Asymptotic Analysis of Differential Equations (1): Linear ODE

We analyze the asymptotic expansion of the solution for linear ODE on the complex plane. Denote the domain $\Omega \subseteq \mathbb{C}$, the meromorphic function domain $m(\Omega)$ and holomorphic function ring $\mathbb{O}(\Omega)$, the $n \times n$ matrix $M(m(\Omega), n)$ with its elements composed of function $f \in m(\Omega)$. For a matrix $A \in M(m(\Omega), n)$, a linear ODE can be represented by

$$y'(z) = A(z)y(z), \qquad y \in \mathbb{C}^n$$

Based on variation of parameters, we only need to study the homogeneous equation. For an initial value problem $y(z_0) = y_0$, we can first solve for the matrix equation

$$Y'(z) = A(z)Y(z), \qquad Y(z_0) = I \tag{$*$}$$

With this fundamental matrix, we have $y = Yy_0$. Hence, we will focus on this matrix equation.

- Qualitative theory of solutions (6.1)

Cauchy Theorem

For $y' = f(z, y)$ with $y(z_0) = y_0$, if f is analytic then there exists a unique analytic solution.

Consider $(*)$ has a solution Y_0 around z_0. For a path γ starting from z_0, we can perform analytic continuation of Y_0 into a neighborhood of γ. When γ goes back to z_0, we can obtain another solution Y_γ. Then we state that there exists an invertible $C_\gamma \in GL(\mathbb{C}, n)$ such that $Y_\gamma = Y_0 C_\gamma$. We define a mapping $\rho_A : \gamma \mapsto C_\gamma$, and when γ_1 and γ_2 are homotopic, we have $C_{\gamma_1} = C_{\gamma_2}$. This implies that $\rho_A : \pi_1(\Omega^*, z_0) \to GL(\mathbb{C}, n)$, with Ω^* is the domain with poles removed. Moreover, if $\gamma = \gamma_1 \circ \gamma_2$, then $C_\gamma = C_{\gamma_1} C_{\gamma_2}$. So ρ_A is a group homomorphism and a representation of π_1.

For the matrix equation $(*)$, consider a transformation $P \in GL(\mathbb{O}(\Omega), n)$ and denote $Z = PY$.

$$Z' = P'Y + PY' = (P'P^{-1} + PAP^{-1})Z = BZ$$

The two mappings ρ_A and ρ_B are equivalent. $P'P^{-1} + PAP^{-1}$ is a meromorphic connection on vector bundles on a complex manifold, an example of Riemann-Hilbert correspondence.

Local problem

Let z_0 is a pole of A, with r denoted as the Poincaré rank. This implies that

$$A(z) = (z - z_0)^{-r}\tilde{A}(z), \qquad \tilde{A}(z) \in M(\mathbb{O}(z_0), n), \qquad \tilde{A}(z_0) \neq 0$$

Without loss of generality, take $z_0 = 0$ and we have

$$z^r Y'(z) = \tilde{A}(z)Y(z)$$

Since z_0 is a pole, $Y(z_0)$ may not exists, and we only focus on the equation. The solution is highly influenced by the Poincaré rank r.

When $r = 1$, consider a constant matrix A and we have

$$zY'(z) = AY(z)$$

Select a branch cut C from $z = 0$, we have $\ln z \in \mathbb{O}(\Omega \setminus C)$ and $Y(z) = e^{A \ln z}$. Consider A has the Jordan normal form $A = PJP^{-1}$ with $J = \Lambda + N$. Then we can write

$$Y(z) = P\left(e^{J \ln z}\right)P^{-1}, \qquad e^{J \ln z} = \Lambda^z \left(\sum_{k=0}^{n} \frac{\ln^k z}{k!} N^k\right)$$

The singularity is regular for $r = 1$. Going around $z = 0$, we obtain C_γ as follows

$$Y_0\left(ze^{2\pi i}\right) = e^{A(\ln z + 2\pi i)} = Y_0(z)e^{2\pi i A}, \qquad C_\gamma = e^{2\pi i A}$$

When $r = 2$, still consider a constant matrix A and we have

$$z^2 Y'(z) = AY(z), \qquad Y(z) = e^{-A/z}$$

Now $z = 0$ becomes an essential singularity, and the solution only exists in a sector. We cannot go around $z = 0$ as in the previous case. For $r \geq 2$, the singularity is irregular.

- Majorant series & Cauchy theorem

Let $\Omega \subseteq \mathbb{C}^{d+1}$ with coordinates $\tilde{y} = (z, y)$ with $z \in \mathbb{C}$, $y \in \mathbb{C}^d$ and a function $f \in \mathbb{O}(\Omega, \mathbb{C}^d)$.

$$y' = f(z, y), \qquad \tilde{y}_0 = (z_0, y_0) \in \Omega$$

The function f is analytic when there exists $r > 0$ such that when $\tilde{y} \in B(\tilde{y}_0, r)$, the following series is convergent

$$f(\tilde{y}) = \sum_{j \geq 0} c_j (\tilde{y} - \tilde{y}_0)^j, \qquad j = \{j_0, j_1, \cdots, j_d\}$$

The neighborhood is $B(\tilde{y}_0, r) = \{\tilde{y} \in \Omega \mid |\tilde{y} - \tilde{y}_0| < r\}$ with the L_∞ norm $|\tilde{y}| = \max|y_i|$. The above notation means

$$f(\tilde{y}) = \sum_{j_0, \cdots, j_d \geq 0} c_{j_0 \cdots j_d} (z - z_0)^{j_0} (y_1 - y_{10})^{j_1} \cdots (y_d - y_{d0})^{j_d}$$

Majorant series

Consider a formal power series $f(\tilde{y})$. If there exists another series $F(\tilde{y})$ such that $\forall j$ we have $|a_j| \leq A_j$, then $F(\tilde{y})$ is a majorant series of $f(\tilde{y})$.

$$f(\tilde{y}) = \sum_{j \geq 0} a_j (\tilde{y} - \tilde{y}_0)^j, \qquad F(\tilde{y}) = \sum_{j \geq 0} A_j (\tilde{y} - \tilde{y}_0)^j$$

If $F(\tilde{y})$ converges in $B(\tilde{y}_0, r)$, then $f(\tilde{y})$ also converges. We can then call $F(\tilde{y})$ as the majorant function of $f(\tilde{y})$.

Corollary. If $f(\tilde{y})$ is analytic around $\tilde{y}_0$, i.e., it can be expanded on $B(\tilde{y}_0, R)$ into a convergent series, then for any $r \in (0, R)$, there exists a constant $M > 0$ such that we can write down the majorant function $F(\tilde{y})$ as

$$F(\tilde{y}) = M \prod_{k=0}^{d} \left(1 - \frac{y_k - y_{k0}}{r}\right)^{-1}, \qquad \tilde{y} \in \bar{B}(\tilde{y}_0, r)$$

Proof. Since f converges in $B(\tilde{y}_0, R)$, then it absolutely converges in $\bar{B}(\tilde{y}_0, r)$. If we select a $\tilde{y} \in \partial B(\tilde{y}_0, r)$ on the boundary, we have the following convergent series

$$\sum_{j \geq 0} |a_j| |\tilde{y} - \tilde{y}_0|^j = \sum_{j \geq 0} |a_j| r^{j_0 + \cdots + j_d}$$

Then there exists $M > 0$ such that

$$|a_j| r^{|j|} \leq M, \qquad |a_j| \leq \frac{M}{r^{|j|}}, \qquad |j| = j_0 + \cdots + j_d$$

Now we can construct a majorant function

$$F(\tilde{y}) = \sum_{j \geq 0} \frac{M}{r^{|j|}} (\tilde{y} - \tilde{y}_0)^j = M \prod_{k=0}^{d} \sum_{j_k \geq 0} \left(\frac{y_k - y_{k0}}{r}\right)^{j_k} = M \prod_{k=0}^{d} \left(1 - \frac{y_k - y_{k0}}{r}\right)^{-1} \qquad \blacksquare$$

Corollary. For $F(\tilde{y})$ defined above, consider the Cauchy problem

$$y_j' = F(z, y), \qquad y_j(z_0) = y_{j0}, \qquad j = 1, 2, \cdots, d$$

There exists $\rho > 0$ such that it has a unique analytic solution on $B(z_0, \rho)$.

Proof. Denote $u(z) = y_1(z) - y_{10}$. Based on the Cauchy problem, we have

$$\left(y_i - y_j\right)' = 0, \qquad u(z) = y_i(z) - y_{i0}, \qquad \forall i, j = 1, 2, \cdots, d$$

The ODE for $u(z)$ can be obtained as

$$u'(z) = F(z, y) = M \left(1 - \frac{z - z_0}{r}\right)^{-1} \left(1 - \frac{u}{r}\right)^{-d}, \qquad u(z_0) = 0$$

The solution is

$$u(z) = r - r\left[1 + (d+1) M \ln\left(1 - \frac{z - z_0}{r}\right)\right]^{\frac{1}{d+1}}$$

To guarantee convergence, we can obtain the radius ρ as

$$\left|\frac{z - z_0}{r}\right| < 1, \qquad \left|(d+1) M \ln\left(1 - \frac{z - z_0}{r}\right)\right| < 1, \qquad \rho = r\left[1 - e^{-\frac{1}{(d+1)M}}\right] \qquad \blacksquare$$

Cauchy theorem

Let $\Omega \subseteq \mathbb{C}^{d+1}$ and denote analytic functions $f: \Omega \to \mathbb{C}^d$ with a point $(z_0, y_0) \in \Omega$. There exists $\rho > 0$ such that the Cauchy problem has a unique analytic solution in $B(z_0, \rho)$.

