Solution to Quantitative Seismology

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Contents

1 Chapter 1: Introduction

The **Introduction** part in *Quantitative Seismology* provides various useful reference books and websites. Hope you'll also find them beneficial.

2 Chapter 2: Basic Theorems in Dynamic Elasticity

2.1 Question 1

The equation of motion is

$$\rho \ddot{u}_i = f_i + \tau_{ij,j} \tag{2.1}$$

Considering the constitutive law and the expression of strain tensor

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl}, \qquad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right),$$

and taking into account the symmetry of the elastic tensor, we could get

$$\rho \ddot{u}_i = f_i + \left(c_{ijkl}u_{k,l}\right)_j. \tag{2.2}$$

For an isotropic and homogeneous medium, the elastic tensor could be expressed as

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{2.3}$$

So the corresponding displacement equation becomes

$$\begin{aligned}\rho \ddot{u}_i &= f_i + \lambda u_{j,ji} + \mu(u_{i,jj} + u_{j,ji}) \\ &= (\lambda+\mu)u_{j,ji} + \mu u_{i,jj} \end{aligned} \tag{2.4}$$

Using the following identities

$$u_{j,ji} = [\nabla(\nabla\cdot\mathbf{u})]_i, \qquad u_{i,jj} = (\nabla^2\mathbf{u})_i, \qquad \nabla\times(\nabla\times\mathbf{u}) = \nabla(\nabla\cdot\mathbf{u}) - \nabla^2\mathbf{u},$$

we could find the vector displacement equation

$$\begin{aligned}\rho\ddot{\mathbf{u}} &= \mathbf{f} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} \\ &= \mathbf{f} + (\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla\times(\nabla\times\mathbf{u}) \end{aligned} \tag{2.5}$$

2.2 Question 2

From definition, we could derive the well-known identity

$$\begin{aligned}\varepsilon_{ijk}\varepsilon_{ilm} &= \begin{vmatrix} \delta_{ii} & \delta_{ji} & \delta_{ki} \\ \delta_{il} & \delta_{jl} & \delta_{kl} \\ \delta_{im} & \delta_{jm} & \delta_{km} \end{vmatrix} = \delta_{ii}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) - \delta_{ji}(\delta_{il}\delta_{km} - \delta_{im}\delta_{kl}) + \delta_{ki}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \\ &= 3\cdot(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) - (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) + (\delta_{kl}\delta_{jm} - \delta_{km}\delta_{jl}) \\ &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \end{aligned} \tag{2.6}$$

and similarly

$$\varepsilon_{ijk}\varepsilon_{jlm} = -\varepsilon_{jik}\varepsilon_{jlm} = \delta_{im}\delta_{kl} - \delta_{il}\delta_{km}. \tag{2.7}$$

2.3 Question 3

All we need to do is express e_{kk} (dilatation) with τ_{ii} (isotropic pressure). Using the constitutive law for an isotropic elastic solid and setting $i = j$, we could get

$$\tau_{ii} = \lambda e_{kk}\delta_{ii} + 2\mu e_{ii} = (3\lambda + 2\mu)e_{kk},$$

which is equal to

$$e_{kk} = \frac{1}{3\lambda + 2\mu}\tau_{kk}. \tag{2.8}$$

Substituting Eq.(2.8) into the stress-strain relation, we could obtain the strain-stress relation

$$2\mu e_{ij} = -\frac{\lambda}{3\lambda + 2\mu}\tau_{kk}\delta_{ij} + \tau_{ij}. \tag{2.9}$$

2.4 Question 4

Reference: A piece of lecture note from Prof. Paul A. Lagace
https://ocw.mit.edu/courses/aeronautics-and-astronautics/16-20-structural-mechanics-fall-2002/lecture-notes/unit9.pdf

If the material is unrestrained and its temperature is raised, we would expect that the material will undergo thermal expansion. However, in this question the strain is fixed, which implies that there must be additional **thermal stress** in the body. Given that the **thermal strain** is expressed by the coefficient tensor of thermal expansion

$$\varepsilon^T_{kl} = \alpha_{kl}\Delta T, \tag{2.10}$$

the corresponding thermal stress should be

$$\sigma^T_{ij} = c_{ijkl} \cdot (-\varepsilon^T_{kl}) = -c_{ijkl}\alpha_{kl}\Delta T. \tag{2.11}$$

The negative sign represents that the mechanical strain should counteract the thermal strain. Taking into account this thermal effect, the modified stress-strain relation should be in the following form

$$\sigma_{ij} = c_{ijkl}(\varepsilon_{kl} - \alpha_{kl}\Delta T). \tag{2.12}$$

2.5 Question 5

Given $\mathbf{u}(\mathbf{x}, t)$, we could obtain the strain and stress

$$\varepsilon(\mathbf{x}, t) = \frac{1}{2}\left[\nabla\mathbf{u}(\mathbf{x}, t) + (\nabla\mathbf{u}(\mathbf{x}, t))^{\mathrm{T}}\right], \tag{2.13}$$

$$\boldsymbol{\sigma}(\mathbf{x}, t) = \mathbf{c}(\mathbf{x}, t) : \varepsilon(\mathbf{x}, t). \tag{2.14}$$

The traction could be derived from the stress

$$\mathbf{T}(\mathbf{x}, t) = \hat{\mathbf{n}}(\mathbf{x}, t) \cdot \boldsymbol{\sigma}(\mathbf{x}, t), \tag{2.15}$$

and the body force could be expressed based on the equation of motion

$$\mathbf{f}(\mathbf{x}, t) = \rho(\mathbf{x}, t)\ddot{\mathbf{u}}(\mathbf{x}, t) - \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}, t) \tag{2.16}$$

2.6 Question 6

Relations (2.21)-(2.25) in the book do not involve the dependency of stress on strain or strain rate, so they should not change.

2.7 Question 7

The traction is obtained by Eq.(2.15). In an isotropic medium,

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} \tag{2.17}$$

$$\boldsymbol{\sigma} = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + \mu[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}]. \tag{2.18}$$

So the traction could be expressed as

$$\begin{aligned} \mathbf{T}(\mathbf{u}, \hat{\mathbf{n}}) &= \hat{\mathbf{n}} \cdot \boldsymbol{\sigma} \\ &= \lambda(\nabla \cdot \mathbf{u})\hat{\mathbf{n}} + \mu[\hat{\mathbf{n}} \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u}) \cdot \hat{\mathbf{n}}]. \end{aligned}$$

Using the following expression

$$\begin{aligned} [\hat{\mathbf{n}} \times (\nabla \times \mathbf{u})]_i &= \varepsilon_{ijk} n_j (\varepsilon_{klm} \partial_l u_m) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) n_j \partial_l u_m \\ &= n_j \partial_i u_j - n_j \partial_j u_i \\ &= [(\nabla \mathbf{u}) \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot (\nabla \mathbf{u})]_i \end{aligned}$$

and its corresponding vector formula

$$\hat{\mathbf{n}} \times (\nabla \times \mathbf{u}) = (\nabla \mathbf{u}) \cdot \hat{\mathbf{n}} - \hat{\mathbf{n}} \cdot (\nabla \mathbf{u}) = (\nabla \mathbf{u}) \cdot \hat{\mathbf{n}} - (\hat{\mathbf{n}} \cdot \nabla)\mathbf{u}, \tag{2.19}$$

we could derive the traction

$$\begin{aligned} \mathbf{T}(\mathbf{u}, \hat{\mathbf{n}}) &= \lambda(\nabla \cdot \mathbf{u})\hat{\mathbf{n}} + \mu[\hat{\mathbf{n}} \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u}) \cdot \hat{\mathbf{n}}] \\ &= \lambda(\nabla \cdot \mathbf{u})\hat{\mathbf{n}} + \mu[2(\hat{\mathbf{n}} \cdot \nabla)\mathbf{u} + \hat{\mathbf{n}} \times (\nabla \times \mathbf{u})] \\ &= \lambda(\nabla \cdot \mathbf{u})\hat{\mathbf{n}} + \mu\left[2\frac{\partial \mathbf{u}}{\partial n} + \hat{\mathbf{n}} \times (\nabla \times \mathbf{u})\right]. \end{aligned} \tag{2.20}$$

