MIT Integration Bee: 2024 Regular Season

Question 1

$$\int_1^{2024} \lfloor \log_{43}(x) \rfloor \, dx \tag{1.1}$$

Solution

$$I = \int_1^{43} 0 \, dx + \int_{43}^{1849} 1 \, dx + \int_{1849}^{2024} 2 \, dx = 1806 + 350 = 2156. \tag{1.2}$$

Question 2

$$\int \frac{dx}{x^{2024} - x^{4047}} \tag{2.1}$$

Solution

$$\begin{aligned} I &= \int \left(\frac{1}{x^{2024}} + \frac{1}{x} \cdot \frac{1}{1 - x^{2023}} \right) dx \\ &= \int \left(\frac{1}{x^{2024}} + \frac{1}{x} + \frac{x^{2022}}{1 - x^{2023}} \right) dx \\ &= -\frac{1}{2023\, x^{2023}} + \ln x - \frac{1}{2023} \ln\left(1 - x^{2023}\right) + C. \end{aligned} \tag{2.2}$$

Question 3

$$\int_0^1 x^2 (1 - x)^{2024} \, dx \tag{3.1}$$

Solution

$$I = B(3, 2025) = \frac{2 \cdot 2024!}{2027!} = \frac{2}{2025 \times 2026 \times 2027}. \tag{3.2}$$

Question 4

$$\int \frac{2023x + 1}{x^2 + 2024}\, \mathrm{d}x \tag{4.1}$$

Solution

$$\begin{aligned} I &= 2023 \int \frac{x}{x^2 + 2024}\, \mathrm{d}x + \int \frac{1}{x^2 + 2024}\, \mathrm{d}x \\ &= \frac{2023}{2} \ln\left(x^2 + 2024\right) + \frac{1}{\sqrt{2024}} \arctan \frac{x}{\sqrt{2024}} + C \end{aligned} \tag{4.2}$$

Question 5

$$\int_0^{\pi/2} \sec^2(x)\, e^{-\sec^2(x)}\, \mathrm{d}x \tag{5.1}$$

Solution Notice that $\sec^2 x = \tan^2 x + 1$. With a change of variable $t = \tan x$, we have

$$I = \int_0^\infty e^{-t^2-1}\, \mathrm{d}t = \frac{\sqrt{\pi}}{2e}. \tag{5.2}$$

Question 6

$$\int \cot x\, \cot 2x\, \mathrm{d}x \tag{6.1}$$

Solution

$$\begin{aligned} I &= \int \frac{\cos 2x}{2\sin^2 x}\, \mathrm{d}x = \int \frac{1 - 2\sin^2 x}{2\sin^2 x}\, \mathrm{d}x \\ &= -\frac{1}{2}\cot x - x + C. \end{aligned} \tag{6.2}$$

Question 7

$$\int \frac{\sinh^2 x}{\tanh 2x}\,\mathrm{d}x \tag{7.1}$$

Solution

$$\begin{aligned} I &= \int \frac{\sinh x \cosh 2x}{2\cosh x}\,\mathrm{d}x = \frac{1}{2}\int \sinh 2x\,\mathrm{d}x - \frac{1}{2}\int \tanh x\,\mathrm{d}x \\ &= \frac{1}{4}\cosh 2x - \frac{1}{2}\ln(\cosh x) + C. \end{aligned} \tag{7.2}$$

Question 8

$$\int \arctan(\sqrt{x})\,\mathrm{d}x \tag{8.1}$$

Solution

$$\begin{aligned} I &= x\arctan\left(\sqrt{x}\right) - \frac{1}{2}\int \frac{\sqrt{x}}{1+x}\,\mathrm{d}x \\ &= x\arctan\left(\sqrt{x}\right) - \int \frac{t^2}{1+t^2}\,\mathrm{d}t \qquad \left(t=\sqrt{x},\quad x=t^2,\quad \mathrm{d}x = 2t\,\mathrm{d}t\right) \\ &= (x+1)\arctan\left(\sqrt{x}\right) - \sqrt{x} + C. \end{aligned} \tag{8.2}$$

Question 9

$$\int_0^\infty \frac{x\ln x}{x^4+1}\,\mathrm{d}x \tag{9.1}$$

Solution With the change of variable $t = 1/x$, we find that

$$I = \int_0^\infty \frac{x\ln x}{x^4+1}\,\mathrm{d}x = -\int_0^\infty \frac{t\ln t}{t^4+1}\,\mathrm{d}t = -I. \tag{9.2}$$

Therefore, the result is $I = 0$.

