MIT Integration Bee: 2024 Final

Question 1

$$\int \frac{e^{x/2}\cos x}{\sqrt[3]{3\cos x + 4\sin x}}\, dx \tag{1.1}$$

Solution   Denote the following integrals

$$I = \int \frac{e^{x/2}\cos x}{\sqrt[3]{3\cos x + 4\sin x}}\, dx, \qquad J = \int \frac{e^{x/2}\sin x}{\sqrt[3]{3\cos x + 4\sin x}}\, dx. \tag{1.2}$$

Now consider the following function $f(x)$ and its derivative

$$f(x) = (3\cos x + 4\sin x)^{2/3} = \frac{3\cos x + 4\sin x}{\sqrt[3]{3\cos x + 4\sin x}}, \qquad f'(x) = \frac{2}{3}\frac{4\cos x - 3\sin x}{\sqrt[3]{3\cos x + 4\sin x}}. \tag{1.3}$$

We can obtain

$$\int f(x)\, de^{x/2} = \int \frac{1}{2}e^{x/2}f(x)\, dx = \frac{3}{2}I + 2J, \tag{1.4}$$

$$\int e^{x/2}\, df(x) = \int e^{x/2}f'(x)\, dx = \frac{8}{3}I - 2J. \tag{1.5}$$

Therefore, based on the **product rule** we have

$$\begin{aligned}
I &= \frac{6}{25}\left[\int f(x)\, de^{x/2} + \int e^{x/2}\, df(x)\right] \\
&= \frac{6}{25}\, e^{x/2}\, (3\cos x + 4\sin x)^{2/3} + C.
\end{aligned} \tag{1.6}$$

Question 2

$$\int_0^\infty \frac{\ln\left(2e^x - 1\right)}{e^x - 1}\, dx \tag{2.1}$$

Solution   With several **changes of variables**, we have

$$
\begin{aligned}
I &= \int_0^\infty \frac{\ln\left(2e^x - 1\right)}{e^x - 1}\, dx \\
&= \int_1^{+\infty} \frac{2 \ln t}{t^2 - 1}\, dt \qquad \left(t = 2e^x - 1, \quad x = \ln\left(\frac{t+1}{2}\right), \quad dx = \frac{dt}{t+1}\right) \\
&= \int_0^1 \frac{2 \ln t}{t^2 - 1}\, dt \qquad \left(t \to \frac{1}{t}, \quad dt \to -\frac{dt}{t^2}\right).
\end{aligned}
\tag{2.2}
$$

Using **Taylor series**, we have

$$I = -2 \int_0^1 \ln t \sum_{n=0}^{+\infty} t^{2n}\, dt = -2 \sum_{n=0}^{+\infty} \int_0^1 t^{2n} \ln t\, dt. \tag{2.3}$$

As we have the following result

$$\int_0^1 t^n \ln t\, dt = \frac{t^{n+1}}{n+1} \ln t \Big|_0^1 - \int_0^1 \frac{t^n}{n+1}\, dt = -\frac{1}{(n+1)^2} \qquad n \in \mathbb{N}^*, \tag{2.4}$$

we finally obtain

$$I = 2 \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = 2 \cdot \frac{\pi^2}{8} = \frac{\pi^2}{4}. \tag{2.5}$$

Note   The **Basel problem** directly gives the result of the infinite sum in Eq. (2.5) as follows

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{24}, \qquad \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2} - \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8}. \tag{2.6}$$

Question 3

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^4 + x^3 + x^2 + x + 1} \tag{3.1}$$

Solution   Directly using the **residue theorem** on the contour in Fig. 1, we have

