Constructing Initial Algebras Using Inflationary Iteration

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An old theorem of Adámek constructs initial algebras for sufficiently cocontinuous endofunctors via transfinite iteration over ordinals in classical set theory. We prove a new version that works in constructive logic, using "inflationary" iteration over a notion of *size* that abstracts from limit ordinals just their transitive, directed and well-founded properties. Borrowing from Taylor's constructive treatment of ordinals, we show that sizes exist with upper bounds for any given signature of indexes. From this it follows that there is a rich collection of endofunctors to which the new theorem applies, provided one admits a weak form of choice (the WISC axiom of van den Berg, Moerdijk, Palmgren and Streicher) that is known to hold, for example, in the internal logic of very many kinds of elementary topos.

1 Introduction

Initial algebras for endofunctors are a simple category-theoretic concept that has proved very useful in logic and computer science. Recall that an *initial algebra* $(\mu F, \iota)$ for an endofunctor $F : \mathscr{C} \to \mathscr{C}$ on a category $\mathscr{C}$, is a morphism $\iota : F(\mu F) \to \mu F$ in $\mathscr{C}$ with the property that for any morphism $a : F(A) \to A$, there is a unique $\hat{a} : \mu F \to A$ that is an F-algebra morphism, that is, satisfies $\hat{a} \circ \iota = a \circ F(\hat{a})$. In functional programming, $\hat{a}$ is sometimes called the *catamorphism* associated with the algebra (A, a) [22]. By varying the choice of $\mathscr{C}$ and F, such initial algebras give semantics for various kinds of inductive (or dually, coinductive) structures and, via their catamorphisms, associated (co)recursion schemes. We refer the reader to the draft book by Adámek, Milius, and Moss [6] for an account of this within classical logic.

Here we make a contribution to the existence of initial algebras within *constructive* logics. Our motivation for doing so is not philosophical, nor motivated by the computational insights that a constructive approach can bring, important though both those thing are. Rather, we are interested in the semantics of dependent type theories with inductive constructions, such as types that are inductive [21], inductive-recursive [11], inductive-inductive [15], quotient (inductive-)inductive [9, 19] and more generally higher-inductive [34]. Toposes are often used when constructing models of such type theories and sometimes the easiest way of doing so is to use their "internal logic" [18, Part D] to express the constructions; see [26, 20], for example. Although there are different candidates for what is the internal logic of toposes, in general they are not classical. So we are led to ask for what categories $\mathscr{C}$ and functors $F : \mathscr{C} \to \mathscr{C}$ that are describable in such an internal logic is it the case that an initial F-algebra can be constructed?

We pursue this question by developing a constructive version of Adámek's classical theorem about existence of initial algebras via transfinite iteration over ordinals [4] (we discuss a different constructive approach [5] in Section 5). Recall, or see Adámek et al. [6, section 6.1] for example, that if $F : \mathscr{C} \to \mathscr{C}$ is an endofunctor on a category $\mathscr{C}$ with all small colimits (colimits of small chains is enough), then we get a

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large chain in $\mathscr{C}$, $(F^{\alpha}0)_{\alpha\in\mathsf{Ord}}$ indexed by the totally ordered class of ordinals Ord, defined by recursion over the ordinals:

$$F^{\alpha}0 = \begin{cases} 0 \quad \text{(initial object in } \mathscr{C}\text{)} & \text{if } \alpha = 0 \text{, the ordinal zero} \\ F(F^{\beta}0) & \text{if } \alpha = \beta^{+} \text{ is a successor ordinal} \\ \mathrm{colim}_{\beta<\lambda} F^{\beta}0 & \text{if } \alpha = \lambda \text{ is a limit ordinal} \end{cases} \tag{1}$$

The links in the chain are $\mathscr{C}$-morphisms $i_{\alpha} : F^{\alpha}0 \to F^{\alpha^{+}}0$ also defined by ordinal recursion:

$$i_{\alpha} = \begin{cases} \text{unique morphism given by initiality of } 0 & \text{if } \alpha = 0 \text{, the ordinal zero} \\ F(i_{\beta}) & \text{if } \alpha = \beta^{+} \text{ is a successor ordinal} \\ \text{induced by the universal property of colimits} & \text{if } \alpha = \lambda \text{ is a limit ordinal} \end{cases} \tag{2}$$

Theorem 1.1 [CLASSICAL] (Adámek [4])**.** If i_{α} is an isomorphism for some $\alpha \in \mathsf{Ord}$, then $(F^{\alpha}0, i_{\alpha}^{-1})$ is an initial algebra for $F : \mathscr{C} \to \mathscr{C}$. So in particular, if F preserves colimits of shape λ for some limit ordinal λ, then (by definition of "preserves colimits") i_{λ} is an isomorphism and $F^{\lambda}0$ is an initial F-algebra. $\triangleleft$

This theorem is labelled [CLASSICAL] because its proof uses classical logic: the properties of ordinal numbers that it relies upon require the Law of Excluded Middle ($\forall p.\ p \vee \neg p$). In Section 3 we show that by replacing the use of ordinals with a weaker notion of "size" and modifying the way F is iterated, one can obtain a constructive version of Adámek's theorem (see Theorem 3.8).

Not only the proof, but also the application of Adámek's theorem can rely upon classical logic: the Axiom of Choice (AC) is often used to see that a particular functor of interest preserves λ-colimits for a suitably large limit ordinal λ. For example, consider the polynominal functor $F_{A,B} : \mathbf{Set} \to \mathbf{Set}$ associated with a container specified by $A \in \mathbf{Set}$ and $B \in \mathbf{Set}^{A}$ [1, 16]. This functor maps a set X to the dependent product of function sets $\sum_{a\in A} X^{B(a)}$ and in the case that the sets $B(a)$ are infinite, we can use AC to prove that $F_{A,B}$ preserves λ-colimits when λ has upper bounds for all $B(a)$-indexed families of ordinals less than λ.[1] We show in Section 4 that a much weaker choice principle than AC, the "Weakly Initial Sets of Covers" (WISC) axiom [30, 23, 35] is enough to ensure that our constructive version of Adámek's theorem applies to a rich class of endofunctors. WISC has been called "constructively acceptable" because it is valid in a wide range of elementary toposes [35]. In particular it holds in presheaf and realizability toposes that have been used to construct models of dependent type theory that mix quotients and inductive constructions, which, as we mentioned above, motivates our desire for a constructive treatment of initial algebras.

2 Constructive meta-theory

The results in this paper are presented in the usual informal language of mathematics, but only making use of intuitionistically valid logical principles (and, to obtain the results of Section 4, extended by the WISC axiom). In particular we avoid use of the Law of Excluded Middle, or more generally the Axiom of Choice.