$$y_j' = f_j(z, y), \qquad y_j(z_0) = y_{j0}, \qquad j = 1,2, \cdots, d$$

Proof. Without loss of generality, assume $z_0 = 0$ and $y_0 = 0$. We consider the solution in the form of a power series

$$f_j(\tilde{y}) = f_j(z, y) = \sum_{J \geq 0} a_{jJ} \tilde{y}^J, \qquad y_j(z) = \sum_{m \geq 0} c_{jm} z^m, \qquad j = 1,2, \cdots, d$$

Then we have

$$y_j' = \sum_{k \geq 0} c_{j(k+1)} (k+1) z^k = \sum_{J \geq 0} a_{jJ} z^{J_0} y_1^{J_1} \cdots y_d^{J_d} = f_j(\tilde{y})$$

Substitute $y_j(z)$ into the RHS and compare the coefficients. We can obtain

$$c_{jm} = P_{jm}\big(a_{jJ} \mid |J| \leq m-1\big)$$

The polynomial P_{jm} has positive coefficients. To prove the solution is convergent, consider

$$\hat{y}_j' = F(z, y) = M \prod_{k=0}^{d} \left(1 - \frac{y_k}{r}\right)^{-1}$$

Here M is sufficiently large such that $F(z, y)$ is the majorant function for all $f_1, \cdots, f_d$. We have

$$\hat{y}_j(z) = \sum_{m \geq 0} \hat{c}_{jm} z^m, \qquad \hat{c}_{jm} = P_{jm}\big(A_J \mid |J| \leq m-1\big)$$

A_J is the coefficient for the majorant series $F(\tilde{y})$. Since

$$|c_{jm}| = |P_{jm}(a_{jJ})| \leq P_{jm}(|a_{jJ}|) \leq P_{jm}(A_J) = \hat{c}_{jm}$$

Therefore, the formal series $y_j(z)$ converges. ∎

Corollary. For the matrix equation

$$Y'(z) = A(z) Y(z), \qquad Y(z_0) = I$$

If A is analytic near z_0, then there exists a unique analytic solution.

Theorem. Consider $F \in M(\mathbb{O}(\Omega), d)$ with the following equation and its formal solution

$$z y' = F y, \qquad y(z) = \sum_{k \geq 0} c_k (z - z_0)^k, \qquad c_k \in \mathbb{C}^n$$

There exists $\rho > 0$ such that $y(z)$ converges in $B(z_0, \rho)$ and thus is an analytic solution.

Proof. Assume $z_0 = 0$. Consider $F(z)$ can be expanded as

$$F(z) = \sum_{k\geq 0} F_k z^k, \qquad F_k \in M(\mathbb{C}, d) = \mathbb{C}^{d\times d}$$

The equation becomes

$$\sum_{m\geq 0} mc_m z^m = \left(\sum_{k\geq 0} F_k z^k\right)\left(\sum_{l\geq 0} c_l z^l\right) = \sum_{m\geq 0}\left(\sum_{k+l=m} F_k c_l\right) z^m$$

Comparing the coefficients gives

$$mc_m = \sum_{k+l=m} F_k c_l, \qquad (F_0 - mI)c_m = -\sum_{k=1}^{m} F_k c_{m-k}$$

For $m = 0$ we have $F_0 c_0 = 0$. While for $m = 1$, we have

$$c_1 = F_0 c_1 + F_1 c_0, \qquad (F_0 - I)c_1 = -F_1 c_0$$

If F_0 does not have 1 as an eigenvalue, we can obtain the unique solution of c_1. We take $k \in \mathbb{N}$ that is sufficiently large such that for all $\lambda > k$, the matrix $F_0 - \lambda I$ is invertible. Denote

$$f(\lambda) = |(F_0 - \lambda I)^{-1}|_\infty, \qquad \lambda > k$$

Then we have $f \in C(k, +\infty)$ continuous, and when $\lambda \to +\infty$ we have $f(\lambda) \to 0$. This implies that there exists $C > 0$ such that $f(m) \leq C$ for all $m > k$. The coefficients c_m are bounded as

$$|c_m|_\infty = \left|-(F_0 - \lambda I)^{-1} \sum_{k=1}^{m} F_k c_{m-k}\right|_\infty \leq C \sum_{k=1}^{m} |F_k|_\infty |c_{m-k}|_\infty$$

Define $v_m = |c_m|_\infty$ for $m \leq k$, and

$$v_m = C \sum_{j=1}^{m} \left|F_j\right|_\infty v_{m-j}, \qquad m > k$$

This guarantees $|c_m| \leq v_m$. We want to show that $\{v_m\}$ corresponds to a convergent series.

$$u(z) = \sum_{m\geq 0} v_m z^m, \qquad \phi(z) = \sum_{m\geq 1} |F_m|_\infty z^m$$

We can show that (all norms are $|\cdot|_\infty$)

$$u(z) = [1 - C\phi(z)]^{-1}\left[|c_0| + \sum_{l=1}^{k}\left(|c_l| - C\sum_{j=1}^{l}\left|F_j\right|\left|c_{l-j}\right|\right) z^l\right]$$

This is proved by comparing the coefficients, after multiplying $1 - c\phi(z)$ to the LHS.

$$[z^m]: \quad v_m - C\sum_{j=1}^{m}\left|F_j\right| v_{m-j} = |c_m| - C\sum_{j=1}^{m}\left|F_j\right|\left|c_{m-j}\right|, \qquad m \leq k$$

$$[z^m]: \quad v_m - C\sum_{j=1}^{m}\left|F_j\right| v_{m-j} = 0, \qquad m > k$$

The numerator of $u(z)$ is a polynomial which is convergent. As $\phi(0) = 0$, there exists $\delta_1 > 0$ such that when $|z| < \delta_1$, we have $1 - C\phi(z) \neq 0$ and $(1 - C\phi(z))^{-1}$ is analytic on $B(0, \delta_1)$. Therefore, we prove the majorant $u(z)$ is analytic, and thus $y(z)$. ■

- Asymptotic behavior near ordinary and regular singular points (6.2)

$$zy'(z) = F(z)y(z), \qquad F \in M(B(0,1), n)$$

Now consider the matrix equation

$$zY'(z) = A(z)Y(z), \qquad A \in M(\mathbb{O}(\Omega), n)$$

We require $A(0) \neq 0$ which implies that $z = 0$ is a singular point. The domain Ω: $|z| < \rho$. Our goal is to find a transform $P \in GL(\mathbb{O}(\Omega), n)$ such that $Y = PX$ and

$$zX'(z) = B(z)X(z), \qquad B = P^{-1}AP - zP^{-1}P'$$

We want to choose B to be as simple as possible. The matrix equation to be solved is

$$zP'(z) = A(z)P(z) - P(z)B(z)$$

With the formal power series, the equation becomes

$$\sum_{m \geq 0} mP_m z^m = \left(\sum_{k \geq 0} A_k z^k\right)\left(\sum_{l \geq 0} P_l z^l\right) - \left(\sum_{l \geq 0} P_l z^l\right)\left(\sum_{k \geq 0} B_k z^k\right)$$

Taking the coefficient of z^m, we obtain

$$mP_m = A_0 P_m - P_m B_0 + \sum_{k=1}^{m} (A_k P_{m-k} - P_{m-k} B_k)$$

$$(A_0 - mI)P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k})$$

For $m = 0$, we have $B_0 = P_0^{-1} A_0 P_0$. One choice is $P_0 = I$ and $B_0 = A_0$. Another better one is to choose P_0 such that $B_0 = J_0$ is the Jordan normal form of A_0.

Corollary 1. For $A, B \in M(\mathbb{C}, n)$, define the following map

$$\varphi_{AB}\colon\ M(\mathbb{C}, n) \to M(\mathbb{C}, n), \qquad P \mapsto AP - PB$$

Then φ_{AB} is injective if and only if A, B do not share the same eigenvalue.

Proof. When φ_{AB} is injective, assume that λ is the common eigenvalue. Then there exist non-zero $v, w \in \mathbb{C}^n$ such that $Av = \lambda v$ and $B^T w = \lambda w$. Take $P = vw^T$, and we obtain

$$AP - PB = Avw^T - vw^T B = \lambda vv^T - \lambda vv^T = O$$

This is contradictory to φ_{AB} being injective. On the other hand, when there are no common eigenvalues between A and B, denote $V = \mathbb{C}^n$ and we can write

$$V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}, \qquad V_{\lambda_j} = \operatorname{Ker}\left(B - \lambda_j I\right)^{m_j}$$

We can obtain a basis for V by picking from each root subspace V_λ. Let v be the basis of V_λ. If P satisfies $AP - PB = O$, we have

$$(B - \lambda I)^m v = 0, \qquad P(B - \lambda I)^m v = (A - \lambda I)^m P v = 0$$

Note that λ is not the eigenvalue of A, so $(A - \lambda I)^m$ is invertible and thus $Pv = 0$. Since v can be any vector of the basis, $P = O$ and thus φ_{AB} is injective. ■

Resonant matrix

With this corollary, we need to see if $A_0 - mI$ and B_0 share the same eigenvalue. We call a matrix A as resonant if there are two eigenvalues λ, μ such that $\lambda - \mu \in \mathbb{Z}_{>0}$.

Theorem 2. For $zY' = AY$, if A_0 is non-resonant, then there exists a transformation $Y = PX$ with $P(0) = I$ and $P(z)$ analytic around $z = 0$ such that

$$zX' = A_0 X, \qquad Y(z) = P(z) z^{A_0}$$

Proof. Since A_0 is non-resonant and $P_0 = I$, we know that $A_0 - mI$ and $B_0 = A_0$ do not share the same eigenvalue. We can then choose $B_m = O$ for $m \geq 1$, and there are corresponding P_m

$$P_m = \varphi_{A_0 - mI, B_0}^{-1}\left(-\sum_{k=1}^{m} A_k P_{m-k}\right), \qquad zX' = A_0 X$$

Therefore, we obtain a formal solution $Y(z) = P(z) z^{A_0}$. The equation for $P(z)$ is

$$zP'(z) = A(z)P(z) - P(z)A_0$$

Now take a basis $e_1, \dots, e_{n^2}$ for $M(\mathbb{C}, n)$, and we have

$$P(z) = \sum_{j=1}^{n^2} p_j e_j\,, \qquad zP'(z) = M(z)P(z), \qquad M(z)\colon \varphi_{A(z), A_0}$$

From the existence of an analytic solution for the matrix equation, $P(z)$ is analytic at $z = 0$. ■

If A_0 is resonant, then $(A_0 - mI)P_m - P_m B_0$ is not an isomorphism, so we cannot ensure the existence of P_m for arbitrarily chosen B_m.

$$(A_0 - mI)P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k})$$

As an example, we can choose

$$\sum_{k=1}^{m-1} (P_{m-k} B_k - A_k P_{m-k}) + B_m - A_m = O, \qquad P_m = O$$

In this case, we can obtain the following solution.

