ADAPTIVE OUTPUT REGULATION OF LINEAR SYSTEMS

1 The linear regulator equations

Consider a problem of output regulation for a linear system modelled by

$$\begin{aligned}\dot{w} &= Sw\\ \dot{x} &= Ax + Bu + Pw\\ e &= Cx + Qw\end{aligned}$$

and assume that the matrix S that models the exosystem has simple eigenvalues on the imaginary axis. For expository reasons we consider first the (non realistic) case in which the control law is a feedback law having access to the full state x of the plant and the full state w of the exosystem, namely is a control law of the form

$$u = Kx + Lw\,. \tag{1}$$

This is called the case of "perfect information". We also assume that all system parameters are known exactly.

Necessity. Suppose the problem is solved by some control law of the form (1). Then, in the corresponding closed loop system

$$\begin{pmatrix}\dot{w}\\ \dot{x}\end{pmatrix} = \begin{pmatrix}0 & S\\ P+BL & A+BK\end{pmatrix}\begin{pmatrix}w\\ x\end{pmatrix}$$

$$e = \begin{pmatrix}Q & C\end{pmatrix}\begin{pmatrix}w\\ x\end{pmatrix}$$

all trajectories are bounded and $\lim_{t\to\infty} e(t) = 0$. If the trajectories are bounded, the matrix $A + BK$ must have all eigenvalues with negative real part. Thus, *necessarily, the matrix pair A, B is stabilizable.* If the matrix $A + BK$ has eigenvalues with negative real part, the Sylvester equation

$$\Pi S = (A + BK)\Pi + (P + BL) \tag{2}$$

has a unique solution Π. Change x into $\tilde{x} = x - \Pi w$, to obtain the system

$$\begin{pmatrix}\dot{w}\\ \dot{\tilde{x}}\end{pmatrix} = \begin{pmatrix}S & 0\\ 0 & A+BK\end{pmatrix}\begin{pmatrix}w\\ \tilde{x}\end{pmatrix}$$

$$e = \begin{pmatrix}C\Pi + Q & C\end{pmatrix}\begin{pmatrix}w\\ \tilde{x}\end{pmatrix}.$$

Hence

$$e(t) = (C\Pi + Q)e^{St}w(0) + Ce^{(A+BK)t}\tilde{x}(0)\,.$$

Since the second term is decaying to 0, and the term $e^{St}w(0)$ does not decay to zero, the error $e(t)$ can decay to zero as $t \to \infty$ only if

$$C\Pi + Q = 0\,. \tag{3}$$

Set now $\Gamma = K\Pi + L$ and combine the two equations (2) and (3) to obtain

$$\begin{array}{rcl} \Pi S & = & A\Pi + B\Gamma + P \\ 0 & = & C\Pi + Q \end{array} \tag{4}$$

These equations are called the *linear regulator equations* (or, also, Francis' equations). We have just proven that, if the problem of output regulation is solved, the equations in question *must* have a solution Π, Γ.

Sufficency. Suppose the equations (4) have a solution Π, Γ, and consider a control law of the form

$$u = \Gamma w + K(x - \Pi w)$$

with K such that the eigenvalues of $A + BK$ have negative real part. The corresponding closed loop system is

$$\begin{pmatrix} \dot{w} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} S & 0 \\ P + B(\Gamma - K\Pi) & A + BK \end{pmatrix} \begin{pmatrix} w \\ x \end{pmatrix}$$

The eingenvalues of the system are those of $(A + BK)$ and those of S. Hence system possesses a stable invariant subspace, which is spanned by the columns of the matrix

$$\begin{pmatrix} 0 \\ I \end{pmatrix}.$$

The system also possesses a "center" invariant subspace, which is complementary to the stable invariant subspace and hence must be spanned by the columns of the matrix of the form

$$\begin{pmatrix} I \\ M \end{pmatrix},$$

for some M. Now, it is simple to check that, because of the first equation of (4), the matrix M is precisely equal to Π. In other words, the "center" invariant subspace is the set of all (w, x) pairs such that $x = \Pi w$.

Now, as $t \to \infty$, all trajectories of the system asymptotically approach the "center" invariant subspace. This means that,

$$\lim_{t\to\infty} [x(t) - \Pi w(t)] = 0\,.$$

This being the case, we see from the second equation of (4) that

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [Cx(t) + Qw(t)] = \lim_{t\to\infty} C[x(t) - \Pi w(t)] = 0\,.$$

Hence, the proposed law solves the problem.

2 The case of error feedback

Suppose the following

- the Francis' equations
$$\begin{array}{rcl} \Pi S & = & A\Pi + B\Gamma + P \\ 0 & = & C\Pi + Q \end{array} \tag{5}$$
have a solution Π, Γ
- there exist matrices Φ, H, Σ such that
$$\begin{array}{rcl} \Sigma S & = & \Phi\Sigma \\ \Gamma & = & H\Sigma \end{array} \tag{6}$$
- there exists a matrix G such that the linear system defined by triplet
$$\mathbf{A} = \begin{pmatrix} A & BH \\ 0 & \Phi \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} B \\ G \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} C & 0 \end{pmatrix}$$
is stabilizable by dynamic output feedback, i.e. that there are matrices K, L, M such that the matrix
$$\begin{pmatrix} \begin{pmatrix} A & BH \\ 0 & \Phi \end{pmatrix} & \begin{pmatrix} B \\ G \end{pmatrix} M \\ K \begin{pmatrix} C & 0 \end{pmatrix} & L \end{pmatrix} \tag{7}$$
has all eigenvalues with negative real part.

Consider now the (dynamic, error driven) control law
$$\begin{array}{rcl} u & = & H\eta + M\xi \\ \dot{\eta} & = & \Phi\eta + GM\xi \\ \dot{\xi} & = & L\xi + Ke \end{array} \tag{8}$$
The corresponding closed loop system is modeled by
$$\begin{pmatrix} \dot{w} \\ \dot{x} \\ \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} S & 0 & 0 & 0 \\ P & A & BH & BM \\ 0 & 0 & \Phi & GM \\ KQ & KC & 0 & L \end{pmatrix} \begin{pmatrix} w \\ x \\ \eta \\ \xi \end{pmatrix}.$$
The lower right 3×3 block coincides exactly with the matrix (7). Hence its eigenvalues have negative real part. The system possesses a stable invariant subspace, which is spanned by the columns of the matrix
$$\begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}.$$
The system also possesses a "center" invariant subspace, which is complementary to the stable invariant subspace and hence must be spanned by the columns of the matrix of the form
$$\begin{pmatrix} I \\ M_x \\ M_\eta \\ M_\xi \end{pmatrix},$$

for some M_x, M_η, M_ξ. Now, it is simple to check that, because of the first equation of (4) and (6), the matrix M_x is precisely equal to Π, the matrix M_η is precisely equal to Σ, and $M_\xi = 0$. That is, the "center" invariant subspace is spanned by the columns of the matrix form

$$\begin{pmatrix} I \\ \Pi \\ \Sigma \\ 0 \end{pmatrix} .$$

In other words, the "center" invariant subspace is the set of all (w, x, η, ξ) quadruplets such that

$$\begin{array}{rcl} x & = & \Pi w \\ \eta & = & \Sigma w \\ \xi & = & 0 \,. \end{array}$$

Now, as $t \to \infty$, all trajectories of the system asymptotically approach the "center" invariant subspace. This means that,

$$\begin{array}{l} \lim_{t\to\infty}[x(t) - \Pi w(t)] = 0 \\ \lim_{t\to\infty}[\eta(t) - \Sigma w(t)] = 0 \,. \end{array}$$

This being the case, we see from the second equation of (5) that

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}[Cx(t) + Qw(t)] = \lim_{t\to\infty} C[x(t) - \Pi w(t)] = 0 \,.$$

Hence, the proposed law solves the problem.

