i.e. $e_{ii} \notin \mathcal{E}$. The *graph adjacency matrix* $A = [a_{ij}]$, $A \in \mathbb{R}^{N \times N}$, is such that $a_{ij} = 1$ if $e_{ij} \in \mathcal{E}$ and $a_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$. If $e_{ji} \in \mathcal{E}$, then v_j is a *parent* of v_i. The number of parents of agent i, also called the *in-degree* of vertex v_i, is denoted by $d_i = \sum_{j=1}^{N} a_{ji}$.

A *directed path* from v_i to v_j is a sequence of edges from $\mathcal{E}$ that takes the following form (v_i, v_{i_1}), $(v_{i_1}, v_{i_2}), \ldots, (v_{i_p}, v_j)$. A *directed cycle* is a directed path that starts and ends at the same vertex. A *directed tree* is a directed graph in which every vertex has exactly one parent except for one node, the so-called *root* v_R. Clearly, there is a directed path from v_R to all other nodes of the directed tree.

A *subgraph* $(\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ of $\mathcal{G}$ is a graph with $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ and $\tilde{\mathcal{E}} \subseteq \mathcal{E}$. If there exists a subgraph $(\mathcal{V}, \tilde{\mathcal{E}})$ of $\mathcal{G}$ that is a directed tree, then we say that $\mathcal{G}$ contains a *directed spanning tree*. Hence, a graph $\mathcal{G}$ contains a directed spanning tree if and only if it contains at least one *root*, i.e. one node with a directed path to all other vertices. We denote the set of all roots of $\mathcal{G}$ as $\mathcal{I}_R$ and $\mathcal{I}_{\overline{R}} = \mathcal{I} \backslash \mathcal{I}_R$. In the following, we also say *spanning tree* when referring to a *directed spanning tree*. The *union graph* of a set of P graphs $\{(\mathcal{V}, \mathcal{E}_p)\}$, $p \in \mathcal{P} = \{1, \ldots, P\}$, is $(\mathcal{V}, \bigcup_{p \in \mathcal{P}} \mathcal{E}_p)$. More details on algebraic graph theory can be found, for example, in Godsil and Royle (2000).

3. Problem setup

With these preliminaries, we can now formulate the problem. Consider N agents with non-linear, locally passive dynamics that exchange information over a packet-switched communication network, and for now assume that the interaction topology is fixed. Suppose agent i 'holds' a scalar state $\tilde{x}_i$ that is updated continuously by comparing its own state with the states of its parent agents $\tilde{x}_j$, $e_{ji} \in \mathcal{E}$. This is summarised in the following RFDE:

$$\dot{\tilde{x}}_i(t) = -k_i \sum_{j=1}^{N} a_{ji} g_{ji}\big(\tilde{x}_i(t) - \tilde{x}_j(t - \tau_{ji}(t))\big), \quad i \in \mathcal{I}, \tag{6}$$

where $k_i > 0$ is the coupling gain and a_{ji} are the elements of the adjacency matrix of the underlying interaction graph. The stochastic packet delays τ_{ji}

are uncorrelated, stationary random processes with probability density f_{ji}. This makes the above RFDE stochastic. Finally, the non-linear dynamics g_{ji} satisfy the following assumption:

Assumption 3.1: *The functions* $g_{ji}: \mathbb{R} \rightarrow \mathbb{R}$ *are locally passive, i.e.*

$$g_{ji}(0) = 0 \text{ and } y g_{ji}(y) > 0 \text{ for all } y \in \left[-\gamma_{ji}^{-}, \gamma_{ji}^{+}\right] \backslash \{0\}, \tag{7}$$

with $\gamma_{ji}^{-}, \gamma_{ji}^{+} > 0$.

This model extends the standard linear MAS, studied, for example, in Jadbabaie et al. (2003), and Bliman and Ferrari-Trecate (2008), with non-linear dynamics g_{ji} and stochastic packet delays τ_{ji}. Non-linear, locally passive dynamics are particularly interesting because they include, for instance, the well-known Kuramoto oscillator as discussed in § 7.2. Furthermore, stochastic delays are a much more accurate model for packet-switched communication channels than constant delays, cf § 4.

The main result of our work is to provide conditions for MAS (6) to reach a consensus asymptotically, i.e. all agents eventually converge to the same state $x_i = x_j$ for all $i, j \in \mathcal{I}$. Therefore, we first embed the stochastic delay model in a distributed delay model in Proposition 5.1. Then, we show that the distributed delay model reaches a consensus if the underlying graph contains a spanning tree, see Theorem 5.5 and Corollary 5.6, respectively. This holds both for the leaderless and leader-driven case. A MAS is leaderless if and only if all nodes of the underlying directed graph have a parent, i.e. $d_i > 0$ for all $i \in \mathcal{I}$. On the contrary, a MAS has a leader v_L if the underlying graph contains a spanning tree with root v_L, where this root does not have a parent, i.e. $d_L = 0$.

Moreover, we generalise these consensus conditions for switching topologies as in Hatano and Mesbahi (2005), and Olfati-Saber and Murray (2004), for example. Switching topologies are an active field of investigation because some important network constraints like packet loss can be modelled this way. We show in Theorem 6.1 and Corollary 6.2 that a non-linear, locally passive MAS reaches a consensus over a packet-switched communication network with arbitrarily switching topology. The only requirement is that the union graph of all graphs that persist over time contains a spanning tree.

Finally, we return to MAS (6) in the simulations, see § 7. There, we show that the results for MAS with distributed delays can be transfered to MAS connected via a PSN.

4. Modeling of communication channels with stochastic packet delays

In this section, we recall a distributed delay model for packet-switched communication channels with stochastic delays. Similar models with distributed delays and a particular gamma distributed delay kernel have been studied in Roesch et al. (2005); Michiels et al. (2007); Morărescu et al. (2007). The novelty of our work is an analytical bound for the modelling error, cf Münz et al. (2007). Moreover, we show in the subsequent sections how this model can be used to solve consensus problems over PSN.

Stochastic transmission delays are usually observed in PSN (cf Salza, Draoli, Gaibisso, Palma, and Puccinelli (2000); Tanenbaum (2002); Lopez et al. (2006)). These delays are the same for all data arriving in the same packet. If we are transmitting data from continuous-time processes over PSN, this requires accumulating the data of some time interval ΔT at the emitter before storing it in a packet and sending it over the network. Clearly, ΔT is an additional time-delay for some of the data in the packet, in particular the information that is generated just after the former packet has been sent. If we assume that the system on the receiver side processes the data of each packet linearly, i.e. in the same time as it is generated, ΔT is an additional delay for *all* the data in the packet. Obviously, a trade-off is necessary: large packets should be sent at low rates, causing less traffic in the network and introducing large delays ΔT, and smaller packets should be sent at higher rates in order to produce less additional delays ΔT.

The model presented in this work is appropriate for the second case, that is, if a stream of many small packets is sent over a network. In the limit $\Delta T \rightarrow 0$, we can assume that every piece of data that is sent over the network has its own stochastic delay. Hence, the stochastic channel output $y(t)$ at time t is given by

$$y(t) = u(t - \tau(t)) \tag{8}$$

where $u(t)$ is the deterministic channel input at time t and $\tau(t) \geq 0$ is the stochastic delay induced by the communication channel at time t. We assume that this delay τ is an uncorrelated, stationary random process with probability density f_τ, i.e. the autocorrelation of τ is assumed to be a Dirac function and $\tau(t_1)$ and $\tau(t_2)$ are uncorrelated for $t_1 \neq t_2$. The probability density

f_τ may be arbitrary, as long as the following assumption holds:

Assumption 4.1: *The probability density f_τ of the random communication delay τ satisfies*

$$\begin{aligned} &f_\tau(\eta) \geq 0 \quad \text{for } T^- \leq \eta \leq T^+,\\ &f_\tau(\eta) = 0 \quad \text{for } \eta < T^- \text{ and } \eta > T^+, \text{ and}\\ &\int_{T^-}^{T^+} f_\tau(\eta)\mathrm{d}\eta = 1, \end{aligned} \tag{9}$$

for some T^-, $T^+ \in \mathbb{R}$ with $0 \leq T^- < T^+$.