More specifically, our results can be soundly interpreted in any elementary topos with natural number object and universes (satisfying WISC, for the last part of the paper). Universes are used to classify

[1] There other ways to see that $F_{A,B}$ has an initial algebra – ones that avoid AC and indeed are constructively valid; see [23, Proposition 3.6].

representable classes of small maps, in the sense of Moerdijk and Palmgren [24]. Thus when we refer to the category **Set** of small sets and functions, we mean the generalised elements of some such universe, which we always assume contains the subobject classifier. In fact, in order to interpret quantification over such small sets in a straightforward way, we tacitly assume there is a nested sequence of such universes. A suitable version of Martin-Löf's Extensional Type Theory [21] extended with an impredicative universe of propositions can be used as the internal language of such toposes.

In fact the use of impredicative quantification is not necessary: we are developing a formalisation of the results of this paper using the Agda [8] proof assistant, which can provide a dependent type theory with a predicative universe of (proof irrelevant) propositions. We then have to postulate as axioms some things which are derivable in the logic of toposes, namely axioms for propositional extensionality, quotient sets and unique choice (and WISC, when we need it).

3 Size-indexed inflationary iteration

Throughout this section we fix a category $\mathscr{C}$ and an endofunctor $F : \mathscr{C} \to \mathscr{C}$. We will consider sequences of objects in $\mathscr{C}$ built up by iterating F while taking certain colimits. For simplicity we assume that $\mathscr{C}$ is cocomplete, that is, has colimits of all small diagrams.[2]

From a constructive point of view, the problem with the sequence (1) is that it makes use of ordinals, which rely on the Law of Excluded Middle (LEM) for their good properties; in particular, the definition in (1) is by cases according to whether an ordinal is zero, or a successor, *or not*. In the case that $\mathscr{C}$ is a complete partially ordered set (with joins denoted by $\bigvee$), Abel and Pientka [3, section 4.5] point out that one can avoid this case distinction, while still achieving within constructive logic the same result in the (co)limit, by instead taking the approach of Sprenger and Dam [29] and using what they term an *inflationary iteration*:

$$\mu_i F = \bigvee\nolimits_{j<i} F(\mu_j F) \tag{3}$$

We only need i to range over the elements of a set equipped with a binary relation $<$ that is well-founded for this definition to make sense. Here we generalise from complete posets to cocomplete categories, replacing joins by colimits. Definition 3.2 sums up what we need of the indexes i and the relation $<$ between them in order to ensure that the inflationary sequence can be defined and yields an initial algebra for F if it becomes stationary.

Definition 3.1. Recall that a *semi-category* is like a category, but lacks identity morphisms. A semi-category is *thin* if there is at most one morphism between any pair of objects. Thus a small thin semi-category is the same thing as a set κ (the set of objects) equipped with a transitive relation $\_ < \_ \subseteq \kappa \times \kappa$ (the existence-of-a-morphism relation). Given such a $(\kappa, <)$, a *diagram* $D : \kappa \to \mathscr{C}$ in a category $\mathscr{C}$ is by definition a semi-functor from κ to $\mathscr{C}$: thus D maps each $i \in \kappa$ to a $\mathscr{C}$-object D_i, each pair (j, i) with $j < i$ to a $\mathscr{C}$-morphism $D_{j,i} : D_j \to D_i$, and these morphisms satisfy $D_{j,i} \circ D_{k,j} = D_{k,i}$ for all $k < j < i$ in κ. $\lhd$

Definition 3.2. A *size* is a small thin semi-category $(\kappa, <)$ that is

- *directed*: every finite subset of κ has an upper bound with respect to $<$; specifically, we assume we are given a distinguished element $0^s \in \kappa$ and a binary operation $\_ \sqcup^s \_ : \kappa \times \kappa \to \kappa$ satisfying $\forall i, j \in \kappa.\ i < i \sqcup^s j \wedge j < i \sqcup^s j$

[2] This means that we are given a function assigning a choice of colimit for each small diagram, since we work in a constructive setting and in particular have to avoid use of the Axiom of Choice.

- *well-founded*: for all $K \subseteq \kappa$, if $\forall i \in \kappa.(\forall j < i.\ j \in K) \Rightarrow i \in K$, then $K = \kappa$. $\lhd$

Note that the directedness property in particular gives a successor operation $\uparrow^s : \kappa \to \kappa$ on the elements of a size, defined by $\uparrow^s i \triangleq i \sqcup^s i$ and satisfying $\forall i \in \kappa.\ i < \uparrow^s i$. (We do not need a successor that also preserves $<$, although the sizes constructed in the next section have one that does so; see [14, Definition 5.1].)

Example 3.3. In the next section we will define a rich class of sizes derived from algebraic signatures (see Proposition 4.2). For now, we note that the natural numbers $\mathbb{N}$ with their usual strict order is a size.[3] In classical logic, an ordinal is a size iff its usual strict total order is directed, which happens iff it is a limit ordinal. $\lhd$

Remark 3.4. Since we are working constructively, the well-foundedness property of a size is stated in a suitably positive form; classically, it is equivalent to the non-existence of infinite descending chains for $<$. Well-foundedness of $<$ allows one to define size-indexed families by *well-founded recursion* [33, section 6.3]: given a size κ and a κ-indexed family of sets $(A_i)_{i\in\kappa}$, from each family of functions $(f_i : (\prod_{j<i} A_j) \to A_i)_{i\in\kappa}$ we get a family of elements $(a_i \in A_i)_{i\in\kappa}$, uniquely defined by the requirement $\forall i \in \kappa.\ a_i = f_i((a_j)_{j<i})$. $\lhd$

Given a size κ, for each element $i \in \kappa$ we get a small thin semi-category[4] $\downarrow(i)$ whose vertices are the elements $j \in \kappa$ with $j < i$ and whose morphisms are the instances of the $<$ relation. Thus a diagram $D : \downarrow(i) \to \mathscr{C}$ maps each $j < i$ to a $\mathscr{C}$-object D_j and each pair (k, j) with $k < j < i$ to a $\mathscr{C}$-morphism $D_{k,j} : D_k \to D_j$, satisfying $D_{k,j} \circ D_{l,k} = D_{l,j}$ for all $l < k < j < i$. We write

$$(\mathrm{inc}^D_j : D_j \to \mathrm{colim}_{j<i} D_j)_{j<i} \tag{4}$$

for the colimit of this diagram (recall that we are assuming $\mathscr{C}$ is cocomplete). Thus for all $k < j < i$ it is the case that $\mathrm{inc}^D_k = \mathrm{inc}^D_j \circ D_{k,j}$; and given any cocone in $\mathscr{C}$

$$(f_j : D_j \to X)_{j<i} \qquad \forall k < j < i.\ f_k = f_j \circ D_{k,j}$$

there is a unique $\mathscr{C}$-morphism $\hat{f} : \mathrm{colim}_{j<i} D_j \to X$ satisfying $\forall j < i.\ \hat{f} \circ \mathrm{inc}^D_j = f_j$.