Note also that

$$\lim_{t\to\infty}[u(t) - \Gamma w(t)] = 0 \,.$$

3 Robust output regulation

Suppose now the plant depends on vector μ of uncertain parameters, namely

$$\begin{array}{rcl} \dot{w} & = & Sw \\ \dot{x} & = & A(\mu)x + B(\mu)u + P(\mu)w \\ e & = & C(\mu)x + Q(\mu)w \end{array}$$

(note that S is assumed to be *independent* of μ).

The previous result can be extended as follows. Assume that:

- (i) the μ-dependent Francis' equations

$$\begin{array}{rcl} \Pi(\mu)S & = & A(\mu)\Pi(\mu) + B(\mu)\Gamma(\mu) + P(\mu) \\ 0 & = & C(\mu)\Pi(\mu) + Q(\mu) \end{array} \qquad (9)$$

  have a μ-dependent solution $\Pi(\mu), \Gamma(\mu)$.

- (ii) there exist matrices $\Phi, H, \Sigma(\mu)$, with Φ, H *independent of* μ, such that

$$\begin{array}{rcl} \Sigma(\mu)S & = & \Phi\Sigma(\mu) \\ \Gamma(\mu) & = & H\Sigma(\mu) \end{array} \qquad (10)$$

- (iii) there exists a matrix G, *independent of* μ, such that the linear system defined by triplet

$$\begin{pmatrix} A(\mu) & B(\mu)H \\ 0 & \Phi \end{pmatrix}, \quad \begin{pmatrix} B(\mu) \\ G \end{pmatrix}, \quad \begin{pmatrix} C(\mu) & 0 \end{pmatrix}$$

is *robustly* stabilizable by dynamic output feedback, i.e. that there are matrices K, L, M, *independent of* μ, such that the matrix

$$\begin{pmatrix} \begin{pmatrix} A(\mu) & B(\mu)H \\ 0 & \Phi \end{pmatrix} & \begin{pmatrix} B(\mu) \\ G \end{pmatrix} M \\ K \begin{pmatrix} C(\mu) & 0 \end{pmatrix} & L \end{pmatrix} \tag{11}$$

has all eigenvalues with negative real part.

Then, the same controller discussed above solves the problem of *robust* output regulation. The proof is identical.

4 A useful lemma

The condition (ii) can always be fulfilled. In fact, let $d(\lambda) = \lambda^d + a_{d-1}\lambda^{d-1} + \ldots + a_1\lambda + a_0$ denote the minimal polynomial of S and recall that

$$S^d + a_{d-1}S^{d-1} + \ldots + a_1 S + a_0 I = 0$$

Define

$$\Phi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1} \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Since S is independent of μ, so is Φ. Then, it is easy to see that the conditions in question are fulfilled by

$$\Sigma(\mu) = \begin{pmatrix} \Gamma(\mu) \\ \Gamma(\mu)S \\ \cdots \\ \Gamma(\mu)S^{d-1} \end{pmatrix}$$

This is just one possible choice of matrices Φ, H. For the purpose of fulfilling also condition (iii), it is convenient to change the two matrices in a suitable way, taking advantage of the following Lemma.

Lemma 1 *Let F be any $r \times r$ Hurwitz matrix and let G be any $r \times 1$ vector such that the pair (F, G) is controllable. Let Φ be any $r \times r$ matrix whose eigenvalues have non-negative real part, and let H be any $1 \times r$ vector such that the pair (H, Φ) is observable. Then, there exist a $1 \times r$ vector Ψ and a nonsingular $r \times r$ matrix T such that*

$$\begin{aligned} (F + G\Psi)T &= T\Phi \\ \Psi T &= H\,. \end{aligned}$$

Proof. Observe, first of all, that the Sylvester equation

$$T\Phi = FT + GH$$

has a unique solution T, because Φ and F have no eigenvalues in common. We prove that T is nonsingular. Suppose, by contradiction, that the kernel of T is non-zero. Let $\{v_1, \ldots, v_k\}$ be a basis for $\ker(T)$. Then

$$T\Phi v_j = GHv_j\,, \qquad \text{for} \qquad j = 1, \ldots, k. \tag{12}$$

As T is square, there exists also a set of independent row vectors $\{w_1, \ldots, w_k\}$ such that $w_i T = 0$ for $i = 1, \ldots, k$. Then

$$w_i GHv_j = 0\,, \qquad \text{for} \qquad i, j = 1, \ldots, k.$$

Suppose $Hv_j = 0$ for all j's. In this case, (12) yields $T\Phi v_j = 0$, i.e.,

$$\Phi v_j \in \ker(T)\,, \qquad \text{for} \qquad j = 1, \ldots, k. \tag{13}$$

We find, in this way, that $\ker(T)$ is invariant under Φ and contained in $\ker(H)$, and this contradicts observability of (H, Φ). If $Hv_j \neq 0$ for at least one value of j, then $w_i G$ must be zero for all i's and, with a dual argument, we can prove that this contradicts controllability of (F, G). Having shown that T is nonsingular, to complete the proof it suffices to set $\Psi = HT^{-1}$. $\triangleleft$

Using this Lemma, it is immediate to show that, given any controllable pair of matrices F, G, with F Hurwitz, the matrix $\tilde{\Sigma}(\mu) = T\Sigma(\mu)$ satisfies

$$\begin{array}{rcl} \tilde{\Sigma}(\mu)S &=& (F + G\Psi)\tilde{\Sigma}(\mu) \\ \Gamma(\mu) &=& \Psi\tilde{\Sigma}(\mu) \end{array} \tag{14}$$

which is indeed a pair of equations of a form identical to (10).

Note that if this particular form is chosen, the controller (8) can be rewritten as

$$\begin{array}{rcl} u &=& \Psi\eta + M\xi \\ \dot{\eta} &=& F\eta + G(\Psi\eta + M\xi) \\ \dot{\xi} &=& K\xi + Le \end{array}$$

that is in the form of the cascade of a system modeled by

$$\begin{array}{rcl} u &=& \Psi\eta + v \\ \dot{\eta} &=& F\eta + G(\Psi\eta + M\xi) \end{array} \tag{15}$$

(which is usually referred to as the "internal model") and of a system modeled by

$$\begin{array}{rcl} v &=& M\xi \\ \dot{\xi} &=& K\xi + Le \end{array} \tag{16}$$

(which is usually referred to as the "stabilizer").

This form is particularly useful in fulfilling condition (iii) for certain classes of systems.