2.8 Question 8

a) Fig. 2.1 is the same with Fig. 2.4 in the book. According to its force equilibrium state, we could find

$$\mathbf{T}(\mathbf{x} + \delta\mathbf{x}, \hat{\mathbf{n}}) + \mathbf{T}(\mathbf{x}, -\hat{\mathbf{n}}) \to 0 \quad \text{as} \quad \delta\mathbf{x} \to 0.$$

Using eq. (2.7) in the book, we could show that traction is a continuous function of position

$$\mathbf{T}(\mathbf{x} + \delta\mathbf{x}, \hat{\mathbf{n}}) - \mathbf{T}(\mathbf{x}, \hat{\mathbf{n}}) \to 0 \quad \text{as} \quad \delta\mathbf{x} \to 0. \tag{2.21}$$

Here, $\delta\mathbf{x}$ is taken parallel to the direction $\hat{\mathbf{n}}$.

b) For the area which the book lies on, the traction is non-zero. However, for the part outside of the above area, the traction is zero. Assume that the z-direction is perpendicular to the flat surface of the table, the traction is not a continuous function in the x- or y-direction.

c) In problem **a)**, $\delta\mathbf{x}$ is taken parallel to the direction $\hat{\mathbf{n}}$. The continuity of traction in this sense (i.e. in the z-direction) still holds true for the table if we analyze the traction inside it, but this does not contradict with the fact that the traction is not continuous in the x- or y-directions.

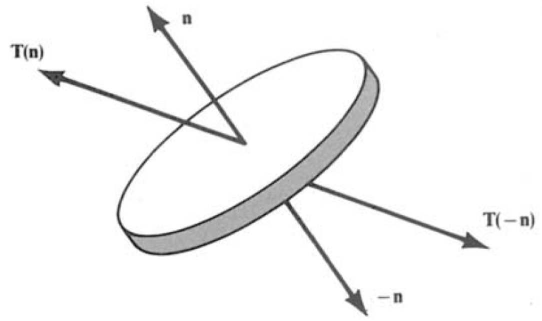

Fig. 2.1 A small disc within a stressed medium (Figure 2.4)

d) First choose $\delta\mathbf{x}$ to be parallel to the z-direction, the continuity of the traction gives

$$\mathbf{T}(\mathbf{x}) = \begin{bmatrix} \tau_{xz}(\mathbf{x}) \\ \tau_{yz}(\mathbf{x}) \\ \tau_{zz}(\mathbf{x}) \end{bmatrix} \text{ is continuous of } z.$$

We could also choose $\delta\mathbf{x}$ to be parallel to the x- or y-directions and obtain that

$$\tau_{ij}(\mathbf{x}) \text{ is continuous in the } i\text{- and } j\text{-directions.}$$

Using the above conclusion, we could know that τ_{zz} need not be continuous in the x- or y-directions, and that τ_{xx}, τ_{yy} and τ_{xy} need not be continuous in the z-direction.

2.9 Question 9

First, we express the stress tensor with isotropic and deviatoric strain, as well as the Lamé parameters:

$$\begin{aligned} \tau_{ij} &= \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \\ &= \lambda e_{kk}\delta_{ij} + 2\mu\left(\frac{1}{3}e_{kk}\delta_{ij} + e'_{ij}\right) \\ &= \left(\lambda + \frac{2}{3}\mu\right)e_{kk}\delta_{ij} + 2\mu e'_{ij}. \end{aligned}$$

Therefore the strain energy density is given by

$$\begin{aligned} \mathcal{U} = \frac{1}{2}\tau_{ij}e_{ij} &= \frac{1}{2}\left[\left(\lambda + \frac{2}{3}\mu\right)e_{kk}\delta_{ij} + 2\mu e'_{ij}\right]\left(\frac{1}{3}e_{kk}\delta_{ij} + e'_{ij}\right) \\ &= \frac{1}{2}\left[\left(\lambda + \frac{2}{3}\mu\right)e_{ii}e_{kk} + 2\mu e'_{ij}e'_{ij}\right], \end{aligned} \tag{2.22}$$

where we use properties $\delta_{ij}\delta_{ij} = 3$ and $\mathrm{tr}(\mathbf{e}') = e'_{ii} = 0$.

Let's see why $\theta = e_{ii}$ is called **dilatation**. Consider a cuboid and the lengths of its three sides are a,b and c separately. The percentage change of its volume after an infinitesimal deformation is

$$\begin{aligned} \frac{\Delta V}{V} = \frac{V + \Delta V}{V} - 1 &= \frac{a(1 + e_{xx}) \cdot b(1 + e_{yy}) \cdot c(1 + e_{zz})}{abc} - 1 \\ &= (1 + e_{xx})(1 + e_{yy})(1 + e_{zz}) - 1 \doteq e_{ii} = \mathrm{tr}(\mathbf{e}). \end{aligned} \tag{2.23}$$

The isotropic and deviatoric strains and stresses, as well as the corresponding moduli, are summarized in Table 2.1, which is modified from Table 6.1 in Dahlen & Tromp (1998). Table 2.1 shows why κ, the **bulk modulus** or **incompressibility**, and μ, the **shear modulus**, are widely used in seismology, since they have clearer physical meanings than Lamé parameters. In addition, from Eq. (2.22) we know that these two moduli should be positive so that the strain energy density is positive.

Variable	Isotropic	Deviatoric
Strain ε	$\theta = \mathrm{tr}(\mathbf{e})$	$\mathbf{e}' = \mathbf{e} - \frac{1}{3}\mathrm{tr}(\mathbf{e})\mathbf{I}$
Stress σ	$-p = \frac{1}{3}\mathrm{tr}(\boldsymbol{\tau})$	$\boldsymbol{\tau}' = \boldsymbol{\tau} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}$
Modulus M	κ	2μ

Table 2.1 Constitutive laws for isotropic and deviatoric components. Modified from Dahlen & Tromp (1998).

Reference: Chapter 6 in Dahlen, F. A., and J. Tromp, Theoretical Global Seismology, Princeton, New Jersey: Princeton University Press, 1998.

2.10 Question 10

We consider the representation theorem in the following form

$$\mathbf{u}(\mathbf{x}, t) = \iiint_V \int_0^t \mathbf{G}(\mathbf{x}, t - \tau; \mathbf{x}', \tau) \cdot \mathbf{f}(\mathbf{x}', \tau)\, d\tau d^3\mathbf{x}'. \tag{2.24}$$

In this problem, the force could be expressed as

$$\mathbf{f}(\mathbf{x}', \tau) = \boldsymbol{\nu}\delta(\mathbf{x}' - \boldsymbol{\xi})\delta(\tau). \tag{2.25}$$

Therefore, we could obtain the displacement vector

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{G}(\mathbf{x}, t; \boldsymbol{\xi}, 0) \cdot \boldsymbol{\nu}, \tag{2.26}$$

and its component in the **n**-direction

$$u_n(\mathbf{x}, t) = \mathbf{n} \cdot \mathbf{u}(\mathbf{x}, t) = n_i G_{ip}(\mathbf{x}, t; \boldsymbol{\xi}, 0)\nu_p. \tag{2.27}$$

Using the reciprocity of the Green tensor, we could get

$$\begin{aligned} u_n(\mathbf{x}, t) &= n_i G_{ip}(\mathbf{x}, t; \boldsymbol{\xi}, 0)\nu_p \\ &= u_\nu(\boldsymbol{\xi}, t) = \nu_p G_{pi}(\boldsymbol{\xi}, t; \mathbf{x}, 0) n_i. \end{aligned}$$