Proposition 3. For $zY' = AY$, we have a resonant A_0. Let M be the largest positive integer such that $M = \lambda - \mu$ for the eigenvalues. Then there exists $Y = PX$ with analytic $P(z)$ such that

$$zX' = \left(A_0 + \sum_{k=1}^{M} B_k z^k \right) X$$

B_k is non-zero only when there are eigenvalues such that $k = \lambda - \mu$.

A better choice is given as follows. For $zY' = AY$, consider $A_0 = P_0 J_0 P_0^{-1}$ with J_0 as the Jordan normal form. Take $Y = P_0 X$, and then we have

$$zX = (P_0^{-1} A P_0) X = (J_0 + A_1 z + \cdots + A_m z^m) X$$

Without loss of generality, assume $A_0 = \Lambda + N_0$ already the Jordan normal form ($P_0 = I$), and its eigenvalues are ordered by decreasing Re λ_α. As N_0 is strictly upper triangular, we have

$$(N_0)_{\alpha\beta} = 0, \qquad \alpha \geq \beta, \qquad (N_0)_{\alpha\beta} \neq 0, \qquad \lambda_\alpha \neq \lambda_\beta$$

When $m = 1$, the matrix equation becomes

$$(A_0 - I) P_1 - P_1 A_0 = B_1 - A_1$$

Using Einstein summation notation, the (α, β) element becomes

$$\Lambda_{\alpha\gamma} (P_1)_{\gamma\beta} + (N_0)_{\alpha\gamma} (P_1)_{\gamma\beta} - (P_1)_{\alpha\beta} - (P_1)_{\alpha\gamma} (\Lambda)_{\gamma\beta} - (P_1)_{\alpha\gamma} (N_0)_{\gamma\beta} = (B_1)_{\alpha\beta} - (A_1)_{\alpha\beta}$$

For the diagonal matrix, we have $\Lambda_{ij} = \lambda_i \delta_{ij}$, which leads to

$$\left(\lambda_\alpha - \lambda_\beta - 1\right) (P_1)_{\alpha\beta} + (N_0)_{\alpha\gamma} (P_1)_{\gamma\beta} - (P_1)_{\alpha\gamma} (N_0)_{\gamma\beta} = (B_1)_{\alpha\beta} - (A_1)_{\alpha\beta}$$

For $(n, 1)$ element, we have $(N_0)_{n\gamma} = (N_0)_{\gamma 1} = 0$, which leads to

$$(\lambda_n - \lambda_1 - 1)(P_1)_{n1} = (B_1)_{n1} - (A_1)_{n1}$$

If $\lambda_n - \lambda_1 \neq 1$, we can choose

$$(B_1)_{n1} = 0, \qquad (P_1)_{n1} = -\frac{(A_1)_{n1}}{\lambda_n - \lambda_1 - 1}$$

If $\lambda_n - \lambda_1 = 1$, we can choose $(B_1)_{n1} = (A_1)_{n1}$, and $(P_1)_{n1}$ is arbitrary. We can continue this process for $(n, 2)$ element and so on using previously determined P_1. This implies that we can find B_1 and P_1, and $(B_1)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = 1$. In general, B_m and P_m exist, and $(B_m)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = m$.

Proposition 3'. With this new choice of B_m (now denoted as N_m) and P_m, we have

$$zX' = (\Lambda + N_0 + N_1 z + \cdots + N_m z^m) X$$

$(N_k)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = k$. This implies that non-zero elements are possible only when $i < j$ since we have ordered the eigenvalues, and N_k are strictly upper triangular.

Corollary 4. With this new choice of Λ and N_k, we have

$$z^\Lambda N_k = N_k z^k z^\Lambda$$

Proof. Note that

$$(\lambda_\alpha - \lambda_\beta - k)(N_k)_{\alpha\beta} = 0, \qquad \Lambda N_k - N_k\Lambda - kN_k = O$$

Therefore, we have

$$z^\Lambda N_k = \sum_{l\geq 0} \frac{(\ln z)^l}{l!}\Lambda^l N_k = \sum_{l\geq 0} \frac{(\ln z)^l}{l!} N_k(\Lambda + k)^l = N_k z^{\Lambda+k} \qquad \blacksquare$$

Corollary 5. For the following equation

$$zX' = (\Lambda + N_0 + N_1 z + \cdots + N_m z^m)X$$

Its solution is

$$\xi = z^\Lambda z^N, \qquad N = N_0 + N_1 + \cdots + N_m$$

Proof. Using the previous Corollary, we can directly calculate

$$z\xi' = (\Lambda z^\Lambda)z^N + z^\Lambda(Nz^N) = \Lambda\xi + (N_0 + N_1 z + \cdots + N_m z^m)\xi \qquad \blacksquare$$

Theorem 6. For matrix equation $zY' = AY$, assume that A_0 has a Jordan normal form $\Lambda + N_0$, with the eigenvalues ordered by $\operatorname{Re}\lambda_\alpha$. Then there exists $P(z) \in GL(\mathbb{O}(\Omega), n)$ and a strictly upper triangular constant matrix $N \in M(\mathbb{C}, n)$ such that

$$Y(z) = P(z)z^\Lambda z^N$$

$(N)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$.

To calculate the solution, since N is a nilpotent matrix with $N^{n+1} = O$, we have

$$z^\Lambda = \operatorname{diag}(z^{\lambda_\alpha}), \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l$$

For any $\theta_0 \in \mathbb{R}$, the solution $Y(z)$ is analytic in $S(\theta_0) = \{z \in \Omega \mid \theta_0 < \arg z < \theta_0 + 2\pi\}$.

Consider $z \to ze^{2\pi i}$, the solution becomes

$$Y(ze^{2\pi i}) = P(z)z^\Lambda e^{2\pi i\Lambda} z^N e^{2\pi i N}$$

From Corollary 4, with $z = e^{2\pi i}$ we have

$$e^{2\pi i\Lambda}N_k = N_k e^{2\pi i k} e^{2\pi i\Lambda} = N_k e^{2\pi i\Lambda}, \qquad e^{2\pi i\Lambda}N = Ne^{2\pi i\Lambda}$$

This shows that $e^{2\pi i\Lambda}$ commutes with N, and thus $M = e^{2\pi i\Lambda}e^{2\pi iN}$. Based on this property, we call M the **monodromic matrix** of the matrix equation, and (Λ, N) the **monodromic data** that determine the multivalued properties of the solution.

Example: Bessel equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

We define the vector Y as

$$Y = \begin{bmatrix} y \\ xy' \end{bmatrix}, \qquad Y' = \begin{bmatrix} y' \\ xy'' + y' \end{bmatrix} = \begin{bmatrix} y' \\ \dfrac{\alpha^2 - x^2}{x} y \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{x} \\ \dfrac{\alpha^2 - x^2}{x} & 0 \end{bmatrix} \begin{bmatrix} y \\ xy' \end{bmatrix}$$

Then we obtain the corresponding matrix equation

$$xY' = A(x)Y, \qquad A(x) = \begin{bmatrix} 0 & 1 \\ \alpha^2 - x^2 & 0 \end{bmatrix}$$

The coefficients of the power series of $A(x)$ are

$$A_0 = \begin{bmatrix} 0 & 1 \\ \alpha^2 & 0 \end{bmatrix}, \qquad A_1 = O, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

To diagonalize A_0 (which also set $P_0 = I$), consider the following transform

$$\Phi = \begin{bmatrix} xy' + \alpha y \\ xy' - \alpha y \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ \alpha & -1 \end{bmatrix} Y, \qquad x\Phi' = \left(\begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} + \frac{x^2}{2\alpha} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right) \Phi$$

When A_0 is non-resonant, we have $2\alpha \notin \mathbb{Z}$ and the solution can be obtained from Theorem 2.

When A_0 is resonant with $2\alpha \in \mathbb{Z}$:

If 2α is odd, as $A_1 = 0$ we can choose $B_1 = P_1 = O$, and then for all $m \geq 2$ we can similarly set $B_m = O$ and solve for P_m. The solution can still be written as $Y(z) = P(z)z^{\Lambda}$.

If 2α is even, for $m = 2$ the equation is

$$(A_0 - 2I)P_2 - P_2A_0 = B_2 - A_2$$

As an example, consider $\alpha = 1$ and we have

$$\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} P_2 - P_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = B_2 - \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Explicitly writing out the elements for P_2, we have

$$P_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{bmatrix} -2a & 0 \\ -4c & -2d \end{bmatrix} = B_2 - \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

This constrains $(B_2)_{12}$ and a valid choice is

$$B_2 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad P_2 = \frac{1}{8} \begin{bmatrix} -2 & 0 \\ -1 & 2 \end{bmatrix}$$

For $m \geq 3$ we can still set $B_m = O$ and solve for P_m. This implies that the final matrix N is

$$N = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l = \begin{bmatrix} 1 & \frac{1}{2}\ln z \\ 0 & 1 \end{bmatrix}, \qquad z^\Lambda = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

We know that for $\alpha = 1$ the solutions are $J_1(z)$ and $Y_1(z)$. The term $\ln z$ contributes to $Y_1(z)$.