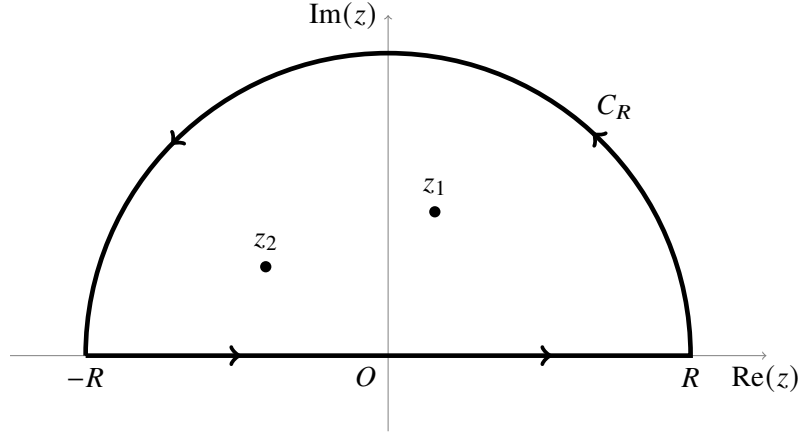**Fig. 1**   Semi-circle contour in the upper plane to evaluate the integral over the real line. z_1 and z_2 are simple poles.

$$I = \int_{-\infty}^{+\infty} \frac{x - 1}{x^5 - 1} \, \mathrm{d}x = 2\pi i \left[\mathrm{Res}\,(f, z_1) + \mathrm{Res}\,(f, z_2) \right] \tag{3.2}$$

where

$$f(z) = \frac{z - 1}{z^5 - 1}, \qquad z_1 = \exp\left(\frac{2\pi i}{5}\right), \qquad z_2 = \exp\left(\frac{4\pi i}{5}\right) = z_1^2. \tag{3.3}$$

The residues are calculated as

$$\mathrm{Res}\,(f, z_1) = \lim_{z \to z_1} \frac{(z - z_1)(z - 1)}{z^5 - 1} = \frac{z_1 - 1}{5 z_1^4} = \frac{z_1 - 1}{5 \bar{z}_1} = \frac{z_1^2 - z_1}{5} = \frac{z_2 - z_1}{5}, \tag{3.4}$$

$$\mathrm{Res}\,(f, z_2) = \lim_{z \to z_2} \frac{(z - z_2)(z - 1)}{z^5 - 1} = \frac{z_2 - 1}{5 z_2^4} = \frac{z_2 - 1}{5 \bar{z}_2} = \frac{z_2^2 - z_2}{5} = \frac{\bar{z}_1 - z_2}{5}. \tag{3.5}$$

Finally, we obtain

$$I = \frac{2\pi i}{5} (\bar{z}_1 - z_1) = \frac{4\pi}{5} \sin\left(\frac{2\pi}{5}\right) = \frac{4\pi}{5} \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \frac{\sqrt{10 + 2\sqrt{5}}}{5} \pi. \tag{3.6}$$

Question 4

$$\int_{-1/3}^{1} \left(\sqrt[3]{1 + \sqrt{1 - x^3}} + \sqrt[3]{1 - \sqrt{1 - x^3}} \right) dx \qquad (4.1)$$

Solution   Denote

$$a(x) = \sqrt[3]{1 + \sqrt{1 - x^3}}, \qquad b(x) = \sqrt[3]{1 - \sqrt{1 - x^3}}, \qquad y(x) = a(x) + b(x). \qquad (4.2)$$

We notice that

$$a^3 + b^3 = 2, \quad ab = x \qquad \Longrightarrow \qquad y^3 = a^3 + b^3 + 3ab(a + b) = 2 + 3xy \qquad (4.3)$$

Therefore, using **integration by parts**, we have

$$I = \int_{-1/3}^{1} y(x)\, dx = xy \Big|_{x=-1/3}^{x=1} - \int_{y(-1/3)}^{y(1)} x(y)\, dy. \qquad (4.4)$$

Based on the following results

$$y^3 - 3xy - 2 = 0 \qquad \Longrightarrow \qquad y(1) = 2, \quad y\left(-\frac{1}{3}\right) = 1, \quad x(y) = \frac{y^3 - 2}{3y}, \qquad (4.5)$$

we finally obtain

$$I = \frac{7}{3} - \int_{1}^{2} \frac{y^3 - 2}{3y}\, dy = \frac{14}{9} + \frac{2}{3} \ln 2. \qquad (4.6)$$

Question 5

$$\int_0^1 \max_{n \in \mathbb{N}} \left[\frac{1}{2^n} \left(\lfloor 2^n x \rfloor - \left\lfloor 2^n x - \frac{1}{4} \right\rfloor \right) \right] \, dx \tag{5.1}$$