3 Chapter 3: Representation of Seismic Sources

3.1 Question 1

a) Generalize eq. (3.26) in the book to a vector equation. Using the following relation

$$\mathbf{M} = M_0\left(\hat{\mathbf{n}}\hat{\mathbf{d}} + \hat{\mathbf{d}}\hat{\mathbf{n}}\right), \qquad M_{ij} = M_0(n_i d_j + n_j d_i) \tag{3.1}$$

and the assumption that S is planar and constant (i.e. $\hat{\mathbf{n}}$ does not change) and that the displacement discontinuity for each event is a shear (i.e. $\hat{\mathbf{n}} \cdot \hat{\mathbf{d}} = 0$), we could obtain

$$\mathbf{M} \cdot \hat{\mathbf{n}} = M_0 \hat{\mathbf{d}}. \tag{3.2}$$

The right part of Eq. (3.2) allows us to obtain the slips in each direction. Therefore, the generalized vector version of eq. (3.26) in the book should be

$$\Delta \mathbf{U} = \frac{\left(\sum_{i=1}^{N} \mathbf{M}^i\right) \cdot \hat{\mathbf{n}}}{\mu S} \tag{3.3}$$

b) Generalize eq. (3.34) in the book to a tensor equation. For an isotropic medium with volume V, the relationship between moment tensors and strains is

$$M_{ij} = c_{ijkl} \Delta e_{kl} V. \tag{3.4}$$

Recall the result in Section 2.3, we could find the total strain, i.e. the generalized tensor version of eq. (3.34) in the book

$$\Delta E_{ij} = \frac{1}{2\mu V} \cdot \sum_{n=1}^{N} \left(-\frac{\lambda}{3\lambda + 2\mu} M_{kk}^n \delta_{ij} + M_{ij}^n \right). \tag{3.5}$$

3.2 Question 2

For eq. (3.2) in the book, we can rewrite it using the reciprocity of Green's function:

$$\begin{aligned} u_n(\mathbf{x}, t) &= \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} [u_i(\boldsymbol{\xi}, \tau)] c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\xi}, 0)\, d\Sigma \\ &= \int_{-\infty}^{\infty} d\tau \iint_{\Sigma} [u_i(\boldsymbol{\xi}, 0)] c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{pn}(\boldsymbol{\xi}, t - \tau; \mathbf{x}, 0)\, d\Sigma. \end{aligned} \tag{3.6}$$

Therefore, the following part in the integrand

$$c_{ijpq} \nu_j \frac{\partial}{\partial \xi_q} G_{pn}(\boldsymbol{\xi}, t - \tau; \mathbf{x}, 0)$$

can be interpreted as a traction on the internal surface Σ. According to the continuity of traction, we can still derive eq. (3.2) from eq. (3.1) in the book, but now $\partial G_{np} / \partial \xi_q$ may not be continuous across the surface.

3.3 Question 3

If $\bar{u}(t)$ is averaged over the area $A(t)$ that has ruptured at time t, then we need to use $A(t)$ to calculate $M_0(t) = \mu \bar{u}(t) A(t)$. Similarly, if $\bar{u}(t)$ is averaged over the area $A(\infty)$, the ultimate ruptured area, then we need to use $A(\infty)$ to calculate $M_0(t) = \mu \bar{u}(t) A(\infty)$. Hence, consistency of the definition of 'average' throughout computation matters.

3.4 Question 4

This problem is equivalent to finding the eigenvalues of matrix $\mathbf{M}$. Since we have

$$|\lambda \mathbf{I} - \mathbf{M}| = \lambda(\lambda^2 - M_0^2), \tag{3.7}$$

the eigenvalues are M_0, 0 and $-M_0$. Consequently, under the principal coordinates a double-couple can be equivalently described by $\mathbf{M} = \mathrm{diag}(M_0, 0, -M_0)$.

3.5 Question 5

A symmetric second-order moment tensor $\mathbf{M}$ can be diagonalized. Therefore, in the principal coordinates, we have

$$\begin{aligned}\mathbf{M} &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \frac{\mathrm{tr}(\mathbf{M})}{3} \cdot \mathbf{I} + \begin{bmatrix} \lambda_1' & 0 & 0 \\ 0 & \lambda_2' & 0 \\ 0 & 0 & \lambda_3' \end{bmatrix} \\ &= \frac{\mathrm{tr}(\mathbf{M})}{3} \cdot \mathbf{I} + \begin{bmatrix} \lambda_1' & 0 & 0 \\ 0 & -\lambda_1' & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3' & 0 \\ 0 & 0 & \lambda_3' \end{bmatrix}, \end{aligned} \tag{3.8}$$

which implies that $\mathbf{M}$ can be thought of as an isotropic point source plus two double couples. This decomposition of a point source is not unique.

We can also write $\mathbf{M}$ in the form

$$\begin{aligned}\mathbf{M} &= \frac{\mathrm{tr}(\mathbf{M})}{3} \cdot \mathbf{I} + \begin{bmatrix} \lambda_1' & 0 & 0 \\ 0 & \lambda_2' & 0 \\ 0 & 0 & \lambda_3' \end{bmatrix} \\ &= \frac{\mathrm{tr}(\mathbf{M})}{3} \cdot \mathbf{I} + \left(\lambda_1' + \frac{\lambda_3'}{2}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda_3' \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{3.9}$$

which implies the decomposition of $\mathbf{M}$ into the best double couple and the associated CLVD component.

3.6 Question 6

Integration by parts in three dimensions can be derived based on the product rule for divergence, which is

$$\nabla \cdot (u\mathbf{V}) = u\nabla \cdot \mathbf{V} + \nabla u \cdot \mathbf{V}. \tag{3.10}$$

This leads to

$$\iint_S u\mathbf{V} \cdot \hat{\mathbf{n}}\, dS = \iiint_V u\nabla \cdot \mathbf{V}\, dV + \iiint_V \nabla u \cdot \mathbf{V}\, dV. \tag{3.11}$$

I will use Eq. (3.11) to derive the equivalent body force.

For the following body force

$$\mathbf{f}(\mathbf{x}, t) = -\mathbf{M}(t) \cdot \nabla\delta(\mathbf{x} - \boldsymbol{\xi}), \tag{3.12}$$

the representation theorem gives

$$\begin{aligned} u_n(\mathbf{x}, t) &= \int_{-\infty}^{+\infty} d\tau \iiint_V f_p(\boldsymbol{\eta}, \tau) G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\eta}, 0)\, dV(\boldsymbol{\eta}) \\ &= -\int_{-\infty}^{+\infty} d\tau \iiint_V M_{pq}(\tau) G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\eta}, 0) \frac{\partial \delta(\boldsymbol{\eta} - \boldsymbol{\xi})}{\partial \eta_q}\, dV(\boldsymbol{\eta}) \\ &= \int_{-\infty}^{+\infty} d\tau \iiint_V M_{pq}(\tau) \left[\frac{\partial}{\partial \eta_q} G_{np}(\mathbf{x}, t - \tau; \boldsymbol{\eta}, 0)\right] \delta(\boldsymbol{\eta} - \boldsymbol{\xi})\, dV(\boldsymbol{\eta}) \\ &= M_{pq}(t) * \frac{\partial}{\partial \xi_q} G_{np}(\mathbf{x}, t; \boldsymbol{\xi}, 0). \end{aligned} \tag{3.13}$$

From the second row to the third row, I use Eq. (3.11) with the scalar function u now being the δ function. Therefore, Eq. (3.12) is the equivalent body force to a point source at $\boldsymbol{\xi}$ with moment tensor $\mathbf{M}$.