In general, for a linear ODE of the form

$$x^n y^{(n)} + p_1(x)x^{n-1}y^{(n-1)} + \cdots + p_n(x)y = 0$$

We can choose vector Y as

$$Y = \left[y, xy', x^2y'', \cdots, x^{n-1}y^{(n-1)}\right]^T$$

For each element y_j, we can obtain the recursive relation

$$y_j = x^{j-1}y^{(j-1)}, \qquad xy_j' = (j-1)y_j + y_{j+1}$$

This leads to the matrix equation

$$xY' = A(x)Y = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ & & & & n-2 & 1 \\ -p_n(x) & \cdots & \cdots & \cdots & -p_2(x) & n-1-p_1(x) \end{bmatrix} Y$$

Global problem

Consider the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and the only meromorphic functions on $\hat{\mathbb{C}}$ are the rational functions, denoted as $\mathbb{K}$. For matrix $A \in M(\mathbb{K}, n)$ and $Y'(z) = A(z)Y(z)$, we want to know when the equation only has regular singular points. For rational functions, we can write the matrix $A(z)$ as

$$A(z) = \sum_{j=1}^{k} \frac{P_j(z)}{\left(z - z_j\right)^{m_j}} + P_0(z), \qquad \deg\left(p_j\right) < m_j$$

If z_j are regular, we have $m_j = 1$ and P_j is a constant matrix. To investigate the behavior at $z = \infty$, consider $w = 1/z$ and

$$\tilde{Y}(w) = Y\left(\frac{1}{w}\right), \qquad \frac{\mathrm{d}\tilde{Y}}{\mathrm{d}w} = -\frac{1}{w^2}Y'\left(\frac{1}{w}\right) = -\frac{1}{w^2}A\left(\frac{1}{w}\right)\tilde{Y}(w)$$

If $z = \infty$ $(w = 0)$ is a regular singularity, we require $w^{-1}A(w^{-1})$ to be analytic at $w = 0$, which is equivalent to $zA(z)$ having a limit as $z \to \infty$, and this requires $P_0(z) = O$. Therefore, if the equation only has regular singularities on $\hat{\mathbb{C}}$, we have $w\tilde{Y}' = \tilde{A}\tilde{Y}$ with

$$A(z) = \sum_{j=1}^{k} \frac{P_j}{z - z_j}, \qquad \tilde{A}(w) = \sum_{j=1}^{k} \frac{P_j}{wz_j - 1}, \qquad \tilde{A}(0) = -\sum_{j=1}^{k} P_j$$

We can then use a linear fractional transformation to obtain

$$Y'(z) = A(z)Y(z), \qquad A(z) = \sum_{j=1}^{N} \frac{A_j}{z - z_j}, \qquad \sum_{j=1}^{N} A_j = 0$$

Now $z = \infty$ is regular. The singularities z_j decompose $\mathbb{C}$ into simply connected polygons U_α, and there is an analytic solution of the equation in each of them. Every side $\overline{z_j z_k}$ corresponds to a monodromic matrix M_{jk} and thus define a map $(A_j) \mapsto (M_{jk})$, which is related to the Riemann-Hilbert problem.

- Asymptotic behavior near irregular singular points (6.3)

$$z^{r+1}Y'(z) = A(z)Y(z), \qquad r \in \mathbb{N}^*, \qquad r \geq 1$$

We call r as the Poincaré rank, and recall the following classification:

$r = -1$: Ordinary point $\qquad r = 0$: Regular singularity $\qquad r \geq 1$: Irregular singularity

First, we can consider the scalar equation with dimension $n = 1$. We have

$$a(z) = \sum_{k\geq 0} a_k z^k, \qquad \frac{y'}{y} = \frac{a(z)}{z^{r+1}} = \sum_{k=0}^{r-1} \frac{a_k}{z^{r+1-k}} + \frac{a_r}{z} + \sum_{k\geq r+1} a_k z^{k-r-1}$$

The solution is

$$\ln y(z) = \sum_{k=0}^{r-1} \frac{a_k}{k-r} \frac{1}{z^{r-k}} + a_r \ln z + \sum_{k\geq r+1} \frac{a_k}{k-r} z^{k-r}, \qquad y(z) = P(z)\, z^\rho\, e^{Q(z^{-1})}$$

The exponent is $\rho = a_r$, and the analytic function $P(z)$ and polynomial $Q(w)$ are defined as

$$P(z) = \exp\left(\sum_{k\geq r+1} \frac{a_k}{k-r} z^{k-r} \right), \qquad Q(w) = \sum_{k=0}^{r-1} \frac{a_k}{k-r} w^{r-k}$$

For the matrix case, we still want to find a transformation P such that $Y = PX$ with

$$z^{r+1}X'(z) = B(z)X(z), \qquad B(z) = P^{-1}AP - z^{r+1}P^{-1}P'$$

The matrix $B(z)$ should be as simple as possible. For the equation of $B(z)$, we similarly obtain

$$z^{r+1}P' = AP - PB$$

Written in formal power series, the coefficients for z^m are

$$[z^m]\, z^{r+1}P' = [z^m] \sum_{k\geq 0} kP_k z^{k+r} = (m-r)P_{m-r}, \qquad P_j = O \text{ for } j < 0$$

$$[z^m]\,(AP - PB) = \sum_{k=0}^{m} (A_k P_{m-k} - P_{m-k} B_k)$$

Therefore, we obtain the following set of equations

$$A_0 P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k}) + (m-r) P_{m-r}$$

We want to properly choose B_m to make the equations simple. Consider A_0 is already reduced to its Jordan normal form, which also gives $P_0 = I$ and $B_0 = A_0$. We need to iteratively solve the matrix equation of the form

$$A_0 P_m - P_m A_0 = B_m - A_m + \sum_{k=1}^{m-1} (P_{m-k} B_k - A_k P_{m-k}) = B_m + F_m$$

The LHS is always resonant. For simplicity, we assume that A_0 has n different eigenvalues and is already diagonalized as $A = \lambda_i \delta_{ij}$. For each element (α, β), we have

$$(\lambda_\alpha - \lambda_\beta)(P_m)_{\alpha\beta} = (B_m)_{\alpha\beta} + (F_m)_{\alpha\beta}$$

When $\alpha \neq \beta$ (off-diagonal elements), we can choose

$$(B_m)_{\alpha\beta} = 0, \qquad (P_m)_{\alpha\beta} = \frac{(F_m)_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta}$$

When $\alpha = \beta$ (diagonal elements), we can choose

$$(B_m)_{\alpha\beta} = -(F_m)_{\alpha\beta}, \qquad (P_m)_{\alpha\beta} = 0$$

Theorem 1. For the matrix equation $z^{r+1} Y'(z) = A(z) Y(z)$, consider that A_0 has n different eigenvalues. There exist an invertible $P(z)$ and a diagonal $B(z)$ such that $Y = PX$ and

$$z^{r+1} X'(z) = B(z) X(z)$$

Corollary 2. With a diagonal $B(z)$, similar to the scalar case, we can define

$$Q(w) = \sum_{k=0}^{r-1} \frac{B_k}{k-r} w^{r-k}, \qquad \rho = B_r, \qquad F'(z) = \left(\sum_{k \geq r+1} B_k z^{k-r-1} \right) F(z), \qquad F(0) = I$$

Note that ρ is a constant diagonal matrix, $Q(w)$ is a diagonal matrix with each element being a polynomial of degree r. Then the solution can be written as

$$X(z) = F(z)\, z^\rho\, e^{Q(z^{-1})}, \qquad Y(z) = P(z)\, F(z)\, z^\rho\, e^{Q(z^{-1})}$$

The result uses the property that ρ and Q are commutable since they are diagonal.

Theorem 3. For an analytic $A(z)$ with rank $r \geq 1$, consider that A_0 has n different eigenvalues. The formal solution of the matrix equation $z^{r+1} Y'(z) = A(z) Y(z)$ is given as

$$Y(z) = \hat{Y}(z)\, z^\rho\, e^{Q(z^{-1})}, \qquad \hat{Y}(0) = I$$

For arbitrary $\theta_1, \theta_2 \in \mathbb{R}$ with $0 < \theta_1 - \theta_2 < \pi/r$, there exists $R > 0$ such that the equation has an analytic solution in $S(\theta_1, \theta_2) \cap B(0, R)$, where $S(\theta_1, \theta_2)$ denotes the sector

$$S(\theta_1, \theta_2) = \{z \in \mathbb{C} \mid \theta_1 < \arg z < \theta_2\}$$

Also, as $z \to 0$ within the domain $S(\theta_1, \theta_2)$, the asymptotic behavior should be interpreted as

$$Y(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \tilde{Y}(z)$$

For an irregular singularity at $z = \infty$, similarly consider $w = 1/z$ and we have

$$\tilde{Y}(w) = Y\left(\frac{1}{w}\right), \qquad w^{-r+1}\, \tilde{Y}'(w) = \tilde{A}(w) Y(w), \qquad \tilde{A}(w) = -A\left(\frac{1}{w}\right)$$

The formal solution can be written as

$$\tilde{Y}(w) = \hat{Y}(w)\, w^{\rho}\, e^{Q(w)}$$

The solution is analytic within the domain $S(\theta_1, \theta_2) \cap \{w \in \mathbb{C} \mid |w| > R\}$.

Theorem (Sibuya 1962). For $\theta_0 \in \mathbb{R}$, there exists a sufficiently small $\delta > 0$ such that there is a solution $Y(z)$ in $S(\theta_0 - \delta, \theta_0 + \pi/r) \cap B(0, R)$.