The channel output y is a random process whereas the input u is treated as a deterministic signal. We can now determine the mean and variance of $y(t)$ for fixed t

$$\mathrm{E}\{y(t)\} = \int_{T^-}^{T^+} f_\tau(\eta)u(t-\eta)\mathrm{d}\eta \tag{10}$$

$$\begin{aligned} \mathrm{Var}\{y(t)\} &= \mathrm{E}\{y^2(t)\} - \mathrm{E}\{y(t)\}^2\\ &= \int_{T^-}^{T^+} f_\tau(\eta)u^2(t-\eta)\mathrm{d}\eta\\ &\quad - \left(\int_{T^-}^{T^+} f_\tau(\eta)u(t-\eta)\mathrm{d}\eta\right)^2. \end{aligned}$$

With $u_{\min} = \min_{\eta\in[T^-,T^+]} |u(t-\eta)|$ and $u_{\max} = \max_{\eta\in[T^-,T^+]} |u(t-\eta)|$, we can calculate a conservative upper bound for the variance $\mathrm{Var}\{y(t)\}$ using (9):

$$\mathrm{Var}\{y(t)\} \leq u_{\max}^2 - u_{\min}^2. \tag{11}$$

With the Chebyshev inequality (see, for example, Papoulis and Pillai (2002)), we obtain the following probability bound:

$$\mathrm{P}\big\{|y(t) - \mathrm{E}\{y(t)\}| \geq \epsilon\big\} \leq \frac{\mathrm{Var}\{y(t)\}}{\epsilon^2} \tag{12}$$

$$\leq \frac{u_{\max}^2 - u_{\min}^2}{\epsilon^2}. \tag{13}$$

Thus, the probability that the difference between the stochastic output y and its expected value is larger than ϵ is small if the input u changes only slowly, i.e. $u_{\max} \approx u_{\min}$. Then we have

$$\bar{y}(t) \approx \mathrm{E}\{y(t)\}, \tag{14}$$

for almost all samples $\bar{y}$ of y. This approximation is trivial for $u_{\max} = u_{\min}$. The novelty of this result is that Equation (13) gives an analytic bound (in a stochastic sense) for the model error if $u_{\max} \neq u_{\min}$.

Summarising, we are approximating a stochastic time delay by a deterministic, distributed delay model. The advantage of this approximation is obvious: the analysis of distributed delay systems is much easier than the analysis of systems with stochastic delays.

5. Consensus over fixed topologies

5.1 *MAS with distributed delays*

In the previous section, we have developed a framework for embedding stochastic delay systems, like packet-switched communication channels, in distributed delay ones. We now apply this result to our problem as it was formulated in § 3, in particular the non-linear MAS (6). For this, we use (14) to approximate

$$\bar{x}_j(t - \bar{\tau}_{ji}(t)) \approx \int_{T_{ji}^-}^{T_{ji}^+} f_{ji}(\eta)\bar{x}_j(t-\eta)\mathrm{d}\eta, \tag{15}$$

for almost all samples $\bar{\tau}_{ji}$ of τ_{ji} and almost all samples $\bar{x}_j$ of any solution $\tilde{x}_j$ of (6), where f_{ji} is the probability density of τ_{ji}. With this, we reformulate (6) in the following way:

$$\dot{x}_i(t) = -k_i \sum_{j=1}^{N} a_{ji} g_{ji}\left(x_i(t) - \int_{T_{ji}^-}^{T_{ji}^+} f_{ji}(\eta)x_j(t-\eta)\mathrm{d}\eta\right), \tag{16}$$

for all $i \in \mathcal{I}$. We write x_i instead of $\tilde{x}_i$ in order to distinguish the random processes $\tilde{x}_i$ that solve (6) from the deterministic signals x_i that solve (16).

We recall from Münz et al. (2007) that, at least for linear $g_{ji}(y)$, the approximation error of (15), i.e. the modelling error when going from (6) to (16), can be made arbitrarily small by decreasing $K = \max_{i\in\mathcal{I}} k_i$. Note however that, for MAS without delay, a small K leads in general to a slow convergence of the agents, cf Olfati-Saber et al. (2007). A similar behaviour is expected for MAS with communication delays, which is studied in our simulations in § 7. Hence, a trade-off is necessary between the accuracy of the model (16) and convergence speed. The following proposition requires that all samples $\bar{x}_i$ of all stochastic processes $\tilde{x}_i$ that solve (6) have a common upper bound $M_x = \max_{i\in\mathcal{I}}\{|\bar{x}_i(t)|\}$ for all t. Clearly, almost all realistic systems exhibit such state constraints.

Proposition 5.1: *Given any $\epsilon > 0$ and stochastic delays $\tau_{ji}, i, j \in \mathcal{I}$, with probability densities f_{ji} that satisfy Assumption 4.1, then*

$$\lim_{K\to 0} \mathrm{P}\left\{\left|\bar{x}_j(t - \tau_{ji}(t)) - \int_{T_{ji}^-}^{T_{ji}^+} f_{ji}(\eta)\bar{x}_j(t-\eta)\mathrm{d}\eta\right| \geq \epsilon\right\} = 0$$

holds for any sample $\bar{x}_j$ of any solution $\tilde{x}_j$, $j \in \mathcal{I}$, of (6) with $g_{ji}(y) = y$.

The proof is presented in the Appendix. An extension to non-linear, locally Lipschitz g_{ij} is straightforward.

At this point, we should emphasise that no assumptions are necessary on the symmetry or quality of the probability densities f_{ji}. In particular, each edge $e_{ji} \in \mathcal{E}$ can have different delay densities f_{ji} and delay bounds T_{ji}^+ and T_{ji}^-. Obviously, the transformation from (6) to (16) only makes sense if Assumption 4.1 holds for all densities f_{ji}. Yet, we can prove consensus even for the following, more general model:

$$\dot{x}_i(t) = -k_i \sum_{j=1}^{N} a_{ji}\, g_{ji}\left(x_i(t) - \frac{1}{\alpha_{ji}} \int_{T_{ji}^-}^{T_{ji}^+} f_{ji}(\eta) x_j(t-\eta)\mathrm{d}\eta \right), \tag{17}$$

where we replace Assumption 4.1 by the following:

Assumption 5.2: *The functions* $f_{ji}: \mathbb{R} \to \mathbb{R}$, $i, j \in \mathcal{I}$, *satisfy*

$$f_{ji}(\eta) \geq 0 \quad \text{for } T_{ji}^- \leq \eta \leq T_{ji}^+,$$
$$\int_{T_{ji}^-}^{T_{ji}^+} f_{ji}(\eta)\mathrm{d}\eta = \alpha_{ji}. \tag{18}$$

for some $T_{ji}^-, T_{ji}^+ \in \mathbb{R}$ *with* $0 \leq T_{ji}^- < T_{ji}^+$ *and* $\alpha_{ji} > 0$.

A parameter $\alpha_{ji} < 1$ is used if we consider sporadic packet loss with probability $1 - \alpha_{ji}$. The opposite case $\alpha_{ji} > 1$ does not make sense if f_{ji} is a probability density. Yet, our result also holds for this case.

The MAS (17) has reached a consensus if all agents constantly hold the same value $x_i = x_j$ for all $i, j \in \mathcal{I}$. For some $x^* \in \mathbb{R}$, a *consensus point* $\phi_{x^*} \in \mathcal{C}([-\mathcal{T}^+, 0], \mathbb{R}^N)$ is such that all components of ϕ_{x^*} satisfy $\phi_{x^*,i}(\eta) = x^*$, $i \in \mathcal{I}$, for all $\eta \in [-\mathcal{T}^+, 0]$ with $\mathcal{T}^+ = \max_{i,j\in\mathcal{I}} T_{ij}^+$. The *consensus set* of the MAS (17) is

$$\Theta = \bigcup_{x^* \in \mathbb{R}} \{\phi_{x^*}\}. \tag{19}$$

It can be easily checked that any element of the consensus set is a steady-state of (17).

We will prove in the following theorems that the consensus set Θ of MAS (17) is asymptotically attracting for appropriate initial conditions $\varphi \in \mathcal{C}_{\mathbb{D}} = \mathcal{C}([-\mathcal{T}^+, 0], \mathbb{D})$, as long as the underlying directed graph contains a spanning tree. The *region of attraction* $\mathbb{D} \subseteq \mathbb{R}^N$ depends on g_{ji} and is given in Equation (20). We consider the leaderless and the leader-driven consensus separately in §§ 5.2 and 5.3. If the MAS does not possess a leader, the exact point in Θ, at which the system will evolve, depends in a complex way on the initial condition φ and the delay densities f_{ji}. The calculation of these exact points of consensus is beyond the scope of this work.

5.2 *Leaderless consensus*

In order to apply Theorem 2.4, we have to find a closed set Ω that is positively invariant with respect to (17). We prove in the next lemma that this is true for $\mathcal{C}_{\mathbb{D}} = \mathcal{C}([-\mathcal{T}^+, 0], \mathbb{D})$ with $\mathbb{D}$ defined as

$$\mathbb{D} = \left\{ x \in \mathbb{R}^N : |x_i| \leq \frac{\gamma}{2} \right\} \tag{20}$$

with $\gamma = \min_{i,j\in\mathcal{I}}\{\gamma_{ij}^-, \gamma_{ij}^+\}$, where $\gamma_{ij}^-, \gamma_{ij}^+$ are the bounds of the locally passive functions g_{ij}, cf (7). Note that $\mathbb{D} = \mathbb{R}^n$ if g_{ji} are globally passive and, in particular, linear.

Lemma 5.3: *If Assumptions* 3.1 *and* 5.2 *hold, then* $\mathcal{C}_{\mathbb{D}} = \mathcal{C}([-\mathcal{T}^+, 0], \mathbb{D})$ *is a positively invariant set of* (17).

