# MIT Integration Bee: 2024 Semifinal

## Semifinal #1

### Question 1

$$\int_{-\infty}^{+\infty} \frac{\left(x^3-4x\right)\sin x + \left(3x^2-4\right)\cos x}{\left(x^3-4x\right)^2+\cos^2 x}\,\mathrm{d}x \tag{1.1}$$

**Solution** Denote the following functions

$$f(x)=\cos x, \qquad g(x)=x^3-4x, \qquad t(x)=\frac{f(x)}{g(x)}. \tag{1.2}$$

We thus have

$$I=-\int_{-\infty}^{+\infty}\frac{f'g-fg'}{f^2+g^2}\,\mathrm{d}x=-\int_{-\infty}^{+\infty}\frac{1}{1+t^2}\,\mathrm{d}t. \tag{1.3}$$

Note that there are **singularities** $x=0,\pm 2$ after the substitution. Because $\cos 2<0$, we have

$$\begin{aligned}
&t(-\infty)\to 0, \quad t(-2^-)\to+\infty, \quad t(-2^+)\to-\infty, \quad t(0^-)\to+\infty,\\
&t(0^+)\to-\infty, \quad t(2^-)\to+\infty, \quad t(2^+)\to-\infty, \quad t(\infty)\to 0.
\end{aligned}$$

Therefore, the proper way to evaluate the integral should be

$$\begin{aligned}
I&=-\left(\arctan t\Big|_{x=-\infty}^{x=-2^-}+\arctan t\Big|_{x=-2^+}^{x=0^-}+\arctan t\Big|_{x=0^+}^{x=2^-}+\arctan t\Big|_{x=2^+}^{x=+\infty}\right)\\
&=-\left[\arctan(+\infty)-0+2\arctan(+\infty)-2\arctan(-\infty)+0-\arctan(-\infty)\right]\\
&=-6\arctan(+\infty)=-3\pi.
\end{aligned} \tag{1.4}$$

## Question 2

$$\int_0^{\infty} \frac{xe^{-2x}}{e^{-x}+1}\,\mathrm{d}x \tag{2.1}$$

**Solution** With a **change of variable**, we have

$$\begin{aligned} I &= \int_0^{\infty} \frac{xe^{-2x}}{e^{-x}+1}\,\mathrm{d}x \\ &= -\int_0^{1} \frac{t\ln t}{t+1}\,\mathrm{d}t \qquad \left(t = e^{-x}, \quad x = -\ln t, \quad \mathrm{d}x = -\frac{\mathrm{d}t}{t}\right). \end{aligned} \tag{2.2}$$

Using **Taylor series**, we have

$$I = -\int_0^1 \ln t\,\mathrm{d}t + \int_0^1 \frac{\ln t}{t+1}\,\mathrm{d}t = 1 + \sum_{n=0}^{+\infty}(-1)^n \int_0^1 t^n \ln t\,\mathrm{d}t. \tag{2.3}$$

Based on the following results (see 2024 Final: Question 2)

$$\int_0^1 t^n \ln t\,\mathrm{d}t = -\frac{1}{(n+1)^2} \qquad n \in \mathbb{N}^*, \tag{2.4}$$

and the **Basel problem**, we have

$$I = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = 1 - \frac{\pi^2}{12}. \tag{2.5}$$

## Question 3

$$\int_0^{\pi/2} \sin\left(\cot^2 x\right) \sec^2 x \, \mathrm{d}x \tag{3.1}$$

**Solution** With a **change of variable**, we have

$$\begin{aligned} I &= \int_0^{\pi/2} \sin\left(\cot^2 x\right) \sec^2 x \, \mathrm{d}x \\ &= \int_0^{+\infty} \frac{\sin t^2}{t^2} \, \mathrm{d}t \qquad \left(t = \cot x, \quad \tan x = \frac{1}{t}, \quad \sec^2 x \, \mathrm{d}x = -\frac{\mathrm{d}t}{t^2}\right). \end{aligned} \tag{3.2}$$

**Integration by parts** leads to the **Fresnel integral**

$$\begin{aligned} I &= \int_0^{+\infty} \frac{\sin t^2}{t^2} \, \mathrm{d}t = -\frac{\sin t^2}{t}\bigg|_0^{+\infty} + 2\int_0^{+\infty} \cos t^2 \, \mathrm{d}t \\ &= 2\int_0^{+\infty} \cos t^2 \, \mathrm{d}t = \sqrt{\frac{\pi}{2}}. \end{aligned} \tag{3.3}$$