Since $<$ is transitive, if $j < i$ in κ, then $\downarrow(j)$ is a sub-semi-category of $\downarrow(i)$ and each diagram $D : \downarrow(i) \to \mathscr{C}$ restricts to a diagram $D|_j : \downarrow(j) \to \mathscr{C}$. We write

$$\mathrm{c}^D_{j,i} : \mathrm{colim}_{k<j} D_k \to \mathrm{colim}_{k<i} D_k \tag{5}$$

for the unique $\mathscr{C}$-morphism satisfying $\forall k < j < i.\ \mathrm{c}^D_{j,i} \circ \mathrm{inc}^{D|_j}_k = \mathrm{inc}^D_k$.

Definition 3.5. Let κ be a size. Given an endofunctor $F : \mathscr{C} \to \mathscr{C}$ on a cocomplete category $\mathscr{C}$, a diagram $D : \kappa \to \mathscr{C}$ is an *inflationary iteration of F over* κ if for all $i \in \kappa$

$$D_i = \mathrm{colim}_{j<i} F(D_j) \ \wedge\ \forall j < i.\ D_{j,i} = \mathrm{c}^{F\circ D}_{j,i} \qquad \lhd$$

Lemma 3.6. Given an endofunctor $F : \mathscr{C} \to \mathscr{C}$ on a cocomplete category $\mathscr{C}$, for each size κ the inflationary iteration of F over κ exists (and is unique). $\lhd$

[3] $\mathbb{N}$ will be the smallest size once one has developed a comparison relation between sizes. To do that one probably has to restrict to sizes that are *extensional*, that is, satisfy $\forall i, j \in \kappa.\ \{k \in \kappa \mid k < i\} = \{k \in \kappa \mid k < j\} \Rightarrow i = j$. However, we have no need of that property for the results in this paper.

[4] Well-foundedness is preserved, but directedness is not, so $\downarrow(i)$ is not necessarily a size.

Proof. Given $i \in \kappa$, say that a diagram $D : \downarrow(i) \to \mathscr{C}$ is an inflationary iteration of F *up to* i if for all $j < i$, $D_j = \mathrm{colim}_{k<j} F(D_k)$ and $\forall k < j.\ D_{k,j} = \mathrm{c}^{F\circ D}_{k,j}$. Note that given such a diagram, for any $j < i$ we have that $D|_j : \downarrow(j) \to \mathscr{C}$ is an inflationary iteration of F up to j. Using well-founded induction for $<$, one can prove that

$$\forall i \in \kappa,\ \text{any two inflationary iterations of } F \text{ up to } i \text{ are equal} \tag{6}$$

Then one can use well-founded recursion for $<$ (Remark 3.4) to define for each $i \in \kappa$ an inflationary iteration of F up to i, $D^{(i)} : \downarrow(i) \to \mathscr{C}$. If $j < i$, then $D^{(j)}$ and $D^{(i)}|_j$ are both inflationary iterations of F up to j and so are equal by (6). It follows that

$$(D_i \triangleq \mathrm{colim}_{j<i} F(D^{(i)}_j))_{i\in\kappa} \wedge (D_{j,i} \triangleq \mathrm{c}^{F\circ D^{(i)}}_{j,i})_{j,i\in\kappa \mid j<i}$$

defines an inflationary iteration of F. (Furthermore, since any such restricts to an up-to-i inflationary iteration, uniqueness follows from (6).) □

Remark 3.7. We record some simple properties of inflationary iteration that we need in the proof of the theorem below. Let $D : \kappa \to \mathscr{C}$ be the inflationary iteration of $F : \mathscr{C} \to \mathscr{C}$ over κ. Note that for all $j < i$ in κ, the components of the colimit cocone $\mathrm{inc}^{F\circ D|_i}_j : F(D_j) \to \mathrm{colim}_{j<i} F(D_j)$ are morphisms $\iota_{j,i} : F(D_j) \to D_i$ satisfying

$$\forall k < j < i.\ D_{j,i} \circ \iota_{k,j} = \iota_{k,i} = \iota_{j,i} \circ F(D_{k,j}) \tag{7}$$

The first equation follows from the fact that $D_{j,i} = \mathrm{c}^{F\circ D}_{j,i}$ and the second from the definition of $\iota_{j,i}$ as a component of a cocone. Since that cocone is colimiting, one also has

$$(\forall j < i.\ f \circ \iota_{j,i} = g \circ \iota_{j,i}) \Rightarrow f = g \tag{8}$$

for all $\mathscr{C}$-morphisms $f, g : D_i \to X$. ◁

The proof of Lemma 3.6 just uses the transitive and well-founded properties of the relation $<$ on a size κ. The directedness property of $<$ comes into the proof of the following theorem.

Theorem 3.8 (Initial algebras via inflationary iteration)**.** Suppose $\mathscr{C}$ is a cocomplete category, $F : \mathscr{C} \to \mathscr{C}$ is an endofunctor and there is a size κ such that F preserves colimits of diagrams $\kappa \to \mathscr{C}$. Then F has an initial algebra whose underlying $\mathscr{C}$-object is the colimit $\mu F = \mathrm{colim}_{i\in\kappa}\, \mu_i F$ of the inflationary iteration $\mu_i F$ (Definition 3.5) of F over κ. ◁

Proof. Let $D : \kappa \to \mathscr{C}$ be the inflationary iteration of $F : \mathscr{C} \to \mathscr{C}$ over κ and define $\mu F \triangleq \mathrm{colim}_{i\in\kappa} D_i$. For each $i \in \kappa$, as in Definition 3.2 we have $\uparrow^s i \in \kappa$ with $i < \uparrow^s i$ and hence a $\mathscr{C}$-morphism

$$\iota_i \triangleq \left(F(D_i) \xrightarrow{\iota_{i,\uparrow^s i}} D_{\uparrow^s i} \xrightarrow{\mathrm{inc}^D_{\uparrow^s i}} \mathrm{colim}_{i\in\kappa} D_i = \mu F \right)$$

By (7), $(\iota_i)_{i\in\kappa}$ is a cocone under the diagram $F \circ D : \kappa \to \mathscr{C}$ and so induces $\hat{\iota} : \mathrm{colim}_{i\in\kappa} F(D_i) \to \mu F$. Then since F preserves the colimit of D, we get a morphism

$$\iota \triangleq \left(F(\mu F) = F(\mathrm{colim}_{i\in\kappa} D_i) \cong \mathrm{colim}_{i\in\kappa} F(D_i) \xrightarrow{\hat{\iota}} \mu F \right) \tag{9}$$

Therefore μF has the structure of an F-algebra. To see that it is initial, suppose we are given $a : F(A) \to A$. We have to show that there is a unique F-algebra morphism $(\mu F, \iota) \to (A, a)$.

If $h : \mu F \to A$ is such an algebra morphism, that is $h \circ \iota = a \circ F(h)$, then by definition of ι in (9) it follows that the associated cocone $(h_i \triangleq h \circ \mathrm{inc}^D_i : D_i \to A)_{i\in\kappa}$ satisfies $h_{\uparrow^s i} \circ \iota_{i,\uparrow^s i} = a \circ F(h_i)$. From this, using the directedness property of sizes, we get

$$\forall i \in \kappa. \forall j < i.\ h_i \circ \iota_{j,i} = a \circ F(h_i \circ D_{j,i}) \tag{10}$$

So if h and h' are both F-algebra morphisms $(\mu F, \iota) \to (A, a)$, one can prove by well-founded induction for $<$, using (8) and (10), that $\forall i \in \kappa.\ h \circ \mathrm{inc}^D_i = h' \circ \mathrm{inc}^D_i$ and hence that $h = h'$.