Proof: We have to show that any trajectory of (17) that starts in $\mathbb{D}$, i.e. $\varphi \in \mathcal{C}_{\mathbb{D}}$, remains in $\mathbb{D}$, i.e. $x(\varphi)(t-\eta) \in \mathbb{D}$ for all $\eta \in [0, \mathcal{T}^+]$ and all $t \geq 0$. For this, we use Theorem 2.1. Consider the Lyapunov–Razumikhin function candidate

$$V(x(t)) = \frac{1}{2} \max_{i\in\mathcal{I}} x_i^2(t).$$

We want to write the upper right-hand Dini derivative of V along solutions of (17) in a compact form. Therefore, we denote I the index that satisfies $x_I^2(t) = \max_{i\in\mathcal{I}} x_i^2(t)$. If there are several possible indices, we choose that one which is the maximal modulo of the derivative $|\dot{x}_I(t)|$. If there are still several possible indices, we choose any one of them but fix the index I as long as it satisfies the maximum conditions. With this notation, the upper right-hand Dini derivative of V along solutions of (17) is

$$\dot{V}(x_t) = x_I(t)\dot{x}_I(t) = -k_I \sum_{j=1}^{N} a_{jI}\, x_I(t) g_{jI} \times \left(x_I(t) - \frac{1}{\alpha_{jI}} \int_{T_{jI}^-}^{T_{jI}^+} f_{jI}(\eta) x_j(t-\eta)\mathrm{d}\eta \right). \tag{21}$$

The condition on $|\dot{x}_I(t)|$ is necessary in order to guarantee that (21) is indeed the upper right-hand derivative.

Following Theorem 2.1, we consider the behaviour of $\dot{V}$ if $V(x(t)) = \max_{\eta\in[0,\mathcal{T}^+]} V(x(t-\eta))$, i.e. $|x_I(t)| = \max_{\eta\in[0,\mathcal{T}^+]} \max_{j\in\mathcal{I}} |x_j(t-\eta)|$. With Assumption 5.2, this condition yields

$$\left| \frac{1}{\alpha_{jI}} \int_{T_{jI}^-}^{T_{jI}^+} f_{jI}(\eta) x_j(t-\eta)\mathrm{d}\eta \right| \leq \frac{1}{\alpha_{jI}} \int_{T_{jI}^-}^{T_{jI}^+} f_{jI}(\eta) |x_I(t)| \mathrm{d}\eta = |x_I(t)|$$

and we conclude using Assumption 3.1 that

$$x_I(t) g_{jI}\left(x_I(t) - \frac{1}{\alpha_{jI}} \int_{T_{jI}^-}^{T_{jI}^+} f_{jI}(\eta) x_j(t-\eta)\mathrm{d}\eta \right) \geq 0$$

for all j with $e_{jI} \in \mathcal{E}$ and all $x_t \in \mathcal{C}_{\mathbb{D}}$. We see that $\dot{V} \leq 0$ if $V(x(t)) = \max_{\eta\in[0,\mathcal{T}^+]} V(x(t-\eta))$. This completes the proof. □

Remark 5.4: If the initial condition φ of the MAS (17) lies outside $\mathcal{C}_{\mathbb{D}}$ but $\varphi \in \mathcal{C}_{\hat{\mathbb{D}}} = \mathcal{C}([-\mathcal{T}^+, 0], \{x \in \mathbb{R}^N | \, |x_i - \breve{x}_i| \leq \frac{\gamma}{2}\})$ for some $\breve{x} \in \mathbb{R}^N$ with $\breve{x}_i = \breve{x}_j$ for all $i, j \in \mathcal{I}$, then a simple time-invariant coordinate translation $\hat{x}(t) = x(t) - \check{x}$ can be applied to show that $\mathcal{C}_{\hat{\mathbb{D}}}$ is positively invariant with respect to (17). Note that (17) does not change because of this coordinate translation.

Now, we prove the attractivity of the consensus set Θ for any initial condition $\varphi \in \mathcal{C}_{\mathbb{D}}$, as long as the directed interaction graph contains a spanning tree.

Theorem 5.5: *Given a leaderless MAS consisting of N agents with dynamics* (17), *where* f_{ji} *satisfy Assumption* 5.2 *and* g_{ji} *satisfy Assumption* 3.1, *and with initial condition* $\varphi \in \mathcal{C}_{\mathbb{D}}$, *as well as an underlying network topology of a directed graph with a spanning tree, then the consensus set* Θ *of this MAS is asymptotically attracting, i.e.* $x_t \to \Theta$ *as* $t \to \infty$.

Proof: This theorem is proven using the invariance principle in Theorem 2.4. Consider the Lyapunov–Razumikhin function candidates

$$V_1 = \max_{i\in\mathcal{I}} x_i(t),$$
$$V_2 = -\min_{i\in\mathcal{I}} x_i(t).$$

As in the former proof, we denote I and J the indices that satisfy $x_I(t) = \max_{i\in\mathcal{I}} x_i(t)$ and $x_J(t) = \min_{i\in\mathcal{I}} x_i(t)$, respectively. If there are several such indices, we choose those with the maximal derivative $\dot{x}_I(t)$ and minimal derivative $\dot{x}_J(t)$, respectively. If there are still several possible indices, we choose any one of them but fix the indices I and J as long as they satisfy the extremum conditions. Using this notation, we can write the right-hand Dini derivatives of V_1 and V_2 along solutions of (17) in a compact form:

$$\dot{V}_1(x_t) = \dot{x}_I(t) = -k_I \sum_{j=1}^{N} a_{jI}\, g_{jI} \times \left(x_I(t) - \frac{1}{\alpha_{jI}} \int_{T_{jI}^-}^{T_{jI}^+} f_{jI}(\eta) x_j(t-\eta)\mathrm{d}\eta \right),$$

$$\dot{V}_2(x_t) = \dot{x}_J(t) = k_J \sum_{j=1}^{N} a_{jJ}\, g_{jJ} \times \left(x_J(t) - \frac{1}{\alpha_{jJ}} \int_{T_{jJ}^-}^{T_{jJ}^+} f_{jJ}(\eta) x_j(t-\eta)\mathrm{d}\eta \right).$$

The conditions on $\dot{x}_I(t)$ and $\dot{x}_J(t)$ are necessary in order to guarantee that $\dot{V}_1$ and $\dot{V}_2$ are indeed the upper right-hand derivatives.

Following Theorem 2.4, we are interested in the behaviour of $\dot{V}_k$, $k = 1, 2$, whenever $V_k(x(t)) = \max_{\eta\in[0,\mathcal{T}^+]} V_k(x(t-\eta))$, i.e. $x_I(t) = \max_{\eta\in[0,\mathcal{T}^+]} \max_{j\in\mathcal{I}} x_j(t-\eta)$ and $x_J(t) = \min_{\eta\in[0,\mathcal{T}^+]} \min_{j\in\mathcal{I}} x_j(t-\eta)$, respectively. A similar argument as in the proof of Lemma 5.3 shows that $\dot{V}_k \le 0$, $k = 1, 2$. Hence, condition (5) in Theorem 2.4 is fulfilled.

Next, we have to find the sets E_{V_k} and M_{V_k}, $k = 1, 2$, according to (3) and (4). The definition of E_{V_k} guarantees that for every $\varphi \in E_{V_k}$, there exists an $x_k^* \in \mathbb{R}$ such that $\max_{\eta\in[0,\mathcal{T}^+]} V_k(x(\varphi)(t-\eta)) = x_k^*$ for all $t \ge 0$. As explained in Haddock and Terjéki (1983), any $\varphi \in E_{V_k}$ has to satisfy $\dot{V}_k(x_t(\varphi)) = 0$ for any $t \ge 0$ whenever $V_k(x(\varphi)(t)) = \max_{\eta\in[0,\mathcal{T}^+]} V_k(x(\varphi)(t-\eta))$.

We first determine E_{V_1} where the condition $V_1(x(\varphi)(t)) = \max_{\eta\in[0,\mathcal{T}^+]} V_1(x(\varphi)(t-\eta))$ transforms into $x_I(t) = \max_{\eta\in[0,\mathcal{T}^+]} \max_{j\in\mathcal{I}} x_j(t-\eta) = x_1^*$. The right-hand Dini derivative $\dot{V}_1$ is zero if

$$x_I(t) - \frac{1}{\alpha_{jI}} \int_{T_{jI}^-}^{T_{jI}^+} f_{jI}(\eta) x_j(t-\eta)\mathrm{d}\eta = 0$$

for all j with $e_{jI} \in \mathcal{E}$. Hence, all parents v_j of v_I must fulfill $x_j(t-\eta) = x_1^*$ for all $\eta \in [T_{jI}^-, T_{jI}^+]$, i.e. they satisfy $\dot{x}_j(t-\eta) = 0$ for all $\eta \in [T_{jI}^-, T_{jI}^+]$. This requires that all the parents of the parents of v_I be also constant and satisfy $x_\kappa(t-\eta) = x_1^*$ for all $e_{jI}, e_{\kappa j} \in \mathcal{E}$ and all $\eta \in [T_{jI}^- + T_{\kappa j}^-, T_{jI}^+ + T_{\kappa j}^+]$. This can be continued up to any root of the underlying graph of the MAS. We know that there exists at least one root because the graph contains a spanning tree. We see that all roots, i.e. all x_i with $i \in \mathcal{I}_R$, must take the same and constant value x_1^*. It follows that

$$E_{V_1} = \bigcup_{x_1^*\in\mathbb{R}} \left\{ \varphi \in \mathcal{C}_{\mathbb{D}} : \begin{Bmatrix} \varphi_i(\eta) = x_1^* \text{ for all } i \in \mathcal{I}_R \\ \varphi_i(\eta) \le x_1^* \text{ for all } i \in \mathcal{I}_{\overline{R}} \end{Bmatrix} \text{ and all } \eta \in [-\mathcal{T}^+, 0] \right\}. \quad (22)$$

