MIT Integration Bee: 2023 Semifinal

Semifinal #1

Question 1

$$\int e^{\cos x} \cos(2x + \sin x)\, \mathrm{d}x \tag{1.1}$$

Solution

$$\begin{aligned}
I &= \int e^{\cos x} \cos x \cos(x + \sin x)\, \mathrm{d}x + \int \sin(x + \sin x)\, \mathrm{d}\left(e^{\cos x}\right) \\
&= e^{\cos x} \sin(x + \sin x) - \int e^{\cos x} \cos(x + \sin x)\, \mathrm{d}x \\
&= e^{\cos x} \sin(x + \sin x) - \int e^{\cos x} \cos x \cos(\sin x)\, \mathrm{d}x - \int \sin(\sin x)\, \mathrm{d}\left(e^{\cos x}\right) \\
&= e^{\cos x} \sin(x + \sin x) - e^{\cos x} \sin(\sin x) + C.
\end{aligned} \tag{1.2}$$

Question 2

$$\int_0^1 \left(9x^9 - x^{90} + 9x^{99} - x^{900} + 9x^{909} - x^{990} + 9x^{999} - x^{9000} + \cdots\right) \mathrm{d}x \tag{2.1}$$

Solution The contribution from the dominant terms in the integrand is

$$I_1 = \int_0^1 \left(9x^9 + 9x^{99} + 9x^{999} + 9x^{9999} + \cdots\right) \mathrm{d}x = \sum_{k=1}^{\infty} \frac{9}{10^k} = 1. \tag{2.2}$$

The remaining part becomes

$$\begin{aligned} I_2 = \int_0^1 (&- x^{90} - x^{900} + 9x^{909} - x^{990} \\ &- x^{9000} + 9x^{9009} - x^{9090} + 9x^{9099} - x^{9900} + 9x^{9909} - x^{9990} - \cdots) \,\mathrm{d}x. \end{aligned} \tag{2.3}$$

This can be considered an alternating series that eventually sums to zero. Therefore, we can say

$$I = I_1 + I_2 = 1. \tag{2.4}$$

Question 3

$$\int_0^{\pi} \frac{2\cos x - \cos(2021x) - 2\cos(2022x) - \cos(2023x) + 2}{1 - \cos 2x}\, \mathrm{d}x \tag{3.1}$$

Solution Using the trigonometric identity, we have

$$\cos(2021x) + \cos(2023x) = 2\cos(2022x)\cos x. \tag{3.2}$$

Therefore, the integral becomes

$$I = \int_0^{\pi} \frac{2(1+\cos x)\left[1 - \cos(2022x)\right]}{2\left(1-\cos^2 x\right)}\, \mathrm{d}x = \int_0^{\pi} \frac{1-\cos(2022x)}{1-\cos x}\, \mathrm{d}x. \tag{3.3}$$

Denote the general result as

$$I_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\, \mathrm{d}x, \qquad n \in \mathbb{N}. \tag{3.4}$$

Obviously we have $I_0 = 0$ and $I_1 = \pi$. The **reduction formula** can be obtained as

$$\begin{aligned} I_{n+1} &= \int_0^{\pi} \frac{1-\cos(n+1)x}{1-\cos x}\, \mathrm{d}x = \int_0^{\pi} \frac{1+\cos(n-1)x - 2\cos nx \cos x}{1-\cos x}\, \mathrm{d}x \\ &= -I_{n-1} + \int_0^{\pi} \frac{2 - 2\cos nx + 2\cos nx - 2\cos nx\cos x}{1-\cos x}\, \mathrm{d}x \\ &= 2I_n - I_{n-1} + \int_0^{\pi} 2\cos nx\, \mathrm{d}x = 2I_n - I_{n-1} \qquad \text{for } n \geq 1. \end{aligned} \tag{3.5}$$

This implies that $\{I_n\}$ is an arithmetic sequence, and thus $I_n = n\pi$. Finally, we have

$$I = I_{2022} = 2022\,\pi. \tag{3.6}$$

Question 4

$$\int \frac{3\ln x - 1 + 2x}{x\ln x + x^2 + 2x^4}\,\mathrm{d}x \tag{4.1}$$

Solution Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\ln x + x + 2x^3\right) = \frac{1 + x + 6x^3}{x\left(\ln x + x + 2x^3\right)}, \qquad \frac{1}{x} = \frac{\ln x + x + 2x^3}{x\left(\ln x + x + 2x^3\right)}. \tag{4.2}$$