Question 10

$$\int_0^{10} \lfloor x \lfloor x \rfloor \rfloor \, \mathrm{d}x \tag{10.1}$$

Solution We first study the integral over a general interval $[n, n+1]$ for any $n \in \mathbb{N}$, which is

$$I_n = \int_n^{n+1} \lfloor x \lfloor x \rfloor \rfloor \, \mathrm{d}x = \int_n^{n+1} \lfloor nx \rfloor \, \mathrm{d}x. \tag{10.2}$$

It is trivial that $I_0 = 0$. For $n \geq 1$, the integrand is a piecewise constant function, with its value ranging from n^2 to $n^2 + n - 1$. The integral is equivalent to the average of all these values, giving

$$I_n = \frac{1}{n}\left[n^2 + (n^2+1) + \cdots + (n^2+n-1)\right] = n^2 + \frac{n-1}{2}, \qquad \text{for } n \geq 1. \tag{10.3}$$

Therefore, we obtain

$$\begin{aligned} I = \sum_{k=0}^{9} I_k &= \left(1^2 + 2^2 + \cdots + 9^2\right) + \frac{1+2+\cdots+8}{2} \\ &= \frac{9 \times 10 \times 19}{6} + 18 = 303. \end{aligned} \tag{10.4}$$

Question 11

$$\int_0^{1} e^{-x} \sqrt{1 + \cot^2\left(\arccos e^{-x}\right)} \, \mathrm{d}x \tag{11.1}$$

Solution With the change of variable $t = \arccos e^{-x}$, we have

$$t(0) = 0, \qquad t(1) = \arccos e^{-1}, \qquad x = -\ln|\cos t|, \qquad \mathrm{d}x = \tan t \, \mathrm{d}t. \tag{11.2}$$

Therefore, the integral becomes

$$\begin{aligned} I &= \int_0^{\arccos e^{-1}} \cos t \sqrt{1 + \cot^2 t} \cdot \tan t \, \mathrm{d}t \\ &= \int_0^{\arccos e^{-1}} \sqrt{\sin^2 t + \cos^2 t} \, \mathrm{d}t = \arccos e^{-1}. \end{aligned} \tag{11.3}$$

Question 12

$$\int_1^3 \frac{1+\frac{1+\cdots}{x+\cdots}}{x+\frac{1+\cdots}{x+\cdots}}\,\mathrm{d}x \tag{12.1}$$

Solution Denote the integrand as $f(x)$. Therefore, we have

$$f(x) = \frac{1+f(x)}{x+f(x)}, \qquad f(x) = \frac{1-x+\sqrt{(x-1)^2+4}}{2}. \tag{12.2}$$

The integral can be evaluated as

$$\begin{aligned} I &= -1 + \frac{1}{2}\int_0^2 \sqrt{x^2+4}\,\mathrm{d}x \\ &= -1 + \frac{1}{2}\left(\frac{1}{2}x\sqrt{x^2+4} + 2\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right|\right)\Bigg|_0^2 \\ &= -1 + \sqrt{2} + \ln\left(1+\sqrt{2}\right). \end{aligned} \tag{12.3}$$

Note A better way is to use **integration by parts**

$$x = \frac{1+f-f^2}{f}, \qquad f(1) = 1, \qquad f(3) = \sqrt{2}-1. \tag{12.4}$$

The integral can be alternatively calculated as

$$I = xf\big|_1^3 - \int_1^{\sqrt{2}-1}\left(\frac{1}{f}+1-f\right)\mathrm{d}f = \sqrt{2}-1-\ln\left(\sqrt{2}-1\right). \tag{12.5}$$

Question 13

$$\int_0^1 \frac{2x(1-x)^2}{1+x^2}\,\mathrm{d}x \tag{13.1}$$

Solution

$$I = \int_0^1 \left(2x - 4 + \frac{4}{1+x^2}\right)\mathrm{d}x = \pi - 3. \tag{13.2}$$