With similar arguments for E_{V_2}, we get

$$E_{V_2} = \bigcup_{x_2^*\in\mathbb{R}} \left\{ \varphi \in \mathcal{C}_{\mathbb{D}} : \begin{Bmatrix} \varphi_i(\eta) = x_2^* \text{ for all } i \in \mathcal{I}_R \\ \varphi_i(\eta) \ge x_2^* \text{ for all } i \in \mathcal{I}_{\overline{R}} \end{Bmatrix} \text{ and all } \eta \in [-\mathcal{T}^+, 0] \right\}, \quad (23)$$

where $x_2^* = \min_{\eta\in[0,\mathcal{T}^+]} \min_{j\in\mathcal{I}} x_j(\varphi)(t-\eta)$ for all $\varphi \in E_{V_2}$ and all $t \ge 0$.

Since both Lyapunov functions V_1 and V_2 satisfy the conditions of Theorem 2.4, we conclude that $E_{V_1} \cap E_{V_2}$ is asymptotically attracting to all solutions $x_t(\varphi)$ of (17) with $\varphi \in \mathcal{C}_{\mathbb{D}}$. Take a $\varphi^* \in E_{V_1}$. For all $\eta \in [-\mathcal{T}^+, 0]$, we have $\varphi_i^*(\eta) = x_1^*$ for all $i \in \mathcal{I}_R$ and $\varphi_i^*(\eta) \le x_1^*$ for all $i \in \mathcal{I}_{\overline{R}}$. Now we have that in addition $\varphi^* \in E_{V_2}$, i.e. for all $\eta \in [-\mathcal{T}^+, 0]$, φ^* satisfies $\varphi_i^*(\eta) = x_2^*$ for all $i \in \mathcal{I}$ and $\varphi_i^*(\eta) \ge x_2^*$ for all $i \in \mathcal{I}_{\overline{R}}$. This obviously requires that $x_1^* = x_2^*$ and moreover $\varphi_i^*(\eta) = x_1^*$ for all $i \in \mathcal{I}$ and all $\eta \in [-\mathcal{T}^+, 0]$, i.e. we have $E_{V_1} \cap E_{V_2} = \Theta$. Remember that every element of Θ is invariant with respect to (17), i.e. $E_{V_1} \cap E_{V_2} = M_{V_1} \cap M_{V_2}$. From this, we conclude that for all $\varphi \in \mathcal{C}_{\mathbb{D}}$, $x_t(\varphi) \to \Theta$ as $t \to \infty$. □

5.3 *Leader-driven consensus*

We now address briefly the leader-driven consensus problem, i.e. the case in which there is exactly one vertex v_L with state x_L and in-degree zero, $d_L = 0$. The invariance of the set $\mathbb{D}$ still holds by the arguments of Lemma 5.3.

Corollary 5.6: *Given a leader-driven MAS consisting of N agents with dynamics* (17), *where f_{ji} satisfy Assumption* 5.2 *and g_{ji} satisfy Assumption* 3.1, *and with initial condition $\varphi \in \mathcal{C}_{\mathbb{D}}$, $\varphi_L(\eta) = x_L$ for all $\eta \in [-\mathcal{T}^+, 0]$, as well as an underlying network topology of a directed graph with a spanning tree, then the initial condition φ_L of the leader is asymptotically attracting, i.e. $x_t \to \phi_{x_L} = \varphi_L$ as $t \to \infty$.*

Proof: The leader-driven MAS has only one possible spanning tree with root v_L, i.e. $\mathcal{I}_R = \{v_L\}$. Moreover, we know that $\dot{x}_L = 0$. Going back to the proof of Theorem 5.5, we see that

$$E_{V_1} = \big\{\varphi \in \mathcal{C}_{\mathbb{D}} : \varphi_L(\eta) = x_L \text{ and } \varphi_i(\eta) \le x_L \\ \text{for all } \eta \in [-\mathcal{T}^+, 0] \text{ and all } i \in \mathcal{I}\big\},$$

$$E_{V_2} = \big\{\varphi \in \mathcal{C}_{\mathbb{D}} : \varphi_L(\eta) = x_L \text{ and } \varphi_i(\eta) \ge x_L \\ \text{for all } \eta \in [-\mathcal{T}^+, 0] \text{ and all } i \in \mathcal{I}\big\}.$$

Hence, ϕ_{x_L} is the only element in $E_{V_1} \cap E_{V_2}$ and we conclude that $x_t \to \phi_{x_L}$ for $t \to \infty$. □

Remark 5.7: Assume that there is more than one leader, i.e. there are at least two nodes v_{L_i}, $i = 1, \ldots, m$ with in-degree zero. In this case, the graph does not have a spanning tree. Yet, if there is a directed path from at least one of the leaders to every agent that is not a leader, then consensus is still asymptotically attracting for all initial conditions $\varphi \in \mathcal{C}_{\mathbb{D}}$, $\varphi_{L_i}(\eta) = x_L$ for all $\eta \in [-\mathcal{T}^+, 0]$ and $i = 1, \ldots, m$, provided that the leaders hold the same initial condition x_L. This result can be shown easily following the proof of Theorem 5.5.

6. Consensus with switching topology

6.1 *MAS with switching topology*

Having established attractivity of the consensus set for fixed graph topologies, we now turn to the case of a dynamic topology. We modify the assumptions on the MAS (17) in the following way. Assume a finite set of directed graphs $\{\mathcal{G}_p\}$ with $p \in \mathcal{P} = \{1, \ldots, P\}$. The communication network between the agents is represented at any time t by one of the graphs $\mathcal{G}_p$. The switching signal $\sigma : [0, \infty) \to \mathcal{P}$ determines the index of the active graph at time t. The signal σ is piecewise constant from the right and non-chattering, i.e. the time between two switching instants t_l is separated by a dwell time h, so that $t_{l+1} - t_l \ge h > 0$ for all $l = 1, 2, \ldots$. We assume that there are infinitely many switching times t_l because otherwise this problem could be solved as in § 5 considering only the last active graph. We denote all switching times when Graph $\mathcal{G}_p$ becomes active t_{p_ν}, $t_{p_{\nu+1}} > t_{p_\nu}$, i.e. $\sigma(t) = p$ for $t \in [t_{p_\nu}, t_{p_\nu+1})$ with $\nu = 1, 2, \ldots$. We define the set $\mathcal{P}_\infty \subseteq \mathcal{P}$ such that every graph $\mathcal{G}_p, p \in \mathcal{P}_\infty$ is infinitely often active, i.e. there are infinitely many switching times t_{p_ν}. Finally, we define the union graph $\mathcal{G}_\infty = (\mathcal{V}, \bigcup_{p \in \mathcal{P}_\infty} \mathcal{E}_p)$. This is the set of graphs that *persist over time*.

In Theorem 6.1, we require that $\mathcal{G}_\infty$ has a spanning tree in order to achieve a consensus. This condition resembles the *connected-over-time* assumption in previous works on switching MAS, see Jadbabaie et al. (2003) for continuous-time MAS and Moreau (2005), and Ren and Beard (2005) for the discrete-time case. For continuous-time MAS, Jadbabaie et al. (2003) require that there exists an infinite sequence of bounded, non-overlapping time intervals $[t_{\gamma_i}, t_{\gamma_i + \delta_i})$, $i = 1, 2, \ldots$, such that the union of the active graphs across each of these intervals contains a spanning tree. For a finite set of graphs $\{\mathcal{G}_p\}, p \in \mathcal{P}$, we know that there is only a finite number of subsets $\{\mathcal{G}_p\}, p \in \tilde{\mathcal{P}}_s \subseteq \mathcal{P}, s \in \mathcal{S} = \{1, \ldots, S\}$, of $\{\mathcal{G}_p\}, p \in \mathcal{P}$, such that the union graph of $\{\mathcal{G}_p\}$, $p \in \tilde{\mathcal{P}}_s$, $s \in \mathcal{S}$, contains a spanning tree. Clearly, the condition in Jadbabaie et al. (2003) requires that all elements of at least one of these subsets $\{\mathcal{G}_p\}$, $p \in \tilde{\mathcal{P}}_s, s \in \mathcal{S}$, appear in each time interval $[t_{\gamma_i}, t_{\gamma_i + \delta_i})$, $i = 1, 2, \ldots$. Since there are infinitely many such time intervals, this means that all elements of at least one of these subsets $\{\mathcal{G}_p\}, p \in \tilde{\mathcal{P}}_s, s \in \mathcal{S}$, appear infinitely often. Hence, our new condition is not more restrictive than the former conditions from Jadbabaie et al. (2003).