The integrand can be written as

$$\frac{3\ln x - 1 + 2x}{x\ln x + x^2 + 2x^4} = \frac{3}{x} - \frac{1 + x + 6x^3}{x\left(\ln x + x + 2x^3\right)}. \tag{4.3}$$

Therefore, the integral is calculated as

$$I = 3\ln x - \ln\left(\ln x + x + 2x^3\right) + C. \tag{4.4}$$

Tiebreakers Question 1

$$\int_0^{\ln 2} \left\{ \frac{1}{e^x - 1} \right\} \mathrm{d}x \tag{5.1}$$

Solution With the **change of variable**, we have

$$t = \frac{1}{e^x - 1}, \qquad x = \ln\left(\frac{t+1}{t}\right), \qquad \mathrm{d}x = -\frac{\mathrm{d}t}{t(t+1)}. \tag{5.2}$$

Therefore, the integral becomes

$$I = \int_1^{+\infty} \frac{\{t\}\,\mathrm{d}t}{t(t+1)} = \sum_{n=1}^{+\infty} \int_0^1 \frac{u\,\mathrm{d}u}{(u+n)(u+n+1)}. \tag{5.3}$$

Each single term can be evaluated as

$$\begin{aligned}\int_0^1 \frac{u\,\mathrm{d}u}{(u+n)(u+n+1)} &= \int_0^1 \left(\frac{n+1}{u+n+1} - \frac{n}{u+n}\right) \mathrm{d}u \\ &= (n+1)\left[\ln(n+2) - \ln(n+1)\right] - n\left[\ln(n+1) - \ln n\right] \\ &= (n+1)\ln(n+2) - n\ln(n+1) + n\ln n - (n+1)\ln(n+1).\end{aligned} \tag{5.4}$$

Now denote the following two series

$$a_n = n\ln(n+1), \qquad b_n = n\ln n, \qquad n = 1, 2, 3, \cdots. \tag{5.5}$$

The integral can thus be calculated as

$$\begin{aligned}I &= \sum_{n=1}^{+\infty} (a_{n+1} - a_n + b_n - b_{n+1}) = b_1 - a_1 + \lim_{n\to\infty}(a_n - b_n) \\ &= -\ln 2 + \lim_{n\to\infty} \ln\left(1 + \frac{1}{n}\right)^n = 1 - \ln 2.\end{aligned} \tag{5.6}$$

Tiebreakers Question 2

$$\int_1^9 \left(\sqrt[3]{x+\sqrt{x^2-1}}+\sqrt[3]{x-\sqrt{x^2-1}}\right) \mathrm{d}x \tag{6.1}$$

Solution Similar approach to 2024 Final: Question 4. Denote

$$a(x)=\sqrt[3]{x+\sqrt{x^2-1}}, \qquad b(x)=\sqrt[3]{x-\sqrt{x^2-1}}, \qquad y(x)=a(x)+b(x). \tag{6.2}$$

Now we can show that

$$a^3+b^3=2x, \qquad ab=1 \qquad \Longrightarrow \qquad y^3=a^3+b^3+3ab(a+b)=2x+3y. \tag{6.3}$$

Using **integration by parts**, we obtain

$$I=\int_1^9 y(x)\,\mathrm{d}x = xy\,\Big|_{x=1}^{x=9} - \int_{y(1)}^{y(9)} x(y)\,\mathrm{d}y. \tag{6.4}$$

Since we have

$$y^3=2x+3y \qquad \Longrightarrow \qquad y(1)=2, \quad y(9)=3, \quad x(y)=\frac{1}{2}y^3-\frac{3}{2}y, \tag{6.5}$$

we finally obtain

$$I=25-\frac{1}{2}\int_2^3 \left(y^3-3y\right)\mathrm{d}y = 25-\frac{35}{8}=\frac{165}{8}. \tag{6.6}$$

Semifinal #2

Question 1

$$\int \left(\sqrt{x+1}-\sqrt{x}\right)^{\pi}\,\mathrm{d}x \tag{7.1}$$

Solution Denote $y=\sqrt{x+1}-\sqrt{x}$, and then we can show that

$$y^2-y^{-2}=\left(\sqrt{x+1}-\sqrt{x}\right)^2-\left(\sqrt{x+1}+\sqrt{x}\right)^2=-4\sqrt{x}\,\sqrt{x+1}. \tag{7.2}$$

The derivative y' can thus be expressed as

$$y'=\frac{\sqrt{x}-\sqrt{x+1}}{2\sqrt{x}\,\sqrt{x+1}}=\frac{2y}{y^2-y^{-2}}. \tag{7.3}$$