5 The case of systems in normal form

Suppose now the controlled system has relative degree r between input u and error e and put it in normal form, which is

$$\begin{array}{rcl} \dot{w} &=& Sw \\ \dot{z} &=& p_0(\mu)w + A_{00}(\mu)z + a_{01}(\mu)e_1 \\ \dot{e}_1 &=& e_2 \\ && \cdots \\ \dot{e}_{r-1} &=& e_r \\ \dot{e}_r &=& p_r(\mu)w + A_{r0}(\mu)z + a_{r1}(\mu)e_1 + \ldots + a_{rr}(\mu)e_r + b(\mu)u \\ e &=& e_1\,. \end{array} \tag{17}$$

Assume that the eigenvalues of $A_{00}(\mu)$ have negative real part. If this is the case, then all assumptions (i), (ii) and (iii) are fulfilled. In fact, as far as assumption (i) is concerned, observe that under this assumption, since the eigenvalues of S have zero real part, the Sylvester equation

$$\Pi_0(\mu)S = p_0(\mu) + A_{00}(\mu)\Pi_0(\mu)$$

has a unique solution $\Pi_0(\mu)$. This being the case, it is immediate to check that the Francis equations have a solution

$$\Pi(\mu) = \begin{pmatrix} \Pi_0(\mu) \\ 0 \\ \cdots \\ 0 \end{pmatrix}$$

and

$$\Gamma(\mu) = -\frac{1}{b(\mu)}[p_r(\mu) + A_{r0}(\mu)\Pi_0(\mu)]\,.$$

Thus, condition (i) above is fulfilled. Condition (ii) can be fulfilled as explained in the previous section. It remains to show how condition (iii) can be fulfilled. To this end, observe that the system considered in condition (iii), namely the system described by the triplet of matrices

$$\mathbf{A}(\mu) = \begin{pmatrix} A(\mu) & B(\mu)\Psi \\ 0 & F + G\Psi \end{pmatrix}, \quad \mathbf{B}(\mu) = \begin{pmatrix} B(\mu) \\ G \end{pmatrix}, \quad \mathbf{C}(\mu) = \begin{pmatrix} C(\mu) & 0 \end{pmatrix}$$

in this case is a system of the form

$$\begin{array}{rcl} \dot{z} &=& A_{00}(\mu)z + a_{01}(\mu)e_1 \\ \dot{e}_1 &=& e_2 \\ && \cdots \\ \dot{e}_{r-1} &=& e_r \\ \dot{e}_r &=& A_{r0}(\mu)z + a_{r1}(\mu)e_1 + \ldots + a_{rr}(\mu)e_r + b(\mu)[\Psi\eta + v] \\ \dot{\eta} &=& F\eta + G[\Psi\eta + v] \\ e &=& e_1 \end{array} \tag{18}$$

It is possible to show that this system is robustly stabilizable. In fact, this system has still relative degree r between input v and output e. To compute its zero dynamics, we set $e = 0$ and obtain

$$e_1 = e_2 = \ldots = e_r = 0$$

and

$$0 = A_{r0}(\mu)z - b(\mu)[\Psi\eta + v]\,.$$

Entering these constraint into the other equations, we see that z and η are necessarily solutions of

$$\begin{array}{rcl} \dot{z} & = & A_{00}(\mu)z \\ \dot{\eta} & = & F\eta + G\frac{1}{b(\mu)}[-A_{r0}(\mu)z] \end{array}$$

that is

$$\begin{pmatrix} \dot{z} \\ \dot{\eta} \end{pmatrix} = \begin{pmatrix} A_{00}(\mu) & 0 \\ -\frac{1}{b(\mu)}GA_{r0}(\mu) & F \end{pmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix}$$

This system has all eigenvalues with negative real part. Hence, robust stabilization by means of a controller of the form (16) is possible.

6 Internal Model Adaptation

The remarkable feature of the controller discussed in the previous section is the ability of securing asymptotic decay of the regulated output $e(t)$ in spite of parameter uncertainties.

Thus, control schemes incorporating a robust controller efficiently address the problem of rejecting all exogenous inputs generated by the exosystem. In this sense, they generalize the classical way in which integral-control based schemes cope with constant but unknown disturbances. There still is a limitation, though, in these schemes: the necessity of a precise model of the exosystem. As a matter of fact, the controller considered above contains a pair of matrices Φ, H whose construction (see above) requires the knowledge of the precise values of the coefficients of the minimal polynomial of S. The reader will have no difficulty in checking that, in general, the internal model property will be lost if inaccurate values of these coefficients are used to construct the matrix Φ. This limitation is not sensed in a problem of set point control, where the uncertain exogenous input is constant and thus obeys a trivial, parameter independent, differential equation, but becomes immediately evident in the problem of rejecting e.g. a *sinusoidal* disturbance of unknown amplitude and phase. A robust controller is able to cope with uncertainties on amplitude and phase of the exogenous sinusoidal signal, but the frequency at which the internal model oscillates must exactly match the frequency of the exogenous signal: any mismatch in such frequencies results in a nonzero steady-state error.

In what follows we show how this limitation can be removed, by automatically tuning the "natural frequencies" of the robust controller. For the sake of simplicity, we limit ourselves to sketch here the main philosophy of the design method.

Consider again the system for which we have learned how to design a robust controller but suppose, now, that the model of exosystem which generates the disturbance w depends on a vector ϱ of uncertain parameters, ranging on a prescribed set $\mathcal{Q}$, as in

$$\dot{w} = S(\varrho)w\,. \tag{19}$$

We retain the assumption that the exosystem is neutrally stable, in which case $S(\varrho)$ can only have eigenvalues on the imaginary axis (with simple multiplicity in the minimal polynomial).

Therefore, uncertainty in the value of ϱ is reflected in the uncertainty in the value of the imaginary part of these eigenvalues.

Let

$$m_\varrho(\lambda) = \lambda^d + d_{s-1}(\varrho)\lambda^{d-1} + \cdots + a_1(\varrho)\lambda + d_0(\varrho)$$

denote the minimal polynomial of $S(\varrho)$ and assume that the coefficients $a_{d-1}(\varrho), \ldots, a_1(\varrho)$, $a_0(\varrho)$ are continuous functions of ϱ. Define a pair of matrices Φ_ϱ, H as

$$\Phi_\varrho = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(\varrho) & -a_1(\varrho) & -a_2(\varrho) & \cdots & -a_{d-1}(\varrho) \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

the former of which is a continuous function of ϱ. Appealing to Lemma 1, it can be asserted that, if F, G is a controllable pair in which F is a Hurwitz matrix, there exists a vector Ψ_ϱ and a matrix T_ϱ such that

$$\begin{array}{rcl} (F + G\Psi_\varrho)T_\varrho & = & T_\varrho\Phi_\varrho \\ \Psi_\varrho T_\varrho & = & H\,. \end{array}$$

Consequently, there exists a matrix $\tilde{\Sigma}(\varrho, \mu)$ such that such that

$$\begin{array}{rcl} \tilde{\Sigma}(\varrho, \mu)S(\varrho) & = & (F + G\Psi_\varrho)\tilde{\Sigma}(\varrho, \mu) \\ \Gamma(\mu) & = & \Psi_\varrho\tilde{\Sigma}(\varrho, \mu) \end{array} \tag{20}$$

If ϱ were known, the controller considered above, with $\Psi = \Psi_\varrho$ would be a robust controller (having assumed, of course, that the remaining assumptions are fulfilled). In case ϱ is not known, one may wish to replace the vector Ψ with an *estimate* $\hat{\Psi}$ of Ψ_ϱ, *to be tuned* by means of an appropriate adaptation law.

We illustrate how this works in the simpler situation in which the system has relative degree 1.