The dynamics of the N agents of the MAS with switching topology are

$$\dot{x}_i(t) = -k_i^{(\sigma)} \sum_{j=1}^{N} a_{ji}^{(\sigma)} g_{ji}\left(x_i(t) - \frac{1}{\alpha_{ji}} \int_{T_{ji}^-}^{T_{ji}^+} f_{ji}(\eta) x_j(t-\eta) \mathrm{d}\eta \right), \tag{24}$$

for all $i \in \mathcal{I}$ and with $k_i^{(p)} > 0$ for all $i \in \mathcal{I}$ and all $p \in \mathcal{P}$. $A^{(p)} = [a_{ij}^{(p)}]$ is the adjacency matrix of graph $\mathcal{G}_p$. The initial condition is $x_0 = \varphi$. We assume that the delay densities f_{ji} satisfy Assumption 5.2. For $\alpha_{ji} = 1$, (24) is the distributed delay analogue of (6) with a switching topology. Hence, our result holds for MAS consensus problems over PSN with switching topologies. The consensus set of the MAS (24) is Θ as defined in (19).

6.2 *Leaderless consensus*

Since the topology is changing in an arbitrary way, we will approach the analysis using a *common Lyapunov* *function* argument. The simplest example is what is known as *quadratic stability* to ensure that a system comprised of P subsystems of the form $\dot{x} = A_p x, p \in \mathcal{P}$, is stable under arbitrary switching. The conditions in this case require the existence of a *common* Lyapunov function $V = x^T Q x$, $Q > 0$, such that $A_p^T Q + Q A_p < 0$ for all $p \in \mathcal{P}$; see Boyd, El Ghaoui, Feron, and Balakrishnan (1994). For consensus problems over network channels with delays, such an argument comes along with many challenges. The main problem is that an invariance principle has to be applied to conclude attractivity of the consensus set. Our solution uses some recent results from Hespanha et al. (2005).

As in § 5, we start with the leaderless consensus problem. Since $\mathcal{C}_{\mathbb{D}}$ is positively invariant with respect to every subsystem of (24), it is also invariant with respect to an arbitrarily switching system (24).

Theorem 6.1: *Given a leaderless MAS consisting of N agents with dynamics* (24), *where f_{ji} satisfy Assumption* 5.2 *and g_{ji} satisfy Assumption* 3.1, *and with initial condition $\varphi \in \mathcal{C}_{\mathbb{D}}$, as well as an underlying switched network topology of directed graphs, such that the union graph $\mathcal{G}_\infty$ has a spanning tree, then the consensus set Θ of this MAS is asymptotically attracting, i.e. $x_t \to \Theta$ as $t \to \infty$.*

Proof: Following Theorem 5.5, we prove the attractivity of the consensus set Θ in the spirit of Theorem 2.4. In this paragraph, we briefly outline the proof. In Part (i) of the proof, we define two common Lyapunov Razumikhin function candidates V_1 and V_2 and prove that they satisfy condition (5). In order to determine the sets E_{V_k} and M_{V_k}, $k = 1, 2$, we adopt an invariance principle from Hespanha et al. (2005). This requires the definition of two functionals $\overline{V}_k$, $k = 1, 2$, which are based on V_1 and V_2. Since the derivatives of $\overline{V}_k$ cannot be determined easily, we just distinguish between the two important cases $\dot{\overline{V}}_k = 0$ and $\dot{\overline{V}}_k < 0$ through simple rules in Part (ii) of the proof. Using these conditions, we apply the result from Hespanha et al. 2005 and determine the sets E_{V_k} and M_{V_k} in Part (iii).

Part (i): We consider the functions V_1 and V_2 from the proof of Theorem 5.5 as common Lyapunov–Razumikhin function candidates. The indices I and J are defined as in Theorem 5.5 to be the maximal and minimal states over all agents. The right-hand Dini derivatives of V_1 and V_2 along solutions of (24) are

$$\dot{V}_1^{(\sigma)}(x_t) = -k_I^{(\sigma)} \sum_{j=1}^{N} a_{jI}^{(\sigma)} g_{jI} \times \left(x_I(t) - \frac{1}{\alpha_{jI}} \int_{T_{jI}^-}^{T_{jI}^+} f_{jI}(\eta) x_j(t-\eta) \mathrm{d}\eta \right),$$

$$\dot{V}_2^{(\sigma)}(x_t) = k_J^{(\sigma)} \sum_{j=1}^{N} a_{jJ}^{(\sigma)} g_{jJ} \times \left(x_J(t) - \frac{1}{\alpha_{jJ}} \int_{T_{jJ}^-}^{T_{jJ}^+} f_{jJ}(\eta) x_j(t-\eta) \mathrm{d}\eta \right).$$

Following the proof of Theorem 5.5, we know that $\dot{V}_k^{(p)} \le 0$, $k = 1, 2$, for all $p \in \mathcal{P}$, whenever $V_k(x(t)) = \max_{\eta \in [0, \mathcal{T}^+]} V_k(x(t-\eta))$. We conclude that condition (5) is satisfied and that the functionals

$$\overline{V}_k(x_t) = \max_{\eta \in [0, \mathcal{T}^+]} V_k(x(t-\eta)),$$

$k = 1, 2$, are non-increasing.

Part (ii): It would be quite difficult to calculate the right-hand Dini derivatives of $\overline{V}_k$ along solutions of (24). Instead, we determine simple rules to distinguish the two main cases $\dot{\overline{V}}_k = 0$ and $\dot{\overline{V}}_k < 0$. Therefore, we use the following notation: let the values I_η and J_η indicate the indices at each time t that satisfy $x_{I_\eta}(t-\eta) = \max_{i \in \mathcal{I}} x_i(t-\eta)$ and $x_{J_\eta}(t-\eta) = \min_{i \in \mathcal{I}} x_i(t-\eta)$, respectively. If there are several possible indices, we chose any one of them. Let $\eta_I, \eta_J \in [0, \mathcal{T}^+]$ be such that

$$x_{I_{\eta_I}}(t-\eta_I) = \max_{\eta \in [0, \mathcal{T}^+]} \max_{i \in \mathcal{I}} x_i(t-\eta), \tag{25}$$

$$x_{J_{\eta_J}}(t-\eta_J) = \min_{\eta \in [0, \mathcal{T}^+]} \min_{i \in \mathcal{I}} x_i(t-\eta), \tag{26}$$

i.e. $\overline{V}_1(x_t) = x_{I_{\eta_I}}(t-\eta_I)$ and $\overline{V}_2(x_t) = -x_{J_{\eta_J}}(t-\eta_J)$. Clearly, η_I and η_J are changing with time and there might be several values $\eta_I, \eta_J \in [0, \mathcal{T}^+]$ that satisfy (25) and (26), respectively. Now, we can state the following about the derivatives of $\overline{V}_k$, $k = 1, 2$, along solutions of (24):

- $\dot{\overline{V}}_k^{(p)}(x_t) \le 0$ for all $p \in \mathcal{P}$.
- $\dot{\overline{V}}_1^{(p)}(x_t) = 0$ if and only if there exists an $\eta_I \in [0, \mathcal{T}^+)$ that satisfies Equation (25). This situation is depicted in Figure 1.
- $\dot{\overline{V}}_1^{(p)}(x_t) < 0$ if and only if $\eta_I = \mathcal{T}^+$ satisfies Equation (25) and there does not exist an $\eta_I^* \in [0, \mathcal{T}^+)$ that satisfies Equation (25). This situation is depicted in Figure 2.
- $\dot{\overline{V}}_2^{(p)}(x_t) = 0$ if and only if there exists an $\eta_J \in [0, \mathcal{T}^+)$ that satisfies Equation (26).
- $\dot{\overline{V}}_2^{(p)}(x_t) < 0$ if and only if $\eta_J = \mathcal{T}^+$ satisfies Equation (26) and there does not exist an $\eta_J^* \in [0, \mathcal{T}^+)$ that satisfies Equation (26).

Part (iii): With these conditions, we can now turn to an invariance principle for switched systems. Barbarlat's Lemma (see, e.g. Khalil 2002) cannot be used in this case because of the derivatives $\dot{\overline{V}}_k^{(\sigma)}$ that are switching with σ. Therefore, our proof goes along Theorem 7 in Hespanha et al. (2005), which addresses

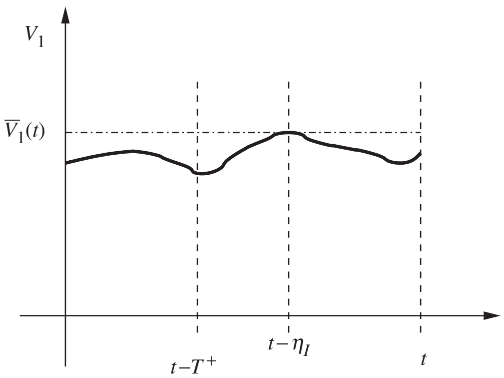

Figure 1. Exemplary Lyapunov function V_1 and exemplary Lyapunov functional $\overline{V}_1$ in a situation where $\dot{\overline{V}}_1^{(p)}=0$, i.e. $\eta_I\in[0,\mathcal{T}^+)$.

