Robust Output Regulation by State Feedback Control for Coupled Linear Parabolic PIDEs

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***Abstract*—This paper is concerned with the backstepping design of state feedback regulators that achieve robust output regulation for coupled linear parabolic partial integro-differential equations (PIDEs) with spatially-varying coefficients. This problem is solved for a general setup, where polynomial and trigonometric reference inputs and disturbances are taken into account by employing a non-diagonalizable signal model. The regulator design is based on the internal model principle, which amounts to stabilize an ODE-PDE cascade, which consists of a finite-dimensional internal model driven by coupled parabolic PIDEs. For this, a systematic backstepping approach is developed and it is shown that the stabilizability depends on the plant transfer behaviour. A simple proof of robust output regulation is given, which does not rely on solving the extended regulator equations. The results of the paper are illustrated by means of an unstable parabolic system described by three coupled parabolic PIDEs with two outputs. The robustness of the proposed state feedback regulator is verified by comparing it with a non-robust feedforward regulator.**

***Index Terms*—Parabolic systems, coupled PDEs, output regulation, backstepping, boundary control.**

I. Introduction

A. Background and Motivation

The *backstepping control* of boundary controlled distributed-parameter systems (DPSs) has attracted the attention of many researchers in the last decades (for an overview see [20], [28]). As far as parabolic systems are concerned, current research is focused on the backstepping control of coupled parabolic PDEs. First solutions of this problem were given in [4], [24] for systems with constant coefficients. Subsequently, these results were extended to the spatially-varying coefficients case. Diffusion-convection-reaction systems are considered in [27], while [10] deals with parabolic partial integro-differential equations (PIDEs). Besides the theoretical appeal to enlarge the applicability of the backstepping method to a wider class of DPSs, the backstepping control of coupled parabolic systems is also of paramount interest for applications. In particular, these systems can be utilized to model technological processes originating in chemical and biochemical engineering (see [2], [16]). Furthermore, parabolic PIDEs arise in the physical modelling of crystallization processes and chemical reactors (see [5]), or result from a singular perturbation of separate subsystems with different time-scales (see [23]).

The aforementioned papers solve the stabilization problem, i. e., they regulate the system state in the presence of non-vanishing initial values to the origin. For a successful controller design, however, exogenous signals, i. e., reference inputs and disturbances, have to be taken into account, too. In particular, the controller should be able to track reference inputs in the presence of disturbances. Typically, the form of these signals is known beforehand. Hence, it is possible to model them by the solutions of ODEs. Consequently, the controller results from solving an *output regulation problem* assuming a finite-dimensional signal model. A basic property of any stabilizing controller is its inherent robustness w. r. t. at least sufficiently small model uncertainty. Hence, it is desirable that the output regulation should share the same robustness property. This leads to the *robust output regulation problem*, which is of fundamental importance in control theory. To achieve the desired robustness, the controller has to contain a model of the exogenous signals, which is driven by the tracking error. This is the well-known *internal model principle* of control theory (see [12]). As far as DPSs with boundary control and pointlike outputs are considered, the general properties of robust regulators are characterized in [25].

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B. Contributions

While many results exist for the regulator design concerning DPS described by a single PDE, the extension to coupled PDEs is still an emerging research topic. Furthermore, recent scientific contributions dealing with this system class mainly focus on regulators in form of observer-based feedforward controllers (see, e. g., [3] for an overview). Backstepping results for their design concerning coupled hyperbolic systems can be found in [1], [9] and in [11], [13], [18] for coupled parabolic systems. Since these controllers do not fulfill the internal model principle, robust output regulation is not ensured. Moreover, only few results can be found concerning robust regulators for coupled DPSs. For 2×2 hyperbolic systems with a single input, the works [21], [22] introduce integral action in the controller, while [7] considers more general signal models. For coupled parabolic PDEs the rejection of boundary input disturbances by making use of sliding mode control was investigated in [14]. Recently, it was demonstrated in [9] that the backstepping methods for general coupled hyperbolic PDEs can also be successfully applied to solve the robust output regulation problem.

In this paper, robust output regulation by state feedback control is considered for coupled parabolic PIDEs with spatially-varying coefficients. This is the extension of the results for single PDEs in [6], [8] to the multivariable case. The considered setup takes n coupled parabolic PIDEs subject to in-domain and boundary disturbances into account. Each component of the output can independently be defined pointwise in-domain or distributed in-domain as well as at the boundaries. Furthermore, an arbitrary number $1 \leq p \leq n$ of outputs is allowed to increase the design flexibility.

The previous works [6]–[9] dealing with the robust output regulation problem only assume diagonalizable signal models, which limits the class of allowed exogenous signals. In order to increase the applicability of the output regulation result, non-diagonalizable signal models are considered in this paper so that polynomial and trigonometric type signals with polynomial amplitudes can also be allowed. This, however, leads to a new challenge for verifying robust output regulation. In particular, the robustness analysis on the basis of the so-called *extended regulator equations* with the approach in [6]–[9] leads to a set of overdetermined equations. This substantially hinders the solvability verification of the extended regulator equations. Therefore, robust output regulation is proved by showing that the closed-loop solution in the steady state always ensures output regulation if the closed-loop system contains an internal model of the exogenous signals. Consequently, a new and simple proof for robust output regulation is obtained.

In comparison to the previous works [6]–[9], an additional challenge appears in the backstepping design of the regulator for the considered system class. In particular, for coupled parabolic PDEs the solution of the boundary value problems (BVPs) to determine the backstepping controller and the plant transfer matrix cannot be obtained explicitly. This is a major difference to [6]–[9], where these BVPs are explicitly solvable. Consequently, a major obstacle results in the derivation of conditions for the stabilizability of the corresponding target system, since the approach in [6]–[9] is based on the explicit solution of these BVPs. In this paper, an interesting and non-trivial relation between these BVPs is unveiled. More precisely, it is shown that the state equations derived from these BVPs are adjoint to each other. Consequently, the corresponding fundamental matrices are related through the well-known *reciprocity relation* (see, e. g., [26, Ch. 4]). With this, the solvability conditions of the stabilization problem needed for output regulation can be verified without the explicit knowledge of the fundamental matrices. This is also of general interest, when dealing with actuator or sensor dynamics described by coupled parabolic PIDEs generalizing the results in [19].

Finally, a simplified analysis of the closed-loop exponential stability is performed by decoupling the ODE-PDE cascade into the final target system consisting of independent ODE and PDE subsystems. This, in addition, provides a sound basis for the backstepping stabilization of coupled parabolic PIDE-ODE cascades, where results cannot be found in the literature so far. Consequently, the paper presents a systematic solution of the robust output regulation problem for a large class of coupled parabolic systems.

C. Organization

The next section introduces the considered output regulation problem. Section III presents the design of the state feedback controller for the plant augmented by a p-copy internal model. Robust output regulation is verified in Section IV. Section V demonstrates the results of the paper for an unstable parabolic system described by three coupled parabolic PIDEs with two outputs. The comparison with a feedforward regulator confirms the robustness properties of the proposed output regulation approach.

II. Problem formulation

Consider the *coupled parabolic PIDEs*

$$\partial_t x(z,t) = \bar{\Lambda}(z)\partial_z^2 x(z,t) + \bar{\mathcal{A}}[x(t)](z) + G_1(z)d(t) \tag{1a}$$
$$\bar{\theta}_0[x(t)](0) = G_2 d(t), \qquad t > 0 \tag{1b}$$
$$\bar{\theta}_1[x(t)](1) = u(t) + G_3 d(t), \qquad t > 0 \tag{1c}$$
$$y(t) = \bar{\mathcal{C}}[x(t)] + G_4 d(t), \qquad t > 0 \tag{1d}$$

with the formal operator

$$\bar{\mathcal{A}}[x(t)](z) = \bar{A}(z)x(z,t) + \bar{A}_0(z)x(0,t) + \int_0^z \bar{F}(z,\zeta)x(\zeta,t)\mathrm{d}\zeta, \tag{2}$$

in which $\bar{A} = A + \Delta A, \bar{A}_0 = A_0 + \Delta A_0 \in (C^1[0,1])^{n\times n}$ and $\bar{F} = F + \Delta F \in (C^1([0,1]^2))^{n\times n}$. The PIDEs in (1a) are defined on $(z,t) \in (0,1)\times\mathbb{R}^+$ with the state $x(z,t) \in \mathbb{R}^n$, $n > 1$, and the input $u(t) \in \mathbb{R}^n$. The matrix $\bar{\Lambda}(z) = \Lambda(z) + \Delta\Lambda(z) \in \mathbb{R}^{n\times n}$ is diagonal, i. e.,

$$\bar{\Lambda}(z) = \mathrm{diag}(\bar{\lambda}_1(z), \ldots, \bar{\lambda}_n(z)), \tag{3}$$

with $\bar{\lambda}_i \in C^2[0,1]$, $i = 1,\ldots,n$, satisfying $\bar{\lambda}_1(z) > \ldots > \bar{\lambda}_n(z) > 0$, which has also to be valid for $\Delta\Lambda = 0$. Furthermore, decoupled Robin boundary conditions (BCs) are considered at $z = 0$, while the boundaries at $z = 1$ can be coupled. This means that the formal *matrix differential operators*

$$\bar{\theta}_i[h] = \mathrm{d}_z h + \bar{B}_i h, \quad i = 0,1 \tag{4}$$

are specified by

$$\bar{B}_0 = B_0 + \Delta B_0 = \mathrm{diag}(\bar{b}_1, \ldots, \bar{b}_n) \tag{5a}$$
$$\bar{B}_1 = B_1 + \Delta B_1 \in \mathbb{R}^{n\times n}. \tag{5b}$$

The disturbance input locations characterized by $G_1 \in (C[0,1])^{n\times q}$ and $G_i \in \mathbb{R}^{n\times q}$, $i = 2,3$, $G_4 \in \mathbb{R}^{p\times q}$ need not be available for the design. The initial conditions (ICs) of the system are $x(z,0) = x_0(z) \in \mathbb{R}^n$.