Solution   For a real number $x \in (0, 1)$, its **binary representation** is

$$x = \sum_{k=1}^{+\infty} \frac{a_k}{2^k} = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_k}{2^k} + \cdots, \qquad \text{with } a_k \in \{0, 1\}. \tag{5.2}$$

We need to evaluate the difference between the integer parts of the following two expressions

$$2^n x = 2^{n-1} a_1 + \cdots + a_n + \frac{a_{n+1}}{2} + \frac{a_{n+2}}{2^2} + \frac{a_{n+3}}{2^3} + \cdots \tag{5.3}$$

$$2^n x - \frac{1}{4} = 2^{n-1} a_1 + \cdots + a_n + \frac{a_{n+1}}{2} + \frac{a_{n+2} - 1}{2^2} + \frac{a_{n+3}}{2^3} + \cdots \tag{5.4}$$

The difference between the integer parts is either 0 or 1. The condition is analyzed as

$$\lfloor 2^n x \rfloor - \left\lfloor 2^n x - \frac{1}{4} \right\rfloor = 1 \qquad \Longleftrightarrow \qquad \frac{a_{n+1}}{2} + \frac{a_{n+2} - 1}{2^2} + \frac{a_{n+3}}{2^3} + \cdots < 0. \tag{5.5}$$

Therefore, we conclude

$$\lfloor 2^n x \rfloor - \left\lfloor 2^n x - \frac{1}{4} \right\rfloor = \begin{cases} 1, & a_{n+1} = a_{n+2} = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}$$

Taking the maximum over $n \in \mathbb{N}$ simply picks out the earliest index n that satisfies $a_{n+1} = a_{n+2} = 0$. Define the following probability

$$P(X_n) = \text{Prob. of first having } a_{n+1} = a_{n+2} = 0 \text{ in an infinite binary string.}$$

We can write down

$$P(X_0) = \frac{1}{4}, \quad P(X_1) = \frac{1}{8}. \tag{5.7}$$

Note that event X_n indicates that $a_n = 1$. The recurrence relation can be obtained as

$$P(X_{n+2}) = \frac{1}{4} P(X_n) + \frac{1}{2} P(X_{n+1}). \tag{5.8}$$

The integral is now equivalent to an **expectation**

$$I = \sum_{n=0}^{+\infty} \frac{P(X_n)}{2^n} = P(X_0) + \frac{1}{2} P(X_1) + \sum_{n=0}^{+\infty} \frac{1}{2^{n+2}} \cdot \frac{1}{4} P(X_n) + \sum_{n=1}^{+\infty} \frac{1}{2^{n+1}} \cdot \frac{1}{2} P(X_n)$$
$$= \frac{1}{4} + \frac{1}{16} + \frac{I}{16} + \frac{1}{4} \left(I - \frac{1}{4} \right). \tag{5.9}$$

Finally, the result is

$$I = \frac{1}{4} + \frac{5}{16} I, \qquad I = \frac{4}{11}. \tag{5.10}$$

Tiebreakers Question 1

$$\int \frac{\mathrm{d}x}{\sqrt[4]{x^4 + 1}} \tag{6.1}$$

Solution   With several **changes of variables**, we have

$$
\begin{aligned}
I &= \int \frac{\mathrm{d}x}{\sqrt[4]{x^4 + 1}} \\
&= -\int \frac{1}{y}\frac{\mathrm{d}y}{\sqrt[4]{y^4 + 1}} && \left(y = \frac{1}{x}, \quad \mathrm{d}x = -\frac{\mathrm{d}y}{y^2}\right) \\
&= -\int \frac{t^2}{t^4 - 1}\,\mathrm{d}t && \left(t = \sqrt[4]{y^4 + 1}, \quad y = \sqrt[4]{t^4 - 1}, \quad \mathrm{d}y = t^3\left(t^4 - 1\right)^{-3/4}\mathrm{d}t\right).
\end{aligned}
\tag{6.2}
$$

This leads to

$$
\begin{aligned}
I &= \frac{1}{2}\int \frac{\mathrm{d}t}{1 - t^2} - \frac{1}{2}\int \frac{\mathrm{d}t}{1 + t^2} \\
&= \frac{1}{4}\ln\left|\frac{1 + t}{1 - t}\right| - \frac{1}{2}\arctan t + C = \frac{1}{2}\operatorname{arctanh} t - \frac{1}{2}\arctan t + C.
\end{aligned}
\tag{6.3}
$$