Consider a control law of the form

$$\begin{array}{rcl} u & = & \hat{\Psi}\eta + v \\ \dot{\eta} & = & F\eta + G(\hat{\Psi}\eta + v) \end{array} \tag{21}$$

in which $\hat{\Psi}$ is a $1 \times d$ vector to be tuned. Then, the controlled system becomes (recall that we are dealing with the relative degree 1 case)

$$\begin{array}{rcl} \dot{w} & = & S(\varrho)w \\ \dot{z} & = & p_0(\mu)w + A_{00}(\mu)z + a_{01}(\mu)e_1 \\ \dot{e}_1 & = & p_1(\mu)w + A_{10}(\mu)z + a_{11}(\mu)e_1 + b(\mu)[\hat{\Psi}\eta + v] \\ \dot{\eta} & = & F\eta + G[\hat{\Psi}\eta + v] \\ e & = & e_1 \end{array}$$

Define an estimation error $\tilde{\Psi} = \hat{\Psi} - \Psi_\varrho$, and rewrite the system in question as

$$\begin{array}{rcl} \dot{w} & = & S(\varrho)w \\ \dot{z} & = & p_0(\mu)w + A_{00}(\mu)z + a_{01}(\mu)e_1 \\ \dot{e}_1 & = & p_1(\mu)w + A_{10}(\mu)z + a_{11}(\mu)e_1 + b(\mu)[\Psi_\varrho\eta + v] + b(\mu)\tilde{\Psi}\eta \\ \dot{\eta} & = & F\eta + G[\Psi_\varrho\eta + v] + G\tilde{\Psi}\eta \\ e & = & e_1 \end{array}$$

Assume, without loss of generality, that $b(\mu) > 0$. We know from the previous analysis that if $\tilde{\Psi}$ were zero, the choice of a stabilizing control

$$v = -ke_1$$

(with $k > 0$ and large) would solve the problem of output regulation. In particular, in the resulting closed loop system, all trajectories would converge to the "center" invariant subspace

$$z = \Pi_0 w\,, \qquad e_1 = 0\,, \qquad \eta = \tilde{\Sigma}(\varrho,\mu)w\,.$$

For convenience, set

$$\bar{z} = z - \Pi_0 w$$

and

$$\bar{\eta} = \eta - \tilde{\Sigma}(\varrho,\mu)w$$

to obtain (we need to use, here, some of the identities established earlier)

$$\begin{array}{rcl} \dot{w} & = & S(\varrho)w \\ \dot{\bar{z}} & = & A_{00}(\mu)\bar{z} + a_{01}(\mu)e_1 \\ \dot{e}_1 & = & A_{10}(\mu)\bar{z} + a_{11}(\mu)e_1 + b(\mu)[\Psi_\varrho\bar{\eta} + v] + b(\mu)[\tilde{\Psi}\eta] \\ \dot{\bar{\eta}} & = & F\eta + G[\Psi_\varrho\eta + v] + G[\tilde{\Psi}\eta] \\ e & = & e_1 \end{array}$$

(note that we *have not* modified the terms $\tilde{\Psi}\eta$ for reasons that will become clear in a moment). The dynamics of w is completely decoupled, so that we can concentrate on the system

$$\begin{array}{rcl} \dot{\bar{z}} & = & A_{00}(\mu)\bar{z} + a_{01}(\mu)e_1 \\ \dot{e}_1 & = & A_{10}(\mu)\bar{z} + a_{11}(\mu)e_1 + b(\mu)[\Psi_\varrho\bar{\eta} + v] + b(\mu)[\tilde{\Psi}\eta] \\ \dot{\bar{\eta}} & = & F\eta + G[\Psi_\varrho\eta + v] + G[\tilde{\Psi}\eta]\,. \end{array}$$

This system has relative degree 1 between input v and output e. To put it in normal form, choose

$$x = \bar{\eta} - \frac{1}{b(\mu)}Ge_1$$

to obtain a system

$$\begin{pmatrix} \dot{\tilde{z}} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} A_{00}(\mu) & 0 \\ \frac{1}{b(\mu)}A_{10}(\mu) & F \end{pmatrix} \begin{pmatrix} \tilde{z} \\ x \end{pmatrix} + \begin{pmatrix} a_{01}(\mu) \\ +\frac{1}{b(\mu)}a_{11}(\mu) \end{pmatrix} e_1$$

$$\dot{e}_1 = \begin{pmatrix} A_{10}(\mu) & b(\mu)\Psi_\varrho \end{pmatrix} \begin{pmatrix} \tilde{z} \\ x \end{pmatrix} + [a_{11}(\mu) + \Psi_\varrho G]e_1 + b(\mu)v + b(\mu)\tilde{\Psi}\eta$$

We know that, if $\tilde{\Psi}\eta$ were zero, the system could be globally robustly stabilized by means of a control $v = -ke_1$ and large k, actually with a Lyapunov function of the form

$$V(\tilde{z}, x, e_1) = (\,\tilde{z}^{\mathrm{T}} \quad x\,)\, Z(\mu) \begin{pmatrix} \tilde{z} \\ x \end{pmatrix} + \frac{1}{2}e_1^2\,,$$

in which $Z(\mu)$ is a positive definite matrix satisying

$$Z(\mu)\begin{pmatrix} A_{00}(\mu) & 0 \\ \frac{1}{b(\mu)}A_{10}(\mu) & F \end{pmatrix} + \begin{pmatrix} A_{00}(\mu) & 0 \\ \frac{1}{b(\mu)}A_{10}(\mu) & F \end{pmatrix}^{\mathrm{T}} Z(\mu) < 0\,.$$

Set

$$\mathbf{x} = \begin{pmatrix} \tilde{z} \\ x \\ e_1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and rewrite the system thus obtained as

$$\dot{\mathbf{x}} = \mathbf{A}(\mu,\varrho)\mathbf{x} + \mathbf{b}\, b(\mu)\tilde{\Psi}\eta$$

in which $\mathbf{A}(\mu,\varrho)$ is the matrix

$$\mathbf{A}(\mu,\varrho) = \begin{pmatrix} \begin{pmatrix} A_{00}(\mu) & 0 \\ \frac{1}{b(\mu)}A_{10}(\mu) & F \end{pmatrix} & \begin{pmatrix} a_{01}(\mu) \\ +\frac{1}{b(\mu)}a_{11}(\mu) \end{pmatrix} \\ \begin{pmatrix} A_{10}(\mu) & b(\mu)\Psi_\varrho \end{pmatrix} & \begin{pmatrix} a_{11}(\mu) + \Psi_\varrho G - b(\mu)k \end{pmatrix} \end{pmatrix}$$

In this notation, the Lyapunov function $V(\tilde{z},x,e_1)$ can be rewritten as

$$V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{x}\,.$$

We know that if k is large

$$\mathbf{Z}(\mu)\mathbf{A}(\mu,\varrho) + \mathbf{A}^{\mathrm{T}}(\mu,\varrho)\mathbf{Z}(\mu) < 0\,.$$

Moreover, it is easy to check that

$$\mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{b} = e_1\,.$$

Compute now the derivative along the trajectories of the closed loop system of the positive definite quadratic form

$$U(\mathbf{x},\tilde{\Psi}) = \mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{x} + b(\mu)\tilde{\Psi}\tilde{\Psi}^{\mathrm{T}}\,.$$