The known *output* $y(t) \in \mathbb{R}^p$, $1 \le p \le n$, *to be controlled* is modeled by the formal *output operator*

$$\bar{\mathcal{C}}[h] = \int_0^1 \bar{C}(z)h(z)\mathrm{d}z + \bar{C}_{b0}h(0) + \bar{C}_{b1}h(1) \tag{6}$$

for $h(z) \in \mathbb{C}^n$ with $\bar{C}_{bi} = C_{bi} + \Delta C_{bi} \in \mathbb{R}^{p\times n}$, $i = 0,1$, and

$$\bar{C}(z) = \bar{C}_0(z) + \sum_{i=1}^{l} \bar{C}_i\delta(z - z_i). \tag{7}$$

This matrix is determined by $\bar{C}_0(z) = C_0(z) + \Delta C_0(z) = [c_{ij}(z)] \in \mathbb{R}^{p\times n}$ with c_{ij} piecewise continuous functions, $\bar{C}_i = C_i + \Delta C_i \in \mathbb{R}^{p\times n}$ and $z_i \in (0,1)$, $i = 1,\ldots,l$. In that way the operator (6) describes arbitrary combinations of distributed in-domain, point-wise in-domain and boundary outputs.

The known *nominal parameters* are Λ, A, A_0, F, B_i, C_i and C_{bi} whereas $\Delta\Lambda$, ΔA, ΔA_0, ΔF, ΔB_i, ΔC_i and ΔC_{bi} represent unknown *model uncertainties*.

Remark 1: General parabolic systems described by $\partial_t x(z,t) = \partial_z(\bar{\Lambda}(z)\partial_z x(z,t)) + \bar{\Phi}(z)\partial_z x(z,t) + \bar{\mathcal{A}}[x(t)](z) + G_1(z)d(t)$ with $\bar{\Phi} = \mathrm{diag}(\bar{\Phi}_1, \ldots, \bar{\Phi}_n) \in (C^1[0,1])^{n\times n}$ can be traced back to (1a) by a Hopf-Cole-type state transformation (see [11]). The BCs (4) arise frequently in applications. Nevertheless, the following results can be applied in the same way for systems with mixed Dirichlet-Robin BCs (see [10]). ◁

It is assumed that the known *reference input* $r(t) \in \mathbb{R}^p$ and the unmeasurable disturbance $d(t) \in \mathbb{R}^q$ can be represented by the solution of the finite-dimensional *signal model*

$$\dot{v}(t) = Sv(t), \qquad t > 0, v(0) = v_0 \in \mathbb{R}^{n_v} \tag{8a}$$
$$d(t) = P_d v(t), \qquad t \ge 0 \tag{8b}$$
$$r(t) = P_r v(t), \qquad t \ge 0 \tag{8c}$$

with $P_d \in \mathbb{R}^{q\times n_v}$ and $P_r \in \mathbb{R}^{p\times n_v}$. Therein, the *spectrum* $\sigma(S)$ of the matrix S only contains eigenvalues on the imaginary axis. This allows the modeling of persistently acting exogenous signals, which frequently appear in applications. In particular, they can be of the general form

$$d(t) = p_0(t) + \sum_{i=1}^{n_d}\Big(p_i^1(t)\cos(\omega_i^d t) + p_i^2(t)\sin(\omega_i^d t)\Big) \tag{9a}$$
$$r(t) = q_0(t) + \sum_{i=1}^{n_r}\Big(q_i^1(t)\cos(\omega_i^r t) + q_i^2(t)\sin(\omega_i^r t)\Big) \tag{9b}$$

with polynomial vector functions $p_0(t), p_i^j(t) \in \mathbb{R}^q$ and $q_0(t), q_i^j(t) \in \mathbb{R}^p$ of known degree as well as with known frequencies ω_i^d and ω_i^r. For the design of the regulator only the knowledge of the matrix S in (8a) is required. This means that only the form of the exogenous signals is known, while their parametrization by $v(0)$ and their generation through (8b) and (8c) need not be available for the controller design.

In this paper, the *robust output regulation problem* using state feedback is solved for (1). This amounts to designing a state feedback regulator such that the nominal closed-loop system is exponentially stable and the *output tracking error* $e_y = y - r$ satisfies

$$\lim_{t\to\infty} e_y(t) = \lim_{t\to\infty}(y(t) - r(t)) = 0 \tag{10}$$

for all ICs. Furthermore, the property (10) should be robust in the sense that it holds despite of all model uncertainties $\Delta\Lambda$, ΔA, ΔA_0, ΔF, ΔB_i, ΔC_i and ΔC_{bi} for which the controller stabilizes the closed-loop system.

III. State Feedback with p-Copy Internal Model

The solution of the posed robust output regulation problem requires that the *p-copy internal model principle* is satisfied (see, e. g., [25]). This means that a *p-copy internal model* has to be included in the controller. For this, the system matrix $\widetilde{S}$ of the controller must contain p copies of the *cyclic part* $S_{\min}$ of the matrix S. In particular, the matrix $S_{\min} \in \mathbb{R}^{n_c \times n_c}$ is obtained by taking only the largest Jordan block of each eigenvalue of S (see [12]). Then, the internal model needs to be driven by $e_{y_i} = y_i - r_i$, $i = 1, \ldots, p$, with $e_y = [e_{y_i}]$. Hence, it has the form

$$\dot{\hat{v}}(t) = \widetilde{S}\hat{v}(t) + B_y e_y(t), \quad t > 0, \quad \hat{v}(0) = \hat{v}_0 \in \mathbb{R}^{pn_c}, \tag{11}$$

with the block diagonal matrices $\widetilde{S} = \mathrm{bdiag}(S_{\min}, \ldots, S_{\min}) \in \mathbb{R}^{pn_c \times pn_c}$, $B_y = \mathrm{bdiag}(b_{y_1}, \ldots, b_{y_p}) \in \mathbb{R}^{pn_c \times p}$ and $b_{y_i} \in \mathbb{R}^{n_c}$, $i = 1, \ldots, p$. A prerequisite for the stabilizability of the resulting augmented system is that the vectors b_{y_i} are chosen to obtain controllable pairs $(S_{\min}, b_{y_i})$, $i = 1, \ldots, p$, which is always possible.

Remark 2: The intuitive interpretation for including the cyclic part $S_{\min}$ in the internal model (11) is that each largest Jordan block of S contained in $\widetilde{S}$ is also capable to generate the signal corresponding to lower order Jordan blocks associated with the same eigenvalue. $\triangleleft$

The regulator design relies on stabilizing the nominal plant characterized by $\Delta\Lambda(z) = \Delta A(z) = \Delta A_0(z) = \Delta F(z) = \Delta B_i = \Delta C_0(z) = \Delta C_i = \Delta C_{bi} = 0$ in (1). For this, the *state feedback controller*

$$u(t) = \mathcal{K}[\hat{v}(t), x(t)] \tag{12}$$

is determined, in which $\mathcal{K}[\cdot,\cdot]$ denotes the *formal feedback operator*

$$\mathcal{K}[\hat{v}(t), x(t)] = -K_{\hat{v}}\hat{v}(t) - R_1 x(1,t) - \int_0^1 R(z)x(z,t)\mathrm{d}z \tag{13}$$

with the *feedback gains* $K_{\hat{v}} \in \mathbb{R}^{n \times pn_c}$ and $R_1, R(z) \in \mathbb{R}^{n \times n}$. Consequently, the *state feedback regulator* is given by the dynamic state feedback controller (11)–(13). Application of this regulator to the nominal plant leads to the *nominal closed-loop system*

$$\dot{\hat{v}}(t) = \widetilde{S}\hat{v}(t) + B_y\mathcal{C}[x(t)], \quad t > 0 \tag{14a}$$
$$\partial_t x(z,t) = \Lambda(z)\partial_z^2 x(z,t) + \mathcal{A}[x(t)](z) \tag{14b}$$
$$\theta_0[x(t)](0) = 0, \quad t > 0 \tag{14c}$$
$$\theta_1[x(t)](1) = \mathcal{K}[\hat{v}(t), x(t)], \quad t > 0, \tag{14d}$$

where (14b) is defined on $(z,t) \in (0,1) \times \mathbb{R}^+$ and the operators $\mathcal{A}[\cdot]$, $\theta_0[\cdot]$, $\theta_1[\cdot]$ as well as $\mathcal{C}[\cdot]$ resulting from (2), (4) and (6) by assuming nominal system parameters. The state feedback controller (12) is obtained from mapping (14) into the stable *ODE-PDE cascade*

$$\dot{e}_{\hat{v}}(t) = (\widetilde{S} - B_{\hat{v}}K_{\hat{v}})e_{\hat{v}}(t), \quad t > 0 \tag{15a}$$
$$\partial_t\tilde{x}(z,t) = \Lambda(z)\partial_z^2\tilde{x}(z,t) - \mu_c\tilde{x}(z,t) - \widetilde{A}_0(z)\tilde{x}(0,t) \tag{15b}$$
$$\partial_z\tilde{x}(0,t) = 0, \quad t > 0 \tag{15c}$$
$$\partial_z\tilde{x}(1,t) = -K_{\hat{v}}e_{\hat{v}}(t), \quad t > 0 \tag{15d}$$

with (15b) for $(z,t) \in (0,1) \times \mathbb{R}^+$, $\mu_c \in \mathbb{R}^+$ and

$$\widetilde{A}_0(z) = \begin{bmatrix} 0 & \ldots & \ldots & 0 \\ \widetilde{A}_{21}(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \widetilde{A}_{n1}(z) & \ldots & \widetilde{A}_{n\,n-1}(z) & 0 \end{bmatrix}. \tag{16}$$