Corollary 4. There exists $\delta > 0, R > 0$ such that there is a solution $Y(z)$ satisfying Theorem 3 in the following domain

$$S_l = \left\{z \in \mathbb{C}^* \mid \frac{\pi}{r}(l-1) - \delta < \arg z < \frac{\pi}{r} l\right\} \cap B(0, R), \qquad l = 1, 2, \cdots, 2r$$

Stokes phenomenon

Now consider the intersection

$$S(l, l+1) = \left\{z \in \mathbb{C}^* \mid \frac{\pi}{r} l - \delta < \arg z < \frac{\pi}{r} l\right\} \cap B(0, R)$$

Corollary 4 indicates that there are solutions Y_l and Y_{l+1} in this domain $S(l, l+1)$. Hence, there is a constant matrix C_l, the **Stokes multiplier**, such that $Y_{l+1}(z) = Y_l(z) C_l$.

In S_l and S_{l+1} respectively, we have

$$Y_l(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \hat{Y}(z), \qquad Y_{l+1}(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \hat{Y}(z), \qquad z \to 0$$

Multiply the second equation with the inverse of the first one, and we have

$$e^{Q(z^{-1})} z^{\rho} C_l\, z^{-\rho} e^{-Q(z^{-1})} \sim I, \qquad z \to 0, \qquad z \in S(l, l+1)$$

For each element (α, β), we have

$$(C_l)_{\alpha\beta}\, e^{q_\alpha(z^{-1}) - q_\beta(z^{-1})}\, z^{\rho_\alpha - \rho_\beta} \sim \delta_{\alpha\beta}, \qquad z \to 0, \qquad z \in S(l, l+1)$$

When $\alpha = \beta$ (diagonal), we have $(C_l)_{\alpha\alpha} = 1$. When $\alpha \neq \beta$ (off-diagonal), note that

$$q_\alpha(z^{-1}) - q_\beta(z^{-1}) = \frac{\lambda_\beta - \lambda_\alpha}{r} z^{-r} + o(z^{-r})$$

Consider a ray $\gamma \in S(l, l+1)$. If there exists a ray γ such that as $z \to 0$ along γ, we have

$$\mathrm{Re}\{(\lambda_\beta - \lambda_\alpha) z^{-r}\} > 0, \qquad \text{then } \ (C_l)_{\alpha\beta} = 0$$

If there does not exist such a ray γ for the exponent, then nothing can be said about $(C_l)_{ij}$. If the eigenvalues λ_n are sorted by lexicographic order (λ_R, λ_I), then C_l must be an upper or lower triangular matrix, dependent on l being odd or even.

We define the **Stokes ray** as those that lead to

$$\mathrm{Re}\{(\lambda_\beta - \lambda_\alpha) z^{-r}\} = 0$$

There are $M = n(n-1)r$ Stokes rays in total. Each ray corresponds to a Stokes factor.

Example: Airy equation

$$y'' = zy$$

The corresponding matrix equation is

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix}, \qquad Y' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ zy \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix} Y$$

To analyze the behavior at $z = \infty$, we rewrite it as

$$z^{-1} Y'(z) = \begin{bmatrix} 0 & 1/z \\ 1 & 0 \end{bmatrix} Y, \qquad r = 2$$

- Exercise

Regular singular point

$$Y(z) = \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}, \qquad zY'(z) = \begin{bmatrix} -1/2 + z & z \\ z & 1/2 + z \end{bmatrix} Y(z) = A(z)Y(z)$$

$z = 0$ is a regular singularity. The coefficients of the power series of $A(z)$ are

$$A_0 = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad A_k = O, \qquad k \geq 2$$

We already have a diagonal A_0, which implies $P_0 = I$ and $B_0 = A_0$. For $m = 1$ we have

$$(A_0 - I)P_1 - P_1 A_0 = B_1 - A_1$$

Explicitly writing out the elements for P_1, we have

$$P_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{bmatrix} -a & -2b \\ 0 & -d \end{bmatrix} = B_1 - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We can obtain a lower triangular B_1, as well as the corresponding P_1 as

$$P_1 = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

For $m \geq 2$, there is no resonance and we can choose $B_m = O$. As an example, for $m = 2$

$$(A_0 - 2I)P_2 - P_2 A_0 = B_2 + P_1 B_1 - A_1 P_1$$

We can solve for P_2 as

$$P_2 = \begin{bmatrix} 1/4 & 1/2 \\ 0 & 3/4 \end{bmatrix}, \qquad B_2 = O$$

Repeat this process and we can also obtain

$$P_3 = \begin{bmatrix} -1/12 & 5/16 \\ -1/4 & 5/12 \end{bmatrix}, \qquad B_3 = O$$

The monodromic data (Λ, N) are then given as

$$\Lambda = \operatorname{diag}\left(-\frac{1}{2}, \frac{1}{2}\right), \qquad N = B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The transformation $P(z)$ is given as

$$P(z) = \sum_{k \geq 0} P_k z^k = \begin{bmatrix} 1 + z + \frac{1}{4}z^2 - \frac{1}{12}z^3 + \cdots & \frac{1}{2}z + \frac{1}{2}z^2 + \frac{5}{16}z^3 + \cdots \\ -\frac{1}{4}z^3 + \cdots & 1 + z + \frac{3}{4}z^2 + \frac{5}{12}z^3 + \cdots \end{bmatrix}$$

The fundamental solution matrix becomes

$$Y(z) = P(z) z^{\Lambda} z^{N}, \qquad z^{\Lambda} = \operatorname{diag}\left(\frac{1}{\sqrt{z}}, \sqrt{z}\right), \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l = \begin{bmatrix} 1 & 0 \\ \ln z & 1 \end{bmatrix}$$

Asymptotic Analysis of DEs (2): Linear ODE with Parameters

For $x \in I \subseteq \mathbb{R}$ and $y \in \Omega \subseteq \mathbb{R}^n$, consider the following ODE with respect to a small parameter $\varepsilon \in B^*(0,\delta)$ given as

$$F(x, y, y', \varepsilon) = 0, \qquad y(x_0) = y_0$$

We want to study the asymptotic behavior of its solution $y(x;\varepsilon)$ as $\varepsilon \to 0^{\pm}$.

- Formal power series expansion (7.1)

Assume that the solution can be written as

$$y(x;\varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$$

The ODE now becomes

$$F(x, y_0 + \varepsilon y_1 + \cdots, y_0' + \varepsilon y_1' + \cdots; \varepsilon) = 0$$

The Taylor expansion with respect to ε around $p_0 = (x, y_0, y_0'; 0)$ is

$$F = F(p_0) + \varepsilon\left[\frac{\partial F}{\partial y}(p_0) y_1 + \frac{\partial F}{\partial y'}(p_0)\, y_1' + \frac{\partial F}{\partial \varepsilon}(p_0)\right]$$

$$+\varepsilon^2\left[\frac{\partial F}{\partial y} y_2 + \frac{\partial F}{\partial y'} y_2' + \frac{1}{2} y_1^T \frac{\partial^2 F}{\partial y \partial y} y_1 + \frac{1}{2} y_1'^T \frac{\partial^2 F}{\partial y' \partial y'} y_1' + \frac{1}{2}\frac{\partial^2 F}{\partial \varepsilon^2}\right] = 0$$

For each order of ε, we have

$$\varepsilon^0:\ F(x, y_0, y_0'; 0) = 0, \qquad y_0 = y_0(x)$$

$$\varepsilon^1:\ y_1' = A(x) y_1 + B_1(x), \qquad A = -\left(\frac{\partial F}{\partial y'}\right)^{-1}\frac{\partial F}{\partial y}, \qquad B_1 = -\left(\frac{\partial F}{\partial y'}\right)^{-1}\frac{\partial F}{\partial \varepsilon}$$

Note that for $[\varepsilon^1]$, the derivatives are evaluated at $(x, y_0(x), y_0'(x), 0)$. For $[\varepsilon^2]$ we have

$$\varepsilon^2:\ y_2' = A(x) y_2 + B_2(x), \qquad B_2(x) = -\left(\frac{\partial F}{\partial y'}\right)^{-1}(\cdots)$$

As long as the fundamental matrix of $y' = A(x) y$ is known, we can recursively solve $y_n(x)$.

There are several issues arising

- F may not be defined at $\varepsilon = 0$.
- $F(x, y_0, y_0', 0)$ may not have a solution (e.g., boundary layer equation).
- The Jacobian $\partial F / \partial y'$ is not invertible at $p_0 = (x, y_0, y_0'; 0)$.
- The properties of the formal power series are bad.

Now simply consider a function $y(x;\varepsilon)$ with its formal power series

$$y(x;\varepsilon) = \sum_{n \geq 0} y_n(x) \varepsilon^n, \qquad \varepsilon \to 0$$

The equivalent statement is that for $\forall N \in \mathbb{N}$ we have

$$\lim_{\varepsilon \to 0} \frac{y(x;\varepsilon) - \sum_{n=0}^{N} y_n(x)\varepsilon^n}{y_N(x)\varepsilon^N} = 0$$

If the function series has pointwise but not uniform convergence, then the remainder depends on x and is unbounded at some points. The partial sum is thus not practical to use.