This yields

$$\begin{aligned}\dot{U} \;&=\; \mathbf{x}^{\mathrm{T}}[\mathbf{Z}(\mu)\mathbf{A}(\mu,\varrho) + \mathbf{A}^{\mathrm{T}}(\mu,\varrho)\mathbf{Z}(\mu)]\mathbf{x} + 2\mathbf{x}^{\mathrm{T}}\mathbf{Z}(\mu)\mathbf{b}\, b(\mu)\tilde{\Psi}\eta + 2b(\mu)\tilde{\Psi}\dot{\tilde{\Psi}}^{\mathrm{T}} \\ &\le\; 2b(\mu)[e_1\tilde{\Psi}\eta + \tilde{\Psi}\dot{\tilde{\Psi}}^{\mathrm{T}}] = 2b(\mu)\tilde{\Psi}[e_1\eta + \dot{\tilde{\Psi}}^{\mathrm{T}}]\,.\end{aligned}$$

Observing that Ψ_ϱ is constant, we see that

$$\dot{\tilde{\Psi}}^{\mathrm{T}} = \dot{\hat{\Psi}}^{\mathrm{T}}\,.$$

Inspection of the previous inequality suggests to choose

$$\dot{\hat{\Psi}}^{\mathrm{T}} = -e_1\eta$$

so as to obtain

$$\dot{U} \le 0\,.$$

Thus, since $U(\mathbf{x},\tilde{\Psi})$ is positive definite, the trajectories of the closed-loop system are bounded. Moreover, the classical arguments of La Salle's invariance principle show that $\lim_{t\to\infty}\mathbf{x}(t) = 0$. In particular, this implies that $\lim_{t\to\infty} e_1(t) = 0$, and therefore the problem is solved.

APPENDIX: Robust stabilization of linear systems having all zeros with negative real part (minimum-phase systems)

1 The Theorems of Lyapunov for linear systems

Theorem 1 [Direct Theorem] *Consider a linear system*

$$\dot{x} = Ax\,.$$

Let P be a symmetric positive definite matrix and suppose that the matrix

$$PA + A^{\mathrm{T}}P$$

is negative definite. Then, all eigenvalues of the matrix A have negative real part.

Theorem 2 [Converse Theorem] *Consider a linear system*

$$\dot{x} = Ax$$

and suppose all eigenvalues of the matrix A have negative real part. Then, for any choice of a symmetric positive definite matrix Q, there exists a unique symmetric positive definite matrix P such that

$$PA + A^{\mathrm{T}}P = -Q\,.$$

2 The normal form

Consider a (single-input single-output) uncertain system

$$\begin{array}{rcl} \dot{x} &=& A(\mu)x + B(\mu)u \\ y &=& C(\mu)x \end{array} \tag{1}$$

with state x of dimension n and in which μ is a vector of uncertain parameters ranging on a known set $\mathbb{M}$. Suppose that the following assumptions hold:

- $A(\mu), B(\mu), C(\mu)$ are matrices of continuous functions of μ.
- The relative degree of the system is the same for all $\mu \in \mathbb{M}$.
- The zeros of the transfer function $C(\mu)(sI - A(\mu))^{-1}B(\mu)$ have negative real part for all $\mu \in \mathbb{M}$.

Letting r denote the relative degree, the system in question can be put in normal form, by means of a change of variables

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2(\mu) \end{pmatrix} x$$

in which

$$T_2(\mu) = \begin{pmatrix} C(\mu \\ C(\mu)A(\mu) \\ \cdots \\ C(\mu)A^{r-1}(\mu) \end{pmatrix}\,.$$

The normal form in question can be written as

$$
\begin{array}{rcl}
\dot{z} & = & A_{11}(\mu)z + A_{12}(\mu)\xi_1 \\
\dot{\xi}_1 & = & \xi_2 \\
 & & \cdots \\
\dot{\xi}_{r-1} & = & \xi_r \\
\dot{\xi}_r & = & A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)u \\
y & = & \xi_1 \,,
\end{array}
$$

with z a vector of dimension $n-r$, or, alternatively, as

$$
\begin{array}{rcl}
\dot{z} & = & A_{11}(\mu)z + A_{12}(\mu)\hat{C}\xi \\
\dot{\xi} & = & \hat{A}\xi + \hat{B}[A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)u] \\
y & = & \hat{C}\xi
\end{array}
$$

in which $\hat{A}, \hat{B}, \hat{C}$ are matrices of dimension $r \times r$, $r \times 1$ and $1 \times r$ respectively and have the form

$$
\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix}, \quad \hat{C} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.
$$

Moreover,

$$
b(\mu) = C(\mu)A^{r-1}(\mu)B(\mu)\,.
$$

As the consequence of the assumptions:

- $b(\mu) \neq 0$ for all $\mu \in \mathbb{M}$. By continuity, $b(\mu)$ it is either positive or negative for all $\mu \in \mathbb{M}$. In what follows, without loss of generality, it will be assumed that

$$
b(\mu) > 0 \qquad \text{for all } \mu \in \mathbb{M}. \tag{2}
$$

- The eigenvalues of $A_{11}(\mu)$ have negative real part for all $\mu \in \mathbb{M}$. This being the case, it is known from the converse Lyapunov Theorem that there exists a symmetric and *positive definite* matrix $P(\mu)$, of dimension $(n-r) \times (n-r)$, of continuous functions of μ such that

$$
P(\mu)A_{11}(\mu) + A_{11}^{\mathrm{T}}(\mu)P(\mu) = -I, \qquad \text{for all } \mu \in \mathbb{M}. \tag{3}
$$

3 The case of relative degree 1

Suppose, for the time being, that $r = 1$, in which case ξ is a vector of dimension 1 (i.e. a scalar quantity) and the normal form reduces to

$$
\begin{array}{rcl}
\dot{z} & = & A_{11}(\mu)z + A_{21}(\mu)\xi \\
\dot{\xi} & = & A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)u \\
y & = & \xi\,.
\end{array}
$$

Consider the control law

$$
u = -ky\,.
$$

This yields a closed loop system of the form

$$
\begin{array}{rcl}
\dot{z} & = & A_{11}(\mu)z + A_{12}(\mu)\xi \\
\dot{\xi} & = & A_{21}(\mu)z + [A_{22}(\mu) - b(\mu)k]\xi\,,
\end{array}
$$

or, what is the same

$$\begin{pmatrix} \dot z \\ \dot \xi \end{pmatrix} = \begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) \\ A_{21}(\mu) & [A_{22}(\mu) - b(\mu)k] \end{pmatrix} \begin{pmatrix} z \\ \xi \end{pmatrix} . \tag{4}$$

We want to prove that if k is large enough, this system is stable for all $\mu \in \mathbb{M}$. To this end, consider the positive definite $n \times n$ matrix

$$\begin{pmatrix} P(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

in which $P(\mu)$ is the matrix defined in (3). If we are able to show that the matrix

$$Q(\mu) = \begin{pmatrix} P(\mu) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) \\ A_{21}(\mu) & [A_{22}(\mu) - b(\mu)k] \end{pmatrix} + \begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) \\ A_{21}(\mu) & [A_{22}(\mu) - b(\mu)k] \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P(\mu) & 0 \\ 0 & 1 \end{pmatrix}$$

is *negative definite*, then by the direct Lyapunov Theorem, we can assert that system (4) has all eigenvalues with negative real part. A simple calculation shows that, because of (3)

$$Q(\mu) = \begin{pmatrix} -I & P(\mu)A_{12}(\mu) + A_{21}^{\mathrm{T}}(\mu) \\ [P(\mu)A_{12}(\mu) + A_{21}^{\mathrm{T}}(\mu)]^{\mathrm{T}} & 2[A_{22}(\mu) - b(\mu)k] \end{pmatrix}$$