The corresponding transformations

$$\tilde{x}(z,t) = x(z,t) - \int_0^z K(z,\zeta)x(\zeta,t)\mathrm{d}\zeta = \mathcal{T}[x(t)](z) \tag{17a}$$
$$e_{\hat{v}}(t) = \hat{v}(t) - \int_0^1 \widetilde{Q}(z)\tilde{x}(z,t)\mathrm{d}z, \tag{17b}$$

with the kernel $K(z,\zeta) \in \mathbb{R}^{n \times n}$ and the matrix $\widetilde{Q}(z) = \mathrm{col}(\widetilde{Q}_1(z), \ldots, \widetilde{Q}_p(z)) \in \mathbb{R}^{pn_c \times n}$ are determined in the following.

A. Backstepping Transformation of the PDE Subsystem

In the first step, the invertible *backstepping transformation* (17a) is utilized to map the PDE subsystem (14b)–(14d) into the intermediate target system

$$\dot{\hat{v}}(t) = \widetilde{S}\hat{v}(t) + B_y\mathcal{C}\mathcal{T}^{-1}[\tilde{x}(t)] \tag{18a}$$
$$\partial_t\tilde{x}(z,t) = \Lambda(z)\partial_z^2\tilde{x}(z,t) - \mu_c\tilde{x}(z,t) - \widetilde{A}_0(z)\tilde{x}(0,t) \tag{18b}$$
$$\partial_z\tilde{x}(0,t) = 0 \tag{18c}$$
$$\partial_z\tilde{x}(1,t) = \mathcal{K}[\hat{v}(t), x(t)] - K(1,1)x(1,t) - \int_0^1 K_z(1,z)x(z,t)\mathrm{d}z \tag{18d}$$

of simpler structure. This significantly facilitates the decoupling into the ODE-PDE cascade in the next step by making use of (17b). For the derivation of the output operator $\mathcal{C}\mathcal{T}^{-1}[\cdot]$ in (18a), the *inverse backstepping transformation*

$$x(z,t) = \tilde{x}(z,t) + \int_0^z K_I(z,\zeta)\tilde{x}(\zeta,t)\mathrm{d}\zeta = \mathcal{T}^{-1}[\tilde{x}(t)](z) \tag{19}$$

is needed, which always exists (see [10]). Inserting (19), (6) in (1d) and considering the nominal system yields

$$\begin{aligned} y(t) &= \mathcal{C}\mathcal{T}^{-1}[\tilde{x}(t)] \\ &= C_{b0}\tilde{x}(0,t) + C_{b1}\tilde{x}(1,t) \\ &\quad + \int_0^1 \underbrace{\Big(C_{b1}K_I(1,z) + \breve{C}(z)\Big)}_{\widetilde{C}(z)}\tilde{x}(z,t)\mathrm{d}z = \widetilde{\mathcal{C}}[\tilde{x}(t)] \end{aligned} \tag{20}$$

for $d(t) \equiv 0$. Herein $\breve{C}(z) = C(z) + \int_z^1 C(\zeta)K_I(\zeta,z)\mathrm{d}\zeta$ follows from changing the order of integration.

In order to determine (17a), differentiate the latter transformation w. r. t. time and insert (14) in the result. After straightforward calculations (see, e. g., [10]) this yields the *kernel equations*

$$\Lambda(z)K_{zz}(z,\zeta) - (K(z,\zeta)\Lambda(\zeta))_{\zeta\zeta} = \mathcal{B}[K](z,\zeta) \tag{21a}$$
$$\begin{aligned} &K_\zeta(z,0)\Lambda(0) + K(z,0)(\Lambda'(0) + \Lambda(0)B_0) \\ &\quad = \widetilde{A}_0(z) + A_0(z) - \textstyle\int_0^z K(z,\zeta)A_0(\zeta)\mathrm{d}\zeta \end{aligned} \tag{21b}$$
$$\begin{aligned} &\Lambda(z)K'(z,z) + \Lambda(z)K_z(z,z) + K_\zeta(z,z)\Lambda(z) + K(z,z)\Lambda'(z) \\ &\quad = -(A(z) + \mu_c I) \end{aligned} \tag{21c}$$
$$K(z,z)\Lambda(z) - \Lambda(z)K(z,z) = 0 \tag{21d}$$
$$K(0,0) = -B_0, \tag{21e}$$

in which (21a) is defined on $0 < \zeta < z < 1$ and the formal operator $\mathcal{B}[K](z,\zeta) = K(z,\zeta)(A(\zeta) + \mu_c I) - F(z,\zeta) + \int_\zeta^z K(z,\bar{\zeta})F(\bar{\zeta},\zeta)\mathrm{d}\bar{\zeta}$ is introduced. It is shown in [10] that (21) has a piecewise C^2-solution, which determines the elements $\widetilde{A}_{ij}(z)$, $i > j$, in (16).

B. Decoupling of the ODE Subsystem

In the second step, the intermediate target system (18) is mapped into the ODE-PDE cascade (15) using (17b). Firstly, the latter transformation is represented in the original coordinates x. To this end, insert (17a) in (17b) and change the order of integration to get

$$e_{\hat{v}}(t) = \hat{v}(t) - \int_0^1 Q(z)x(z,t)\mathrm{d}z \tag{22}$$

where $Q(z) = \widetilde{Q}(z) - \int_z^1 \widetilde{Q}(\zeta)K(\zeta,z)\mathrm{d}\zeta$. With this, the BC (15d) of the ODE-PDE cascade is obtained by solving (22) for $\hat{v}$ and inserting the result in (18d) yielding the feedback gains

$$R_1 = -B_1 - K(1,1) \quad \text{and} \quad R(z) = -K_{\hat{v}}Q(z) - K_z(1,z). \tag{23}$$

For determining (17b), differentiate it w. r. t. time, insert (18) with (20) in the result and take the BC (15d) into account. This gives

$$\begin{aligned}
\dot{e}_{\hat{v}}(t) &= \dot{\hat{v}}(t) - \int_0^1 \widetilde{Q}(z)\partial_t\tilde{x}(z,t)\mathrm{d}z = (\widetilde{S} - B_{\hat{v}}K_{\hat{v}})e_{\hat{v}}(t)\\
&+ \int_0^1 \Big(\widetilde{S}\widetilde{Q}(z) - (\widetilde{Q}(z)\Lambda(z))'' + \mu_c\widetilde{Q}(z) + B_y\widetilde{C}(z)\Big)\tilde{x}(z,t)\mathrm{d}z\\
&+ \Big(\int_0^1 \widetilde{Q}(z)\widetilde{A}_0(z)\mathrm{d}z - \widetilde{Q}'(0)\Lambda(0) - \widetilde{Q}(0)\Lambda'(0) + B_yC_{b0}\Big)\tilde{x}(0,t)\\
&+ \Big(\widetilde{Q}'(1)\Lambda(1) + \widetilde{Q}(1)\Lambda'(1) + B_yC_{b1}\Big)\tilde{x}(1,t)
\end{aligned} \tag{24}$$

with

$$B_{\hat{v}} = -\widetilde{Q}(1)\Lambda(1). \tag{25}$$

Hence, (15a) is obtained if the last three lines in (24) vanish. This leads to the *decoupling equations*

$$(\widetilde{Q}(z)\Lambda(z))'' = (\widetilde{S} + \mu_cI)\widetilde{Q}(z) + B_y\widetilde{C}(z), \quad z \in (0,1) \tag{26a}$$

$$(\widetilde{Q}\Lambda)'(0) = \int_0^1 \widetilde{Q}(z)\widetilde{A}_0(z)\mathrm{d}z + B_yC_{b0} \tag{26b}$$

$$(\widetilde{Q}\Lambda)'(1) = -B_yC_{b1} \tag{26c}$$

to be fulfilled by $\widetilde{Q}(z)$ for decoupling (18) into the ODE-PDE cascade (15). The next lemma clarifies the solvability of (26).

Lemma 1 (Solvability of the decoupling equations): Denote the spectrum of the PDE subsystem (15b)–(15d) by σ_c. The *decoupling equations* (26) have a unique piecewise C^2-solution $\widetilde{Q}(z) \in \mathbb{R}^{pn_c\times n}$ if $\sigma_c \cap \sigma(S) = \emptyset$.