Example: Duffing equation

$$y'' + y + \varepsilon y^3 = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

Multiplying y' gives

$$\left(\frac{1}{2}y'^2 + \frac{1}{2}y^2 + \frac{\varepsilon}{4}y^4\right)' = 0, \qquad (y')^2 + y^2 + \frac{\varepsilon}{2}y^4 = 1 + \frac{\varepsilon}{2}$$

The constant is determined from the initial conditions. This leads to an elliptical integral

$$x = \pm \int_1^y \frac{\mathrm{d}y}{\sqrt{\left(1 + \frac{\varepsilon}{2}\right) - y^2 - \frac{\varepsilon}{2}y^4}}$$

We notice that $y_{\min} = -1$ and $y_{\max} = 1$. The period of the oscillator is

$$T = 2\int_{y_{\min}}^{y_{\max}} \frac{\mathrm{d}y}{\sqrt{\left(1 + \frac{\varepsilon}{2}\right) - y^2 - \frac{\varepsilon}{2}y^4}}$$

If we directly expand it into a formal power series, we have

$$(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) + \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)^3 = 0$$

The initial conditions are

$$y_0(0) = 1, \qquad y_0'(0) = 0, \qquad y_k(0) = 0, \qquad y_k'(0) = 0, \qquad k \geq 1$$

For each order of ε, we have

$$\varepsilon^0\!: \; y_0'' + y_0 = 0, \qquad y_0 = \cos x$$

$$\varepsilon^1\!: \; y_1'' + y_1 + y_0^3 = 0, \qquad y_1 = \frac{1}{32}(\cos 3x - \cos x) - \frac{3}{8}x \sin x$$

The $x \sin x$ term gives an increasing amplitude with x. We can similarly obtain

$$y_2 = -\frac{9}{128}x^2 \cos x + \frac{3}{32}x \sin x - \frac{9}{256}x \sin 3x + \cdots$$

The **secular terms** such as $x^n \cos x$ make the partial sum useless for computation. The reason for this behavior is the resonance with the forcing term involving y_0 to y_{n-1}. Now we consider a simpler version of the Duffing equation

$$y'' + y + \varepsilon y = 0, \qquad y(x;\varepsilon) = \cos\left(\sqrt{1 + \varepsilon}\, x\right)$$

The period deviates slightly from 2π, and the Taylor expansion will lead to secular terms. This shows the limitation of the method of direct expansion.

➢ Poincaré-Lindstedt, Poincaré-Lighthill-Kuo (PLK), Strained coordinate method (9.3)

Consider the following example (Tsien, 1956)

$$(x + \varepsilon u)u' + u = 0, \qquad u(1) = 1$$

First we try using the formal power series

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

For each order of ε, we have

$$\varepsilon^0\colon\ xu_0' + u_0 = 0, \qquad u_0(1) = 1, \qquad u_0 = \frac{1}{x}$$

$$\varepsilon^1\colon\ xu_1' + u_0u_0' + u_1 = 0, \qquad u_1(1) = 0, \qquad u_1 = \frac{1}{2x}\left(1 - \frac{1}{x^2}\right)$$

We can similarly obtain

$$\varepsilon^2\colon\ xu_2' + u_2 + u_0u_1' + u_1u_0' = 0, \qquad u_2 = -\frac{1}{2x^3}\left(1 - \frac{1}{x^2}\right)$$

The solution is ordinary around $x = 1$, but is singular at $x = 0$. In other words, the solution is uniformly convergent in $[a, +\infty)$ for any $a > 0$, but not in $(0, +\infty)$.

Strained coordinate (9.3.3)

We introduce the strained coordinate $x = x(\xi)$ with the formal power series

$$u(x;\varepsilon) = u_0(\xi) + \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + \cdots$$

$$x(x;\varepsilon) = \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \cdots$$

Now we denote u' and x' as the derivatives with respect to ξ. The operator becomes

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{1}{x'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}$$

Then the ODE becomes

$$(x + \varepsilon u)u' + x'u = 0$$

For each order of ε, we have

$$\varepsilon^0\colon\ \xi u_0' + u_0 = 0, \qquad u_0 = \frac{1}{\xi}$$

$$\varepsilon^1\colon\ \xi u_1' + u_1 = -x_1u_0' - x_1'u_0 - u_0u_0', \qquad x_1(1) = u_1(1) = 0$$

Here both x_1 and u_1 are unknowns. We require that the singularity of u_1 at $\xi = 0$ is not higher than the singularity of u_0. We want to find x_1 such that the RHS is ordinary at $\xi = 0$. A simple choice is to let the RHS be zero, which gives

$$(\xi u_1)' = -\left(x_1u_0 + \frac{1}{2}u_0^2\right)' = 0, \qquad x_1 = \frac{1}{2}\left(\xi - \frac{1}{\xi}\right), \qquad u_1 = 0$$

We can similarly obtain

$$\varepsilon^2\colon\ (x_2 + u_1)u_0' + (x_1 + u_0)u_1' + \xi u_2' + u_2 + u_0x_2' + u_1x_1' = 0$$

$$\xi u_2' + u_2 = \frac{x_2}{\xi^2} - \frac{x_2'}{\xi}, \qquad x_2(1) = u_2(1) = 0$$

The choice $x_2 = u_2 = 0$ is valid. For $n \geq 2$, the equation is homogeneous with respect to x_n, and we can always choose $x_n = u_n = 0$. Hence, we obtain an exact solution

$$u(\xi;\varepsilon) = \frac{1}{\xi}, \qquad x(\xi;\varepsilon) = \xi + \frac{\varepsilon}{2}\left(\xi - \frac{1}{\xi}\right)$$

Writing as $u = u(x;\varepsilon)$, we have

$$u = -\frac{x}{\varepsilon} + \sqrt{\left(\frac{x}{\varepsilon}\right)^2 + \frac{2}{\varepsilon} + 1}$$

Example: Duffing equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y + \varepsilon y^3 = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

Now we consider the solution

$$y(x;\varepsilon) = y_0(\xi) + \varepsilon y_1(\xi) + \varepsilon^2 y_2(\xi) + \cdots$$
$$x(x;\varepsilon) = \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \cdots$$

The second-order derivative operator becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{x'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{y'(\xi)}{x'(\xi)}\right) = \frac{y''x' - y'x''}{(x')^3}$$

The equation then becomes

$$y''x' - y'x'' + (x')^3(y + \varepsilon y^3) = 0$$

The initial conditions are

$$y_0(0) = 1, \qquad y_0'(0) = 0, \qquad y_k(0) = y_k'(0) = 0, \qquad x_k(0) = 0, \qquad k \geq 1$$

There is no constraint on $x_k'(0)$ and we can set $x_k'(0) = 0$. For each order of ε, we have

$$\varepsilon^0:\ y_0'' + y_0 = 0, \qquad y_0 = \cos\xi$$
$$\varepsilon^1:\ y_1'' + y_1 = y_0'x_1'' + 2y_0''x_1' - y_0^3$$

Using the solution y_0, we have

$$y_1'' + y_1 = -\sin\xi\, x_1'' - 2\cos\xi\, x_1' - \frac{3}{4}\cos\xi - \frac{1}{4}\cos 3\xi$$

The forcing term $\cos\xi$ leads to resonance, and we want to remove the secular term by setting

$$\sin\xi\, x_1'' + 2\cos\xi\, x_1' + \frac{3}{4}\cos\xi = 0, \qquad x_1 = -\frac{3}{8}\xi$$

Then the equation for y_1 becomes

$$y_1'' + y_1 = -\frac{1}{4}\cos 3\xi\,, \qquad y_1 = \frac{1}{32}(\cos 3\xi - \cos\xi)$$

We can further obtain (e.g., Fourier series expansion)

$$\begin{aligned}\varepsilon^2:\ y_2'' + y_2 &= y_1'x_1'' + y_0'x_2'' + 2y_0''x_2' + 2y_1''x_1' - 3y_0''(x_1')^2 - 3y_0'x_0'x_1'' - 3y_0^2 y_1 \\ &= -\sin\xi\, x_2'' - 2\cos\xi\, x_2' + \frac{57}{128}\cos\xi + \frac{1}{16}\cos 3\xi - \frac{3}{128}\cos 5\xi\end{aligned}$$

To remove the secular term, we need to set

$$-\sin\xi\, x_2'' - 2\cos\xi\, x_2' + \frac{57}{128}\cos\xi = 0, \qquad x_2 = \frac{57}{256}\xi$$

With this choice of x_2, we can solve for y_2. Eventually, the solution is

$$x = \xi\left(1 - \frac{3}{8}\varepsilon + \frac{57}{256}\varepsilon^2 - \cdots\right) = \xi\left(1 + \sum_{k\geq 1}\omega_k\varepsilon^k\right)$$

This is the typical form of the strained coordinate for weakly nonlinear oscillations.

- Method of multiple scales (9.3.4)

$$y'' + 2\varepsilon y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

The exact solution is obtained from the characteristic equation

$$\lambda^2 + 2\varepsilon\lambda + 1 = 0, \qquad \lambda_{1,2} = -\varepsilon \pm i\sqrt{1-\varepsilon^2}$$

$$y(x) = e^{-\varepsilon x}\cos\left(\sqrt{1-\varepsilon^2}\, x + \theta_0\right), \qquad \theta_0 = -\arctan\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}$$

First we try using the strained coordinate method

$$y''x' - y'x'' + 2\varepsilon(x')^2 y' + (x')^3 y = 0$$

For each order of ε, we have

$$\varepsilon^0:\ y_0'' + y_0 = 0, \qquad y_0 = \cos\xi$$

$$\varepsilon^1:\ y_1'' + y_1 = -\sin\xi\, x_1'' - 2\cos\xi\, x_1' - 2\sin\xi$$

We still want to remove the secular term, but now $x_1(\xi)$ becomes singular at $\xi = \pi$

$$x_1 = 1 - \xi\cot\xi$$

The issue is due to the lack of amplitude information ($e^{-\varepsilon x}$) in the strained coordinate method.