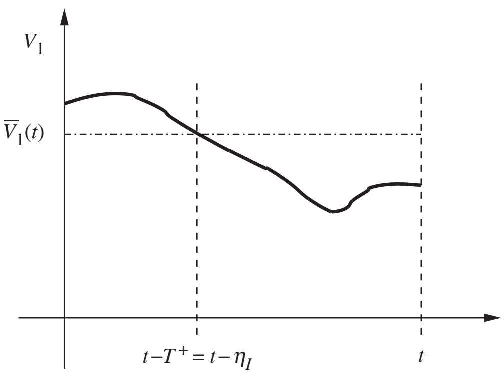

Figure 2. Exemplary Lyapunov function V_1 and exemplary Lyapunov functional $\overline{V}_1$ in a situation where $\dot{\overline{V}}_1^{(p)}<0$, i.e. $\eta_I=\mathcal{T}^+$.

the global asymptotical stability of a switched system with arbitrary switching. However, we have to adjust this result at some points because we are considering switching RFDEs instead of switching ODEs.

Recall the definition of the switching times t_{p_ν} given in §6.1, such that $\sigma(t)=p$ for $t\in[t_{p_\nu},t_{p_\nu+1})$ with $\nu=1,2,\ldots$. Since two switching times t_l are separated by a dwell time h, we have

$$\overline{V}_k(x(t_l))-\overline{V}_k(x(0))=\sum_{p=1}^{P}\sum_{\nu=1}^{\nu_p^*}\int_{t_{p_\nu}}^{t_{p\nu+1}}\dot{\overline{V}}_k^{(p)}(x_t)\mathrm{d}t, \quad (27)$$

with ν_p^* such that $t_{p_{\nu_p^*}+1}\le t_l$ and $t_{p_{\nu_p^*+1}}>t_l$. From our former arguments, we know that $\dot{\overline{V}}_k^{(p)}(x_t)\le 0$. Moreover, we know that the left-hand side of (27) converges to a finite value for $t_l\to\infty$ because $\overline{V}_k$ is nonincreasing and bounded from below. Following the proof of Theorem 7 in Hespanha et al. (2005), we conclude that $\dot{\overline{V}}_k^{(p)}(x_t)\to 0$ as $t\to\infty$ for all $p\in\mathcal{P}_\infty$ and $k=1,2$. Clearly, $\dot{\overline{V}}_k^{(p^*)}(x_t)\to 0$ is not necessary for those $p^*\in\mathcal{P}\backslash\mathcal{P}_\infty$ because these graphs are only active for a finite number of times, i.e. $\nu_{p^*}^*$ does not go to infinity for $t_l\to\infty$.

Now, we have to determine the sets

$$E_{V_k}=\left\{\varphi\in\mathcal{C}_{\mathbb{D}}:\dot{\overline{V}}_k^{(p)}(x_t(\varphi))=0\quad\text{for all } p\in\mathcal{P}_\infty,\ t\ge 0\right\},$$
$$k=1,2,$$

in order to conclude that $x_t\to E_{V_k}$ as $t\to\infty$. We first consider E_{V_1}. For every $\varphi\in E_{V_1}$, there exists an $x_1^*\in\mathbb{R}$ such that $\overline{V}_1(x_t(\varphi))=x_1^*$, i.e. $x_1^*=\max_{\eta\in[0,\mathcal{T}^+]}\max_{i\in\mathcal{I}}x_i(\varphi)(t-\eta)$, for all $t\ge 0$. Following our former arguments, we know that $\dot{\overline{V}}_1^{(p)}(x_t)=0$ if and only if there exists an $\eta_I\in[0,\mathcal{T}^+)$ that satisfies (25). Hence, we know that the right-hand Dini derivative of the Razumikhin candidates V_1 satisfy $\dot{V}_1^{(p)}(x(t-\eta_I))=0$ (Figure 1) and this requires that

$$x_{I_{\eta_I}}(t-\eta_I)-\frac{1}{\alpha_{jI_{\eta_I}}}\int_{T_{jI_{\eta_I}}^-}^{T_{jI_{\eta_I}}^+}f_{jI_{\eta_I}}(\eta)x_j(t-\eta_I-\eta)\mathrm{d}\eta=0, \quad (28)$$

for all j with $e_{jI_{\eta_I}}\in\mathcal{E}_p$. Since $x_{I_{\eta_I}}(t-\eta_I)=x_1^*$, we conclude that $x_j(t-\eta_I-\eta)=x_1^*$ for all $\eta\in[T_{jI_{\eta_I}}^-,T_{jI_{\eta_I}}^+]$ and $e_{jI\eta_I}\in\mathcal{E}_p$. The same arguments hold for all $p\in\mathcal{P}_\infty$. Note that x_1^* depends on φ but not on p. Following the proof of Theorem 5.5, we conclude that, if the union graph $\mathcal{G}_\infty$ has at least one root $v_i, i\in\mathcal{I}_R$, then E_{V_1} is given by (22). We know that $\mathcal{G}$ has at least one root because it contains a spanning tree. We use similar arguments for E_{V_2} and obtain (23). As in Theorem 5.5, we conclude that $x_t\to\Theta=E_{V_1}\cap E_{V_2}$ for $t\to\infty$. □

This result shows that a consensus is reached among non-linear, locally passive MAS even if they exchange information over PSN with a switching network topology.

6.3 *Leader-driven consensus*

Finally, we consider the leader-driven consensus problem with switching topologies. The leader is denoted by v_L and its constant state is x_L. Again, $\mathcal{C}_{\mathbb{D}}$ is positively invariant with respect to an arbitrarily switching system (24).

Corollary 6.2: *Given a leader-driven MAS consisting of N agents with dynamics* (24), *where* f_{ji} *satisfy Assumption* 5.2 *and* g_{ji} *satisfy Assumption* 3.1, *and with initial condition* $\varphi\in\mathcal{C}_{\mathbb{D}}$, $\varphi_L(\eta)=x_L$ *for all* $\eta\in[-\mathcal{T}^+,0]$, *as well as an underlying switched network topology of directed graphs, such that the union graph* $\mathcal{G}_\infty$ *has a spanning tree, then the initial condition* φ_L *of the leader is asymptotically attracting*, i.e. $x_t\to\phi_{x_L}=\varphi_L$ *as* $t\to\infty$.

The proof can be derived in a way similar to the proofs of Corollary 5.6 and Theorem 6.1.

Remark 6.3: Our results in Theorem 6.1 and Corollary 6.2 are also valid for switching delay distributions $f_{ji}^{(\sigma)}$ with

$$\int_{\bar{T}_{ij}^{(p)}}^{\overset{+}{T}_{ij}^{(p)}} f_{ji}^{(p)}(\eta)\mathrm{d}\eta = \alpha_{ji}^{(p)},$$

where $\alpha_{ji}{}^{(p)} > 0$ appear in (24) and $\mathcal{T}^+ = \max_{i,j\in\mathcal{I},p\in\mathcal{P}} \overset{+}{T}_{ij}^{(p)}$. Switching delay distributions can model PSN with time-varying stochastic properties due to changing background traffic. Even the non-linear, locally passive functions $g_{ji}^{(p)}$ may switch over time if for all $p \in \mathcal{P}$

$$g_{ji}^{(p)}(0) = 0 \text{ and } y g_{ji}^{(p)}(y) > 0 \text{ for all } y \in [-\gamma_{ij}^-, \gamma_{ij}^+]\backslash\{0\}.$$

The only restriction is that there are some $f_{ji}^{(p)}$ and $g_{ji}^{(p)}$ that are infinitely many times active. We skip this generalisation to simplify the notation. The proof follows directly from Theorem 6.1 with appropriate modifications in Equation (28).

7. Agreement and synchronisation of Kuramoto oscillators via PSN

So far, we presented a very general result for consensus of MAS over PSN which has been obtained by modelling the network channel dynamics using distributed delays. This simplification is necessary in order to obtain analytical results. Now, we return to the MAS communicating over PSN. We study the agreement and synchronisation of identical Kuramoto oscillators (Kuramoto 1984). First, we apply our results to a network of Kuramoto oscillators with distributed delays. These results are compared to simulations of Kuramoto oscillators exchanging state information over a PSN. This example illustrates that the results obtained for distributed delay MAS can be directly linked to MAS consensus over a PSN. This also clarifies the relation between accuracy of the simplified distributed delay model (14) and the maximum coupling gain K as described in Proposition 5.1. Moreover, we will see that the distributed delay model is useful to describe the steady-state frequency of the coupled Kuramoto oscillators.