The proof of this result can be found in Appendix B.

C. Stabilizability of the ODE-PDE Cascade

To obtain a stable ODE-PDE cascade (15), each subsystem needs to be stable. The exponential stability of the PDE subsystem can always be ensured by a suitable choice of μ_c (for details see Theorem 2). This, in addition, implies $\mathrm{Re}_{\lambda\in\sigma_c}\,\lambda < 0$ so that the condition of Lemma 1 is fulfilled as $\sigma(S) \subset j\mathbb{R}$. The ODE subsystem (15a) has to be stabilized by an eigenvalue assignment for the matrix $\widetilde{S} - B_{\hat{v}}K_{\hat{v}}$, which requires the controllability of the matrix pair $(\widetilde{S}, B_{\hat{v}})$. The next theorem shows that this is a property of the plant. To this end, it is convenient to introduce the matrices

$$T_1 = \begin{bmatrix} I_n \\ 0 \end{bmatrix} \in \mathbb{R}^{2n\times n} \quad \text{and} \quad T_2 = \begin{bmatrix} 0 \\ I_n \end{bmatrix} \in \mathbb{R}^{2n\times n}. \tag{27}$$

Theorem 1 (Stabilizability of the ODE-PDE cascade): Let $N(s) \in \mathbb{C}^{p\times n}$ be the numerator of the transfer matrix $F(s) = N(s)D^{-1}(s)$ from u to y w. r. t. the nominal system (1). The pair $(\widetilde{S}, B_{\hat{v}})$ is controllable iff the pairs $(S_{\min}, b_{y_i})$, $i = 1,\ldots,p$, are controllable and $\operatorname{rank} N(\mu) = p$, $\forall\mu \in \sigma(S)$, holds. Furthermore, the numerator is given by

$$\begin{aligned}
N(s) &= C_{b0} + C_{b1}T_1^\top\Psi^\top(0,1,s)T_1\\
&+ \int_0^1 \Big(\widetilde{C}(z)T_1^\top\Psi^\top(0,z,s)T_1 + \Big(C_{b1}T_1^\top\Psi^\top(z,1,s)T_2\\
&- \int_1^z \widetilde{C}(\zeta)T_1^\top\Psi^\top(z,\zeta,s)T_2\mathrm{d}\zeta\Big)\Lambda^{-1}(z)\widetilde{A}_0(z)\Big)\mathrm{d}z
\end{aligned} \tag{28}$$

in view of (20), (27) and (29), where $\Psi(z,\zeta,s) : (0,1)^2 \times \mathbb{C} \to \mathbb{C}^{2n\times 2n}$ is the solution of

$$\partial_z\Psi(z,\zeta,s) = -\Upsilon^\top(z,s)\Psi(z,\zeta,s), \quad \Psi(\zeta,\zeta,s) = I \tag{29}$$

with

$$\Upsilon(z,s) = \begin{bmatrix} 0 & I \\ (s+\mu_c)\Lambda^{-1}(z) & 0 \end{bmatrix}. \tag{30}$$

For the proof, see Appendix C.

Remark 3: If $\operatorname{rank} N(\mu) = p$, $\forall\mu \in \sigma(S)$, then the eigenmodes of (8a) can be transferred from the input u to the output y of the plant (1) in the steady state. This is a prerequisite for the solution of the output regulation problem. ◁

D. Stability of the ODE-PDE Cascade

The stability of the ODE-PDE cascade (15) and thus the stability of the nominal closed-loop system can be readily verified by mapping it into the *final target system*

$$\dot{e}_{\hat{v}}(t) = (\widetilde{S} - B_{\hat{v}}K_{\hat{v}})e_{\hat{v}}(t), \qquad t > 0 \tag{31a}$$

$$\partial_t\tilde{e}(z,t) = \Lambda(z)\partial_z^2\tilde{e}(z,t) - \mu_c\tilde{e}(z,t) - \widetilde{A}_0(z)\tilde{e}(0,t) \tag{31b}$$

$$\partial_z\tilde{e}(0,t) = 0, \qquad t > 0 \tag{31c}$$

$$\partial_z\tilde{e}(1,t) = 0, \qquad t > 0 \tag{31d}$$

with (31b) for $(z,t) \in (0,1)\times\mathbb{R}^+$. To this end, introduce the change of coordinates

$$\tilde{e}(z,t) = \tilde{x}(z,t) - \Sigma(z)e_{\hat{v}}(t) \tag{32}$$

with $\Sigma(z) \in \mathbb{R}^{n\times pn_c}$. Taking the time derivative of (32) and inserting (15), results in the BVP

$$\Sigma(z)(\widetilde{S} - B_{\hat{v}}K_{\hat{v}}) - \Lambda(z)\Sigma''(z) + \mu_c\Sigma(z) = -\widetilde{A}_0(z)\Sigma(0) \tag{33a}$$

$$\Sigma'(0) = 0 \tag{33b}$$

$$\Sigma'(1) = -K_{\hat{v}} \tag{33c}$$

for $\Sigma(z)$ with (33a) defined on $z \in (0,1)$. Postmultiplying (33) with the generalized eigenvectors of $\widetilde{S} - B_{\hat{v}}K_{\hat{v}}$ and applying the same reasoning as in the proof of Lemma 1 shows that (33) has a piecewise C^2-solution if $\sigma_c \cap \sigma(\widetilde{S} - B_{\hat{v}}K_{\hat{v}}) = \emptyset$. This can always be fulfilled if the conditions of Theorem 1 hold, because then the eigenvalues of $\widetilde{S} - B_{\hat{v}}K_{\hat{v}}$ are arbitrarily assignable. Hence, the stability analysis of (15) is traced back to the well-posedness and stability proof for the usual target system (31b)–(31d), which can be found in [10], [11]. This and the bounded invertibility of (32) directly lead to the stability result of the next theorem.

Theorem 2 (Nominal closed-loop stability): Let the feedback gains R_1 and $R(z)$ be given by (23) and denote by σ_c the spectrum of (31b)–(31d). Assume that $\mu_c > 0$ and that $\widetilde{S} - B_{\hat{v}}K_{\hat{v}}$ is a Hurwitz matrix with $\sigma_c \cap \sigma(\widetilde{S} - B_{\hat{v}}K_{\hat{v}}) = \emptyset$ implying $\alpha = \max(\alpha_v, -\mu_c) < 0$ where $\alpha_v = \max_{\lambda\in\sigma(\widetilde{S}-B_{\hat{v}}K_{\hat{v}})}\mathrm{Re}\,\lambda$. Then, the resulting closed-loop system with the state $x_c(t) = \operatorname{col}(\hat{v}(t), x(t))$ and $x(t) = \{x(z,t), z \in [0,1]\}$ is well-posed in the state space $X = \mathbb{C}^{pn_c} \oplus (L_2(0,1))^n$ with the usual weighted inner product. Furthermore, the system is exponentially stable in the norm $\|\cdot\| = (\|\cdot\|^2_{\mathbb{C}^{pn_c}} + \|\cdot\|^2_{L_2})^{1/2}$ where $\|h\|^2_{L_2} = \int_0^1 \|\Lambda^{-\frac{1}{2}}(z)h(z)\|^2_{\mathbb{C}^n}\mathrm{d}z$. In particular, $\|x_c(t)\| \leq M\mathrm{e}^{(\alpha+c)t}\|x_c(0)\|$, $t \geq 0$ holds for all $x_c(0) \in \mathbb{C}^{pn_c} \oplus (H^2(0,1))^n \subset X$ satisfying the BCs of the closed-loop system, an $M \geq 1$ and any $c > 0$ such that $\alpha + c < 0$.

IV. Robust Output Regulation

As the designed controller (11), (12) is able to stabilize the nominal system extended by the internal model, it remains to show that it achieves output regulation (10) and that this property is robust w. r. t. model uncertainties that do not destabilize the closed-loop system. In particular, the uncertain closed-loop system should at least be *strongly asymptotically stable* (see, e. g., [15]), i. e., it is the generator of a strongly asymptotically stable C_0-semigroup $\mathcal{T}_c(t)$, $t \geq 0$, satisfying

$\lim_{t\to\infty}\|\mathcal{T}_c(t)x_c(0)\|=0$, for all ICs $x_c(0)$ characterized in Theorem 2. This is the result of the next theorem.

Theorem 3 (Robust output regulation): Assume that the nominal system parameters $(\Lambda, A, A_0, F, B_i, C_{bi}, C_i)$ are perturbed to $(\bar{\Lambda}, \bar{A}, \bar{A}_0, \bar{F}, \bar{B}_i, \bar{C}_{bi}, \bar{C}_i)$ such that the resulting closed-loop system is strongly asymptotically stable and the output operator $\bar{\mathcal{C}}$ in (6) is relatively bounded w. r. t. the closed-loop system operator. Then, the controller (11), (12) achieves output regulation, i. e., $\lim_{t\to\infty} e_y(t) = 0$, independently of the disturbance input locations characterized by G_i, $i = 1, 2, 3, 4$, and the generation of the disturbance and reference signals by the matrices P_d, P_r.