The matrix in question is negative definite if its opposite, namely

$$-Q(\mu) = \begin{pmatrix} I & -[P(\mu)A_{12}(\mu) + A_{21}^{\mathrm{T}}(\mu)] \\ -[P(\mu)A_{12}(\mu) + A_{21}^{\mathrm{T}}(\mu)]^{\mathrm{T}} & -2[A_{22}(\mu) - b(\mu)k] \end{pmatrix} \tag{5}$$

is positive definite. Thus, we apply to this matrix the Sylvester criterion and we check the sign of all leading principal minor. Because of the special form of this matrix, the leading principal minors of order $1, 2, \ldots, n-1$ are all 1 (and hence positive). Thus the matrix in question is positive definite if (and only if) its determinant is positive. To compute the determinant, we observe that the matrix in question has the form

$$-Q(\mu) = \begin{pmatrix} I & d(\mu) \\ d^{\mathrm{T}}(\mu) & q(\mu) \end{pmatrix} .$$

Hence, by a well known formula

$$\det[-Q(\mu)] = \det[I]\det[q(\mu) - d^{\mathrm{T}}(\mu)I^{-1}d(\mu)] = q(\mu) - d^{\mathrm{T}}(\mu)d(\mu) = q(\mu) - \|d(\mu)\|^2 .$$

Thus, we can conclude that $Q(\mu)$ is negative definite for all $\mu \in \mathbb{M}$ if

$$q(\mu) - \|d(\mu)\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Reverting to the notations of (5) we can say that the $Q(\mu)$ is negative definite, or – what is the same – system (4) has all eigenvalues with negative real part - for all $\mu \in \mathbb{M}$ if

$$-2[A_{22}(\mu) - b(\mu)k] - \|[P(\mu)A_{12}(\mu) + A_{21}^{\mathrm{T}}(\mu)]\|^2 > 0 \qquad \text{for all } \mu \in \mathbb{M}.$$

Since $b(\mu) > 0$, the inequality is equivalent to

$$k > \frac{1}{2b(\mu)}\Big(2A_{22}(\mu) + \|[P(\mu)A_{12}(\mu) + A_{21}^{\mathrm{T}}(\mu)]\|^2\Big) \qquad \text{for all } \mu \in \mathbb{M}.$$

Now, set

$$k^* := \max_{\mu\in\mathbb{M}}\Big(2A_{22}(\mu) + \|[P(\mu)A_{12}(\mu) + A_{21}^{\mathrm{T}}(\mu)]\|^2\Big) .$$

This maximum exists because the functions are continuous functions of μ and $\mathbb{M}$ is closed and bounded. Then we are able to conclude that if

$$k > k^*$$

the control law

$$u = -ky$$

stabilizes the closed loop system, regardless of what the particular value of $\mu \in \mathbb{M}$ is. In other words, the control *robustly* stabilizes the system.

4 The case of higher relative degree: partial state feedback

Consider now the case of a higher relative degree r. Let the variable ξ_r of the normal form be replaced by a new state variable defined as

$$\theta = \xi_r + a_0\xi_1 + a_1\xi_2 + \cdots a_{r-2}\xi_{r-1} \tag{6}$$

in which $a_0, a_1, \ldots, a_{r-2}$ are design parameters. With this change of variable (it's only a change of variables, no control has been chosen yet !), we obtain a system of the form

$$\begin{array}{rcl} \dot{z} &=& A_{11}(\mu)z + A_{12}(\mu)\xi_1 \\ \dot{\xi}_1 &=& \xi_2 \\ && \cdots \\ \dot{\xi}_{r-1} &=& -(a_0\xi_1 + a_1\xi_2 + \cdots a_{r-2}\xi_{r-1}) + \theta \\ \dot{\theta} &=& A_{21}(\mu)z + \alpha_{21}(\mu)\xi_1 + \cdots + \alpha_{2,r-1}(\mu)\xi_{r-1} + \alpha_{2r}(\mu)\theta + b(\mu)u \\ y &=& \xi_1\,, \end{array}$$

in which $\alpha_{21}(\mu), \ldots, \alpha_{2r}(\mu)$ are appropriate coefficients. This system can be formally viewed as a system having relative degree 1, with input u and output θ. To this end, in fact, it suffices to set

$$\eta = \begin{pmatrix} z \\ \xi_1 \\ \cdots \\ \xi_{r-1} \end{pmatrix}$$

and rewrite the system as

$$\begin{array}{rcl} \dot{\eta} &=& \begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-1} \end{pmatrix} \eta + \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} \theta \\ \dot{\theta} &=& \begin{pmatrix} A_{21}(\mu) & \alpha_{21}(\mu) & \alpha_{22}(\mu) & \cdots & \alpha_{2,r-1}(\mu) \end{pmatrix} \eta + \alpha_{2r}(\mu)\theta + b(\mu)u \end{array}$$

The latter has the structure of a normal form

$$\begin{array}{rcl} \dot{\eta} &=& F_{11}(\mu)\eta + F_{21}(\mu)\theta \\ \dot{\theta} &=& F_{21}(\mu)\eta + F_{22}(\mu)\theta + b(\mu)u \end{array} \tag{7}$$

in which $F_{11}(\mu)$ is a block-triangular matrix

$$F_{11}(\mu) = \begin{pmatrix} A_{11}(\mu) & A_{12}(\mu) & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & -a_0 & -a_1 & \cdots & -a_{r-1} \end{pmatrix} := \begin{pmatrix} A_{11}(\mu) & * \\ 0 & A_0 \end{pmatrix}$$

By hypothesis, all eigenvalues of the submatrix $A_{11}(\mu)$ have negative real part for all μ. On the other hand, the characteristic polynomial of the submatrix A_0, which is a matrix in companion form, coincides with the polynomial

$$p_a(\lambda) = a_0 + a_1\lambda + \cdots + a_{r-2}\lambda^{r-2} + \lambda^{r-1}\,. \tag{8}$$

Thus, the coefficients $a_0, a_1, \ldots, a_{r-2}$ can be chosen in such a way that all eigenvalues of A_0 have negative real part. If this is the case, we can conclude that *all* the eigenvalues of $F_{11}(\mu)$ have negative real part for all μ.

Thus, system (7) can be seen as a system having relative degree 1 which satisfies the assumptions used in the previous section to obtain robust stability. In view of this, it is immediately concluded that there exists a number k^* such that, if $k > k^*$, the control law

$$u = -k\theta$$

robustly stabilizes the system.

Note that the control thus found, expressed in the original coordinates, reads as

$$u = -k[a_0\xi_1 + a_1\xi_2 + \cdots + a_{r-2}\xi_{r-1} + \xi_r]$$

that is as a linear combination of the components of the vector ξ. This is a *partial state* feedback, which can be written, in compact form, as

$$u = H\xi \,. \tag{9}$$

5 The case of higher relative degree: output feedback

We have seen earlier that a system in normal form can be semiglobally stabilized by means of a feedback law which is a linear form of the states $\xi_1, \dots, \xi_r$, that is a by means of a law of the form (9) in which H is a vector of "gain coefficients", which depend on the assigned set of uncertain parameters.