Networks of Kuramoto oscillators are often used to model synchronisation of oscillators in different fields of biology, physics and engineering. Here, we consider oscillators with identical natural frequency $\omega \in \mathbb{R}$. The phase $\tilde{\theta}_i$ of agent i, $i \in \mathcal{I}$, is continuously updated according to the following stochastic RFDE:

$$\dot{\tilde{\theta}}_i(t) = \omega - \frac{K}{d_i}\sum_{j=1}^{N} a_{ji} \sin\big(\tilde{\theta}_i(t) - \tilde{\theta}_j(t - \tau_{ji}(t))\big), \qquad (29)$$

where $K > 0$ is the coupling gain. The topology between the agents is given by a directed graph $\mathcal{G}$ with adjacency matrix $A = [a_{ij}]$. If $a_{ji} > 0$, data can be sent from agent j to agent i with a stochastic delay τ_{ji} induced by the PSN. The in-degree of agent i is d_i. Note that $\tilde{\theta}_i$ are stochastic processes. We can reformulate (29) with distributed delays, using (14), and obtain the deterministic RFDE for the deterministic phase θ_i

$$\dot{\theta}_i(t) = \omega - \frac{K}{d_i}\sum_{j=1}^{N} a_{ji} \sin\left(\theta_i(t) - \int_{T_{ji}^-}^{T_{ji}^+} f_{ji}(\eta)\theta_j(t-\eta)\mathrm{d}\eta\right), \qquad (30)$$

where f_{ji} describe the delay distribution of the communication channels. For simplicity, we assume all channels to have the same statistics, i.e. $f_{ji} = f$, $T_{ji}^- = T^-$, and $T_{ji}^+ = T^+$ for all $i, j \in \mathcal{I}$.

The MAS we will be looking at consists of $N = 4$ agents. We compare two graphs, $\mathcal{G}_1$ and $\mathcal{G}_2$ given in Figure 3, where $\mathcal{G}_2$ contains a leader and $\mathcal{G}_1$ does not have a leader. Both graphs contain a spanning tree. We consider two constant initial conditions for RFDE (29), namely $\varphi^{(1)}(s) = \pi[0.45 \;\; 0.2 \;\; -0.43 \;\; -0.15]^T$ and $\varphi^{(2)}(s) = \pi[0.95 \;\; 0.2 \;\; -0.43 \;\; -0.15]^T$ for $s \in [-1, 0]$. All simulations are performed with MATLAB SIMULINK®. The PSN is modelled using SimEvents (MathWorks 2006). In order to transmit continuous-time signals over PSN, they are first sampled with a sampling period of $T_S = 0.005$ s. Then, every 10 consecutive sampled values are stored in one packet, and every $T_P = 0.05$ s, a packet is sent over the network. At the receiver side, we reconstruct a continuous-time piecewise constant signal by extracting the 10 sampled values one after the other from the packets. If the delay between two consecutive packets is more than T_P, the last value from the old packet is held until the new packet arrives. If the delay is less than T_P, then the remaining data from the old packet is dismissed and the data from the new packet is given out immediately. The processing time of the server is chosen such that the packet-delay τ_{ji} from sender to

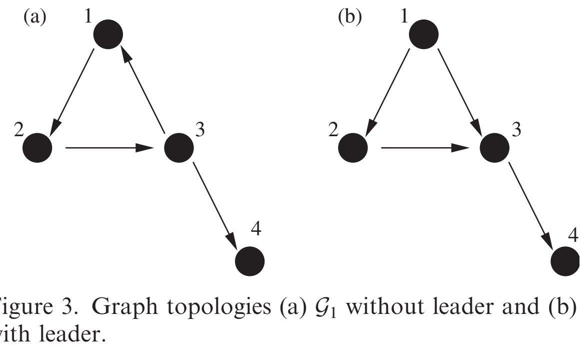

Figure 3. Graph topologies (a) $\mathcal{G}_1$ without leader and (b) $\mathcal{G}_2$ with leader.

receiver satisfies in all simulations $\tau_{ji} < 0.1$ s and the delays are approximately gamma distributed as proposed in Salza et al. (2000). Between every two connected agents, there is an individual PSN channel. All channels have the same delay distribution f but different realisations of the random delay processes τ_{ji}. Summarising, this is a typical scenario for a PSN like the Internet.

In § 7.1, we first study the degenerate case $\omega = 0$, i.e. the agents try to agree on a common phase. The resulting model fits perfectly in our problem statement in § 3, in particular Equation (3). In § 7.2, we study the synchronisation of Kuramoto oscillators with $\omega > 0$, i.e. the agents have to agree on a common phase that rotates with a steady-state frequency $\overline{\omega}$. This requires further analysis of the coupled Kuramoto oscillators to explain the simulation results.

7.1 *Agreement of Kuramoto oscillators*

First, we consider Equation (30) for $\omega = 0$. Note that $g_{ji}(y) = \sin(y)$ is locally passive with $-\gamma_{ji}^- = \gamma_{ji}^+ = \gamma < \pi$ for all $i, j \in \mathcal{I}$. Consequently, the first initial condition satisfies $\varphi^{(1)} \in \mathcal{C}_{\mathbb{D}}$ whereas $\varphi^{(2)} \notin \mathcal{C}_{\mathbb{D}}$ (cf (20)). Therefore, Theorem 5.5 and Corollary 5.6 guarantee consensus of MAS (30) for both graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ for initial condition $\varphi^{(1)}$ but not for $\varphi^{(2)}$.

We illustrate this result in our simulations using MAS (29). Exemplarily, we show first the result of two simulations with initial condition $\varphi^{(1)}$ and $K = 1$ for both topologies, see Figure 4. Consensus is achieved in both cases. For the leader-driven MAS, the state of the leader is adopted by the other agents, see Figure 4(b). Note that the leader-less MAS converges faster than the leader-driven MAS.

Next, we study the speed of convergence. As pointed out in § 5.1, the speed of convergence depends in general on K for MAS without delay. Here, we study the MAS (29) for $K \in \{0.5, 1, 2\}$. To allow for the stochastic behaviour of the PSN, we performed a Monte Carlo test with 100 simulations. For each simulation, i.e for each realisation of the stochastic processes $\tilde{\theta}$, we calculate the standard deviation (Std) of the four states 5 s after the onset of the consensus algorithm. Note that the Std has to decrease to 0 when consensus is achieved. The Std of the initial condition $\varphi^{(1)}$ is 1.215. It turns out that a consensus is always achieved and that the Std after 5 s is roughly the same for all simulations for a specific K. Table 1 shows for both graphs and $K \in \{0.5, 1, 2\}$ the range of the Std for 100 simulations. Note that a larger K leads to a smaller Std, i.e. a consensus is reached faster for larger K.

With the second initial condition $\varphi^{(2)}$, none of the simulations achieved a consensus, neither for $\mathcal{G}_1$ nor for $\mathcal{G}_2$. As explained before, the Kuramoto oscillator models the phase of coupled oscillators. Hence, $\theta_i + 2\pi$ is the same phase as θ_i. It is interesting to remark that all simulations of the MAS with initial condition $\varphi^{(2)}$ end up in a steady-state where all states are identical except for an integer multiple of 2π (not shown here).

7.2 *Synchronisation of Kuramoto oscillators*

Now, we study the more difficult problem of synchronisation of identical Kuramoto oscillators over PSN, i.e. (29), (30) with $\omega > 0$. Synchronisation means that the agents rotate in unison with the same phase and frequency. For identical constant delays, this model has been studied in Yeung and Strogatz (1999), and Earl and Strogatz (2003).

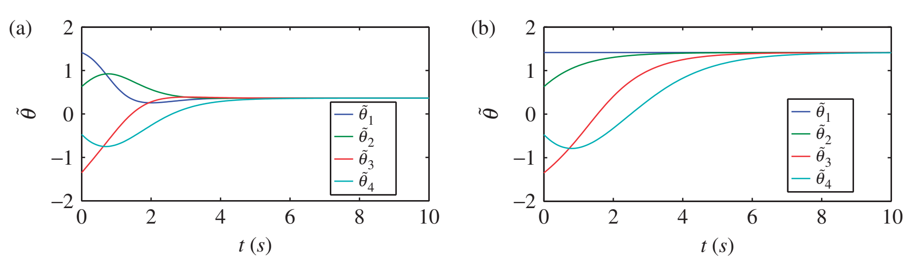

Figure 4. Simulation result for agreement with initial condition $\varphi^{(1)}$, (a) Topology $\mathcal{G}_1$ without leader and (b) Topology $\mathcal{G}_2$ with leader.

Table 1. Standard deviation of the states of the MAS (29) 5 s after the onset of the consensus algorithm.