3.7 Question 7

a) In the spherical coordinate, the curl of a vector $\mathbf{A}$ is calculated as

$$\nabla \times \mathbf{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta \mathbf{e}_\varphi \\ \partial_r & \partial_\theta & \partial_\varphi \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix}. \tag{3.14}$$

Therefore, when the displacement is only in the radial direction, we have $\nabla \times \mathbf{u} = 0$. For the final static displacement, the inertial term and the external force term are also zero. Based on the vector wave equation in Eq. (2.5), the final static displacement satisfies $\nabla(\nabla \cdot \mathbf{u}) = 0$.

b) Since we have

$$\nabla(\nabla \cdot \mathbf{u}) = \frac{d}{dr}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 u_r\right)\right] = 0, \tag{3.15}$$

its general solution is

$$u_r = Ar + \frac{B}{r^2}. \tag{3.16}$$

For the external solution when $r \geq a$, the constraint $u_r \to 0$ for $r \to \infty$ guarantees that the radial displacement is proportional to $1/r^2$.

c) The stress-strain relation for τ_{rr}, based on eq. (2.50) in the book, can be written as

$$\tau_{rr} = \lambda(e_{rr} + e_{\theta\theta} + e_{\varphi\varphi}) + 2\mu e_{rr}. \tag{3.17}$$

From eq. (2.45) in the book, with $h^1 = 1, h^2 = r$ and $h^3 = r\sin\theta$ for the spherical coordinate, we have

$$\begin{aligned} e_{rr} &= \frac{u_r}{r}, \\ e_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r}, \\ e_{\varphi\varphi} &= \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{\cot\theta}{r}u_\theta. \end{aligned} \tag{3.18}$$

Therefore, when we only have radial displacement u_r, Eq. (3.17) leads to

$$\tau_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \frac{2\lambda}{r}u_r. \tag{3.19}$$

d) The external solution has the form $u_r = B/r^2$. We also know that the walls of the cavity at $r = a$ experience $\tau_{rr} = -\delta p$. With this condition, we have

$$-\delta p = (\lambda + 2\mu)\cdot\frac{-2B}{a^3} + 2\lambda\cdot\frac{B}{a^3} = -4\mu\cdot\frac{B}{a^3}, \tag{3.20}$$

which gives $B = \delta p \cdot a^3/4\mu$. Therefore, the final outward static displacement at $r = a$ is

$$\delta a = u_r(a) = \frac{\delta p \cdot a}{4\mu}, \tag{3.21}$$

which is equivalent to

$$\delta p = 4\mu\frac{\delta a}{a}. \tag{3.22}$$

3.8 Question 8

a) Following the procedure in Question 7, the internal solution has the form $u_r = Ar$. With Eq. (3.19) we have

$$\delta p = -\tau_{rr} = -(3\lambda + 2\mu)A. \tag{3.23}$$

b) The effects of confinement reduces the static displacement at $r = a$ from Δa to δa, which can be expressed as

$$u_r(a) = Aa = -(\Delta a - \delta a), \tag{3.24}$$

from which we can determine the constant A. This leads to

$$\delta p = \frac{3\lambda + 2\mu}{a}(\Delta a - \delta a). \tag{3.25}$$

Using Eq. (3.22) from Question 7, we obtain

$$\begin{aligned} 4\mu\delta a &= (3\lambda + 2\mu)(\Delta a - \delta a) \\ \Delta a &= \frac{\lambda + 2\mu}{\lambda + \frac{2}{3}\mu}\delta a. \end{aligned} \tag{3.26}$$

c) Because $\Delta V = 4\pi a^2 \Delta a$ and $\delta V = 4\pi a^2 \delta a$, the proportionality between ΔV and δV is the same as in Eq. (3.26). Therefore,

$$M_0(\infty) = \left(\lambda + \frac{2}{3}\mu\right)\Delta V = (\lambda + 2\mu)\,\delta V. \tag{3.27}$$

d) The external solution gives that $u_r = B/r^2$ where B is a constant determined by δp and the source region radius a_0. Therefore, when we evaluate the outward actual static displacement for a new surface at r, the corresponding actual final (static) volume change is

$$\delta V = 4\pi r^2 u_r = 4\pi B = \pi a_0^3\,\frac{\delta p}{\mu}. \tag{3.28}$$

Hence, δV is unchanged in value. In other words, it is independent of where the outward actual static displacement is measured.

4 Chapter 4: Elastic Waves from a Point Dislocation Source

4.1 Question 1

When the source time function is $S(t) = \delta(t)$, we have

$$G_{np}(\mathbf{x}, t; \mathbf{0}, 0) = \frac{3\gamma_n\gamma_p - \delta_{np}}{4\pi\rho r^3} t\,[H(t - t_P) - H(t - t_S)] + \frac{\gamma_n\gamma_p}{4\pi\rho\alpha^2 r}\delta(t - t_P) - \frac{\gamma_n\gamma_p - \delta_{np}}{4\pi\rho\beta^2 r}\delta(t - t_S). \tag{4.1}$$

Therefore, the area under the near-field pulse can be calculated as

$$S^{NF} \propto \frac{1}{r^3}\int_{t_P}^{t_S} t\,dt = \frac{1}{2r}\left(\frac{1}{\beta^2} - \frac{1}{\alpha^2}\right), \tag{4.2}$$

which is proportional to $1/r$. It is obvious that for the far-field terms, the area under each pulse is proportional to $1/r$.

In the frequency domain, the P-wave term is

$$G^P_{np}(\mathbf{x}, \omega; \mathbf{0}, 0) = \frac{e^{i\omega r/\alpha}}{4\pi\rho\alpha^2 r}\left[\gamma_n\gamma_p + (3\gamma_n\gamma_p - \delta_{np})\left(-\frac{\alpha}{i\omega r}\right) + (3\gamma_n\gamma_p - \delta_{np})\left(-\frac{\alpha}{i\omega r}\right)^2\right], \tag{4.3}$$

and similarly, the S-wave term is

$$G^S_{np}(\mathbf{x}, \omega; \mathbf{0}, 0) = -\frac{e^{i\omega r/\beta}}{4\pi\rho\beta^2 r}\left[(\gamma_n\gamma_p - \delta_{np}) + (3\gamma_n\gamma_p - \delta_{np})\left(-\frac{\beta}{i\omega r}\right) + (3\gamma_n\gamma_p - \delta_{np})\left(-\frac{\beta}{i\omega r}\right)^2\right]. \tag{4.4}$$

As $\omega \to 0$, we have

$$e^{i\omega r/\alpha} \approx 1 + \left(\frac{i\omega r}{\alpha}\right) + \frac{1}{2}\left(\frac{i\omega r}{\alpha}\right)^2. \tag{4.5}$$

Therefore, Eqs. (4.3) and (4.4) can be written as

$$G^P_{np}(\mathbf{x}, \omega; \mathbf{0}, 0) = \frac{1}{4\pi\rho r}\left(\frac{\delta_{np} - \gamma_n\gamma_p}{2\alpha^2} - \frac{3\gamma_n\gamma_p - \delta_{np}}{\omega^2 r^2}\right) + O(\omega^0), \tag{4.6}$$

$$G^S_{np}(\mathbf{x}, \omega; \mathbf{0}, 0) = \frac{1}{4\pi\rho r}\left(\frac{\delta_{np} + \gamma_n\gamma_p}{2\beta^2} + \frac{3\gamma_n\gamma_p - \delta_{np}}{\omega^2 r^2}\right) + O(\omega^0), \tag{4.7}$$

which leads to

$$\lim_{\omega\to 0} G_{np}(\mathbf{x}, \omega; \mathbf{0}, 0) = \lim_{\omega\to 0}\left(G^P_{np} + G^S_{np}\right) = \frac{1}{8\pi\rho r}\left(\frac{\delta_{np} - \gamma_n\gamma_p}{\alpha^2} + \frac{\delta_{np} + \gamma_n\gamma_p}{\beta^2}\right). \tag{4.8}$$

Hence, the distance dependence as $\omega \to 0$ is indeed like $1/r$. Besides, Eq. (4.8) is also the static solution as $t \to \infty$ for the Heaviside source time function $S(t) = H(t)$.

Fig. 4.1 shows the three component seismograms generated by a vertical point-force with a sharp Gaussian source time function. The near-field and far-field terms are clearly separated. However, when the source time function has a longer duration, as shown in Fig. 4.2, all terms are equally important in the near field. This can be related to a seismometer that is sensitive only to periods comparable to (or much longer than) the S-P time.