Remark 4: In the nominal case, the closed-loop system is exponentially stable, i. e., the closed-loop system is the generator of an exponentially stable C_0-semigroup (see Theorem 2 and the cited references) and thus also strongly asymptotically stable. Moreover, it can be verified that the output operator is relatively bounded w. r. t. the closed-loop system operator (see [6] for details). Hence, nominal output regulation is ensured. Assuming that the output operator is still relatively bounded despite the model uncertainties is a standing assumption for robust output regulation (see [25]). ◁

The proof of Theorem 3 follows from determining the steady state solution of the closed-loop system. Since this solution solely depends on the eigenmodes of (8a) and thus also on the modes of the internal model (11), only $e_y(t) \equiv 0$ is possible in the steady state implying output regulation. Consequently, as long as the uncertain closed-loop system remains stable, this steady state exists and robust output regulation is achieved.

Proof: Consider the closed-loop system (1), (8), (11) and (13). Apply the transformations

$$\tilde{x}(z,t) = x(z,t) - \int_0^z \bar{K}(z,\zeta)x(\zeta,t)\mathrm{d}\zeta = \bar{\mathcal{T}}[x(t)](z) \tag{34a}$$

$$e_{\hat{v}}(t) = \hat{v}(t) - \int_0^1 \bar{\widetilde{Q}}(z)\tilde{x}(z,t)\mathrm{d}z \tag{34b}$$

(see (17)) to this system and assume that $\bar{K}(z,\zeta)$ and $\bar{\widetilde{A}}_0(z)$ satisfy (21) for $(\Lambda, \mathcal{A}, B_0, \widetilde{A}_0) \to (\bar{\Lambda}, \bar{\mathcal{A}}, \bar{B}_0, \bar{\widetilde{A}}_0)$ and $\bar{\widetilde{Q}}(z)$ satisfies (26) for $(\Lambda, C, \widetilde{A}_0, C_{b0}, C_{b1}) \to (\bar{\Lambda}, \bar{C}, \bar{\widetilde{A}}_0, \bar{C}_{b0}, \bar{C}_{b1})$. Using the same calculations as in Section III yields

$$\dot{v}(t) = Sv(t) \tag{35a}$$

$$\dot{e}_{\hat{v}}(t) = (\widetilde{S} - \bar{B}_{\hat{v}}K_{\hat{v}})e_{\hat{v}}(t) + H_1 v(t) \tag{35b}$$

$$\partial_t\tilde{x}(z,t) = \bar{\Lambda}(z)\partial_z^2\tilde{x}(z,t) - \mu_c\tilde{x}(z,t) - \bar{\widetilde{A}}_0(z)\tilde{x}(0,t) + \widetilde{G}_1(z)v(t) \tag{35c}$$

$$\partial_z\tilde{x}(0,t) = \widetilde{G}_2 v(t) \tag{35d}$$

$$\partial_z\tilde{x}(1,t) = \widetilde{\mathcal{K}}[e_{\hat{v}}(t), \tilde{x}(t)] + \widetilde{G}_3 v(t) \tag{35e}$$

$$e_y(t) = \bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\tilde{x}(t)] - \widetilde{P}_r v(t) \tag{35f}$$

with $\bar{B}_{\hat{v}} = -\bar{\widetilde{Q}}(1)\bar{\Lambda}(1)$, $H_1 = -\int_0^1 \bar{\widetilde{Q}}(z)\widetilde{G}_1(z)\mathrm{d}z$, $\widetilde{G}_1(z) = \bar{\mathcal{T}}[G_1](z)P_d$, $\widetilde{G}_i = G_i P_d$, $i = 2, 3$, $\widetilde{P}_r = P_r - G_4 P_d$ and $\widetilde{\mathcal{K}}[e_{\hat{v}}(t), \tilde{x}(t)] = -K_{\hat{v}}e_{\hat{v}}(t) - (R_1 + \bar{B}_1 + \bar{K}(1,1))\tilde{x}(1,t) - \int_0^1(R(z) + K_{\hat{v}}\bar{Q}(z) + \bar{K}_z(1,z))\mathrm{d}z$, where $\bar{Q}(z) = \bar{\widetilde{Q}}(z) - \int_z^1 \bar{\widetilde{Q}}(\zeta)\bar{K}(\zeta,z)\mathrm{d}\zeta$.

In order to verify output regulation on the basis of the closed-loop stability, introduce the change of coordinates

$$\varepsilon_{\hat{v}}(t) = e_{\hat{v}}(t) - \Pi_{e_{\hat{v}}}v(t) \tag{36a}$$

$$\varepsilon(z,t) = \tilde{x}(z,t) - \Pi(z)v(t) \tag{36b}$$

with $\Pi_{e_{\hat{v}}} \in \mathbb{R}^{pn_c \times n_v}$ and $\Pi(z) \in \mathbb{R}^{n \times n_v}$. This leads to

$$\dot{v}(t) = Sv(t) \tag{37a}$$

$$\dot{\varepsilon}_{\hat{v}}(t) = \bar{F}\varepsilon_{\hat{v}}(t) \tag{37b}$$

$$\partial_t\varepsilon(z,t) = \bar{\Lambda}(z)\partial_z^2\varepsilon(z,t) - \mu_c\varepsilon(z,t) - \bar{\widetilde{A}}_0(z)\varepsilon(0,t) \tag{37c}$$

$$\partial_z\varepsilon(0,t) = 0 \tag{37d}$$

$$\partial_z\varepsilon(1,t) = \widetilde{\mathcal{K}}[\varepsilon_{\hat{v}}(t), \varepsilon(t)] \tag{37e}$$

$$e_y(t) = \bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\varepsilon(t)] + (\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\Pi] - \widetilde{P}_r)v(t) \tag{37f}$$

with $\bar{F} = \widetilde{S} - \bar{B}_{\hat{v}}K_{\hat{v}}$ if $(\Pi_{e_{\hat{v}}}, \Pi(z))$ is the solution of

$$\Pi_{e_{\hat{v}}}S - \bar{F}\Pi_{e_{\hat{v}}} = H_1 \tag{38a}$$

$$\Pi(z)S - \bar{\Lambda}(z)\Pi''(z) + \mu_c\Pi(z) + \bar{\widetilde{A}}_0(z)\Pi(0) = \widetilde{G}_1(z) \tag{38b}$$

$$\Pi'(0) = \widetilde{G}_2 \tag{38c}$$

$$\Pi'(1) = \widetilde{\mathcal{K}}[\Pi_{e_{\hat{v}}}, \Pi] + \widetilde{G}_3, \tag{38d}$$

with (38b) defined on $z \in (0,1)$. Since the closed-loop system in the presence of model uncertainties is assumed to be strongly asymptotically stable, the eigenvalues of $\bar{F}$ in (37b) must lie in the left half plane. Thus, the assumption $\sigma(S) \subset j\mathbb{R}$ implies $\sigma(\bar{F}) \cap \sigma(S) = \emptyset$ so that (38a) has a unique solution $\Pi_{e_{\hat{v}}}$. Moreover, solving the BVP (38b)–(38d) with the same methods as in the proof of Lemma 1 shows that it has a unique piecewise C^2-solution $\Pi(z)$, if $\bar{\sigma}_p \cap \sigma(S) = \emptyset$, where $\bar{\sigma}_p$ is the *point spectrum* of the closed-loop system (37c)–(37e) with model uncertainties. The strong asymptotic stability of the closed-loop system ensures $\mathrm{Re}_{\lambda\in\bar{\sigma}_p}\,\lambda < 0$ (see [15]) so that the mapping (36) into (37) exists. The stability of (37b)–(37e) implies that $e_{\hat{v}}(t)$ and $\tilde{x}(t)$ converge to their steady state solutions $e_{\hat{v},\infty}(t) = \Pi_{e_{\hat{v}}}v(t)$ and $\tilde{x}_\infty(z,t) = \Pi(z)v(t)$ according to (36). What remains to be shown is that the solution $\Pi(z)$ ensures the convergence of the output tracking error $e_y(t)$ to zero for $t \to \infty$. According to (37f), this requires $(\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\Pi] - \widetilde{P}_r)v(t) = 0$. To prove this, consider the internal model (11) with the excitation (37f), yielding

$$\dot{\hat{v}}(t) = \widetilde{S}\hat{v}(t) + B_y\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\varepsilon(t)] + B_y(\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\Pi] - \widetilde{P}_r)v(t). \tag{39}$$