In this damped oscillation, there are two (time) scales for the fast oscillation and slow damping, respectively. We introduce a number of scales

$$T_0 = x, \qquad T_1 = \varepsilon x, \qquad T_k = \varepsilon^k x$$

Consider the solution of the form

$$y(x;\varepsilon) = Y(T_0, T_1, \cdots, T_k; \varepsilon) = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots$$

The goal is to convert the original ODE into PDEs with more degrees of freedom introduced. With the notations D_x and ∂_k, the derivative operator becomes

$$D_x y = \frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{k\geq 0}\frac{\partial Y}{\partial T_k}\frac{\partial T_k}{\partial x} = \sum_{k\geq 0}\varepsilon^k\frac{\partial Y}{\partial T_k} = \sum_{k\geq 0}\varepsilon^k\partial_k Y$$

The damped oscillation equation then becomes

$$[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2) + 2\varepsilon(\partial_0 + \varepsilon\partial_1) + 1](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2) = o(\varepsilon^2)$$

The initial conditions are analyzed as

$$y(0) = Y(T_0(0), T_1(0), \cdots; \varepsilon) = Y(\mathbf{0}; \varepsilon) = Y_0(\mathbf{0}) + \varepsilon Y_1(\mathbf{0}) + \cdots = 1$$

$$y'(0) = \left(\sum_{k\geq 0} \varepsilon^k \partial_k\right)\left(\sum_{l\geq 0} \varepsilon^l Y_l(\mathbf{0})\right) = 0$$

This leads to the initial conditions for $Y_k(\mathbf{0})$ given as

$$Y_0(\mathbf{0}) = 1, \qquad Y_k(\mathbf{0}) = 0, \qquad k \geq 1$$

For the derivatives, the first several orders give the initial conditions

$$\partial_0 Y_0(\mathbf{0}) = 0, \qquad \partial_1 Y_0(\mathbf{0}) + \partial_0 Y_1(\mathbf{0}) = 0, \qquad \partial_0 Y_2(\mathbf{0}) + \partial_1 Y_1(\mathbf{0}) + \partial_2 Y_0(\mathbf{0}) = 0$$

For ε^0 term, we have

$$\varepsilon^0\colon\ \partial_0^2 Y_0 + Y_0 = 0, \qquad Y_0 = A_0(T_1, \cdots)\cos T_0 + B_0(T_1, \cdots)\sin T_0$$

For ε^1 term, we have

$$\varepsilon^1\colon\ \partial_0^2 Y_1 + Y_1 = -2\partial_1\partial_0 Y_0 - 2\partial_0 Y_0 = 2(\partial_1 A_0 + A_0)\sin T_0 - 2(\partial_1 B_0 + B_0)\cos T_0$$

To remove the secular terms, we set the coefficients of the resonant forcing as zero

$$\partial_1 A_0 + A_0 = 0, \qquad \partial_1 B_0 + B_0 = 0$$

We can update the general solutions for A_0 and B_0 as

$$A_0(T_1, \cdots) = e^{-T_1} A_0(T_2, \cdots), \qquad B_0(T_1, \cdots) = e^{-T_1} B_0(T_2, \cdots)$$

The equation of Y_1 then gives

$$\partial_0^2 Y_1 + Y_1 = 0, \qquad Y_1 = A_1(T_1, \cdots)\cos T_0 + B_1(T_1, \cdots)\sin T_0$$

For ε^2 terms, we can further obtain

$$\partial_0^2 Y_2 + Y_2 + 2(\partial_0\partial_1 + \partial_0)Y_1 + (2\partial_0\partial_2 + \partial_1^2 + 2\partial_1)Y_0 = 0$$

Note that from previous results, we already have

$$Y_0 = e^{-T_1} A_0(T_2, \cdots)\cos T_0 + e^{-T_1} B_0(T_2, \cdots)\sin T_0$$

$$\partial_2\partial_0 Y_0 = -e^{-T_1}(\partial_2 A_0)\sin T_0 + e^{-T_1}(\partial_2 B_0)\cos T_0\,, \qquad (\partial_1^2 + 2\partial_1)Y_0 = -Y_0$$

$$(\partial_0\partial_1 + \partial_0)Y_1 = -(\partial_1 A_1 + A_1)\sin T_0 + (\partial_1 B_1 + B_1)\cos T_0$$

The equation for Y_2 is thus obtained as

$$\partial_0^2 Y_2 + Y_2 = (2\partial_2 A_0 + B_0)e^{-T_1}\sin T_0 + (-2\partial_2 B_0 + A_0)e^{-T_1}\cos T_0$$
$$-(\partial_1 A_1 + A_1)\sin T_0 + (\partial_1 B_1 + B_1)\cos T_0$$

To remove these secular terms, we require

$$\partial_2 A_0 = -\frac{1}{2}B_0, \qquad \partial_2 B_0 = \frac{1}{2}A_0, \qquad \partial_2^2 A_0 + \frac{1}{4}A_0 = 0$$

$$\partial_1 A_1 + A_1 = 0, \qquad \partial_1 B_1 + B_1 = 0$$

We can update the general solutions of the coefficients as

$$A_0(T_1, \cdots) = A_0(T_3, \cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right) + B_0(T_3, \cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right)$$

$$B_0(T_1, \cdots) = A_0(T_3, \cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right) - B_0(T_3, \cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right)$$

$$A_1(T_1,\cdots) = e^{-T_1}A_1(T_2,\cdots), \qquad B_1(T_1,\cdots) = e^{-T_1}B_1(T_2,\cdots)$$

The equation of Y_2 then gives

$$\partial_0^2 Y_2 + Y_2 = 0, \qquad Y_2 = A_2(T_1,\cdots)\cos T_0 + B_2(T_1,\cdots)\sin T_0$$

As a summary, now we obtain

$$Y_0 = \left[A_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right) + B_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right)\right]\cos T_0$$
$$+\left[A_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right) - B_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right)\right]\sin T_0$$
$$Y_1 = e^{-T_1}A_1(T_2,\cdots)\cos T_0 + e^{-T_1}B_1(T_2,\cdots)\sin T_0$$

At the initial $x = 0$, we have

$$Y_0(\mathbf{0}) = A_0(T_3,\cdots) = 1, \qquad \partial_0 Y_0(\mathbf{0}) = -B_0(T_3,\cdots) = 0$$
$$Y_1(\mathbf{0}) = A_1(T_2,\cdots) = 0, \qquad \partial_1 Y_0(\mathbf{0}) + \partial_0 Y_1(\mathbf{0}) = -A_0(T_3,\cdots) + B_1(T_2,\cdots) = 0$$

With these coefficients, we have

$$Y_0 = e^{-T_1}\cos\left(T_0 - \frac{1}{2}T_2\right), \qquad Y_1 = e^{-T_1}\sin T_0$$

The summary of the current solution is

$$y(x;\varepsilon) = Y_0 + \varepsilon Y_1 + \cdots = e^{-\varepsilon x}\left[\cos\left(x - \frac{1}{2}\varepsilon^2 x + \cdots\right) + \varepsilon\sin(x + \cdots)\right] + \cdots$$

Example 1: Van der Pol oscillator (p397)

$$y'' + \varepsilon(y^2 - 1)y' + y = 0$$

We want to obtain a general solution. The equation becomes

$$[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2) + \cdots](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots)$$
$$+ \varepsilon(Y_0^2 + 2\varepsilon Y_0 Y_1 - 1)(\partial_0 + \varepsilon\partial_1)(Y_0 + \varepsilon Y_1) + (Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots) = 0$$

For ε^0 term, we still have

$$\varepsilon^0\!: \ \partial_0^2 Y_0 + Y_0 = 0, \qquad Y_0 = A_1(T_1,T_2,\cdots)\cos\big(T_0 + B_1(T_1,T_2,\cdots)\big)$$

For ε^1 term, denote $\theta = T_0 + B_1$ and we have

$$\varepsilon^1\!: \ \partial_0^2 Y_1 + Y_1 = -2\partial_0\partial_1 Y_0 - (Y_0^2 - 1)\partial_0 Y_0$$
$$= 2(\partial_1 A_1)\sin\theta + 2A_1\cos\theta\,(\partial_1 B_1) + (A_1^2\cos^2\theta - 1)A_1\sin\theta$$
$$= \left(2\partial_1 A_1 - A_1 + \frac{1}{4}A_1^3\right)\sin\theta + 2A_1(\partial_1 B_1)\cos\theta + \frac{1}{4}A_1^3\sin 3\theta$$

To remove the secular terms, we require

$$2\partial_1 A_1 - A_1 + \frac{1}{4}A_1^3 = 0, \qquad \partial_1 B_1 = 0$$

We can then solve for A_1 and B_1 as

$$\frac{1}{A_1^2} = \frac{1}{4}(C_1 e^{-T_1} + 1), \qquad A_1 = \frac{2}{\sqrt{1 + C_1(T_2,\cdots)e^{-T_1}}}, \qquad B_1 = B_2(T_2,\cdots)$$

Now the equation for Y_1 can be solved as

$$\partial_0^2 Y_1 + Y_1 = \frac{1}{4} A_1^3 \sin 3\theta\,, \qquad Y_1 = -\frac{A_1^3}{32}\sin(3\theta) + C_2 \cos\theta$$

The summary of the current solution is

$$y(x;\varepsilon) = \frac{2}{\sqrt{1 + C_1(\varepsilon^2 x, \cdots)e^{-\varepsilon x}}}\cos\big(x + B_2(\varepsilon^2 x, \cdots)\big) + \varepsilon Y_1 + o(\varepsilon)$$

Example 2: Mathieu equation (9.2)

$$y'' + (\delta(\varepsilon) + \varepsilon\cos x)y = 0$$

We want to properly choose $\delta(\varepsilon)$ such that the solution still has a period of 2π. Consider

$$\delta(\varepsilon) = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots$$

Directly expanding $y(x;\varepsilon)$ into the formal power series, we have

$$(y_0'' + \varepsilon y_1'' + \cdots) + (\delta_0 + \varepsilon\delta_1 + \cdots + \varepsilon\cos x)(y_0 + \varepsilon y_1 + \cdots) = 0$$

For ε^0 term, to keep the 2π-periodicity we have

$$\varepsilon^0\!: \; y_0'' + \delta_0 y_0 = 0, \qquad y_0 = A_0\cos\big(\sqrt{\delta_0}x\big) + B_0\sin\big(\sqrt{\delta_0}x\big), \qquad \delta_0 = n^2, \quad n \in \mathbb{N}^*$$

Take $\delta_0 = 1$, and for ε^1 term we have

$$\varepsilon^1\!: \; y_1'' + \delta_0 y_1 = -\delta_1 y_0 - y_0\cos x$$
$$y_1'' + y_1 = -(\delta_1 + \cos x)(A_0\cos x + B_0\sin x)$$

To remove the secular terms, we require $\delta_1 = 0$ and then y_1 is solved as

$$y_1 = -\frac{A_0}{2} + \frac{A_0}{6}\cos 2x + \frac{B_0}{6}\sin 2x + A_1\cos x + B_1\sin x$$

For ε^2 term, we have

$$\begin{aligned}\varepsilon^2\!: \; y_2'' + y_2 = &-\delta_2(A_0\cos x + B_0\sin x)\\ &-\cos x\left(-\frac{A_0}{2} + \frac{A_0}{6}\cos 2x + \frac{B_0}{6}\sin 2x + A_1\cos x + B_1\sin x\right)\end{aligned}$$

To remove the secular terms, we have

$$A_0\left(-\delta_2 + \frac{5}{12}\right) = 0, \qquad -B_0\left(\delta_2 + \frac{1}{12}\right) = 0$$

Therefore, since the initial conditions determine A_0 and B_0, not all conditions will lead to the same period of 2π. When A_0 or B_0 is zero, it is possible to keep the same period.