So it just remains to prove that there is such an h. It suffices to construct a cocone $(h_i : D_i \to A)_{i\in\kappa}$ satisfying (10) and then take h to be the morphism given by the universal property of the colimit; for then we have $\forall i \in \kappa.\ h_{\uparrow^s i} \circ \iota_{i,\uparrow^s i} = a \circ F(h_i)$ and hence $h \circ \iota = a \circ F(h)$, as required. For each $i \in \kappa$, say that a morphism $h' : D_i \to A$ is an *up-to-i algebra morphism* if $\forall j < i.\ h' \circ \iota_{j,i} = a \circ F(h' \circ D_{j,i})$ (*cf.* (10)). Given such a morphism, then for any $j < i$, $h' \circ D_{j,i} : D_j \to A$ is an up-to-j algebra morphism. From this it follows by well-founded induction for $<$ that any two up-to-i algebra morphisms are equal. A well-founded recursion for $<$ allows one to construct an up-to-i algebra morphism $h_i : D_i \to A$ for each $i \in \kappa$; and the uniqueness of up-to algebra morphisms implies that $h_j = h_i \circ D_{j,i}$ when $j < i$. Thus $(h_i)_{i\in\kappa}$ is the required cocone satisfying (10). □

Corollary 3.9. With the same assumptions on $\mathscr{C}$, F and κ as in Theorem 3.8, then free F-algebras exist, that is, the forgetful functor from the category of F-algebras to $\mathscr{C}$ has a left adjoint. ◁

Proof. The free F-algebra on an object $X \in \mathscr{C}$ is the same thing as an initial algebra for the endofunctor $F(\_) + X$. So by the theorem, it suffices to check that $F(\_) + X$ preserves colimits of diagrams $\kappa \to \mathscr{C}$. It does so because F does by assumption and because κ is directed (cf. Proposition 4.6(4) below). □

4 Initial algebras for sized endofunctors

In classical set theory with the Axiom of Choice, given a set of operation symbols $A \in \mathbf{Set}$ with associated arities $B \in \mathbf{Set}^A$, the associated polynomial endofunctor $X \mapsto \sum_{a\in A} X^{B(a)}$ on $\mathbf{Set}$ preserves λ-colimits when the ordinal λ is large enough; specifically it does so if for all $a \in A$, λ has upper bounds (with respect to the strict total order given by membership) for all $B(a)$-indexed families of ordinals less than λ. We will see that this notion of "large enough" is also the right one for sizes in our constructive setting.

Definition 4.1. A *signature* (also known as a *container* [1, 16]) is specified by a set $A \in \mathbf{Set}$ and an A-indexed family of sets $B \in \mathbf{Set}^A$. Given such a signature $\Sigma = (A, B)$, we say that a size $(\kappa, <)$ is Σ-*filtered* if for all $a \in A$ and every function $f : B(a) \to \kappa$, there exists $i \in \kappa$ with $\forall x \in B(a).\ f(x) < i$. ◁

We can deduce the existence of Σ-filtered sizes by abstracting from the constructive analysis of Conway's surreal numbers by Shulman [28], which in turn is inspired by Taylor's constructive notion of "plump" ordinal [32]. For each signature $\Sigma = (A, B)$, let W_Σ be the initial algebra for the associated polynominal endofunctor $F_{A,B} : \mathbf{Set} \to \mathbf{Set}$, $F_{A,B}(X) = \sum_{a\in A} X^{B(a)}$. Thus W_Σ is an example of a W-type [25, Chapter 15]. They exist in our constructive setting, because they can be constructed in elementary toposes with natural number objects [23, Proposition 3.6]: one can take the elements of W_Σ to be well-founded trees representing the algebraic terms inductively generated by the signature Σ. Each such term t is uniquely of the form $\sup_a f$ where $\sup_a$ is the $B(a)$-arity operation symbol named by $a \in A$ and, inductively, $f = (t_x)_{x\in B(a)}$ is a $B(a)$-tuple of well-founded algebraic terms over Σ. The *plump* ordering on W_Σ is given by the least relations $\_ < \_ \subseteq W_\Sigma \times W_\Sigma$ and $\_ \leq \_ \subseteq W_\Sigma \times W_\Sigma$ satisfying for all $a \in A$, $f : B(a) \to W_\Sigma$ and $t \in W_\Sigma$

$$(\forall x \in B(a).\, f(x) < t) \Rightarrow \sup_a f \leq t \quad \text{and} \quad (\exists x \in B(a).\, t \leq f(x)) \Rightarrow t < \sup_a f \tag{11}$$

As noted in [14, Example 5.4], $<$ is transitive and well-founded, and $\leq$ is a preorder (reflexive and transitive). In particular, since $\leq$ is reflexive, from (11) we deduce that $\forall x \in B(a).\, f(x) < \sup_a f$, in other words for each arity set $B(a)$ in the signature, any function $f : B(a) \to W_\Sigma$ is bounded above in the $<$ relation by $\sup_a f$. This allows us to construct Σ-filtered sizes:

Proposition 4.2. There is a Σ-filtered size $(\kappa_\Sigma, <)$ for every signature Σ. $\triangleleft$

Proof. Given a signature $\Sigma = (A, B)$, we extend it to a signature (A', B') by adding fresh nullary and binary operation symbols. Thus $A' \triangleq A \uplus \{n, b\}$ and $B' \in \mathbf{Set}^{A'}$ satisfies $B'(a) \triangleq B(a)$ for $a \in A$, $B'(n) \triangleq \emptyset$ and $B'(b) \triangleq \{0, 1\}$. Let set κ_Σ be the W-type $W_{(A',B')}$ and let $<$ be the plump order given by (11). As noted above, $<$ is transitive and well-founded and has upper bounds for any arity-indexed family and hence in particular it is Σ-filtered. It just remains to see that it is directed (Definition 3.2). Since A' contains the nullary operation symbol n, κ_Σ contains $0^s \triangleq \sup_n \emptyset$; and given $i, j \in \kappa_\Sigma$, letting $f : B'(b) = \{0, 1\} \to \kappa_\Sigma$ map 0 to i and 1 to j, then $i \sqcup^s j \triangleq \sup_b f$ is an upper bound for i and j with respect to $<$. $\square$

Definition 4.3. Given a signature Σ, a functor $F : \mathscr{C} \to \mathscr{D}$ between cocomplete categories is Σ-*sized* if it preserves colimits of all diagrams $\kappa \to \mathscr{C}$ for any Σ-filtered size κ. A *sized functor* is a functor together with a signature Σ for which it is Σ-sized. $\triangleleft$