The asymptotic stability of (37b) and the strong asymptotic stability of (37c)–(37e) lead to $\lim_{t\to\infty}\varepsilon_{\hat{v}}(t) = 0$ and $\lim_{t\to\infty}\varepsilon(z,t) = 0$. Hence, (36) implies $e_{\hat{v}}(t) \to \Pi_{e_{\hat{v}}}v(t)$ and $\tilde{x}(z,t) \to \Pi(z)v(t)$ as $t \to \infty$. Then, by (22) the solution $\hat{v}(t)$ of the internal model (11) converges to its steady state response $\hat{v}_\infty(t) = (\Pi_{e_{\hat{v}}} + \int_0^1 Q(z)\bar{\mathcal{T}}^{-1}[\Pi](z)\mathrm{d}z)v(t)$ in view of (34a). The assumed relative boundedness of $\bar{\mathcal{C}}$ and the boundedness of $\bar{\mathcal{T}}^{-1}$ imply a relatively bounded operator $\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}$. Hence, $\lim_{t\to\infty}\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\varepsilon(t)] = 0$ follows (see [6]) and the steady state response $\hat{v}_\infty(t)$ is the solution of $\dot{\hat{v}}_\infty(t) = \widetilde{S}\hat{v}_\infty(t) + B_y(\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\Pi] - \widetilde{P}_r)v_\infty(t)$ for some IC $\hat{v}_\infty(0)$ according to (39). Obviously, $v_\infty(t)$ contains the modes of $\hat{v}_\infty(t)$, as S and $\widetilde{S}$ share the same eigenvalues. Thus, if $B_y(\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\Pi] - \widetilde{P}_r)v_\infty(t) \neq 0$, the internal model is in resonance and a bounded steady state response $\hat{v}_\infty(t)$ does not exist, contradicting its already verified existence. This implies that $(\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\Pi] - \widetilde{P}_r)v_\infty(t) = 0$, because $\operatorname{rank} B_y = p$. Consequently, (37f) shows that $\lim_{t\to\infty} e_y(t) = \lim_{t\to\infty}\bar{\mathcal{C}}\bar{\mathcal{T}}^{-1}[\varepsilon(t)] = 0$, which verifies robust output regulation. ■

Remark 5: It is verified in [25] that the obtained output regulation result is robust w. r. t. sufficiently small model uncertainties. Furthermore, the robustness of the regulator can be tested for given perturbations with the methods presented in [25]. As the internal model principle ensures that a robustly stabilizing controller implies robust output regulation, the robustness of the output regulation result can be increased by determining robust backstepping controllers. The latter problem, however, is not the topic of this paper. Nevertheless, the inherent robustness of backstepping controllers for parabolic systems justifies the proposed regulator design based on the nominal case. ◁

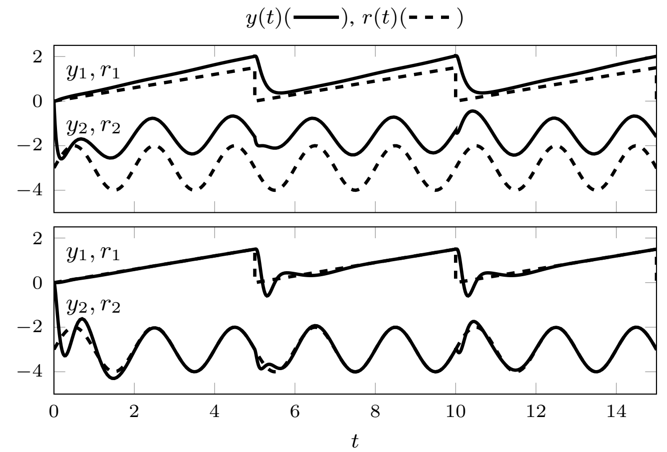

Fig. 1. Closed-loop reference behaviour for $r_1(t) = \frac{3}{10}\,\mathrm{mod}(t, 5)$, $r_2(t) = -3 + \sin(\pi t)$ and $d(t) \equiv 0$ using feedforward control (upper plot) and the internal model principle (lower plot) in the presence of model uncertainty.

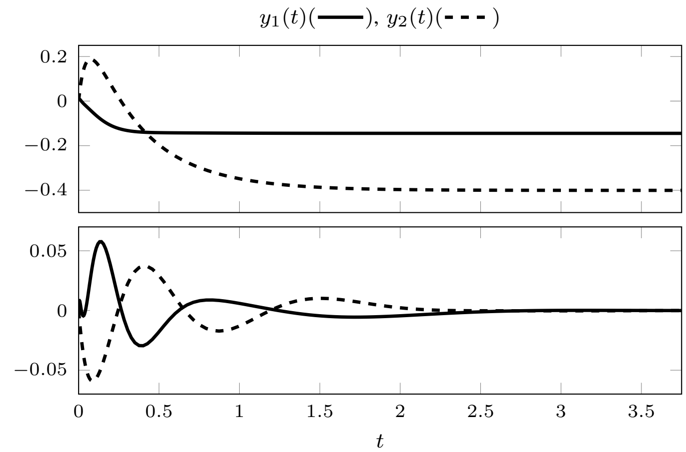

Fig. 2. Closed-loop disturbance behaviour for $d(t) = 1$ and $r(t) \equiv 0$ using feedforward control (upper plot) and the internal model principle (lower plot) in the presence of model uncertainty.

V. Example

To illustrate the results of this paper, an unstable system consisting of $n = 3$ coupled parabolic PIDEs is considered. The nominal parameters of the plant (1) are $\lambda_1(z) = \frac{3}{2} + z^2\cos(2\pi z)$, $\lambda_2(z) = \frac{1}{2} - \frac{1}{4}\sin(2\pi z)$, $\lambda_3(z) = \frac{1}{4} - \frac{1}{8}\sin(2\pi z)$,

$$A(z) = \begin{bmatrix} 1 & 1+z & \frac{1}{2}+z \\ \frac{1}{2}+z & 1 & 1+z \\ 1+z & 1 & z \end{bmatrix},\ A_0(z) = \begin{bmatrix} z & 1-z & 1 \\ z & 1-z & 1+z \\ z & 1-z & 1+z \end{bmatrix},$$

$$F(z,\zeta) = \begin{bmatrix} \mathrm{e}^{z+\zeta} & \mathrm{e}^{z-\zeta} & \mathrm{e}^{z} \\ 1-\mathrm{e}^{\zeta-z} & \mathrm{e}^{-(z+\zeta)} & 1+\mathrm{e}^{z+\zeta} \\ \mathrm{e}^{\zeta-z} & \mathrm{e}^{-(z+\zeta)} & \mathrm{e}^{z+\zeta} \end{bmatrix}, B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (40)$$

with $B_0 = \mathrm{diag}(-3, -2, -1)$ resulting in an unstable system with its largest eigenvalue at 9.85. A disturbance $d(t) \in \mathbb{R}$ acts in-domain as well as at the boundaries which is described by the disturbance input vectors $G_1(z) = \begin{bmatrix} 2 & 2 & 2 \end{bmatrix}^\top$, $G_2 = G_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$, $G_4 = 0$. The $p = 2$ outputs (1d) are defined in-domain pointwise as well as distributed, leading to the output operator

$$\mathcal{C}[x(t)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(0.25, t) - \int_0^1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(z, t)\mathrm{d}z. \quad (41)$$

Output regulation problem. The state feedback regulator shall achieve the robust asymptotic tracking of the ramp reference signal $r_1(t) = q_{01} + q_{11}t$, $t > 0$, $q_{i1} \in \mathbb{R}$, $i = 0, 1$, which can be periodically reset to achieve sawtooth waves as well as the sinusoidal reference signal $r_2(t) = q_{02} + q_{22}\sin(\pi t + \varphi)$, $t > 0$, with $q_{i2}, \varphi \in \mathbb{R}$, $i = 0, 2$. Furthermore, steplike disturbances $d(t) = d_0 \in \mathbb{R}$, $t > 0$, shall be rejected. To this end, the spectrum of the matrix S in (8a) has to be $\sigma(S) = \{0, 0, 0, \pm j\pi\}$. Consequently,

$$S = \mathrm{bdiag}\left(0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}\right) \quad (42)$$

allows to model the exogenous signals.

State feedback regulator design. The construction of the internal model (11) requires to include the largest Jordan block of each eigenvalue of S, i. e., the cyclic part $S_{\min}$. Thus,

$$S_{\min} = \mathrm{bdiag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \pi \\ -\pi & 0 \end{bmatrix}\right) \quad (43)$$

is a sufficient choice determining the dynamics of the internal model. With this and setting $b_{y_i} = [1\ 1\ 1\ 1]^\top$, $i = 1, 2$, in (11), the pairs $(S_{\min}, b_{y_i})$ are controllable. In addition, the numerator $N(\mu)$ from u to y has full rank $p = 2$, $\forall \mu \in \sigma(S)$ verifying the solvability of the posed output regulation problem in view of Theorem 1. Consequently, the pair $(\widetilde{S}, B_{\hat{v}})$ is controllable so that the matrix $K_{\hat{v}}$ can be determined to place the eigenvalues of $\widetilde{S} - B_{\hat{v}}K_{\hat{v}}$ at $-4.5 \pm j\pi$, -4.5, -4.5, $-4 \pm j\pi$, -4 and -4. The design parameter of the backstepping-based state feedback controller is chosen as $\mu_c = 5$ and the IC of the internal model is set to $\hat{v}(0) = 0$.