**Note** One particular result for the definite Fresnel integral is

$$\int_0^{+\infty} \cos\left(x^\alpha\right) \mathrm{d}x = \Gamma\left(1 + \frac{1}{a}\right) \cos\left(\frac{\pi}{2\alpha}\right), \tag{3.4}$$

$$\int_0^{+\infty} \sin\left(x^\alpha\right) \mathrm{d}x = \Gamma\left(1 + \frac{1}{a}\right) \sin\left(\frac{\pi}{2\alpha}\right), \qquad \text{for } \alpha > 1. \tag{3.5}$$

These are obtained by evaluating the integral of $e^{-z^\alpha}$ using the contour in Fig. 1.

$$e^{i\theta} \int_0^{+\infty} e^{-ir^\alpha} \, \mathrm{d}r = \int_0^{+\infty} e^{-x^\alpha} \, \mathrm{d}x = \frac{1}{\alpha} \int_0^{+\infty} t^{\frac{1}{\alpha}-1} e^{-t} \, \mathrm{d}t = \Gamma\left(1 + \frac{1}{\alpha}\right). \tag{3.6}$$

The real and imaginary parts correspond to Eqs (3.4) and (3.5), respectively.

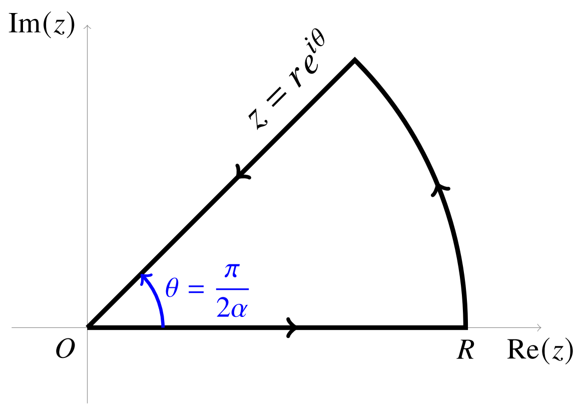


**Fig. 1** Sector-shaped contour in the upper plane to evaluate the Fresnel integral.

## Question 4

$$\int \cosh^2(3x)\tanh(2x)\,\mathrm{d}x \tag{4.1}$$

**Solution** Using the following identities for the hyperbolic functions

$$\cosh^2 x = \frac{1+\cosh(2x)}{2}, \qquad \cosh(3x) = 4\cosh^3 x - 3\cosh x \tag{4.2}$$

we have

$$\begin{aligned} I &= \frac{1}{2}\int \tanh(2x)\,\mathrm{d}x + \frac{1}{2}\int \cosh(6x)\tanh(2x)\,\mathrm{d}x \\ &= \frac{1}{4}\ln|\cosh(2x)| + \int \left[2\cosh^2(2x) - \frac{3}{2}\right]\sinh(2x)\,\mathrm{d}x \\ &= \frac{1}{4}\ln|\cosh(2x)| + \frac{1}{3}\cosh^3(2x) - \frac{3}{4}\cosh(2x) + C \\ &= \frac{1}{4}\ln|\cosh(2x)| + \frac{1}{12}\cosh(6x) - \frac{1}{2}\cosh(2x) + C. \end{aligned} \tag{4.3}$$

## Tiebreakers Question 1

$$\int \sec^5 x \, \mathrm{d}x \tag{5.1}$$

**Solution** Note that

$$(\tan x)' = \sec^2 x, \qquad (\sec x)' = \tan x \sec x, \qquad \tan^2 x = \sec^2 x - 1. \tag{5.2}$$

We can first obtain the **reduction formula** for the general integral. Since we have

$$\begin{aligned} I_n = \int \sec^n x \, \mathrm{d}x &= \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x \, \mathrm{d}x \\ &= \sec^{n-2} x \tan x - (n-2)\left(I_n - I_{n-2}\right), \end{aligned} \tag{5.3}$$

the **recurrence relation** is

$$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}, \qquad n \geq 3. \tag{5.4}$$

When $n = 1, 2$, we have

$$I_1 = \int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C, \qquad I_2 = \int \sec^2 x \, \mathrm{d}x = \tan x + C. \tag{5.5}$$

Therefore, we can obtain

$$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln|\sec x + \tan x| + C. \tag{5.6}$$

## Tiebreakers Question 2

$$\int_{-\infty}^{+\infty} \operatorname{sech}\left(2x + 1 - \frac{1}{x-1} - \frac{2}{x+1}\right) \mathrm{d}x \tag{6.1}$$