4.2 Question 2

For a general point-force $S(t)$ in the p-th direction at the origin, we have

$$u_n(\mathbf{x}, t) = \frac{3\gamma_n\gamma_p - \delta_{np}}{4\pi\rho r^3}\int_{t_P}^{t_S} \tau S(t - \tau)\,d\tau + \frac{\gamma_n\gamma_p}{4\pi\rho\alpha^2 r}S(t - t_P) - \frac{\gamma_n\gamma_p - \delta_{np}}{4\pi\rho\beta^2 r}S(t - t_S). \tag{4.9}$$

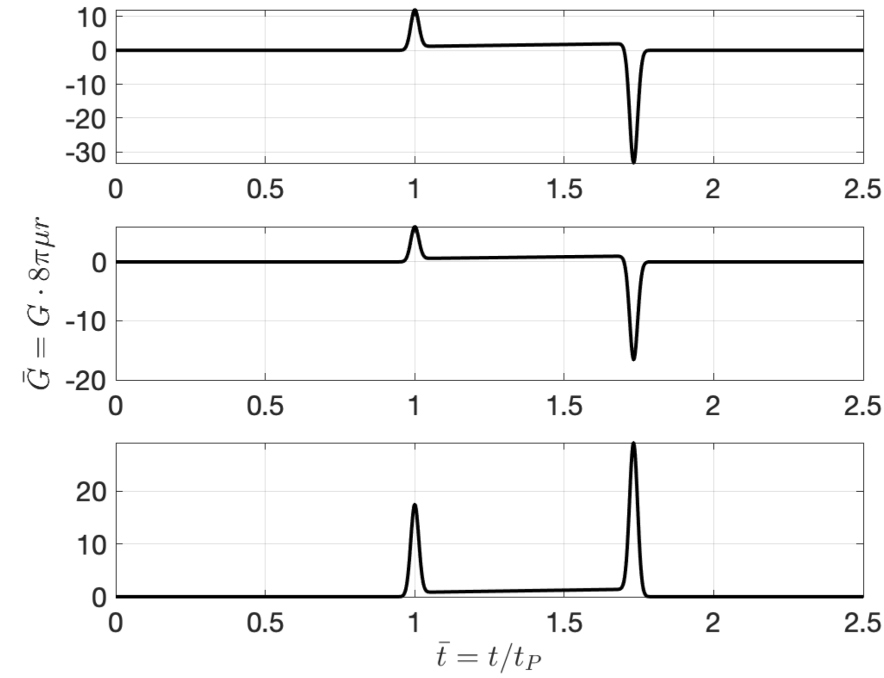

Fig. 4.1 Three component seismograms generated by a vertical point-force at receiver $\mathbf{x} = (2, 1, 3)$ km. The time axis is normalized by the P-wave travel time $t_P = r/\alpha$, and the displacement is multiplied by factor $8\pi\mu r$, which also helps visualize the dependence on $1/r$. The source time function is a Gaussian with $\sigma = 0.01$ s.

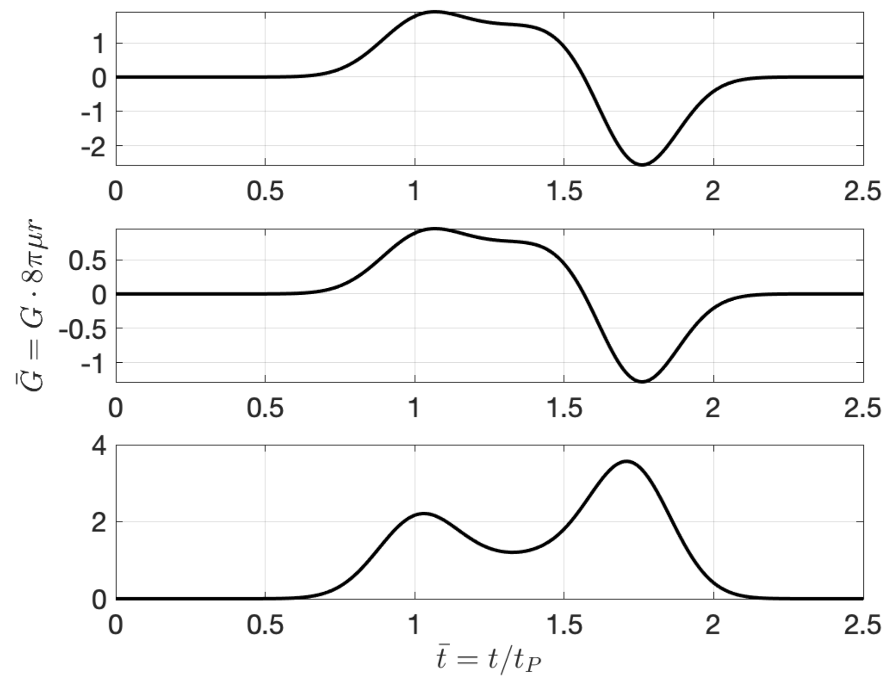

Fig. 4.2 Three component seismograms generated by a vertical point-force at receiver $\mathbf{x} = (2, 1, 3)$ km. The source time function is a Gaussian with $\sigma = 0.1$ s.

When the force is constant, we have $S(t) = H(t)$, which leads to

$$u_n(\mathbf{x},t) = \frac{3\gamma_n\gamma_p - \delta_{np}}{8\pi\rho r^3}\left[(t^2 - t_P^2)H(t - t_P) - (t^2 - t_S^2)H(t - t_S)\right] + \frac{\gamma_n\gamma_p}{4\pi\rho\alpha^2 r}H(t - t_P) - \frac{\gamma_n\gamma_p - \delta_{np}}{4\pi\rho\beta^2 r}H(t - t_S). \tag{4.10}$$

Hence, the static solution at $t \to \infty$ is

$$u_n^{\text{static}}(\mathbf{x}) = \frac{1}{8\pi\rho r}\left(\frac{\delta_{np} - \gamma_n\gamma_p}{\alpha^2} + \frac{\delta_{np} + \gamma_n\gamma_p}{\beta^2}\right), \tag{4.11}$$

which is the same as derived in Section 4.1.

4.3 Question 3

Under the principal coordinates, the moment tensor can be described as $\mathbf{M} = \text{diag}(M_0, 0, -M_0)$. Therefore, ξ_1' marks the tension axis (T), while ξ_3' marks the pressure axis (P).

The tension axis (T) corresponds to maximum outward particle motion, while the pressure axis (P) corresponds to maximum inward particle motion.

4.4 Question 4

Note that $\delta\Delta$ should be measured at the same radius for two ray paths close to each other. Now consider a point $\mathbf{x}$ on the ray path in a **spherically symmetric medium**. In the azimuth direction, the elementary area has a side length of $|\mathbf{x}| \sin\Delta\, \delta\phi$. For the take-off angle direction, when the rays are going down, the elementary area has a side length of $|\mathbf{x}| \cos i_x\, \delta\Delta$, with $i_x < 90°$ and $\delta\Delta > 0$. On the contrary, when the rays are going up, the length is also $|\mathbf{x}| \cos i_x\, \delta\Delta$, with $i_x > 90°$ and $\delta\Delta < 0$. This case is shown in Fig. 4.3. Therefore, the cross-sectional area of the ray tube at $\mathbf{x}$ is

$$\delta A = |\mathbf{x}|^2 \cos i_x \sin\Delta\, \delta\Delta\, \delta\phi. \tag{4.12}$$