Simulation results. To investigate the robustness of the achieved regulator, model deviations in form of $\Delta\Lambda(z) = 0.2\Lambda(z)$, $\Delta A(z) = 0.2A(z)$, $\Delta B_0 = -0.2B_0$, $\Delta F(z,\zeta) = 0.2F(z,\zeta)$, $\Delta A_0(z) = 0.2A_0(z)$, $\Delta C_1(z) = 0.2C_1(z)$ and $\Delta C_0(z) = -0.2C_0(z)$ are considered in the simulation. In order to investigate the robustness property of the proposed state feedback regulator, the achieved output regulation result is compared with the feedforward controller designed in [11]. The latter design solves the regulator equations for the nominal case and is therefore sensitive to parameter variations. The upper plot in Fig. 1 shows the resulting tracking behaviour for the reference inputs $r_1(t) = \frac{3}{10}\,\mathrm{mod}(t, 5)$, $t > 0$, where $\mathrm{mod}(t, 5)$ denotes the modulo operator with the numerator t and denominator 5 and $r_2(t) = -3 + \sin(\pi t)$, $t > 0$, with $d(t) \equiv 0$. Due to the parameter variations the output regulation result is significantly deteriorated and shows a persisting steady state deviation. In contrast, the internal model principle controller still achieves the asymptotic tracking of the reference signals for the same uncertain plant depicted in the lower plot. Here, deviations only appear at the discontinuities of r_1 at $t \in \{5, 10\}$. This due to the fact that the latter lead to steps in the initial values of the signal model and thus require the internal model states to converge to new values. The upper plot in Fig. 2 shows the disturbance behaviour (i. e., $r(t) \equiv 0$) for $d(t) = 1$, $t > 0$, for the feedforward controller design. The parameter perturbations cause severe steady state deviations. In the lower plot, the disturbance rejection achieved by the robust regulator is depicted. Here, the outputs contain oscillations resulting from the eigenvalues of the internal model but asymptotically converge to zero verifying robust disturbance rejection.

VI. Concluding remarks

An obvious extension of the presented results is to design an observer for implementing the state feedback regulator. In [11] it is demonstrated that the related observer design for the considered class of parabolic systems can be based on the backstepping approach if boundary measurements are available. Furthermore, an interesting topic for future research is to extend the class of exogenous signals by considering signal models which are infinite-dimensional or finite-dimensional with time-varying coefficients.

APPENDIX

A. Reciprocity of the Fundamental Matrices

For the proof of Lemma 1 the following preliminary result is needed.

Lemma 2: Consider the state equation

$$\dot{x}(t) = A(t)x(t), \quad t > 0, \quad x(0) \in \mathbb{R}^n \tag{44}$$

with the state $x(t) \in \mathbb{R}^n$ and the matrix $A \in (C(\mathbb{R}_0^+))^{n\times n}$. The corresponding fundamental matrix $\Phi(t,\tau) : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^{n\times n}$ is the solution of the IVP

$$\partial_t \Phi(t,\tau) = A(t)\Phi(t,\tau), \quad \Phi(\tau,\tau) = I. \tag{45}$$

The *adjoint state equation* reads

$$\dot{\bar{x}}(t) = -A^\top(t)\bar{x}(t), \quad t > 0, \quad \bar{x}(0) \in \mathbb{R}^n \tag{46}$$

with the state $\bar{x}(t) \in \mathbb{R}^n$. Here, the fundamental matrix $\Psi(t,\tau)$: $\mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}^{n\times n}$ results from solving the IVP

$$\partial_t \Psi(t,\tau) = -A^\top(t)\Psi(t,\tau), \quad \Psi(\tau,\tau) = I. \tag{47}$$

Then, the fundamental matrices are related by

$$\Phi(t,\tau) = \Psi^\top(\tau,t). \tag{48}$$

Proof: In order to prove this result treat τ as parameter and consider $\mathrm{d}_t(\Psi^\top\Phi) = \Psi_t^\top\Phi + \Psi^\top\Phi_t = -\Psi^\top A\Phi + \Psi^\top A\Phi = 0$ in view of (47) and (45). Hence, $\Psi^\top\Phi = const.$ holds, which is the well-known *reciprocity relation* between the fundamental matrices w. r. t. the state equation (44) and the related adjoint state equations (46) (see, e. g, [26, Ch. 4]). From this, $\int_\tau^t \mathrm{d}_{\bar{t}}(\Psi^\top(\bar{t},\tau)\Phi(\bar{t},\tau))\mathrm{d}\bar{t} = \Psi^\top(t,\tau)\Phi(t,\tau) - I = 0$ is readily obtained in light of $\Psi^\top\Phi = const.$. Using this result and the well-known property $(\Psi(t,\tau))^{-1} = \Psi(\tau,t)$ of a fundamental matrix yields the relationship (48). ■

Since the fundamental matrices appearing in the solution of the decoupling equations (26) and for the determination of the transfer matrix $F(s)$ in Theorem 1 are, in general, not attainable in closed form, the result of Lemma 2 is fundamental for establishing a relation between the solution of these problems. This is needed to obtain a condition for the controllability of the matrix pair $(\widetilde{S}, B_{\tilde{v}})$.

B. Proof of Lemma 1

In order to simplify (26), consider the transformation $R(z) = \widetilde{Q}(z)\Lambda(z)$ to obtain

$$R''(z) = (\widetilde{S} + \mu_c I)R(z)\Lambda^{-1}(z) + B_y\widetilde{C}(z) \tag{49a}$$

$$R'(0) = \int_0^1 R(z)\Lambda^{-1}(z)\widetilde{A}_0(z)\mathrm{d}z + B_y C_{b0} \tag{49b}$$

$$R'(1) = -B_y C_{b1}. \tag{49c}$$

In view of $R(z) = \mathrm{col}(R_1(z), \ldots, R_p(z))$, $R_i(z) \in \mathbb{R}^{n_c\times n}$, $\widetilde{C}(z) = \mathrm{col}(\tilde{c}_1^\top(z), \ldots, \tilde{c}_p^\top(z))$, $C_{bi} = \mathrm{col}(c_{bi1}^\top, \ldots, c_{bip}^\top)$, $i = 0, 1$ as well as the block diagonal matrices $\widetilde{S}$ and B_y, the latter result can be equivalently rewritten as

$$R_i''(z) = (S_{\min} + \mu_c I)R_i(z)\Lambda^{-1}(z) + b_{y_i}\widetilde{c}_i^\top(z) \tag{50a}$$

$$R_i'(0) = \int_0^1 R_i(z)\Lambda^{-1}(z)\widetilde{A}_0(z)\mathrm{d}z + b_{y_i}c_{b0i}^\top \tag{50b}$$

$$R_i'(1) = -b_{y_i}c_{b1i}^\top \tag{50c}$$

for $i = 1, \ldots, p$. The matrix $S_{\min}$ contains $s_{\min}$ Jordan blocks J_j of length l_j, corresponding to the eigenvalue μ_j. Denote by $w_j^{(k)\top}$, $j = 1, \ldots, s_{\min}$, $k = 1, \ldots, l_j$ the *generalized left eigenvectors* of $S_{\min}$. Now, premultiplying (50) by $w_j^{(k)\top}$ and introducing $w_j^{(k)\top}R_i(z) = r_{ij}^{(k)\top}(z) = \boldsymbol{r}^\top(z)$, as well as $w_j^{(k)\top}b_{y_i} = b_{ij}^{(k)} = \boldsymbol{b}$ gives

$$\mathrm{d}_z^2\boldsymbol{r}^\top(z) = \boldsymbol{r}^\top(z)(\mu_j + \mu_c)\Lambda^{-1}(z) + \underbrace{\boldsymbol{b}\widetilde{c}_i^\top(z) + r_{ij}^{(k-1)\top}(z)}_{\bar{\boldsymbol{c}}^\top(z)} \tag{51a}$$

$$\mathrm{d}_z\boldsymbol{r}^\top(0) = \int_0^1 \boldsymbol{r}^\top(z)\Lambda^{-1}(z)\widetilde{A}_0(z)\mathrm{d}z + \boldsymbol{b}c_{b0i}^\top \tag{51b}$$

$$\mathrm{d}_z\boldsymbol{r}^\top(1) = -\boldsymbol{b}c_{b1i}^\top, \tag{51c}$$

where $r_{ij}^{(0)\top} = 0^\top$. In what follows, the ODE (51a) is reformulated in form of a state space representation. The particular choice $\boldsymbol{\rho}_1^\top = \rho_{ij,1}^{(k)\top} = -\mathrm{d}_z\boldsymbol{r}^\top$ and $\boldsymbol{\rho}_2^\top = \rho_{ij,2}^{(k)\top} = \boldsymbol{r}^\top$ of the state variables ensures that the resulting state equation is adjoint to the state equation that will be derived in the proof of Theorem 1 (see Appendix A). The corresponding state space representation reads

$$\mathrm{d}_z\boldsymbol{\rho}^\top(z) = -\boldsymbol{\rho}^\top(z)\Upsilon(z,\mu_j) - \boldsymbol{b}\widetilde{c}_i^\top(z)T_1^\top \tag{52}$$

where $\boldsymbol{\rho}^\top = [\boldsymbol{\rho}_1^\top \;\; \boldsymbol{\rho}_2^\top]$ and $\Upsilon(z,s)$ in (30). With the fundamental matrix $\Psi(z,\zeta,s)$ introduced in Theorem 1, the solution of (52) and (51c) is $\boldsymbol{\rho}^\top(z) = \boldsymbol{\rho}_2^\top(1)T_2^\top\Psi^\top(z,1,\mu_j) + \boldsymbol{b}c_{b1i}^\top T_1^\top\Psi^\top(z,1,\mu_j) - \int_1^z(\boldsymbol{b}\widetilde{c}_i^\top(\zeta) + r_{ij}^{(k-1)\top}(\zeta))T_1^\top\Psi^\top(z,\zeta,\mu_j)\mathrm{d}\zeta$. Inserting this in (51b) results in

$$\boldsymbol{\rho}_2^\top(1)M(\mu_j) = h_{ij}^{(k)\top} \tag{53}$$

with $M(\mu_j) = T_2^\top(\Psi^\top(0,1,\mu_j)T_1 + \int_0^1\Psi^\top(z,1,\mu_j)T_2\Lambda^{-1}(z) \cdot \widetilde{A}_0(z)\mathrm{d}z)$ and

$$\begin{aligned} h_{ij}^{(k)\top} &= -\boldsymbol{b}(c_{b0i}^\top + c_{b1i}^\top T_1^\top\Psi^\top(0,1,\mu_j)T_1) \\ &+ \int_0^1 \Big(\bar{\boldsymbol{c}}^\top(z)T_1^\top\Psi^\top(0,z,\mu_j)T_1 + \Big(\boldsymbol{b}c_{b1i}^\top T_1^\top\Psi^\top(z,1,\mu_j)T_2 \\ &- \int_1^z \bar{\boldsymbol{c}}^\top(\zeta)T_1^\top\Psi^\top(z,\zeta,\mu_j)T_2\mathrm{d}\zeta\Big)\Lambda^{-1}(z)\widetilde{A}_0(z)\Big)\mathrm{d}z. \end{aligned} \tag{54}$$