Now we study the Mathieu equation using the method of multiple scales. Directly set $\delta_0 = 1$ and $\delta_1 = 0$, and we have

$$\begin{aligned}&[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2)](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2)\\ &\qquad + (1 + \varepsilon\cos T_0 + \varepsilon^2\delta_2)(Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2) = o(\varepsilon^2)\end{aligned}$$

For ε^0 and ε^1 terms, we have

$$\varepsilon^0\text{:}\ \partial_0^2 Y_0 + Y_0 = 0, \qquad Y_0 = A_0(T_1, \cdots) \cos T_0 + B_0(T_1, \cdots) \sin T_0$$

$$\varepsilon^1\text{:}\ \partial_0^2 Y_1 + Y_1 = -2(-\partial_1 A_0 \sin T_0 + \partial_1 B_0 \cos T_0) - A_0 \frac{1 + \cos 2T_0}{2} - \frac{B_0}{2} \sin 2T_0$$

To remove the secular terms, we require

$$\partial_1 A_0 = 0, \qquad \partial_1 B_0 = 0, \qquad A_0 = A_0(T_2, \cdots), \qquad B_0 = B_0(T_2, \cdots)$$

The general solution to Y_1 is the same as previous

$$Y_1 = -\frac{A_0}{2} + \frac{A_0}{6} \cos 2T_0 + \frac{B_0}{6} \sin 2T_0 + A_1(T_1, \cdots) \cos T_0 + B_1(T_1, \cdots) \sin T_0$$

For ε^2 term, we have

$$\begin{aligned}\varepsilon^2\text{:}\ \partial_0^2 Y_2 + Y_2 = {} & -2(-\partial_1 A_1 \sin T_0 + \partial_1 B_1 \cos T_0) - 2(-\partial_2 A_0 \sin T_0 + \partial_2 B_0 \cos T_0) \\ & + \frac{1}{2} A_0 \cos T_0 - \frac{1}{12} A_0 \cos T_0 - \frac{B_0}{12} \sin T_0 - \delta_2 A_0 \cos T_0 - \delta_2 B_0 \sin T_0 + \cdots\end{aligned}$$

The non-resonant forcing terms are neglected. To remove the secular terms, we require

$$2\partial_1 A_1 + 2\partial_2 A_0 = \left(\frac{1}{12} + \delta_2\right) B_0, \qquad -2\partial_1 B_1 - 2\partial_2 B_0 = \left(-\frac{5}{12} + \delta_2\right) A_0$$

Note that from ε^1 term, we have A_0 and B_0 depending on T_2 and further. Consider a simpler case with $A_1 = B_1 = 0$, which correspond to specific initial conditions. This gives

$$\partial_2 A_0 = \frac{1}{2}\left(\frac{1}{12} + \delta_2\right) B_0, \qquad \partial_2 B_0 = \frac{1}{2}\left(\frac{5}{12} - \delta_2\right) A_0$$

This leads to a second-order equation for A_0 as

$$\partial_2^2 A_0 + K_2 A_0 = 0, \qquad K_2 = \frac{1}{4}\left(\delta_2 + \frac{1}{12}\right)\left(\delta_2 - \frac{5}{12}\right)$$

Depending on the sign of K_2, we have

$$A_0 = C_1 \cos \sqrt{K_2} T_2 + C_2 \sin \sqrt{K_2} T_2\,, \qquad \delta_2 < -\frac{1}{12} \text{ or } \delta_2 > \frac{5}{12}$$

$$A_0 = C_1 e^{\sqrt{-K_2} T_2} + C_2 e^{-\sqrt{-K_2} T_2}, \qquad -\frac{1}{12} < \delta_2 < \frac{5}{12}$$

The summary of the current solution is

$$y(x; \varepsilon) = A_0 \cos T_0 + B_0 \sin T_0 + \varepsilon Y_1 + \cdots$$

For the exponential case, the finite energy of the system implies $C_1 = 0$, while the exponential decay cannot be observed. This corresponds to the band gap.

Asymptotic Analysis of Differential Equations (3): WKBJ method

For $x \in I = [\alpha, \beta] \subseteq \mathbb{R}$ and $y \in \Omega \subseteq \mathbb{R}^n$, consider the following ODE with respect to a large parameter λ given as

$$y''(x) + f(x;\lambda)\, y(x) = 0, \qquad \lambda \to +\infty$$

The function $f(x;\lambda)$ has the asymptotic expansion

$$f(x;\lambda) \sim \lambda^2 \sum_{n\geq 0} f_n(x) a_n(\lambda), \qquad \lambda \to +\infty, \qquad x \in I$$

This method originates from the analysis of Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi, \qquad \psi'' + \frac{2m(V-E)}{\hbar^2}\psi = 0$$

The classical limit corresponds to $\hbar \to 0^+$ ($\lambda = \hbar^{-1} \to +\infty$). The issue of this problem lies in the function $f(x;\lambda) \to \infty$ as $\lambda \to +\infty$. Note that

$$\frac{y''}{f} + y = 0 \qquad \Longrightarrow \qquad y = 0 \quad \text{when} \quad \lambda \to +\infty$$

We cannot obtain a useful solution from ε^0 term, since $\varepsilon = \lambda^{-1} \to 0^+$ is in the highest-order derivative term, unlike the ODE with parameters analyzed by previous methods.

- WKBJ method

We first transform the ODE into the **Riccati equation**

$$u = (\ln y)' = \frac{y'}{y}, \qquad u' + u^2 + f = 0$$

From the solution of the Riccati equation, the solution of the original equation is

$$y = \exp\left(\int_{x_0}^{x} u(s;\lambda)\, \mathrm{d}s\right)$$

Consider the asymptotic series

$$u(x;\lambda) \sim \sum_{n\geq 0} u_n(x) b_n(\lambda), \qquad f(x;\lambda) \sim \lambda^2 \sum_{n\geq 0} f_n(x) a_n(\lambda), \qquad \lambda \to +\infty$$

This implies the following constraints

$$a_0(\lambda) = 1, \qquad a_{n+1}(\lambda) = o\big(a_n(\lambda)\big), \qquad b_{n+1}(\lambda) = o\big(b_n(\lambda)\big), \qquad \lambda \to +\infty$$

The Riccati equation becomes

$$[u_0(x)b_0(\lambda) + u_1(x)b_1(\lambda) + \cdots]' + [u_0(x)b_0(\lambda) + u_1(x)b_1(\lambda) + \cdots]^2 + \lambda^2[f_0(x)a_0(\lambda) + f_1(x)a_1(\lambda) + \cdots] = 0$$

The leading order terms are

$$u_0'(x)b_0(\lambda) + u_0^2(x)b_0^2(\lambda) + \lambda^2 f_0(x) = o\big(b_0(\lambda)\big) + o\big(b_0^2(\lambda)\big) + o(\lambda^2)$$

We analyze the dominant balance among these three terms, and we need to set

$$b_0(\lambda) = \lambda, \qquad u_0^2(x) + f_0(x) = o(\lambda^2), \qquad u_0 = \pm\sqrt{-f_0(x)}$$

The **turning points** at which $f_0(x) = 0$ govern the behavior of the solution in different regimes. The next order terms give

$$u_0'(x)\lambda + 2u_0(x)u_1(x)\lambda b_1(\lambda) + \lambda^2 f_1(x)a_1(\lambda) = o(\lambda) + o\big(\lambda b_1(\lambda)\big) + o\big(\lambda^2 a_1(\lambda)\big)$$

There are several cases depending on the order of $a_1(\lambda)$:

- Case 1: Dominant balance of term I and II (special $C = 0$ of Case 3)

$$\lambda^2 a_1(\lambda) = o(\lambda), \qquad a_1(\lambda) = o\left(\frac{1}{\lambda}\right), \qquad b_1(\lambda) = 1, \qquad u_1(x) = -\frac{u_0'}{2u_0}$$

- Case 2: Dominant balance of term II and III

$$\lambda = o\big(\lambda^2 a_1(\lambda)\big), \qquad b_1(\lambda) = \lambda a_1(\lambda), \qquad u_1(x) = -\frac{f_1}{2u_0}$$

- Case 3: Dominant balance of all three terms

$$\lim_{\lambda\to+\infty} \lambda a_1(\lambda) = C, \qquad b_1(\lambda) = 1, \qquad u_1(x) = -\frac{u_0' + Cf_1}{2u_0}$$

This process ends when we reach $b_N(\lambda) = O(1)$, and this gives

$$u(x;\lambda) = \sum_{n=0}^{N} u_n(x)b_n(\lambda) + o(1), \qquad \lambda \to +\infty$$

$$y(x;\lambda) = \exp\left(\sum_{n=0}^{N} b_n(\lambda)\int_{x_0}^{x} u_n(s)\,\mathrm{d}s\right)\big(1 + o(1)\big), \qquad \lambda \to +\infty$$