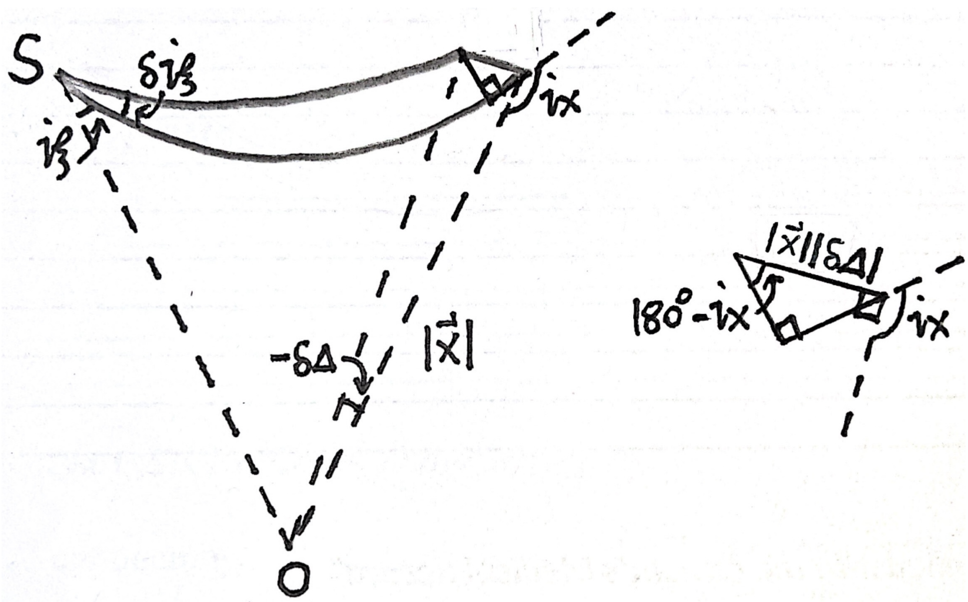

Fig. 4.3 Illustration of the side length in the take-off angle direction.

Given that the solid angle $\delta\Omega = \sin i_\xi \, \delta i_\xi \, \delta\phi$, the geometrical spreading function is

$$\mathcal{R}(\mathbf{x}, \boldsymbol{\xi}) = \sqrt{\frac{\delta A}{\delta\Omega}} = |\mathbf{x}| \sqrt{\frac{\cos i_x \sin\Delta \, \delta\Delta}{\sin i_\xi \, \delta i_\xi}}. \tag{4.13}$$

From the ray parameter, we have

$$p = \frac{|\boldsymbol{\xi}| \sin i_\xi}{c(\xi)}, \qquad \delta p = \frac{|\boldsymbol{\xi}| \cos i_\xi}{c(\xi)} \delta i_\xi = p \cot i_\xi \, \delta i_\xi. \tag{4.14}$$

Using Eq. (4.14), we have $\delta p > 0$ (assuming $i_\xi < 90°$), and thus we obtain

$$\begin{aligned} \mathcal{R}(\mathbf{x}, \boldsymbol{\xi}) \, c(\boldsymbol{\xi}) &= |\mathbf{x}| \, \frac{|\boldsymbol{\xi}| \sin i_\xi}{p} \sqrt{\frac{p \cos i_x \cos i_\xi \sin\Delta \, \delta\Delta}{\sin^2 i_\xi \, \delta p}} \\ &= |\mathbf{x}| \, |\boldsymbol{\xi}| \left[\frac{\cos i_x \cos i_\xi \sin\Delta}{p} \, \frac{\partial\Delta}{\partial p} \right]^{1/2}. \end{aligned} \tag{4.15}$$

Eq. (4.15) directly implies the reciprocity $\mathcal{R}(\mathbf{x}, \boldsymbol{\xi}) \, c(\boldsymbol{\xi}) = \mathcal{R}(\boldsymbol{\xi}, \mathbf{x}) \, c(\mathbf{x})$.

4.5 Question 5

From the following far-field P-wave Green's function

$$G_{ij}^P = \frac{\gamma_i \gamma_j}{4\pi\rho\alpha^2 r} \delta\left(t - \frac{r}{\alpha}\right), \tag{4.16}$$

we need to generalize it into the form

$$G_{ij}^{P,\mathrm{ray}} = \frac{C(\boldsymbol{\xi}) \, \mathcal{F}^P}{4\pi\sqrt{\rho(\mathbf{x}) \, \alpha(\mathbf{x})} \, \mathcal{R}^P(\mathbf{x}, \boldsymbol{\xi})} \, \delta\left[t - T^P(\mathbf{x}, \boldsymbol{\xi})\right]. \tag{4.17}$$

Here, we already identify and generalize r/α as the ray travel time T^P, $1/r$ as the geometrical spreading factor $1/\mathcal{R}^P(\mathbf{x}, \boldsymbol{\xi})$, and $1/\rho\,\alpha^2$ as the factor $C(\boldsymbol{\xi})/\sqrt{\rho(\mathbf{x}) \, \alpha(\mathbf{x})}$, which implies that the constant $C(\boldsymbol{\xi}) = 1/\sqrt{\rho(\boldsymbol{\xi}) \, \alpha^3(\boldsymbol{\xi})}$. The radiation pattern $\gamma_i \gamma_j$, similarly, can be generalized by replacing the source-receiver direction unit vectors $\hat{\gamma}$ with the ray direction unit vector $\hat{\mathbf{l}}(\boldsymbol{\xi})$ and $\hat{\mathbf{l}}(\mathbf{x})$. Therefore, we obtain

$$G_{ij}^{P,\mathrm{ray}} = \frac{l_i(\mathbf{x}) \, l_j(\boldsymbol{\xi})}{4\pi\sqrt{\rho(\mathbf{x}) \, \alpha(\mathbf{x}) \, \rho(\boldsymbol{\xi}) \, \alpha^3(\boldsymbol{\xi})} \, \mathcal{R}^P(\mathbf{x}, \boldsymbol{\xi})} \, \delta\left[t - T^P(\mathbf{x}, \boldsymbol{\xi})\right]. \tag{4.18}$$

Reciprocity states that

$$G_{ij}(\mathbf{x}; \boldsymbol{\xi}) = G_{ji}(\boldsymbol{\xi}; \mathbf{x}), \tag{4.19}$$

which, using Eq. (4.18), gives

$$\frac{l_i(\mathbf{x}) \, l_j(\boldsymbol{\xi})}{\sqrt{\rho(\mathbf{x}) \, \alpha(\mathbf{x}) \, \rho(\boldsymbol{\xi}) \, \alpha^3(\boldsymbol{\xi})} \, \mathcal{R}^P(\mathbf{x}, \boldsymbol{\xi})} = \frac{l_j(\boldsymbol{\xi}) \, l_i(\mathbf{x})}{\sqrt{\rho(\boldsymbol{\xi}) \, \alpha(\boldsymbol{\xi}) \, \rho(\mathbf{x}) \, \alpha^3(\mathbf{x})} \, \mathcal{R}^P(\boldsymbol{\xi}, \mathbf{x})}. \tag{4.20}$$

Now we prove that

$$\mathcal{R}^P(\mathbf{x}, \boldsymbol{\xi}) \, \alpha(\boldsymbol{\xi}) = \mathcal{R}^P(\boldsymbol{\xi}, \mathbf{x}) \, \alpha(\mathbf{x}) \tag{4.21}$$

4.6 Question 6

The far-field radiation patterns for P and S waves from **a point source of fault slip** are

$$\mathbf{A}^{FP} = \sin 2\theta \cos\phi \, \hat{\mathbf{r}}, \qquad \mathbf{A}^{FS} = \cos 2\theta \cos\phi \, \hat{\theta} - \cos\theta \sin\phi \, \hat{\phi}. \tag{4.22}$$

The RMS values for these radiation patterns, averaged over the focal sphere, are calculated as

$$
\begin{aligned}
a^{FP} &= \sqrt{\frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\, |\mathbf{A}^{FP}(\theta,\phi)|^2 \sin\theta} = \sqrt{\frac{1}{4\pi}\int_0^{2\pi} \cos^2\phi \,\mathrm{d}\phi \cdot \int_0^{\pi} \sin^2 2\theta \sin\theta \,\mathrm{d}\theta} \\
&= \sqrt{\frac{1}{4\pi}\cdot \pi \cdot 8\left(\frac{2}{3} - \frac{4}{5}\cdot\frac{2}{3}\right)} = \sqrt{\frac{4}{15}}.
\end{aligned}
\tag{4.23}
$$

$$
\begin{aligned}
a^{FS} &= \sqrt{\frac{1}{4\pi}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\, |\mathbf{A}^{FS}(\theta,\phi)|^2 \sin\theta} \\
&= \sqrt{\frac{1}{4\pi}\int_0^{2\pi} \cos^2\phi \,\mathrm{d}\phi \cdot \int_0^{\pi} \cos^2 2\theta \sin\theta \,\mathrm{d}\theta + \sqrt{\frac{1}{4\pi}\int_0^{2\pi} \sin^2\phi \,\mathrm{d}\phi \cdot \int_0^{\pi} \cos^2 \theta \sin\theta \,\mathrm{d}\theta}} \\
&= \sqrt{\frac{1}{4\pi}\cdot \pi \cdot \left(\frac{14}{15} + \frac{2}{3}\right)} = \sqrt{\frac{2}{5}}.
\end{aligned}
\tag{4.24}
$$

