

MIT Integration Bee: 2023 Final

Question 1

$$\int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\sin x + \cos x)^2} dx \quad (1.1)$$

Solution With the **change of variable** $t = \sqrt[3]{\tan x}$, we have

$$t = \sqrt[3]{\tan x}, \quad 3t^2 dt = \sec^2 x dx. \quad (1.2)$$

Therefore, the integral becomes

$$\begin{aligned} I &= \int_0^{\pi/2} \frac{\sqrt[3]{\tan x}}{(\sin x + \cos x)^2} dx = \int_0^{\infty} \frac{3t^3 dt}{(\sin x + \cos x)^2 \sec^2 x} \\ &= \int_0^{\infty} \frac{3t^3 dt}{(t^3 + 1)^2} = -\frac{t}{t^3 + 1} \Big|_0^{\infty} + \int_0^{\infty} \frac{dt}{t^3 + 1} \\ &= \int_0^{\infty} \frac{dt}{t^3 + 1} = \frac{2\pi}{3\sqrt{3}}. \end{aligned} \quad (1.3)$$

Note The following integral can be calculated using the **residue theorem**.

$$I_n = \int_0^{\infty} \frac{dx}{x^n + 1}, \quad \text{for } n \geq 2. \quad (1.4)$$

Based on the contour in Fig. 1, we have

$$(1 - e^{i2\pi/n}) I_n = 2\pi i \cdot \text{Res} \left(\frac{1}{z^n + 1}, z_1 \right), \quad I_n = \frac{\pi}{n \sin(\pi/n)}. \quad (1.5)$$

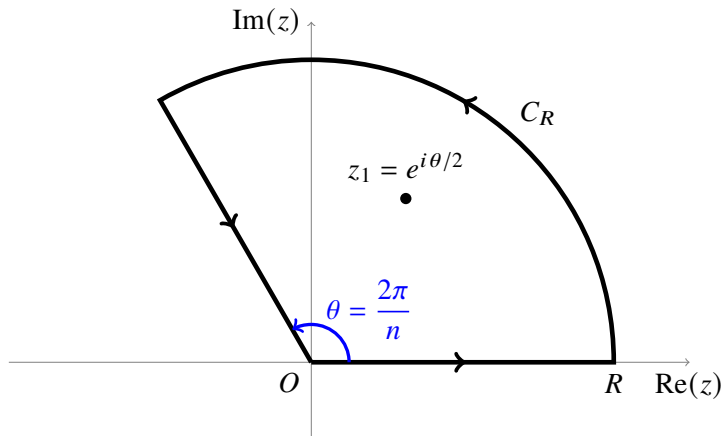


Fig. 1 Sector-shaped contour in the upper plane for $n = 3$. There is one simple pole z_1 within the contour.

Question 2

$$\int_0^\pi \left(\frac{\sin 2x \sin 3x \sin 5x \sin 30x}{\sin x \sin 6x \sin 10x \sin 15x} \right)^2 dx \quad (2.1)$$

Solution Using double- and triple-angle formulae, we have

$$\begin{aligned} I &= \int_0^\pi \left(\frac{\cos x \cos 15x}{\cos 3x \cos 5x} \right)^2 dx = \int_0^\pi \left(\frac{4 \cos^2 5x - 3}{4 \cos^2 x - 3} \right)^2 dx \\ &= \int_0^\pi \left(\frac{2 \cos 10x - 1}{2 \cos 2x - 1} \right)^2 dx = \frac{1}{2} \int_0^{2\pi} \left(\frac{2 \cos 5x - 1}{2 \cos x - 1} \right)^2 dx \end{aligned} \quad (2.2)$$

We consider the following **Fourier series**

$$\frac{2 \cos 5x - 1}{2 \cos x - 1} = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + a_4 \cos 4x. \quad (2.3)$$

This leads to

$$\begin{aligned} 2 \cos 5x &= (-a_0 + a_1 + 1) + (2a_0 - a_1 + a_2) \cos x + (a_1 - a_2 + a_3) \cos 2x \\ &\quad + (a_2 - a_3 + a_4) \cos 3x + (a_3 - a_4) \cos 4x + a_4 \cos 5x \end{aligned} \quad (2.4)$$

and matching the coefficients gives

$$\frac{2 \cos 5x - 1}{2 \cos x - 1} = -1 - 2 \cos x + 2 \cos 3x + 2 \cos 4x. \quad (2.5)$$

Based on the orthogonality of the Fourier basis, the integral is calculated as

$$\begin{aligned} I &= \frac{1}{2} \int_0^{2\pi} (-1 - 2 \cos x + 2 \cos 3x + 2 \cos 4x)^2 dx \\ &= \frac{1}{2} (2\pi + 2^2 \cdot \pi + 2^2 \cdot \pi + 2^2 \cdot \pi) = 7\pi. \end{aligned} \quad (2.6)$$

Question 3

$$\int_{-1/2}^{1/2} \sqrt{x^2 + 1 + \sqrt{x^4 + x^2 + 1}} \, dx \quad (3.1)$$

Solution We notice that

$$x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 + x + 1)(x^2 - x + 1). \quad (3.2)$$