Going back to the original variable, we have

$$
t = \frac{\sqrt[4]{x^4 + 1}}{x}, \qquad I = -\frac{1}{2}\arctan\left(\frac{\sqrt[4]{x^4 + 1}}{x}\right) + \frac{1}{4}\ln\left|\frac{\sqrt[4]{x^4 + 1} + x}{\sqrt[4]{x^4 + 1} - x}\right| + C. \tag{6.4}
$$

Note   Using the identity $\arctan x + \arctan x^{-1} = \pi/2$, we can also write

$$
I = \frac{1}{2}\arctan\left(\frac{x}{\sqrt[4]{x^4 + 1}}\right) + \frac{1}{4}\ln\left|\frac{\sqrt[4]{x^4 + 1} + x}{\sqrt[4]{x^4 + 1} - x}\right| + C \tag{6.5}
$$

Tiebreakers Question 2

$$\int_0^{2\pi} \frac{(\sin 2x - 5\sin x)\sin x}{\cos 2x - 10\cos x + 13}\, dx \tag{7.1}$$

Solution   Note that

$$\begin{aligned}
I &= \int_0^{2\pi} \frac{(\sin 2x - 5\sin x)\sin x}{\cos 2x - 10\cos x + 13}\, dx = \int_0^{2\pi} \frac{(2\cos x - 5)\left(1 - \cos^2 x\right)}{2\left(\cos x - 2\right)\left(\cos x - 3\right)}\, dx \\
&= -\int_0^{2\pi} \left(\cos x + \frac{5}{2} + \frac{3}{2}\frac{1}{\cos x - 2} + \frac{4}{\cos x - 3}\right)dx \\
&= -5\pi - \frac{3}{2}\int_0^{2\pi} \frac{dx}{\cos x - 2} - 4\int_0^{2\pi} \frac{dx}{\cos x - 3}.
\end{aligned} \tag{7.2}$$

For the integral of **rational functions of cosine and sine**, we have the general result

$$\int_0^{2\pi} R\left(\cos\theta, \sin\theta\right) d\theta = \oint_{|z|=1} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}. \tag{7.3}$$

Therefore, we have

$$F(a) = \int_0^{2\pi} \frac{dx}{\cos x + a} = -2i \oint_{|z|=1} \frac{dz}{z^2 + 2az + 1}. \tag{7.4}$$

Using the **residue theorem**, we have

$$F(-2) = -\frac{2\pi}{\sqrt{3}}, \qquad F(-3) = -\frac{\pi}{\sqrt{2}}. \tag{7.5}$$

Finally, we obtain

$$I = -5\pi - \frac{3}{2}F(-2) - 4F(-3) = \left(\sqrt{3} + 2\sqrt{2} - 5\right)\pi. \tag{7.6}$$

Note   The poles of function $f(z) = \left(z^2 + 2az + 1\right)^{-1}$ are

$$z_1 = -a - \sqrt{a^2 - 1}, \qquad z_2 = -a + \sqrt{a^2 - 1}. \tag{7.7}$$

Therefore, we need to discuss different cases to apply the residue theorem.

(a) When $a > 1$, we have $|z_1| > 1$ and $|z_2| < 1$.

$$F(a) = 4\pi \operatorname{Res}(f, z_2) = \frac{4\pi}{z_2 - z_1} = \frac{2\pi}{\sqrt{a^2 - 1}} \qquad \text{when } a > 1. \tag{7.8}$$

(b) When $a < -1$, we have $|z_1| < 1$ and $|z_2| > 1$.

$$F(a) = 4\pi \operatorname{Res}(f, z_1) = \frac{4\pi}{z_1 - z_2} = -\frac{2\pi}{\sqrt{a^2 - 1}} \qquad \text{when } a < -1. \tag{7.9}$$

(c) When $-1 \le a \le 1$, the integral does not converge.