The energy radiated seismically from **a double-couple point source in a homogeneous full-space**, as P waves for example, is

$$
E_P = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi \int_0^{+\infty} \mathrm{d}t\, \rho\alpha\, |\mathbf{u}_P(r,\theta,\phi,t)|^2 . \tag{4.25}
$$

Note that only far-field P waves behave as r^{-1} decay, which is able to cancel out the factor r^2 from the spherical surface area in Eq. (4.25). Therefore, the total radiated energy we will obtain later is far-field result in the limit of $r \to \infty$. Through calculation, we have

$$
E_P = 4\pi \cdot (a^{FP})^2 \cdot \frac{\int_0^{\infty}\left[\ddot{M}_0(t)\right]^2 \,\mathrm{d}t}{16\pi^2\rho\alpha^5} = \frac{\int_0^{\infty}\left[\ddot{M}_0(t)\right]^2 \,\mathrm{d}t}{15\pi\rho\alpha^5}, \tag{4.26}
$$

$$
E_S = 4\pi \cdot (a^{FS})^2 \cdot \frac{\int_0^{\infty}\left[\ddot{M}_0(t)\right]^2 \,\mathrm{d}t}{16\pi^2\rho\beta^5} = \frac{\int_0^{\infty}\left[\ddot{M}_0(t)\right]^2 \,\mathrm{d}t}{10\pi\rho\beta^5}. \tag{4.27}
$$

These formulae indeed represent the source (i.e., its integrated source time function), but to convert observed E_P or E_S to the source property, we need to use the radiated energy measured at the far-field.

4.7 Question 7

The point force solution described in the Cartesian coordinates, only showing the far-field components, is

$$
u_i = F_j * G_{ij} = \frac{\gamma_i\gamma_j}{4\pi\rho\alpha^2 r} F_j\left(t - \frac{r}{\alpha}\right) - \frac{\gamma_i\gamma_j - \delta_{ij}}{4\pi\rho\beta^2} F_j\left(t - \frac{r}{\beta}\right). \tag{4.28}
$$

In vector form, with $\hat{\gamma} = \hat{\mathbf{r}}$, we have

$$
\mathbf{u} = \frac{\hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\mathbf{F})}{4\pi\rho\alpha^2 r} + \frac{\mathbf{F} - \hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\mathbf{F})}{4\pi\rho\beta^2 r}. \tag{4.29}
$$

Since in the spherical polar coordinates, there is identity $\mathbf{F} = (\hat{\mathbf{r}}\cdot\mathbf{F})\,\hat{\mathbf{r}} + (\hat{\boldsymbol{\theta}}\cdot\mathbf{F})\,\hat{\boldsymbol{\theta}} + (\hat{\boldsymbol{\phi}}\cdot\mathbf{F})\,\hat{\boldsymbol{\phi}}$, which is simply the projection of a vector onto the basis vectors, we thus obtain

$$
\mathbf{u} = \frac{\hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\mathbf{F})}{4\pi\rho\alpha^2 r} + \frac{\hat{\boldsymbol{\theta}}\,(\hat{\boldsymbol{\theta}}\cdot\mathbf{F}) + \hat{\boldsymbol{\phi}}\,(\hat{\boldsymbol{\phi}}\cdot\mathbf{F})}{4\pi\rho\beta^2 r}. \tag{4.30}
$$

Each component can be written as

$$
u_i = F_j * G_{ij} = \frac{1}{4\pi\rho\alpha^2}\,\frac{\hat{r}_i\hat{r}_j}{r}\, F_j\left(t - \frac{r}{\alpha}\right) + \frac{1}{4\pi\rho\beta^2}\,\frac{\hat{\theta}_i\hat{\theta}_j + \hat{\phi}_i\hat{\phi}_j}{r}\, F_j\left(t - \frac{r}{\beta}\right). \tag{4.31}
$$

4.8 Question 8

Start from the ray equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = c\nabla T. \tag{4.32}$$

We choose a scalar variable σ to define the position along the ray with

$$\frac{\mathrm{d}\sigma}{\mathrm{d}s} = c(\mathbf{x}(s)), \qquad \sigma(s) = \int_0^s c(\mathbf{x}(s))\,\mathrm{d}s, \tag{4.33}$$

which represents the integral of wave speed along the ray path. Therefore, we have

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\sigma}\frac{\mathrm{d}\sigma}{\mathrm{d}s} = c\nabla T, \qquad \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\sigma} = \nabla T. \tag{4.34}$$

When deriving the differential equation for a ray, we have the following result

$$\frac{\mathrm{d}}{\mathrm{d}s}\nabla T = \frac{c}{2}\nabla\left(\frac{1}{c^2}\right). \tag{4.35}$$

Similarly, we can obtain

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}\sigma^2} = \frac{\mathrm{d}}{\mathrm{d}\sigma}\nabla T = \frac{1}{2}\nabla\left(\frac{1}{c^2}\right), \tag{4.36}$$

which demonstrates that solving for a ray path is equivalent to solving for the motion of a particle moving in a force field with potential $1/(2c^2)$.

4.9 Question 9

As the wave speed $c(z)$ only depends on depth z, the ray paths are within the xz-plane. From Snell's law, we know

$$p = \frac{\sin i}{c(z)} = \text{const}. \tag{4.37}$$

The angle i is the one between the z-direction (downward) and the ray. It also depends on ray distance s (although the function dependence is not explicitly written out). The ray equation is

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \sin i, \qquad \frac{\mathrm{d}z}{\mathrm{d}s} = \cos i. \tag{4.38}$$

With $c(z) = az + b$, Eqs (4.37) and (4.38) lead to

$$\sin i = p(az + b), \qquad \cos i \cdot \frac{\mathrm{d}i}{\mathrm{d}s} = pa \cdot \frac{\mathrm{d}z}{\mathrm{d}s} = pa \cdot \cos i. \tag{4.39}$$

Therefore, we obtain the following result

$$\frac{1}{R} = \left|\frac{\mathrm{d}i}{\mathrm{d}s}\right| = |pa| = \text{const}, \tag{4.40}$$

which indicates that the radius of curvature R is a constant, and the ray path is thus a circular arc with radius $R = |pa|^{-1}$. To find the center of the circle, we can first solve for the ray turning depth z_m, which is

$$c(z_m) = az_m + b = \frac{1}{p}, \qquad z_m = \frac{1}{pa} - \frac{b}{a} = R - \frac{b}{a}. \tag{4.41}$$

Hence, the center of the circle lies at the depth $z = -b/a$.

A more general way is to directly apply the definition of the radius of curvature R. Denoting the ray path as $z = z(x)$

for the portion with well-defined $z(x)$, we have

$$R = \frac{\left[1 + (z')^2\right]^{\frac{3}{2}}}{|z''|}. \tag{4.42}$$

The derivatives can be obtained as

$$z' = \frac{\mathrm{d}z}{\mathrm{d}x} = \cot i, \qquad z'' = \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} = -\frac{1}{\sin^2 i} \cdot \frac{\mathrm{d}i}{\mathrm{d}x}. \tag{4.43}$$

From the Snell's law in Eq. (4.37), we have

$$\sin i = pc(z), \qquad \cos i \cdot \frac{\mathrm{d}i}{\mathrm{d}x} = pc'(z)\,\frac{\mathrm{d}z}{\mathrm{d}x}, \qquad \frac{\mathrm{d}i}{\mathrm{d}x} = \frac{pc'(z)}{\sin i}. \tag{4.44}$$

Therefore, the radius of curvature is evaluated as

$$R = \frac{\left(1 + \cot^2 i\right)^{\frac{3}{2}}}{\left|pc'(z)/\sin^3 i\right|} = \frac{1}{|pc'(z)|}. \tag{4.45}$$

With $c(z) = az + b$, we have a constant $R = |pa|^{-1}$.