In general, the components of the state ξ are not available for feedback, nor they can be retrieved from the original state x, since the transformation that defines ξ in terms of x depends on the uncertain parameter μ. We see now how this problem can be overcome, by designing a dynamic controller that provides appropriate "replacements" for the components of ξ in the control law (9). Observing that these variables coincide, by definition, with the measured output y and with its first $r-1$ derivatives with respect to time, it seems reasonable to try to generate them by means of a dynamical system of the form

$$\begin{array}{rcl} \dot{\hat{\xi}}_1 &=& \hat{\xi}_2 + gc_{r-1}(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_2 &=& \hat{\xi}_3 + g^2 c_{r-2}(y - \hat{\xi}_1) \\ && \cdots \\ \dot{\hat{\xi}}_{r-1} &=& \hat{\xi}_r + g^{r-1} c_1(y - \hat{\xi}_1) \\ \dot{\hat{\xi}}_r &=& g^r c_0(y - \hat{\xi}_1)\,. \end{array} \tag{10}$$

In fact, if $\hat{\xi}_1$ where identical to y, it would follow that $\hat{\xi}_i$ coincides with $y^{(i-1)}$ for all $i = 1, 2, \dots, i$, that is with ξ_i. In compact form, this system will be rewritten as

$$\dot{\hat{\xi}} = F\hat{\xi} + Gy\,,$$

in which

$$F = \begin{pmatrix} -gc_{r-1} & 1 & 0 & \cdots & 0 \\ -g^2 c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -g^{r-1}c_1 & 0 & 0 & \cdots & 1 \\ -g^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} gc_{r-1} \\ g^2 c_{r-2} \\ \cdots \\ g^{r-1} c_1 \\ g^r c_0 \end{pmatrix}.$$

Let now ξ be replaced by $\hat{\xi}$ in the expression of the control law (9). In this way, we obtain a dynamic controller, described by equations of the form

$$\begin{array}{rcl} \dot{\hat{\xi}} &=& F\hat{\xi} + Gy \\ u &=& H\hat{\xi}\,. \end{array} \tag{11}$$

We will show in what follows that, if the coefficients g and $c_0, \dots, c_{r-1}$ which characterize (10) are chosen appropriately, this dynamic – output feedback – control law does actually robustly stabilizes the system.

Controlling the system (assumed to be expressed in normal form) by means of the control (11) yields a closed loop system

$$\begin{array}{rcl} \dot{z} &=& A_{11}(\mu)z + A_{12}(\mu)\hat{C}\xi \\ \dot{\xi} &=& \hat{A}\xi + \hat{B}[A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)H\hat{\xi}] \\ \dot{\hat{\xi}} &=& F\hat{\xi} + G\hat{C}\xi\,. \end{array}$$

To analyze this closed-loop system, we first perform a change of coordinates, defining

$$e_i = \frac{1}{g^i}\,(\hat{\xi}_i - \xi_i), \qquad i = 1, \dots, r\,.$$

In order to express the relation between $\xi, \hat{\xi}$ and e in compact form, we set

$$D_g = \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & g^2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdot & g^r \end{pmatrix}$$

and we observe that

$$D_g e = \hat{\xi} - \xi\,,$$

that is

$$\hat{\xi} = \xi + D_g e\,.$$

The next step in the analysis is to determine the differential equations for the new variables e_i, for $i = 1, \dots, r$. Simple calculations yield

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \cdots \\ \dot{e}_{r-1} \\ \dot{e}_r \end{pmatrix} = g \begin{pmatrix} -c_{r-1} & 1 & 0 & \cdots & 0 & 0 \\ -c_{r-2} & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ -c_1 & 0 & 0 & \cdots & 0 & 1 \\ -c_0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ e_{r-1} \\ e_r \end{pmatrix} + g^{-r} \begin{pmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 1 \end{pmatrix} [A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)H\hat{\xi}]. \tag{12}$$

In compact form, remembering the definition of $\hat{A}, \hat{B}, \hat{C}$ given before, this can be written as

$$\dot{e} = g(\hat{A} - G_0\hat{C})e + g^{-r}\hat{B}[A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)H\hat{\xi}]\,.$$

in which

$$G_0 = \begin{pmatrix} c_{r-1} \\ c_{r-2} \\ \cdots \\ c_1 \\ c_0 \end{pmatrix}.$$

Replacing also $\hat{\xi}$ with its expression in terms of ξ and e we obtain, at the end, a description of the closed loop system in the form

$$\begin{array}{rcl} \dot{z} &=& A_{11}(\mu)z + A_{12}(\mu)\hat{C}\xi \\ \dot{\xi} &=& \hat{A}\xi + \hat{B}[A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)H(\xi + D_g e)] \\ \dot{e} &=& g(\hat{A} - G_0\hat{C})e + g^{-r}\hat{B}[A_{21}(\mu)z + A_{22}(\mu)\xi + b(\mu)H(\xi + D_g e)]\,. \end{array}$$

To simplify this system further, we set

$$\tilde{x} = \begin{pmatrix} z \\ \xi \end{pmatrix}$$

and define the matrices

$$\begin{aligned} F_{11}(\mu) &= \begin{pmatrix} A_{11}(\mu) & A_{12}(\mu)\hat{C} \\ \hat{B}A_{21}(\mu) & \hat{A} + \hat{B}[A_{22}(\mu) + b(\mu)H] \end{pmatrix} & F_{12}(\mu, g) &= \begin{pmatrix} 0 \\ \hat{B}b(\mu)H \end{pmatrix} D_g \\ F_{21}(\mu, g) &= g^{-r}\,(\,\hat{B}A_{21}(\mu) \quad \hat{B}[A_{22}(\mu) + b(\mu)H]\,) & F_{22}(\mu, g) &= g^{-r}\hat{B}b(\mu)HD_g \end{aligned}$$

in which case the equations of the closed-loop system will be rewritten as

$$\begin{aligned} \dot{\tilde{x}} &= F_{11}(\mu)\tilde{x} + F_{12}(\mu, g)e \\ \dot{e} &= F_{21}(\mu, g)\tilde{x} + [g(\hat{A} - G_0\hat{C}) + F_{22}(\mu, g)]e\,. \end{aligned}$$

The advantage of having the system written in this form is that we know that the matrix $F_{11}(\mu)$, if H has been chosen as described in the earlier section, has eigenvalues with negative real part for all μ. Hence, there is a positive definite symmetric matrix $P(\mu)$ such that

$$P(\mu)F_{11}(\mu) + F_{11}(\mu)^{\mathrm{T}}P(\mu) = -I\,.$$

Moreover, it is readily seen that the characteristic polynomial of the matrix $(\hat{A} - G_0\hat{C})$ coincides with the polynomial

$$p_c(\lambda) = c_0 + c_1\lambda + \cdots + c_{r-1}\lambda^{r-1} + \lambda^{r-1}\,. \tag{13}$$