Tiebreakers Question 3

$$\int \sqrt{x^4 - 4x + 3}\,\mathrm{d}x \tag{8.1}$$

Solution   We first have

$$
\begin{aligned}
I &= \int (x-1)\sqrt{x^2 + 2x + 3}\,\mathrm{d}x \\
&= \frac{1}{2}\int \sqrt{x^2 + 2x + 3}\,\mathrm{d}\left(x^2 + 2x + 3\right) - 2\int \sqrt{(x+1)^2 + 2}\,\mathrm{d}x.
\end{aligned}
\tag{8.2}
$$

Note that

$$\int \sqrt{x^2 + a^2}\,\mathrm{d}x = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + C. \tag{8.3}$$

Therefore, we obtain

$$I = \frac{1}{3}\left(x^2 + 2x + 3\right)^{\frac{3}{2}} - (x+1)\sqrt{x^2 + 2x + 3} - 2\ln\left|x + 1 + \sqrt{x^2 + 2x + 3}\right| + C. \tag{8.4}$$

Tiebreakers Question 4

$$\int_{-\infty}^{+\infty} \sin^2\left(2^x\right) \cos^2\left(3^x\right) \left[4\cos^2\left(2^x\right)\left(4\cos^2\left(3^x\right) - 3\right)^2 - 1\right] dx \tag{9.1}$$

Solution   Using the **triple-angle formula** $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, we have

$$
\begin{aligned}
I &= \int_{-\infty}^{+\infty} \left[4\sin^2\left(2^x\right)\cos^2\left(2^x\right)\cos^2\left(3^{x+1}\right) - \sin^2\left(2^x\right)\cos^2\left(3^x\right)\right] dx \\
&= \int_{-\infty}^{+\infty} \left[\sin^2\left(2^{x+1}\right)\cos^2\left(3^{x+1}\right) - \sin^2\left(2^x\right)\cos^2\left(3^x\right)\right] dx.
\end{aligned}
\tag{9.2}
$$

Now denote

$$
\begin{aligned}
I_N &= \int_{-N}^{N} \left[\sin^2\left(2^{x+1}\right)\cos^2\left(3^{x+1}\right) - \sin^2\left(2^x\right)\cos^2\left(3^x\right)\right] dx \\
&= \int_{N}^{N+1} \sin^2\left(2^x\right)\cos^2\left(3^x\right) dx - \int_{-N}^{-N+1} \sin^2\left(2^x\right)\cos^2\left(3^x\right) dx.
\end{aligned}
\tag{9.3}
$$

The second part goes to 0 as $N \to +\infty$ according to the **mean value theorem**. For the first part, we can evaluate it as

$$
\begin{aligned}
A_N &= \int_{N}^{N+1} \sin^2\left(2^x\right)\cos^2\left(3^x\right) dx \\
&= \frac{1}{4} + \frac{1}{4}\int_{N}^{N+1} \left[\cos\left(2\cdot 3^x\right) - \cos\left(2^{x+1}\right) - \cos\left(2\cdot 3^x\right)\cos\left(2^{x+1}\right)\right] dx.
\end{aligned}
\tag{9.4}
$$

Therefore, we conclude

$$I = \lim_{N\to+\infty} I_N = \lim_{N\to+\infty} A_N = \frac{1}{4}. \tag{9.5}$$

Tiebreakers Question 5

$$\int_2^\infty \frac{\lfloor x \rfloor x^2}{x^6 - 1}\,dx \tag{10.1}$$

Solution   Note that

$$
\begin{aligned}
I &= \int_2^\infty \frac{\lfloor x \rfloor x^2}{x^6 - 1}\,dx = \sum_{n=2}^{+\infty} \int_n^{n+1} \frac{n x^2}{x^6 - 1}\,dx \\
&= \sum_{n=2}^{+\infty} \frac{n}{3} \int_n^{n+1} \frac{dx^3}{x^6 - 1} = \sum_{n=2}^{+\infty} \frac{n}{6} \ln\left(\frac{t^3 - 1}{t^3 + 1}\right)\Big|_n^{n+1}.
\end{aligned}
\tag{10.2}
$$

Denote the following functions $g(n)$ and $h(n)$ for convenience

$$g(n) = \ln\left(\frac{n-1}{n+1}\right), \qquad h(n) = \ln\left(\frac{n^2 + n + 1}{n^2 - n + 1}\right) = \ln\left[\frac{n(n+1)+1}{(n-1)n+1}\right]. \tag{10.3}$$