Hence, the decoupling equations (26) are solvable if $\det M(\mu_j) \neq 0$, $j = 1, \ldots, s_{\min}$, in view of (53). A simple calculation shows that $\det M(s) = 0$, $s \in \mathbb{C}$, is the characteristic equation related to the PDE subsystem (15b)–(15d). As $S_{\min}$ contains one Jordan block for each eigenvalue of S, this leads to the condition of the lemma. Moreover, the piecewise C^2-solution of (26) follows from the fact that the elements in $\tilde{c}_i^\top(z)$ in (52) are piecewise continuous.

C. Proof of Theorem 1

In order to determine the transfer matrix $F(s)$, apply the backstepping transformation $\tilde{x}(z,t) = \mathcal{T}[x(t)](z)$ to the nominal system (1) resulting in

$$\partial_t\tilde{x}(z,t) = \Lambda(z)\partial_z^2\tilde{x}(z,t) - \mu_c\tilde{x}(z,t) - \widetilde{A}_0(z)\tilde{x}(0,t) \tag{55a}$$

$$\partial_z\tilde{x}(0,t) = 0 \tag{55b}$$

$$\partial_z\tilde{x}(1,t) = \mathcal{Q}[\tilde{x}(t), u(t)] \tag{55c}$$

$$y(t) = \widetilde{\mathcal{C}}[\tilde{x}(t)] \tag{55d}$$

with the formal operator $\mathcal{Q}[\tilde{x}(t), u(t)] = (-B_1 - K(1,1))\mathcal{T}^{-1}[\tilde{x}(t)](1) + u(t) - \int_0^1 K_z(1,\zeta)\mathcal{T}^{-1}[\tilde{x}(t)](\zeta)\mathrm{d}\zeta$. The transfer matrix in the sense of [29] is obtained by introducing $\tilde{x}(z,t) = \check{x}(z)\,\mathrm{e}^{st}$, $u(t) = \check{u}\,\mathrm{e}^{st}$ and $y(t) = \check{y}\,\mathrm{e}^{st}$, $s \in \mathbb{C}$, in (55) to get

$$\check{x}''(z) = \Lambda^{-1}(z)\Big((s+\mu_c)\check{x}(z) + \widetilde{A}_0(z)\check{x}(0)\Big) \quad (56a)$$
$$\check{x}'(0) = 0 \quad (56b)$$
$$\check{x}'(1) = \mathcal{Q}[\check{x}, \check{u}] \quad (56c)$$
$$\check{y} = \widetilde{\mathcal{C}}[\check{x}] \quad (56d)$$

after a simple rearrangement. With the state $\varrho = \mathrm{col}(\varrho_1, \varrho_2) = \mathrm{col}(\check{x}, \check{x}')$ the ODE (56a) can be represented by

$$\varrho'(z) = \Upsilon(z,s)\varrho(z), \quad (57)$$

in which $\Upsilon(z,s)$ is given by (30). Hence, the related fundamental matrix $\Phi(z,\zeta,s)$ is the solution of the IVP

$$\partial_z\Phi(z,\zeta,s) = \Upsilon(z,s)\Phi(z,\zeta,s), \quad \Phi(\zeta,\zeta,s) = I. \quad (58)$$

Hence, application of Lemma 2 in Appendix A to (29) and (58) yields

$$\Phi(z,\zeta,s) = (\Psi^\top(z,\zeta,s))^{-1} = \Psi^\top(\zeta,z,s). \quad (59)$$

With this, the solution of the IVP (57) and (56b) can be represented in terms of the fundamental matrix $\Psi(z,\zeta,s)$ related to the decoupling equations (26) by

$$\varrho(z) = \bigg(\Psi^\top(0,z,s)T_1 + \int_0^z \Psi^\top(\zeta,z,s)T_2\Lambda^{-1}(\zeta)\widetilde{A}_0(\zeta)\mathrm{d}\zeta\bigg)\varrho_1(0). \quad (60)$$

By inserting this in (56c) one can compute $\varrho_1(0)$. After substituting the corresponding expression for $\varrho_1(0)$ in (60) the solution of (56a)–(56c) is determined. Then, the transfer matrix $F(s)$ follows from inserting the solution in (56d) resulting in (28).

Let $\tilde{\boldsymbol{w}}^\top = \tilde{w}_{ij}^\top = [\alpha_{ij}^1 w_j^{(1)\top}, \ldots, \alpha_{ij}^p w_j^{(1)\top}]$, $\alpha_{ij}^l = \boldsymbol{\alpha}^l \in \mathbb{C}$, $i = 1,\ldots,p$, $j = 1,\ldots,s_{\min}$, with $|\boldsymbol{\alpha}^1| + \ldots + |\boldsymbol{\alpha}^n| > 0$ be the left eigenvectors of $\widetilde{S}$ w. r. t. the eigenvalue μ_j (see proof of Lemma 1). Consequently, the pair $(\widetilde{S}, B_{\hat{v}})$ is controllable iff

$$\begin{aligned}
\tilde{\boldsymbol{w}}^\top B_{\hat{v}} &= -\tilde{\boldsymbol{w}}^\top \widetilde{Q}(1)\Lambda(1)\\
&= -\Big(\boldsymbol{\alpha}^1 w_j^{(1)\top}\widetilde{Q}_1(1) + \ldots + \boldsymbol{\alpha}^p w_j^{(1)\top}\widetilde{Q}_p(1)\Big)\Lambda(1)\\
&= -\Big(\boldsymbol{\alpha}^1 w_j^{(1)\top}R_1(1) + \ldots + \boldsymbol{\alpha}^p w_j^{(1)\top}R_p(1)\Big)\\
&= -\Big(\boldsymbol{\alpha}^1 r_{1j}^{(1)\top}(1) + \ldots + \boldsymbol{\alpha}^p r_{pj}^{(1)\top}(1)\Big) \neq 0^\top, \quad (61)
\end{aligned}$$

$i = 1,\ldots,p$, $j = 1,\ldots,s_{\min}$, in view of (25) and the proof of Lemma 1, because of the *PBH eigenvector tests* (see [17, Th. 6.2-5]). Considering (54) and (28), the result $h_{ij}^{(1)\top} = -\boldsymbol{b}e_i^\top N(\mu_j)$ follows. With this and $r_{ij}^{(1)\top}(1) = \rho_2^{(1)\top}(1) = h_{ij}^{(1)\top}M^{-1}(\mu_j)$ (see (53)), the expression (61) can be rewritten as

$$\begin{aligned}
\tilde{\boldsymbol{w}}^\top B_{\hat{v}} &= \Big(\boldsymbol{\alpha}^1 b_{1j}^{(1)}e_1^\top + \ldots + \boldsymbol{\alpha}^p b_{pj}^{(1)}e_p^\top\Big)N(\mu_j)M^{-1}(\mu_j)\\
&= [\boldsymbol{\alpha}^1 b_{1j}^{(1)} \;\; \ldots \;\; \boldsymbol{\alpha}^p b_{pj}^{(1)}]N(\mu_j)M^{-1}(\mu_j). \quad (62)
\end{aligned}$$

Therein, the premultiplying matrix on the right hand side cannot vanish, because $b_{ij}^{(1)} = w_j^{(1)\top}b_{y_i} \neq 0$ is implied by the controllability of $(S_{\min}, b_{y_i})$ and $|\boldsymbol{\alpha}^1| + \ldots + |\boldsymbol{\alpha}^p| > 0$. Hence, the condition $\tilde{\boldsymbol{w}}^\top B_{\hat{v}} \neq 0^\top$ is ensured for $\mathrm{rank}\, N(\mu) = p$, $\forall\mu \in \sigma(S)$. On the contrary, if $\mathrm{rank}\, N(\mu_j) < p$, $\boldsymbol{\alpha}^i$ can always be chosen such that $\tilde{\boldsymbol{w}}^\top B_{\hat{v}} = 0^\top$. Hence, $(\widetilde{S}, B_{\hat{v}})$ is not controllable.

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