MIT Integration Bee: 2024 Qualifying Round

Question 1

$$\int_{2023}^{2025} 2024\,\mathrm{d}x = 4048. \tag{1}$$

Question 2

$$\int \frac{(x-1)^{\ln(x+1)}}{(x+1)^{\ln(x-1)}}\,\mathrm{d}x = \int \frac{e^{\ln(x-1)\ln(x+1)}}{e^{\ln(x-1)\ln(x+1)}}\,\mathrm{d}x = \int 1\,\mathrm{d}x = x + C. \tag{2}$$

Question 3

$$\int (x\ln x + 2x)\,\mathrm{d}x = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,\mathrm{d}x + x^2 = \frac{1}{2}x^2\ln x + \frac{3}{4}x^2 + C. \tag{3}$$

Question 4

$$\int \frac{\mathrm{d}x}{x\ln x + 2x} = \int \frac{1}{x}\cdot\frac{1}{\ln x + 2}\,\mathrm{d}x = \ln(\ln x + 2) + C. \tag{4}$$

Question 5

$$\begin{aligned}\int_0^{2\pi} \arccos(\sin x)\,\mathrm{d}x &= \int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)\mathrm{d}x + \int_{\pi/2}^{3\pi/2}\left(x - \frac{\pi}{2}\right)\mathrm{d}x + \int_{3\pi/2}^{2\pi}\left(\frac{5\pi}{2} - x\right)\mathrm{d}x \\ &= \frac{\pi^2}{8} + \frac{\pi^2}{2} + \frac{3\pi^2}{8} = \pi^2. \end{aligned} \tag{5}$$

Note The integral is in fact calculated from its graph. The function $y = \arccos x \in [0, \pi]$.

Question 6

$$\begin{aligned}\int \frac{\cos x+\cot x+\csc x+1}{\sin x+\tan x+\sec x+1}\,\mathrm{d}x &= \int \cot x\cdot\frac{(\sin x+1)(\cos x+1)}{(\sin x+1)(\cos x+1)}\,\mathrm{d}x\\ &= \int \cot x\,\mathrm{d}x = \ln|\sin x|+C. \end{aligned} \tag{6}$$

Question 7

$$\int \frac{x^{2024}-1}{x^{506}-1}\,\mathrm{d}x = \frac{x^{1519}}{1519}+\frac{x^{1013}}{1013}+\frac{x^{507}}{507}+x+C. \tag{7}$$

Question 8

$$\int_{-1}^{1}\left(5x^3-3x\right)^2\,\mathrm{d}x = 2\int_0^1\left(25x^6-30x^4+9x^2\right)\,\mathrm{d}x=\frac{8}{7}. \tag{8}$$

Question 9

$$\int_0^{2\pi}(\sin x+\cos x)^{11}\,\mathrm{d}x = 0. \tag{9}$$

Note This can be seen from the complex expression, where the constant term in the integrand is 0.

Question 10

$$\int_0^{2\pi}(\sinh x+\cosh x)^{11}\,\mathrm{d}x = \int_0^{2\pi}e^{11x}\,\mathrm{d}x=\frac{e^{22\pi}-1}{11}. \tag{10}$$

Question 11

$$\int \csc^2 x\tan^{2024}x\,\mathrm{d}x = \int \sec^2 x\tan^{2022}x\,\mathrm{d}x=\frac{1}{2023}\tan^{2023}x+C. \tag{11}$$

Question 12

$$\int \cos^x x \left(\ln\cos x - x\tan x\right)\, \mathrm{d}x = \cos^x x + C = e^{x\ln\cos x} + C. \tag{12}$$

Question 13

$$\int_{-\infty}^{+\infty} e^{-(x-2024)^2/4}\, \mathrm{d}x = 2\int_{-\infty}^{+\infty} e^{-t^2}\, \mathrm{d}t = 2\sqrt{\pi}. \tag{13}$$

Question 14

$$\int_{1/e}^{e} \left(1 - \frac{1}{x^2}\right) e^{e^{x+1/x}}\, \mathrm{d}x = \int_{1/e}^{e} \left(\frac{1}{t^2} - 1\right) e^{e^{t+1/t}}\, \mathrm{d}t = 0. \tag{14}$$

Note With the **change of variable** $t = 1/x$, the integral satisfies $I = -I$.

Question 15

$$\int (x + 1 - e^{-x})\, e^{xe^x}\, \mathrm{d}x = \int e^{-x}\, \mathrm{d}e^{xe^x} + \int e^{xe^x}\, \mathrm{d}e^{-x} = e^{xe^x - x} + C. \tag{15}$$

Question 16

$$\int \left(\frac{\arctan x}{1-x^2} + \frac{\operatorname{arctanh} x}{1+x^2}\right) \mathrm{d}x = \arctan x \operatorname{arctanh} x + C. \tag{16}$$

Question 17

$$\int \left[\sum_{k=0}^{+\infty} \sin\left(\frac{k\pi}{2}\right) x^k \right] \mathrm{d}x = \int \left[\sum_{k=1}^{+\infty} (-1)^{k+1} x^{2k-1} \right] \mathrm{d}x$$
$$= \frac{1}{2} \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} x^{2k} + C = \frac{1}{2} \ln\left(1 + x^2\right) + C. \tag{17}$$

Question 18

$$\int_0^1 \left(\sum_{n=0}^{2024} x^{2^{n-1012}} \right) \mathrm{d}x = \sum_{n=-1012}^{1012} \frac{1}{2^n + 1} = \frac{1}{2} + \sum_{n=1}^{1012} \left(\frac{1}{2^n + 1} + \frac{1}{2^{-n} + 1} \right) = \frac{2025}{2}. \tag{18}$$

Question 19

$$\int \frac{x^4}{3 - 6x + 6x^2 - 4x^3 + 2x^4} \mathrm{d}x = \frac{1}{2}x + \frac{1}{4} \int \frac{8x^3 - 12x^2 + 12x - 6}{3 - 6x + 6x^2 - 4x^3 + 2x^4}$$
$$= \frac{1}{2}x + \frac{1}{4} \ln\left(3 - 6x + 6x^2 - 4x^3 + 2x^4\right) + C. \tag{19}$$

Question 20

$$\int_1^3 \frac{x + \frac{x+\cdots}{1+\cdots}}{1 + \frac{x+\cdots}{1+\cdots}} \mathrm{d}x \tag{20}$$

Solution Denote the integrand as $f(x)$, we have

$$f(x) = \frac{x + f(x)}{1 + f(x)}, \qquad f(x) = \sqrt{x}, \qquad \text{for } x > 0. \tag{21}$$

Therefore, the result is obtained as

$$I = \int_1^3 \sqrt{x}\, \mathrm{d}x = 2\sqrt{3} - \frac{2}{3}. \tag{22}$$