Theorem 4.4 (Sized endofunctors have initial algebras)**.** Assuming $\mathscr{C}$ is a cocomplete category, if $F : \mathscr{C} \to \mathscr{C}$ is sized, then it has an initial algebra. $\triangleleft$

Proof. If F is Σ-sized, then F preserves colimits of diagrams for the Σ-filtered size κ_Σ from Proposition 4.2. Hence by Theorem 3.8, it has an initial algebra. $\square$

To apply this theorem one needs a rich collection of sized functors. The rest of the section is devoted to exploring closure properties of sized functors. To do so we use the following operation on signatures:

Definition 4.5. Suppose $\Sigma_c = (A_c, B_c)$ is a family of signatures indexed by the elements c of some set C. Then the *signature sum* $\bigoplus_{c \in C} \Sigma_c$ is the signature (A, B) where $A \triangleq \sum_{c \in C} A_c = \{(c, a) \mid c \in C \wedge a \in A_c\}$ and $B \in \mathbf{Set}^A$ maps each (c, a) to the set $B_c(a)$. Note that if a size is $(\bigoplus_{c \in C} \Sigma_c)$-filtered, it is also Σ_c-filtered for each $c \in C$. Conversely, give a single signature Σ, if a size is Σ-filtered, it is also $(\bigoplus_{c \in C} \Sigma)$-filtered.

As a special case when $I = \{0, 1\}$, we have the *binary sum* $\Sigma_0 \oplus \Sigma_1$. There is also an *empty signature* $0 = (\emptyset, \emptyset)$ which acts as a unit for $\oplus$ up to isomorphism (for a suitable notion of signature morphism). $\triangleleft$

Proposition 4.6. Suppose that $\mathscr{C}$,$\mathscr{D}$ and $\mathscr{E}$ are cocomplete categories.

1. Any cocontinuous functor $\mathscr{C} \to \mathscr{D}$ is sized.
2. Identity functors are sized. If $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ are sized, so is $G \circ F : \mathscr{C} \to \mathscr{E}$.
3. The terminal functor $\mathscr{C} \to 1$ and the projection functors $\pi_1 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $\pi_2 : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$ are sized; if $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{E}$ are sized, then so is $\langle F, G \rangle : \mathscr{C} \to \mathscr{D} \times \mathscr{E}$.
4. For any $X \in \mathscr{C}$ the constant functor $1 \to \mathscr{C}$ with value X is sized.
5. If $\mathbf{C}$ is a small category and the functor $F : \mathbf{C} \times \mathscr{C} \to \mathscr{D}$ has the property that for each $c \in \mathbf{C}$ $F(c, \_) : \mathscr{C} \to \mathscr{D}$ is a sized functor, then $\mathrm{colim}_{c \in \mathbf{C}} F(c, \_) : \mathscr{C} \to \mathscr{D}$ is also sized.

6. If $F : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ is sized, then for each $X \in \mathscr{C}$ the functor $F(X, \_) : \mathscr{D} \to \mathscr{D}$ is sized and hence by Theorem 4.4 has an initial algebra, which we write as $\mu Y.F(X,Y)$. Then $X \mapsto \mu Y.F(X,Y)$ is the object part of a sized functor $\mathscr{C} \to \mathscr{D}$. $\lhd$

Proof. For part 1, if $F : \mathscr{C} \to \mathscr{D}$ is cocontinuous, then it is Σ-sized for any Σ and in particular for the empty signature.

The first sentence of part 2 follows from part 1. If F is Σ-sized and G is Σ'-sized, then F and G both preserve colimits over any $\Sigma \oplus \Sigma'$-filtered size, because such a size is also Σ- and Σ'-filtered. The composition $G \circ F$ preserves such a colimit because F and G do. Therefore $G \circ F$ is $\Sigma \oplus \Sigma'$-sized.

For part 3 we use the fact that colimits in a product category are computed componentwise. Thus the terminal and projection functors are sized by part 1; and if $F : \mathscr{C} \to \mathscr{D}$ is Σ-sized and $G : \mathscr{C} \to \mathscr{E}$ is Σ'-sized, then $\langle F, G \rangle$ is $\Sigma \oplus \Sigma'$-sized.

For part 4, note that each size κ is directed and hence in particular is a connected semi-category; therefore $\mathrm{colim}_{i\in\kappa} X$ is canonically isomorphic to X. So the constant functor with value X is Σ-sized for any Σ and in particular for the empty signature.

For part 5, we are given a function from $\mathbf{C}$-objects to signatures, $c \mapsto \Sigma_c$, so that $F(c, \_)$ is Σ_c-sized. Let $F' \triangleq \mathrm{colim}_{c\in\mathbf{C}} F(c, \_)$ and consider $\Sigma' \triangleq \bigoplus_{c\in C} \Sigma_c$, the signature from Definition 4.5. If $D : \kappa \to \mathscr{C}$ is a diagram on a Σ'-filtered size, since κ is Σ_c-filtered for each $c \in \mathbf{C}$ and $F(c, \_)$ is Σ_c-sized, we have a canonical isomorphism $F(c, \mathrm{colim}_{i\in\kappa} D_i) \cong \mathrm{colim}_{i\in\kappa} F(c, D_i)$, natural in c. Taking the colimit over $c \in \mathbf{C}$ we get $F'(\mathrm{colim}_{i\in\kappa} D_i) = \mathrm{colim}_{c\in\mathbf{C}} F(c, \mathrm{colim}_{i\in\kappa} D_i) \cong \mathrm{colim}_{c\in\mathbf{C}} \mathrm{colim}_{i\in\kappa} F(c, D_i)$. Since colimits commute with each other, it follows that the canonical morphism $F'(\mathrm{colim}_{i\in\kappa} D_i) \to \mathrm{colim}_{i\in\kappa} F'(D_i)$ is an isomorphism. Therefore F' is Σ'-sized.

For part 6, suppose $F : \mathscr{C} \times \mathscr{D} \to \mathscr{D}$ is Σ-sized. It follows from parts 2–4 that for each $X \in \mathscr{C}$, the functor $F_X \triangleq F(X, \_) : \mathscr{D} \to \mathscr{D}$ is sized; indeed it is Σ-sized. From Theorems 4.4 and 3.8 we have that the functor given on objects by taking the initial algebra of F_X is obtained as a colimit of an inflationary iteration: $\mu Y.F(X,Y) = \mathrm{colim}_{i\in\kappa_\Sigma} \mu_i F_X$. Since each $\mu_i F_X$ is $\mathrm{colim}_{j<i} F(X, \mu_j F_X)$, it follows by well-founded induction on i that each $\mu_i F_X$ is sized, using part 5 (taking $\mathbf{C}$ to be the category generated by the thin semi-category $\downarrow(i)$). Then by part 5 again (taking $\mathbf{C}$ to be the category generated by κ) we have that $\mu Y.F(\_, Y) = \mathrm{colim}_{i\in\kappa_\Sigma} \mu_i F\_$ is sized. (In fact it is Σ-sized, because of the observation about $(\bigoplus_{c\in C} \Sigma)$-filtered sizes in Definition 4.5.) $\square$

Although the above proposition establishes quite a rich collection of closure properties of sized functors, what is lacking at the moment is any closure under taking limits, assuming the target category has them; in other words the dual of Proposition 4.6(5). We consider this for the case $\mathscr{D} = \mathbf{Set}$, leaving consideration of more general complete and cocomplete categories for future work. First note that if $F, G : \mathscr{C} \to \mathbf{Set}$ are sized functors, the equalizer of any parallel pair $F \rightrightarrows G$ of natural transformations is also a sized functor (it is $(\Sigma \oplus \Sigma')$-sized if F is Σ-sized and G is Σ'-sized). This is because each size κ is directed and so taking κ-colimits in $\mathbf{Set}$ commutes with finite limits and hence in particular with equalizers. So to get closure of sized functors under all small limits it suffices to consider small products.