Therefore, we have

$$f(n) = g(n) + h(n) = \ln\left(\frac{n^3 - 1}{n^3 + 1}\right) \tag{10.4}$$

and the integral becomes

$$
\begin{aligned}
I &= \frac{1}{6} \sum_{n=2}^{+\infty} [n f(n+1) - n f(n)] \\
&= \frac{1}{6}\left[\lim_{n\to+\infty} n f(n+1) - f(2) - \sum_{n=2}^{+\infty} f(n) \right].
\end{aligned}
\tag{10.5}
$$

We can first calculate the limit as

$$\lim_{n\to+\infty} n f(n+1) = \lim_{n\to+\infty} \frac{f(n)}{n^{-1}} = \lim_{n\to+\infty} -n^2 f'(n) = \lim_{n\to+\infty} \frac{6n^4}{n^6 - 1} = 0. \tag{10.6}$$

Furthermore, the infinite sum is

$$\sum_{n=2}^{+\infty} f(n) = \sum_{n=2}^{+\infty} [g(n) + h(n)] = \ln 2 - \ln 3. \tag{10.7}$$

Finally, we have

$$I = -\frac{1}{6}[f(2) + \ln 2 - \ln 3] = \frac{1}{6} \ln\left(\frac{27}{14}\right). \tag{10.8}$$

Lightning Question 1

$$\int_0^1 \left[\left(1 - x^{\frac{3}{2}}\right)^{\frac{3}{2}} - \left(1 - x^{\frac{2}{3}}\right)^{\frac{2}{3}} \right] \, dx \qquad (11.1)$$

Solution 1   Using the **Beta function**, we have

$$
\begin{aligned}
F(\alpha) &= \int_0^1 (1 - x^\alpha)^\alpha \, dx \\
&= \frac{1}{\alpha} \int_0^1 t^{1/\alpha - 1} (1 - t)^\alpha \, dt \qquad \left(t = x^\alpha, \quad dx = \frac{t^{1/\alpha - 1}}{\alpha} \, dt \right) \\
&= \frac{1}{\alpha} B\left(\alpha^{-1}, \alpha + 1\right) = \frac{\Gamma(\alpha) \Gamma\left(\alpha^{-1}\right)}{\Gamma\left(\alpha + \alpha^{-1} + 1\right)}.
\end{aligned}
\qquad (11.2)
$$

The property $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$ is applied. Therefore, we obtain a more general result

$$I = F(\alpha) - F(\alpha^{-1}) = 0 \qquad \text{for } \alpha > 0. \qquad (11.3)$$

Solution 2   Note that the two functions are inverses of each other

$$y = f(x) = (1 - x^\alpha)^\alpha, \qquad x = f^{-1}(y) = \left(1 - y^{\frac{1}{\alpha}}\right)^{\frac{1}{\alpha}}. \qquad (11.4)$$

Therefore, using **integration by parts**, because $y(0) = 1$ and $y(1) = 0$, we have

$$F(\alpha) = \int_0^1 y \, dx = xy \Big|_{x=0}^{x=1} - \int_{y(0)}^{y(1)} x(y) \, dy = \int_0^1 x \, dy = F(\alpha^{-1}). \qquad (11.5)$$

Lightning Question 2

$$\int \left(\frac{x}{x-1}\right)^4 dx \tag{12.1}$$

Solution

$$I = \int \left(1 + \frac{1}{x-1}\right)^4 dx$$
$$= x + 4\ln|x-1| - \frac{6}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{3(x-1)^3} + C. \tag{12.2}$$

Lightning Question 3

$$\int \frac{[\tan(1012x) + \tan(1013x)]\cos(1012x)\cos(1013x)}{\cos(2025x)} dx \tag{13.1}$$

Solution

$$I = \int \frac{\sin(1012x)\cos(1013x) + \sin(1013x)\cos(1012x)}{\cos(2025x)} dx$$
$$= \int \tan(2025x)\, dx = -\frac{1}{2025}\ln|\cos(2025x)| + C. \tag{13.2}$$