Graph	$K = 0.5$	$K = 1$	$K = 2$
$\mathcal{G}_1$	2.036×10^{-1} to 2.046×10^{-1}	1.28×10^{-2} to 1.32×10^{-2}	9.09×10^{-5} to 1.00×10^{-4}
$\mathcal{G}_2$	6.466×10^{-1} to 6.481×10^{-1}	1.346×10^{-1} to 1.356×10^{-1}	2.335×10^{-3} to 2.372×10^{-3}

In order to adapt Equation (30) to our problem setup in § 3, we decompose the states of the agents into the constant steady-state frequency $\overline{\omega} \in \mathbb{R}$ and the relative phase ξ, i.e. $\theta_i(t) = \overline{\omega} t + \xi_i(t)$ for all $i \in \mathcal{I}$. For the transformed system, we show that ξ achieves a consensus for appropriate initial conditions, coupling gain K, and delay distribution f. As we will see later on, the steady-state frequency $\overline{\omega}$ satisfies

$$\overline{\omega} = \omega - K \sin(\overline{\omega}\tilde{\tau}), \tag{31}$$

where $\tilde{\tau} = \int_{T^-}^{T^+} \eta f(\eta) \mathrm{d}\eta$ is the expected value of the packet delay. Remember that we assume identical delay distributions in all channels. The interested reader is referred to Yeung and Strogatz (1999) for a detailed study of the solutions of (31), in particular with respect to the stability of synchronised and incoherent states for specific values of $\tilde{\tau}$ and K. Here, we assume that the parameters $\tilde{\tau}$ and K are such that the synchronised states are stable and the incoherent states are unstable.

Note that $\theta_i(t) = \overline{\omega} t + \xi_i(t)$ has to be satisfied for all agents $i \in \mathcal{I}$. Therefore, synchronisation for leader-driven MAS is only possible if $\omega\tilde{\tau} = k\pi$, $k \in \mathbb{N}$. This can be easily verified with (31) if we recall that the leader of the MAS satisfies $\theta_i(t) = \omega t$. Here, we only study graph $\mathcal{G}_1$ without a leader.

Next, we apply the transformation $\xi_i(t) = \theta_i(t) - \overline{\omega} t$ to (30) and obtain

$$\begin{aligned} \dot{\xi}_i(t) &= -\frac{K}{d_i} \sum_{j=1}^{N} a_{ji} \sin\left(\xi_i(t) - \int_{T^-}^{T^+} f(\eta) \xi_j(t-\eta) \mathrm{d}\eta + \overline{\omega}\tilde{\tau} \right) \\ &\quad + K \sin(\overline{\omega}\tilde{\tau}) \\ &= -\frac{K}{d_i} \sum_{j=1}^{N} a_{ji} \left(\sin\left(\xi_i(t) - \int_{T^-}^{T^+} f(\eta) \xi_j(t-\eta) \mathrm{d}\eta + \overline{\omega}\tilde{\tau} \right) \right. \\ &\quad \left. - \sin(\overline{\omega}\tilde{\tau}) \right). \end{aligned} \tag{32}$$

Note that

$$g(y) = \sin(y + \overline{\omega}\tilde{\tau}) - \sin(\overline{\omega}\tilde{\tau}) \tag{33}$$

is locally passive if $\frac{\mathrm{d}g}{\mathrm{d}y}|_{y=0} > 0$, i.e. $\cos(\overline{\omega}\tilde{\tau}) > 0$. Moreover, $g(y) = 0$ if $y = 2k\pi$ or $y = (2k+1)\pi - 2\overline{\omega}\tilde{\tau}$ for all $k \in \mathbb{Z}$. With this result, we specify the domain $y \in [-\gamma, \gamma]$ such that $yg(y) > 0$:

$$\gamma < \min_{k \in \mathbb{Z}} \left| (2k+1)\pi - 2\overline{\omega}\tilde{\tau} \right|, \tag{34}$$

see also Papachristodoulou and Jadbabaie (2006). Summarising, the distributed delay model guarantees synchronisation if $\cos(\overline{\omega}\tilde{\tau}) > 0$ and if the initial condition satisfies $\varphi \in \mathcal{C}_{\mathbb{D}}$, with $\mathcal{C}_{\mathbb{D}}$ given in (20) with γ from (34).

We use the following simulation parameters: $\omega = 2$, $K \in \mathcal{K} = \{1, 10\}$, and $\tilde{\tau} \approx 0.0525$ s. From this, the steady-state frequency results $\overline{\omega} \in \{1.9004, 1.3118\}$, depending on the choice of K. Clearly, we have $\cos(\overline{\omega}\tilde{\tau}) > 0$ and $\gamma \in \{2.942, 3.003\}$. Hence, the oscillators with distributed delay (30) synchronise for all values of $K \in \mathcal{K}$ and initial condition $\varphi^{(1)}$. This is verified by the oscillators with PSN (29) in all simulations. For two simulations, we depict exemplarily the Std of all four states *versus* time for $K = 1$ and $K = 10$ in Figure 5. We see immediately that the oscillators synchronise faster on the right-hand side, i.e. for larger K. However, we also see that the Std does not vanish completely when synchrony is achieved. This remaining noise of synchrony is due to the agreement on a rotating phase and a stochastic packet delay; it is much less intense in the simulations for $\omega = 0$, see also Table 1. Remember that the states $\tilde{\theta}_i$ of (29) are stochastic processes. Their mean reaches consensus but their variance does not vanish. Interestingly, this variance is smaller in Figure 5(a), i.e. for smaller K. This illustrates nicely the trade-off between fast convergence and accuracy of the distributed delay MAS as indicated in Proposition 5.1. In the presented example, the inaccuracy leads only to a very small synchronisation error, the Std remains below 0.02.

Summarising, the simulations illustrate that our results obtained for MAS with distributed delays can be transfered to packet-switched MAS.

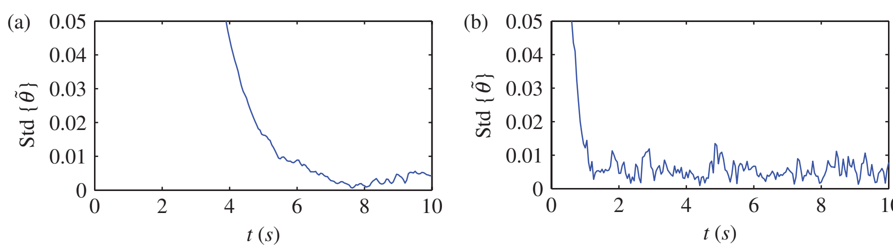

Figure 5. Simulation result for synchronisation with initial condition $\varphi^{(1)}$, (a) $K = 1$ and (b) $K = 10$.

8. Conclusions

In this contribution, we have provided conditions for a class of non-linear, locally passive MAS to reach a consensus even if the agents communicate over a packet-switched communication network with switching topology. For this, we used a distributed delay model to approximate a stochastic model of packet-switched communication channels. We provided an analytic bound for the error between the stochastic delay and distributed delay model. Using the distributed delay model, it was shown that a non-linear, locally passive MAS achieves a consensus over PSN for arbitrary delay distributions and delay sizes as well as both for fixed and switching network topologies. We only require that the underlying graph contains a spanning tree for the fixed topology case; in the case of switching graphs, we require that the union graph of the set of graphs that persist over time contains a spanning tree. We illustrated that the results obtained MAS with distributed delays are also valid for packet-switched MAS simulating the synchronisation of Kuramoto oscillators via PSN.

The tools we used to derive these results are based on an invariance principle for time-delay systems using Razumikhin functions. Future research work will focus on the analysis of consensus algorithms among agents with yet more complicated dynamics and communication constraints.

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Appendix A: Proof of Proposition 5.1

First, we define the following values:

$$T_{j,\max} = \max_{i\in\mathcal{I}}\left(T_{ji}^{+} - T_{ji}^{-}\right),$$
$$\bar{x}_{j,\max} = \max_{\eta\in[T_{ji}^{-},\,T_{ji}^{+}],\,i\in\mathcal{I}} \left|\bar{x}_j(t-\eta)\right|,$$
$$\bar{x}_{j,\min} = \min_{\eta\in[T_{ji}^{-},\,T_{ji}^{+}],\,i\in\mathcal{I}} \left|\bar{x}_j(t-\eta)\right|.$$

With the upper bound M_x, it follows directly from (6) that $\max_{j\in\mathcal{I}} |\dot{\bar{x}}_j(t)| \leq 2K\bar{d}M_x$ with $\bar{d} = \max_{i\in\mathcal{I}} d_i$ and we conclude $\bar{x}_{j,\min} \geq \bar{x}_{j,\max} - 2K\bar{d}M_xT_{j,\max}$. For sufficiently small K, we may assume $M_x \geq \bar{x}_{j,\max} \geq 2K\bar{d}M_xT_{j,\max}$. Then, a conservative bound for the variance of $\bar{x}_j(t-\tau_{ji}(t))$ is (cf (11))

$$\begin{aligned}\mathrm{Var}\{\bar{x}_j(t-\tau_{ji}(t))\} &\leq \bar{x}_{j,\max}^2 - \bar{x}_{j,\min}^2\\ &\leq 4K\bar{d}M_xT_{j,\max}(\bar{x}_{j,\max} - K\bar{d}M_xT_{j,\max})\\ &\leq 4K\bar{d}M_x^2T_{j,\max}(1 - K\bar{d}T_{j,\max})\\ &\leq 2K\bar{d}M_x^2T_{j,\max}.\end{aligned}$$

This proves Proposition 5.1 with the Chebyshev inequality (12) and

$$\lim_{K\to 0}\mathrm{Var}\{\bar{x}_j(t-\tau_{ji}(t))\} \leq \lim_{K\to 0} 2K\bar{d}M_x^2T_{j,\max} = 0.$$