Thus, the coefficients $c_0, c_1, \ldots, c_{r-1}$ can be chosen in such a way that all eigenvalues of $(\hat{A} - G_0\hat{C})$ have negative real part. If this is done, there exists a positive definite symmetric matrix $\hat{P}$ such that

$$\hat{P}(\hat{A} - G_0\hat{C}) + (\hat{A} - G_0\hat{C})^{\mathrm{T}}\hat{P} = -I\,.$$

This being the case, we proceed now to show that the direct criterion of Lyapunov is fulfilled, for the positive definite matrix

$$\begin{pmatrix} P(\mu) & 0 \\ 0 & g^{2r}\hat{P} \end{pmatrix}$$

if the number g is large enough. To this end, we need to check that the matrix

$$\begin{aligned} Q = &\begin{pmatrix} P(\mu) & 0 \\ 0 & g^{2r}\hat{P} \end{pmatrix} \begin{pmatrix} F_{11}(\mu) & F_{12}(\mu, g) \\ F_{21}(\mu, g) & [g(\hat{A} - G_0\hat{C}) + F_{22}(\mu, g)] \end{pmatrix} \\ &+ \begin{pmatrix} F_{11}(\mu) & F_{12}(\mu, g) \\ F_{21}(\mu, g) & [g(\hat{A} - G_0\hat{C}) + F_{22}(\mu, g)] \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} P(\mu) & 0 \\ 0 & g^{2r}\hat{P} \end{pmatrix} \end{aligned}$$

is negative definite. In view of the definitions of $P(\mu)$ and $\hat{P}$, we see that

$$Q = \begin{pmatrix} -I & P(\mu)F_{12}(\mu, g) + F_{21}^{\mathrm{T}}(\mu, g)\hat{P}g^{2r} \\ [P(\mu)F_{12}(\mu, g) + F_{21}^{\mathrm{T}}(\mu, g)\hat{P}g^{2r}]^{\mathrm{T}} & -g^{2r+1}I + [g^{2r}\hat{P}F_{22}(\mu, g) + F_{22}^{\mathrm{T}}(\mu, g)\hat{P}g^{2r}] \end{pmatrix}.$$

The issue is to show that the quadratic form

$$\begin{aligned} (\,\tilde{x}^{\mathrm{T}} \quad e^{\mathrm{T}}\,)\,Q\begin{pmatrix} \tilde{x} \\ e \end{pmatrix} &= -\|\tilde{x}\|^2 + 2\tilde{x}[P(\mu)F_{12}(\mu, g) + F_{21}^{\mathrm{T}}(\mu, g)\hat{P}g^{2r}]e \\ &\quad - g^{2r+1}\|e\|^2 + e^{\mathrm{T}}[g^{2r}\hat{P}F_{22}(\mu, g) + F_{22}^{\mathrm{T}}(\mu, g)\hat{P}g^{2r}]e\,, \end{aligned}$$

is negative definite. This form can be bounded as

$$\begin{aligned}
(\,\tilde{x}^{\mathrm{T}} \quad e^{\mathrm{T}}\,)\, Q \begin{pmatrix} \tilde{x} \\ e \end{pmatrix} &\le -\|\tilde{x}\|^2 + 2\|P(\mu)F_{12}(\mu,g) + F_{21}^{\mathrm{T}}(\mu,g)\hat{P}g^{2r}\| \cdot \|\tilde{x}\| \cdot \|e\| \\
&\quad - g^{2r+1}\|e\|^2 + 2\|g^{2r}\hat{P}F_{22}(\mu,g)\|\|e\|^2 \\
&= (\,\|\tilde{x}\| \quad \|e\|\,) \begin{pmatrix} -1 & a(\mu,g) \\ a(\mu,g) & -g^{2r+1} + 2d(\mu,g) \end{pmatrix} \begin{pmatrix} \|\tilde{x}\| \\ \|e\| \end{pmatrix}
\end{aligned}$$

in which

$$\begin{aligned}
a(\mu,g) &= \|P(\mu)F_{12}(\mu,g) + F_{21}^{\mathrm{T}}(\mu,g)\hat{P}g^{2r}\| \\
d(\mu,g) &= 2\|g^{2r}\hat{P}F_{22}(\mu,g)\|\,.
\end{aligned}$$

The quadratic in form on the right-hand side is negative definite if so is the matrix

$$\begin{pmatrix} -1 & a(\mu,g) \\ a(\mu,g) & -g^{2r+1} + 2d(\mu,g) \end{pmatrix}$$

and this is the case if

$$g^{2r+1} > 2d(\mu,g) + a^2(\mu,g)\,. \tag{14}$$

Observe now that (recall that $\|\hat{B}\| = 1$)

$$d(\mu,g) \le g^{2r}\|\hat{P}\|\|F_{22}(\mu,g)\| \le g^r b(\mu)\|\hat{P}\|\|H\|\|D_g\|$$

If $g > 1$, then $\|D_g\| = g^r$ and hence

$$d(\mu,g) \le d_0(\mu)g^{2r}$$

for some $d_0(\mu) > 0$ depending only on μ and not on g. Similarly

$$a(\mu,g) \le \|P(\mu)\|b(\mu)\|H\|\|D_g\| + g^{-r}(\|A_{21}(\mu)\| + \|A_{22}(\mu) + b(\mu)H\|)g^{2r}\|\hat{P}\| \le a_0(\mu)g^r$$

for some $a_0(\mu) > 0$ depending only on μ and not on g. As a consequence

$$a^2(\mu,g) \le a_0^2(\mu)g^{2r}$$

Thus, a sufficient condition for the inequality (14) to be fulfilled is that

$$g^{2r+1} > 2d_0(\mu)g^{2r} + a_0^2(\mu)g^{2r}\,,$$

which indeed is the case if

$$g > 2d_0(\mu) + a_0^2(\mu)\,.$$

Since μ ranges on a closed and bounded set, this is occurs if $g > g^*$ with

$$g^* = \max_{\mu\in\mathbb{M}}[2d_0(\mu) + a_0^2(\mu)]\,.$$

In summary, we have shown that the uncertain system (1), under the under the assumption that

- $A(\mu), B(\mu), C(\mu)$ are matrices of continuous functions of μ
- the relative degree of the system is the same for all $\mu \in \mathbb{M}$, with $\mathbb{M}$ a closed and bounded set
- the zeros of the transfer function $C(\mu)(sI - A(\mu))^{-1}B(\mu)$ have negative real part for all $\mu \in \mathbb{M}$

can be *robustly stabilized* by means of a dynamic output-feedback control law of the form

$$
\begin{aligned}
\dot{\hat{\xi}} &= \begin{pmatrix} -gc_{r-1} & 1 & 0 & \cdots & 0 \\ -g^2c_{r-2} & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ -g^{r-1}c_1 & 0 & 0 & \cdots & 1 \\ -g^r c_0 & 0 & 0 & \cdots & 0 \end{pmatrix} \hat{\xi} + \begin{pmatrix} gc_{r-1} \\ g^2c_{r-2} \\ \cdots \\ g^{r-1}c_1 \\ g^r c_0 \end{pmatrix} y \\
u &= -k \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{r-2} & 1 \end{pmatrix} \hat{\xi},
\end{aligned}
$$

in which $c_0, c_1, \ldots, c_{r-1}$ and, respectively, $a_0, a_1, \ldots, a_{r-2}$ are such that the polynomials (13) and (8) have negative real part, and g and k are large (positive) parameters.