Hence, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^{n+1}}{n+1}\right)=y^n\,y'=\frac{2y^{n+1}}{y^2-y^{-2}}. \tag{7.4}$$

Based on Eq. (7.4), when $n\neq\pm2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y^{n+2}}{n+2}-\frac{y^{n-2}}{n-2}\right)=\frac{2\left(y^{n+2}-y^{n-2}\right)}{y^2-y^{-2}}=2y^n. \tag{7.5}$$

Now, the original integral is calculated as

$$I=\frac{1}{2}\left[\frac{(\sqrt{x+1}-\sqrt{x})^{\pi+2}}{\pi+2}-\frac{(\sqrt{x+1}-\sqrt{x})^{\pi-2}}{\pi-2}\right]+C. \tag{7.6}$$

Question 2

$$\int_{-2}^{2} ((((x^2-2)^2-2)^2-2)^2-2)\,\mathrm{d}x \tag{8.1}$$

Solution With the **change of variable** $x = 2\cos\theta$ with $\theta \in [0, \pi]$, we have

$$x^2 - 2 = 4\cos^2\theta - 2 = 2\cos 2\theta, \qquad \mathrm{d}x = -2\sin\theta\,\mathrm{d}\theta. \tag{8.2}$$

Then we know that

$$f(x) = x^2 - 2 = 2\cos 2\theta, \quad f^2(x) = (f \circ f)(x) = 2\cos 4\theta, \quad f^4(x) = 2\cos 16\theta. \tag{8.3}$$

Therefore, the integral is evaluated as

$$I = \int_0^{\pi} 4\cos 16\theta \sin\theta\,\mathrm{d}\theta = 2\int_0^{\pi} (\sin 17\theta - \sin 15\theta)\ \mathrm{d}\theta = \frac{4}{17} - \frac{4}{15} = -\frac{8}{255}. \tag{8.4}$$

Question 3

$$\int_0^{+\infty} \frac{\tanh x}{x \cosh 2x} \, \mathrm{d}x \tag{9.1}$$

Solution The integrand can be written as

$$\begin{aligned} \frac{\tanh x}{x \cosh 2x} &= \frac{1}{x} \cdot \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{2}{e^{2x} + e^{-2x}} \\ &= \frac{2}{x} \cdot \frac{e^{2x}\left(e^{2x} - 1\right)}{\left(e^{2x} + 1\right)\left(e^{4x} + 1\right)} \\ &= \frac{2}{x} \left(\frac{1}{e^{2x} + 1} - \frac{1}{e^{4x} + 1} \right). \end{aligned} \tag{9.2}$$

For the **Frullani integrals**, we have the following property

$$\int_0^{+\infty} \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x = [f(\infty) - f(0)] \ln \frac{a}{b}. \tag{9.3}$$

The integral can now be written in the form of

$$f(x) = \frac{2}{e^x + 1}, \qquad I = \int_0^{+\infty} \frac{f(2x) - f(4x)}{x} \, \mathrm{d}x. \tag{9.4}$$

Finally, we can evaluate it as

$$I = [f(\infty) - f(0)] \ln \frac{1}{2} = \ln 2. \tag{9.5}$$

Question 4

$$\int \sin\left(4\arctan x\right) \mathrm{d}x \tag{10.1}$$

Solution With the **change of variable** $t = \arctan x$, we have

$$\tan t = x, \qquad \sin t = \frac{x}{\sqrt{x^2+1}}, \qquad \cos t = \frac{1}{\sqrt{x^2+1}}, \qquad \mathrm{d}x = \frac{\mathrm{d}t}{\cos^2 t}. \tag{10.2}$$

The integral can thus be computed as

$$\begin{aligned} I &= \int \frac{\sin 4t}{\cos^2 t}\,\mathrm{d}t = \int \frac{2\sin 2t\left(2\cos^2 t - 1\right)}{\cos^2 t}\,\mathrm{d}t \\ &= \int \left(4\sin 2t - 4\tan t\right)\,\mathrm{d}t \\ &= -2\cos 2t - 4\ln|\sec t| + C. \end{aligned} \tag{10.3}$$

We also know that

$$\cos 2t = \frac{1-x^2}{x^2+1} = -1 + \frac{2}{x^2+1}, \qquad \sec t = \sqrt{x^2+1}. \tag{10.4}$$

Therefore, the final result becomes

$$I = -\frac{4}{x^2+1} - 2\ln\left(x^2+1\right) + C. \tag{10.5}$$