**Solution** It turns out that the integral is equivalent to the following one (but I do not know how to prove it yet)

$$I = \int_{-\infty}^{+\infty} \operatorname{sech}(2x) \, \mathrm{d}x. \tag{6.2}$$

Therefore, the result is

$$I = \int_{-\infty}^{+\infty} \operatorname{sech}(2x) \, \mathrm{d}x = \int_{-\infty}^{+\infty} \frac{2e^{2x}}{e^{4x}+1} \mathrm{d}x = (\arctan e^{x}) \Big|_{-\infty}^{+\infty} = \frac{\pi}{2}. \tag{6.3}$$

# Semifinal #2

## Question 1

$$\int_0^{+\infty} \frac{\sin(x)\sin(2x)\sin(3x)}{x^3}\,\mathrm{d}x \tag{7.1}$$

**Solution** Using the trigonometric identities, we have

$$I = \frac{1}{2}\int_0^{+\infty} \frac{(\cos x - \cos 3x)\sin 3x}{x^3}\,\mathrm{d}x = \frac{1}{4}\int_0^{+\infty} \frac{\sin 2x + \sin 4x - \sin 6x}{x^3}\,\mathrm{d}x. \tag{7.2}$$

We define the following integral

$$F(\alpha) = \int_0^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3}\,\mathrm{d}x. \tag{7.3}$$

Using **integration by parts**, we have

$$\begin{aligned} F(\alpha) &= -\frac{\alpha - \sin \alpha x}{2x^2}\bigg|_0^{+\infty} + \frac{\alpha}{2}\int_0^{+\infty} \frac{1-\cos \alpha x}{x^2}\,\mathrm{d}x \\ &= -\frac{\alpha}{2}\frac{1-\cos \alpha x}{x}\bigg|_0^{+\infty} + \frac{\alpha^2}{2}\int_0^{+\infty} \frac{\sin \alpha x}{x}\,\mathrm{d}x = \frac{\pi\alpha^2}{4}. \end{aligned} \tag{7.4}$$

Finally, we obtain

$$I = \frac{1}{4}\left[-F(2) - F(4) + F(6)\right] = \pi. \tag{7.5}$$

## Question 2

$$\int (1 + \ln x)(1 + \ln \ln x)\, \mathrm{d}x \tag{8.1}$$

**Solution** Note that

$$(x \ln x)' = 1 + \ln x, \qquad (1 + \ln \ln x)' = \frac{1}{x \ln x}. \tag{8.2}$$

We have

$$\begin{aligned} I &= \int (1 + \ln \ln x)\, \mathrm{d}(x \ln x) \\ &= x \ln x\, (1 + \ln \ln x) - \int 1\, \mathrm{d}x \\ &= -x + x \ln x + (x \ln x) \ln \ln x + C. \end{aligned} \tag{8.3}$$

## Question 3

$$\int_0^{+\infty} \frac{e^{-x^2}}{\left(x^2+\frac{1}{2}\right)^2}\,\mathrm{d}x \tag{9.1}$$

**Solution** It turns out that the integral should be directly related to the Gaussian integral (but I do not know how to prove it yet)

$$I = 2\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}. \tag{9.2}$$

## Question 4

$$\int \tan x \ \sec^2 x \ \cos(2x)\, e^{2\cos x}\, \mathrm{d}x \tag{10.1}$$

**Solution** With a **change of variable** $t = \cos x$, we have

$$I = -\int \frac{2t^2 - 1}{t^3} e^{2t}\, \mathrm{d}t \tag{10.2}$$

We can also obtain the **reduction formula** for the following integral

$$I_n = \int \frac{e^{2t}}{t^n}\, \mathrm{d}t = -\frac{1}{n-1} \frac{e^{2t}}{t^{n-1}} + \frac{2}{n-1} I_{n-1}. \tag{10.3}$$

Finally, we have

$$\begin{aligned} I = I_3 - 2I_1 &= -\frac{1}{2}\frac{e^{2t}}{t^2} + I_2 - 2I_1 \\ &= -\frac{1}{2}\frac{e^{2t}}{t^2} - \frac{e^{2t}}{t} + 2I_1 - 2I_1 = -\left(\frac{1}{2t^2} + \frac{1}{t}\right) e^{2t} + C \\ &= -\left(\frac{\sec^2 x}{2} + \sec x\right) e^{2\cos x} + C. \end{aligned} \tag{10.4}$$