We can deduce preservation of the sized property under taking products, even if the products are not finite, so long as we assume the *Weakly Initial Set of Covers* (WISC) axiom, introduced independently by Streicher [30] (who calls it TTCA_f) and van den Berg and Moerdijk [35] (who call it the "Axiom of Multiple Choice" and build upon previous work by Moerdijk and Palmgren [24]).

Axiom 4.7 [WISC]**.** The *Weakly Initial Set of Covers* axiom states that for every set $X \in \mathbf{Set}$, there is a set $\mathrm{Cov}_X \in \mathbf{Set}$ and a family of surjections $(e_c : \mathrm{Dom}_X(c) \twoheadrightarrow X)_{c\in\mathrm{Cov}_X}$ in $\mathbf{Set}$ that is weakly initial for surjections with codomain X: for any surjection $f : Y \twoheadrightarrow X$, there exist $c \in \mathrm{Cov}_X$ and $g : \mathrm{Dom}_X(c) \to Y$ such that $e_c = f \circ g$. $\lhd$

Classically, WISC is implied by the Axiom of Choice (AC), since with AC once can take $\mathrm{Cov}_X = \{0\}$, $\mathrm{Dom}_X\, 0 = X$ and $e_0 = \mathrm{id}_X$. WISC has been called "constructively acceptable", because if any elementary topos $\mathscr{E}$ satisfies it, then so do toposes of (pre)sheaves and realizability toposes built from $\mathscr{E}$ [35]. Thus starting from the category of sets in classical set theory with AC, it holds in the kinds of topos that have been used to model type theory with various kinds of higher inductive types, whose semantics motivates the work presented here. (However, WISC does not hold in all toposes, as Roberts [27] shows.)

For the theorem below we need to use the *double cover* signature Δ_X of a set $X \in \mathbf{Set}$, inspired by the use that Swan [31] makes of WISC; see also [14, Definition 5.6]. Assuming WISC holds, the double cover is the signature $\Delta_X = (A,B)$ with $A = \mathrm{Cov}_X \uplus \mathrm{Cov}^2_X$ where $\mathrm{Cov}^2_X \triangleq \{(c,c') \mid c \in \mathrm{Cov}_X \wedge c' \in \mathrm{Cov}_{\ker(e_c)}\}$ with $\ker(e_c) \triangleq \{(d,d') \in (\mathrm{Dom}_X(c))^2 \mid e_c(d) = e_c(d')\}$ the equivalence relation induced by the surjection $e_c : \mathrm{Dom}_X(c) \twoheadrightarrow X$. The function $B \in \mathbf{Set}^A$ maps $c \in \mathrm{Cov}_X$ to $\mathrm{Dom}_X(c)$ and maps $(c,c') \in \mathrm{Cov}^2_X$ to $\mathrm{Dom}_{\ker(e_c)}(c')$.

Theorem 4.8 [WISC] (Products of sized functors are sized)**.** Suppose that $\mathscr{C}$ is a cocomplete category. Assuming the WISC axiom holds, if $(F_x : \mathscr{C} \to \mathbf{Set})_{x\in X}$ is a family of sized functors indexed by some set $X \in \mathbf{Set}$, then the functor $\prod_{x\in X} F_x : \mathscr{C} \to \mathbf{Set}$ given by taking products in $\mathbf{Set}$ is also sized. $\lhd$

Proof. We are given a function from X to signatures, $x \mapsto \Sigma_x$, so that each F_x is Σ_x-sized. We claim that the functor $F' \triangleq \prod_{x\in X} F_x$ is Σ'-sized when $\Sigma' = \Delta_X \oplus (\bigoplus_{x\in X} \Sigma_x)$, where Δ_X is the double cover signature defined above using WISC. For if $D : \kappa \to \mathscr{C}$ is a diagram on a Σ'-filtered size κ, then since κ is Σ_x-filtered for each $x \in X$ and F_x is Σ_x-sized, we have a canonical isomorphism $\mathrm{colim}_{i\in\kappa} F_x(D_i) \cong F_x(\mathrm{colim}_{i\in\kappa} D_i)$. Taking the product over $x \in X$, we get $\prod_{x\in X} \mathrm{colim}_{i\in\kappa} F_x(D_i) \cong \prod_{x\in X} F_x(\mathrm{colim}_{i\in\kappa} D_i) = (\prod_{x\in X} F_x)(\mathrm{colim}_{i\in\kappa} D_i)$. So it just remains to show that the canonical function

$$\mathrm{can}_{F,D} : \mathrm{colim}_{i\in\kappa}\,((\textstyle\prod_{x\in X} F_x)\, D_i) = \mathrm{colim}_{i\in\kappa} \prod_{x\in X} F_x(D_i) \to \prod_{x\in X} \mathrm{colim}_{i\in\kappa} F_x(D_i)$$

is an isomorphism, that is, both an injection and a surjection. To do that we use the fact that the colimit in $\mathbf{Set}$ of a directed diagram $D : \kappa \to \mathbf{Set}$ can be described explicitly as the quotient $(\sum_{i\in\kappa} D_i)/{\approx}$ where the equivalence relation $\approx$ identifies $(i,d),(i',d') \in \sum_{i\in\kappa} D_i$ if there is some $j \in \kappa$ with $i < j$, $i' < j$ and $D_{i,j}(d) = D_{i',j}(d')$. The summand Δ_X in Σ' ensures that κ has upper bounds for $\mathrm{Dom}_X(c)$-indexed families for any $c \in \mathrm{Cov}_X$; and for $\mathrm{Dom}_{\ker(e_c)}(c')$-indexed families for any $c' \in \mathrm{Cov}_{\ker(e_c)}$. The first kind of upper bound, together with the weak initiality property of Cov_X comes into play in proving $\mathrm{can}_{F,D}$ is injective; and both kinds of upper bound and the weak initiality properties of Cov_X and $\mathrm{Cov}_{\ker(e_c)}$ come into play in proving $\mathrm{can}_{F,D}$ is surjective. The details are similar to the proof of [14, Theorem 5.8] and we postpone them to the full version of this paper. $\square$

Example 4.9. The *symmetric containers* of Gylterud [17] generalize ordinary signatures by replacing the set of operation symbols by a small groupoid $\mathbf{A}$ and the arity function by a functor $B : \mathbf{A} \to \mathbf{Set}$. The associated endofunctor $S_{\mathbf{A},B} : \mathbf{Set} \to \mathbf{Set}$ maps each set $X \in \mathbf{Set}$ to the colimit

$$S_{\mathbf{A},B}(X) \triangleq \mathrm{colim}_{a\in\mathbf{A}}\, X^{B(a)} \tag{12}$$

Applying Theorem 4.4, Proposition 4.6 and Theorem 4.8 we have that any topos satisfying WISC has initial algebras for symmetric containers.