Therefore, denote $u = x^2 + x + 1$ and $v = x^2 - x + 1$, we have

$$\begin{aligned} I &= \frac{1}{\sqrt{2}} \int_{-1/2}^{1/2} \sqrt{u + v + 2\sqrt{uv}} \, dx = \frac{1}{\sqrt{2}} \int_{-1/2}^{1/2} (\sqrt{u} + \sqrt{v}) \, dx \\ &= \sqrt{2} \int_{-1/2}^{1/2} \sqrt{x^2 + x + 1} \, dx = \sqrt{2} \int_0^1 \sqrt{x^2 + \frac{3}{4}} \, dx. \end{aligned} \quad (3.3)$$

Based on the following indefinite integral

$$\int \sqrt{x^2 + a^2} \, dx = \frac{1}{2}x\sqrt{x^2 + a^2} + \frac{a^2}{2} \ln \left| \frac{x + \sqrt{x^2 + a^2}}{a} \right| + C, \quad (3.4)$$

we can see that the function evaluation at $x = 0$ becomes 0, and the result is

$$I = \frac{\sqrt{2}}{2} \left(\sqrt{\frac{7}{4}} + \frac{3}{4} \ln \left| \frac{2 + \sqrt{7}}{\sqrt{3}} \right| \right) = \frac{\sqrt{14}}{4} + \frac{3\sqrt{2}}{8} \ln \left(\frac{2 + \sqrt{7}}{\sqrt{3}} \right). \quad (3.5)$$

Question 4

$$\left| 10^{20} \int_2^\infty \frac{x^9}{x^{20} - 48x^{10} + 575} dx \right| \quad (4.1)$$

Solution With $t = x^{10}$, we notice that

$$\int_2^\infty \frac{x^9}{x^{20} - 48x^{10} + 575} dx = \frac{1}{10} \int_{2^{10}}^\infty \frac{dt}{(t - 24)^2 - 1} = \frac{1}{20} \ln \left(\frac{1001}{999} \right). \quad (4.2)$$

Based on the following **Taylor expansion**

$$\ln \left(\frac{a+x}{a-x} \right) = 2 \left(\frac{x}{a} \right) + \frac{2}{3} \left(\frac{x}{a} \right)^3 + \frac{2}{5} \left(\frac{x}{a} \right)^5 + \dots, \quad (4.3)$$

we finally obtain

$$I = \left| \frac{10^{19}}{2} \ln \left(\frac{10^3 + 1}{10^3 - 1} \right) \right| = 10^{16} + \frac{10^{10} - 1}{3} + \frac{10^4}{5}. \quad (4.4)$$

Question 5

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{\lfloor 2^n x \rfloor}{3^n} \right)^2 dx \quad (5.1)$$

Solution For a real number $x \in (0, 1)$, its **binary representation** is (see 2024 Final: Question 5)

$$x = \sum_{k=1}^{+\infty} \frac{a_k}{2^k} = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_k}{2^k} + \cdots, \quad \text{with } a_k \in \{0, 1\}. \quad (5.2)$$

When $x \in (0, 1)$, we have i.i.d. random variables a_k following the two-point distribution with equal probability. With this representation, we have

$$\lfloor 2^n x \rfloor = 2^{n-1} a_1 + 2^{n-2} a_2 + \cdots + a_n, \quad (5.3)$$

and the summation can be computed as

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lfloor 2^n x \rfloor}{3^n} &= \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{2^{n-m} a_m}{3^n} = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \frac{2^{n-m} a_m}{3^n} \\ &= \sum_{m=1}^{\infty} \sum_{l=0}^{\infty} \frac{2^l a_m}{3^{l+m}} = \sum_{m=1}^{\infty} \frac{a_m}{3^{m-1}}. \end{aligned} \quad (5.4)$$

The integral is now equivalent to an **expectation**, which is shown as follows

$$\begin{aligned} I &= \sum_{m=1}^{\infty} \frac{\mathbb{E}(a_m^2)}{9^{m-1}} + 2 \sum_{1 \leq i < j} \frac{\mathbb{E}(a_i a_j)}{3^{i+j-2}} \\ &= \frac{9}{8} \mathbb{E}(a_m^2) + 2 \mathbb{E}(a_i a_j) \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{3^{2i+l-2}} \\ &= \frac{9}{8} \mathbb{E}(a_m^2) + 2 \mathbb{E}(a_i a_j) \cdot \frac{9}{8} \cdot \frac{1}{2} = \frac{9}{8} \left[\mathbb{E}(a_m^2) + \mathbb{E}(a_i a_j) \right]. \end{aligned} \quad (5.5)$$

Since a_k are i.i.d. and each follows the two-point distribution, we have

$$\mathbb{E}(a_m^2) = \frac{1}{2}, \quad \mathbb{E}(a_i a_j) = \mathbb{E}(a_i) \mathbb{E}(a_j) = \frac{1}{4}. \quad (5.6)$$

Finally, we obtain

$$I = \frac{9}{8} \left[\mathbb{E}(a_m^2) + \mathbb{E}(a_i a_j) \right] = \frac{27}{32}. \quad (5.7)$$