Question 14

$$\int e^{e^x+3x}\,\mathrm{d}x \tag{14.1}$$

Solution With the change of variable $t = e^x$, we have

$$\begin{aligned} I &= \int e^{e^x+3x}\,\mathrm{d}x = \int e^{t+3\ln t}\,\frac{\mathrm{d}t}{t} \\ &= \int t^2 e^t\,\mathrm{d}t = (t^2 - 2t + 2)e^t + C \\ &= \left(e^{2x} - 2e^x + 2\right)e^{e^x} + C. \end{aligned} \tag{14.2}$$

Question 15

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} 2\left(1 - \frac{|x|}{\sqrt{3}}\right)\mathrm{d}x \tag{15.1}$$

Solution

$$I = 2\sqrt{3} - \frac{4}{\sqrt{3}}\int_0^{\sqrt{3}/2} x\,\mathrm{d}x = \frac{3\sqrt{3}}{2}. \tag{15.2}$$

Question 16

$$\int \frac{\ln\left(1+x^2\right)}{x^2}\,\mathrm{d}x \tag{16.1}$$

Solution

$$\begin{aligned} I &= -\frac{\ln\left(1+x^2\right)}{x} + 2\int \frac{\mathrm{d}x}{1+x^2} \\ &= -\frac{\ln\left(1+x^2\right)}{x} + 2\arctan x + C. \end{aligned} \tag{16.2}$$

Question 17

$$\int 2^x x^2\,\mathrm{d}x \tag{17.1}$$

Solution

$$\begin{aligned} I &= \int 2^x x^2\,\mathrm{d}x = \frac{1}{\ln 2}\left(2^x x^2 - 2\int 2^x x\,\mathrm{d}x\right) \\ &= \frac{2^x x^2}{\ln 2} - \frac{2}{\ln^2 2}\left(2^x x - \frac{2^x}{\ln 2}\right) + C \\ &= \frac{2^x}{\ln^3 2}\left(x^2\ln^2 2 - 2x\ln 2 + 2\right) + C. \end{aligned} \tag{17.2}$$

Question 18

$$\int_0^1 \sqrt{x^8 - x^6 + x^4}\cdot\sqrt{1+x^2}\,\mathrm{d}x \tag{18.1}$$

Solution With the change of variable $t = x^3$, we have

$$\begin{aligned} I &= \int_0^1 x^2\sqrt{1+x^6}\,\mathrm{d}x = \frac{1}{3}\int_0^1 \sqrt{1+t^2}\,\mathrm{d}t \\ &= \frac{1}{6}\left[t\sqrt{1+t^2} + \ln\left(t+\sqrt{1+t^2}\right)\right]_0^1 = \frac{1}{6}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]. \end{aligned} \tag{18.2}$$

Question 19

$$\int_1^\infty \frac{e^x + xe^x}{x^2 e^{2x} - 1}\,\mathrm{d}x \tag{19.1}$$

Solution With the change of variable $t = xe^x$, we have

$$\begin{aligned} I &= \int_1^\infty \frac{\mathrm{d}\,(xe^x)}{x^2 e^{2x} - 1} = \int_e^\infty \frac{\mathrm{d}t}{t^2 - 1} \\ &= \frac{1}{2}\ln\left(\frac{t-1}{t+1}\right)\bigg|_e^\infty = \frac{1}{2}\ln\left(\frac{e+1}{e-1}\right). \end{aligned} \tag{19.2}$$

Question 20

$$\int_0^\infty \left(80x^3 - 60x^4 + 14x^5 - x^6\right) e^{-x}\,\mathrm{d}x \tag{20.1}$$

Solution For each term, we can obtain

$$\begin{aligned} I_n &= \int_0^\infty a_n\, x^n e^{-x}\,\mathrm{d}x = \int_0^\infty n a_n\, x^{n-1} e^{-x}\,\mathrm{d}x = \cdots \\ &= \int_0^\infty n! \cdot a_n\, e^{-x}\,\mathrm{d}x = n! \cdot a_n. \end{aligned} \tag{20.2}$$

Therefore, we have

$$I = 80 \times 6 - 60 \times 24 + 14 \times 120 - 720 = -80 \times 12 + 120 \times 8 = 0. \tag{20.3}$$