In fact these initial algebras are special cases of QW-types [14]: they can be seen as sets of terms quotiented by the symmetries given by the groupoid structure on the arguments of an operation symbol.

So their existence in toposes with WISC follows from the results of that paper. However, the construction here in terms of a colimit of an inflationary iteration gives a simpler description than for the general case of a QW-type. $\lhd$

5 Related and future work

The results in this paper make use of the constructive techniques introduced by the authors and Fiore in our prior paper [14]: the use of sizes given by "plump" well-founded orders on W-types and the use of WISC to see that certain functors preserve colimits of that shape. That paper constructs a large class of quotient-inductive types, called *QWI-types*, which by definition are initial among algebras for indexed containers [1] satisfying a given system of equations. Although the construction proceeds by forming a size-indexed family of objects in the case $\mathscr{C}$ is $\mathbf{Set}^I$ (with $I \in \mathbf{Set}$) and taking its colimit, it does not appear to be a direct corollary of Theorem 3.8. Conversely, the results here do not follow from the ones in [14], since for one thing here we consider general cocomplete categories $\mathscr{C}$, rather than just products of **Set**. In this respect we are closer to the approach of Fiore and Hur [12] and it would be interesting to see whether our techniques can be extended to give constructive proofs of existence of free algebras for the very general notion of equational system on a category that is introduced in that paper. This may involve investigating the extent to which our approach allows a constructive treatment of some of the classical theory of locally presentable and accessible categories [7], which is future work.

The inflationary iteration indexed by a notion of size that we have introduced in this paper generalises from complete posets to cocomplete categories aspects of Abel and Pientka's work [2, 3]. These papers develop a theory of *sized types* and its semantics. Abel has added a version of this to the type theory provided by the Agda proof assistant [8]. Originally, together with Fiore, we used Agda's implementation of sized types to construct the infinitary quotient inductive types called QW-types [13]. Unfortunately recent versions of Agda contain features that allow one to use sized types to prove a logical contradiction. The problem is that, in contrast to the notion of size used in [14] and now here, the one by Abel et al. [2, 3] features a generic size ∞ at which sized-indexed sequences become stationary. Currently in Agda (version 2.6.1) one both has $\infty < \infty$ and can prove that $<$ is well-founded, leading to a contradiction. For us, the intuitive and important aspect of "size" is that there is well-founded ordering, thus permitting definitions by well-founded recursion on a size set. Then having a single size ∞ at which all sequences become stationary is semantically problematic. So we avoid having an explicit stationary size ∞, at the expense of having to take a colimit to obtain an initial algebra, rather that just instantiating an inflationary iteration at ∞.

We hope Agda's sized types will get fixed, since they are useful in practice; they are most often used (together with copatterns) to demonstrate that recursively defined functions on a coinductively defined record type are well-defined (that is, are "productive") [3]. Here, while avoiding sized types, we can still dualise Theorem 3.8: applying it to the opposite category $\mathscr{C}^{\mathrm{op}}$, we have that if $\mathscr{C}$ is complete and $F : \mathscr{C} \to \mathscr{C}$ preserves limits of diagrams $\kappa \to \mathscr{C}$ for some size κ, then *F has a final coalgebra νF given by the limit of a deflationary iteration* $(\nu_i F = \lim_{j<i} F(\nu_j F))_{i\in\kappa}$. We have yet to investigate whether this is useful, that is, how rich is the class of such endofunctors in a constructive setting.

Adámek, Milius and Moss [5] take a different approach to constructive initial algebra theorems than the one here, avoiding iteration of the endofunctor. They consider categories $\mathscr{C}$ with colimits of diagrams of monomorphisms (from some well-behaved class) and endofunctors $F : \mathscr{C} \to \mathscr{C}$ that preserve those monomorphisms. Using the intuitionistically valid fixed point theorem of Pataraia (see [10, Theorem 3.2]), they prove that such an F has an initial algebra iff it has a prefixed point (an algebra whose structure morphism is a monomorphism). Preserving monomorphisms seems less of a condition on a functor than the one we need for Theorem 3.8, preserving colimits of some size κ, although the two conditions are independent. However, as we saw in Proposition 4.6(5), our class of sized endofunctors is closed under taking coequalizers (so that we get initial algebras for constructs involving quotients, such as Example 4.9); whereas endofunctors preserving monomorphisms are not in general closed under taking coequalizers. Another difference to [5] is that it uses impredicative principles (the proof of Pataraia's fixed point theorem uses impredicative quantification); whereas our Agda development shows that our initial algebra theorem (Theorem 3.8) is valid in a predicative constructive logic.

References

[1] M. Abbott, T. Altenkirch & N. Ghani (2005): *Containers: Constructing Strictly Positive Types*. *Theoretical Computer Science* 342(1), pp. 3–27, doi:10.1016/j.tcs.2005.06.002.

[2] A. Abel (2012): *Type-Based Termination, Inflationary Fixed-Points, and Mixed Inductive-Coinductive Types*. *Electronic Proceedings in Theoretical Computer Science* 77, pp. 1–11, doi:10.4204/EPTCS.77.1.

[3] A. Abel & B. Pientka (2016): *Well-Founded Recursion with Copatterns and Sized Types*. *Journal of Functional Programming* 26, p. 61, doi:10.1017/S0956796816000022.

[4] J. Adámek (1974): *Free Algebras and Automata Realizations in the Language of Categories*. *Commentationes Mathematicae Universitatis Carolinae* 15(4), pp. 589–602.

[5] J. Adámek, S. Milius & L. S. Moss (2021): *An Initial Algebra Theorem Without Iteration*. *ArXiv e-prints* arXiv:2104.09837 [cs.LO]. Available at `https://arxiv.org/abs/2104.09837`.

[6] J. Adámek, S. Milius & L. S. Moss (2021): *Initial Algebras, Terminal Coalgebras, and the Theory of Fixed Points of Functors*. Available at `http://www.stefan-milius.eu`. Draft book.

[7] J. Adámek & J. Rosický (1994): *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series, Cambridge University Press, doi:10.1017/CBO9780511600579.

[8] Agda v2.6.1 (2021): Available at `https://agda.readthedocs.io/en/v2.6.1.3/index.html`.

[9] T. Altenkirch, P. Capriotti, G. Dijkstra, N. Kraus & F. N. Forsberg (2018): *Quotient Inductive-Inductive Types*. In C. Baier & U. Dal Lago, editors: *Foundations of Software Science and Computation Structures, FoSSaCS 2018*, *Lecture Notes in Computer Science* 10803, Springer International Publishing, pp. 293–310, doi:10.1007/978-3-319-89366-2_16.

[10] A Bauer & P. Lumsdaine (2013): *On the Bourbaki–Witt Principle in Toposes*. *Mathematical Proceedings of the Cambridge Philosophical Society* 155(1), pp. 87–99, doi:10.1017/S0305004113000108.

[11] P. Dybjer (2000): *A General Formulation of Simultaneous Inductive-Recursive Definitions in Type Theory*. *Journal of Symbolic Logic* 65(2), pp. 525–549.

[12] M. P. Fiore & C.-K. Hur (2008): *On the Construction of Free Algebras for Equational Systems*. *Theoretical Computer Science* 410, pp. 1704–1729, doi:10.1016/j.tcs.2008.12.052.

[13] M. P. Fiore, A. M. Pitts & S. C. Steenkamp (2020): *Constructing Infinitary Quotient-Inductive Types*. In J. Goubault-Larrecq & B. König, editors: *23rd International Conference on Foundations of Software Science and Computation Structures (FoSSaCS 2020)*, *Lecture Notes in Computer Science* 12077, Springer, pp. 257–276, doi:10.1007/978-3-030-45231-5_14.

[14] M. P. Fiore, A. M. Pitts & S. C. Steenkamp (2021): *Quotients, Inductive Types and Quotient Inductive Types*. *ArXiv e-prints* arXiv:2101.02994 [cs.LO]. Available at `https://arxiv.org/abs/2101.02994`.

[15] F. N. Forsberg (2013): *Inductive-Inductive Definitions*. Ph.D. thesis, Swansea University.

[16] N. Gambino & M. Hyland (2004): *Wellfounded Trees and Dependent Polynomial Functors*. In S. Berardi, M. Coppo & F. Damiani, editors: *Types for Proofs and Programs*, Lecture Notes in Computer Science, Springer Berlin Heidelberg, Berlin, Heidelberg, pp. 210–225, doi:10.1007/978-3-540-24849-1_14.

[17] H. R. Gylterud (2011): *Symmetric Containers*. Master of Science, Department of Mathematics, University of Oslo. Available at `https://www.duo.uio.no/bitstream/handle/10852/10740/thesisgylterud.pdf`.

[18] P. T. Johnstone (2002): *Sketches of an Elephant, A Topos Theory Compendium, Volumes 1 and 2*. *Oxford Logic Guides* 43–44, Oxford University Press.

[19] A. Kovács & A. Kaposi (2020): *Large and Infinitary Quotient Inductive-Inductive Types*. In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '20, Association for Computing Machinery, New York, NY, USA, p. 648–661, doi:10.1145/3373718.3394770.

[20] D. R. Licata, I. Orton, A. M. Pitts & B. Spitters (2018): *Internal Universes in Models of Homotopy Type Theory*. In H. Kirchner, editor: *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*, *Leibniz International Proceedings in Informatics (LIPIcs)* 108, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany, pp. 22:1–22:17, doi:10.4230/LIPIcs.FSCD.2018.22.

[21] P. Martin-Löf (1984): *Intuitionistic Type Theory*. Bibliopolis, Napoli.

[22] E. Meijer, M. Fokkinga & R. Paterson (1991): *Functional Programming with Bananas, Lenses, Envelopes and Barbed Wire*. In J. Hughes, editor: *Functional Programming Languages and Computer Architecture, FPCA 1991*, *Lecture Notes in Computer Science* 523, Springer, Berlin, Heidelberg, pp. 124–144, doi:10.1007/3540543961_7.

[23] I. Moerdijk & E. Palmgren (2000): *Wellfounded Trees in Categories*. *Annals of Pure and Applied Logic* 104(1), pp. 189–218, doi:10.1016/S0168-0072(00)00012-9.

[24] I. Moerdijk & E. Palmgren (2002): *Type theories, Toposes and Constructive Set Theory: Predicative Aspects of AST*. *Annals of Pure and Applied Logic* 114(1), pp. 155–201, doi:10.1016/S0168-0072(01)00079-3.

[25] B. Nordström, K. Petersson & J. M. Smith (1990): *Programming in Martin-Löf's Type Theory*. Oxford University Press.

[26] I. Orton & A. M. Pitts (2016): *Axioms for Modelling Cubical Type Theory in a Topos*. In J.-M. Talbot & L. Regnier, editors: *25th EACSL Annual Conference on Computer Science Logic (CSL 2016)*, *Leibniz International Proceedings in Informatics (LIPIcs)* 62, Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Dagstuhl, Germany, pp. 24:1–24:19, doi:10.4230/LIPIcs.CSL.2016.24.

[27] D. M. Roberts (2015): *The Weak Choice Principle WISC may Fail in the Category of Sets*. *Studia Logica* 103, pp. 1005–1017, doi:10.1007/s11225-015-9603-6.

[28] M. Shulman (2014): *The surreals contain the plump ordinals*. Available at `https://homotopytypetheory.org/2014/02/22/surreals-plump-ordinals/`. Homotopy Type Theory blog.

[29] C. Sprenger & M. Dam (2003): *On the Structure of Inductive Reasoning: Circular and Tree-Shaped Proofs in the μCalculus*. In A. D. Gordon, editor: *Foundations of Software Science and Computation Structures (FoSSaCS 2003)*, *Lecture Notes in Computer Science* 2620, Springer, Berlin, Heidelberg., pp. 425–440, doi:10.1007/3-540-36576-1_27.

[30] T. Streicher (2005): *Realizability Models for CZF + $\neg$ Pow*. Available at `http://www2.mathematik.tu-darmstadt.de/~streicher/CIZF/rmczfnp.pdf`. Unpublished note.

[31] A. Swan (2018): *W-Types with Reductions and the Small Object Argument*. *ArXiv e-prints* arXiv:1802.07588 [math.CT]. Available at `https://arxiv.org/abs/1802.07588`.

[32] P. Taylor (1996): *Intuitionistic Sets and Ordinals*. *Journal of Symbolic Logic* 61, pp. 705–744, doi:10.2307/2275781.

[33] P. Taylor (1999): *Practical Foundations of Mathematics*. *Cambridge Studies in Advanced Mathematics* 59, Cambridge University Press.

[34] The Univalent Foundations Program (2013): *Homotopy Type Theory: Univalent Foundations for Mathematics*. Institute for Advanced Study. Available at `http://homotopytypetheory.org/book`.

[35] B. van den Berg & I. Moerdijk (2014): *The Axiom of Multiple Choice and Models for Constructive Set Theory*. *Journal of Mathematical Logic* 14(01), p. 1450005, doi:10.1142/S0219061314500056.