For a spherically symmetric medium, Snell's law and the ray equation now become

$$p = \frac{r \sin i}{c(r)} = \text{const.} \qquad \frac{r\mathrm{d}\theta}{\mathrm{d}s} = \sin i, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = -\cos i, \tag{4.46}$$

where r and θ are the polar coordinates for the ray plane. Now denoting the ray path as $r = r(\theta)$ for the portion with well-defined $r(\theta)$, we have the following equation for the radius of curvature

$$R = \frac{\left(r^2 + r_\theta^2\right)^{\frac{3}{2}}}{\left|r^2 + 2r_\theta^2 - r r_{\theta\theta}\right|}. \tag{4.47}$$

The derivatives can be similarly obtained as

$$r_\theta = \frac{\mathrm{d}r}{\mathrm{d}\theta} = -r \cot i, \qquad r_{\theta\theta} = \frac{\mathrm{d}^2 r}{\mathrm{d}\theta^2} = -\cot i \cdot \frac{\mathrm{d}r}{\mathrm{d}\theta} + \frac{r}{\sin^2 i} \cdot \frac{\mathrm{d}i}{\mathrm{d}\theta} = r \cot^2 i + \frac{r}{\sin^2 i} \cdot \frac{\mathrm{d}i}{\mathrm{d}\theta}. \tag{4.48}$$

From the Snell's law, we have

$$\sin i = \frac{pc(r)}{r}, \qquad \cos i \cdot \frac{\mathrm{d}i}{\mathrm{d}\theta} = \frac{rpc'(r) - pc(r)}{r^2} \cdot \frac{\mathrm{d}r}{\mathrm{d}\theta}, \qquad \frac{\mathrm{d}i}{\mathrm{d}\theta} = -\frac{rpc'(r) - pc(r)}{r \sin i}. \tag{4.49}$$

Therefore, the radius of curvature is evaluated as

$$\left(r^2 + r_\theta^2\right)^{\frac{3}{2}} = \frac{r^3}{\sin^3 i}, \qquad \left|r^2 + 2r_\theta^2 - r r_{\theta\theta}\right| = \frac{r^2}{\sin^2 i}\left|1 - \frac{\mathrm{d}i}{\mathrm{d}\theta}\right|, \tag{4.50}$$

$$R = \frac{r^2}{\left|r \sin i - r \sin i \cdot \dfrac{\mathrm{d}i}{\mathrm{d}\theta}\right|} = \frac{r}{|pc'(r)|}. \tag{4.51}$$

With $c(r) = a - br^2$, we have a constant $R = |2pb|^{-1}$, and thus the ray paths are circular arcs. We can also solve for the ray turning radius r_m, which satisfies

$$\frac{pc(r_m)}{r_m} = 1, \qquad r_m^2 + \frac{r_m}{pb} - \frac{a}{b} = 0. \tag{4.52}$$

4.10 Question 10

In this problem, the unit tangent along a ray is $\mathbf{l}$, and the travel time gradient is $\nabla T = \mathbf{l}/c$.

a) Since the curl of a gradient is zero, we have

$$\nabla \times \left(\frac{\mathbf{l}}{c}\right) = \nabla \times (\nabla T) = \mathbf{0}. \tag{4.53}$$

b) Using the identity $\nabla \times (\psi \mathbf{A}) = \psi\,(\nabla \times \mathbf{A}) + (\nabla \psi) \times \mathbf{A}$, we have

$$\nabla \times \mathbf{l} = \nabla \times (c \nabla T) = \nabla c \times \left(\frac{\mathbf{l}}{c}\right) = \left(\frac{\nabla c}{c}\right) \times \mathbf{l} = -c\,\mathbf{l} \times \left(\frac{\nabla c}{c^2}\right) = c\,\mathbf{l} \times \nabla \left(\frac{1}{c}\right). \tag{4.54}$$

c) We know the differential equation for a ray (eq. 4.44 in the book) is

$$\mathbf{l} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\right) = \nabla \left(\frac{1}{c}\right). \tag{4.55}$$

This leads to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\right) = \frac{1}{c}\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}s} + \mathbf{l}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c}\right) = \nabla\left(\frac{1}{c}\right), \qquad \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}s} = c\nabla\left(\frac{1}{c}\right) - c\mathbf{l}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{c}\right). \tag{4.56}$$

On the other hand, using the triple product expansion $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}\,(\mathbf{c} \cdot \mathbf{a}) - \mathbf{a}\,(\mathbf{b} \cdot \mathbf{c})$, we have

$$(\nabla \times \mathbf{l}) \times \mathbf{l} = \left[c\,\mathbf{l} \times \nabla\left(\frac{1}{c}\right)\right] \times \mathbf{l} = c\nabla\left(\frac{1}{c}\right) - c\mathbf{l}\left[\mathbf{l} \cdot \nabla\left(\frac{1}{c}\right)\right]. \tag{4.57}$$

Finally, note that the derivative along the ray path $\mathrm{d}/\mathrm{d}s$ can be expanded as

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} \cdot \nabla = \mathbf{l} \cdot \nabla. \tag{4.58}$$

Therefore, we prove that

$$(\nabla \times \mathbf{l}) \times \mathbf{l} = \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}s}. \tag{4.59}$$

d) Based on the following result

$$\nabla(\ln c) = \frac{\nabla c}{c} = -c\nabla\left(\frac{1}{c}\right), \tag{4.60}$$

Eq. (4.57) can be modified to

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}s} = c\nabla\left(\frac{1}{c}\right) - \left[c\mathbf{l} \cdot \nabla\left(\frac{1}{c}\right)\right]\mathbf{l} = [\mathbf{l} \cdot \nabla(\ln c)]\,\mathbf{l} - \nabla(\ln c). \tag{4.61}$$

e) Using the forward Euler scheme, we have

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{l} \qquad \rightarrow \qquad \mathbf{x}_{m+1} = \mathbf{x}_m + \Delta s\,\mathbf{l}_m, \tag{4.62}$$

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}s} = [\mathbf{l} \cdot \nabla(\ln c)]\,\mathbf{l} - \nabla(\ln c) \qquad \rightarrow \qquad \mathbf{l}_{m+1} = \mathbf{l}_m + \Delta s\,[(\mathbf{l}_m \cdot \mathbf{g}_m)\,\mathbf{l}_m - \mathbf{g}_m], \tag{4.63}$$

where $\mathbf{g} = \nabla(\ln c)$ and the subscript m denotes variables evaluated at step m.

f) We can further simplify Eq. (4.61) as

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}s} = -\frac{\nabla c}{c} + \left(\mathbf{l} \cdot \frac{\nabla c}{c}\right)\mathbf{l}. \tag{4.64}$$

Using the orthogonal unit vectors $\mathbf{l}$ and $\mathbf{n}$, the gradient of the wave speed can be described as $\nabla c = \alpha\mathbf{l} + \beta\mathbf{n}$ with $\alpha, \beta \in \mathbb{R}$. Therefore, the change of ray direction becomes

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}s} = -\frac{\alpha\mathbf{l} + \beta\mathbf{n}}{c} + \frac{\alpha\mathbf{l}}{c} = \frac{\beta\mathbf{n}}{c}. \tag{4.65}$$

If $\mathbf{l}$ is parallel to ∇c, we have $\beta = 0$, $\mathrm{d}\mathbf{l}/\mathrm{d}s = \mathbf{0}$ and thus $\mathbf{l}$ does not change direction along the ray. If $\mathbf{l}$ is perpendicular to ∇c, we have $\alpha = 0$ and β reaches the largest magnitude at this location, which implies that the ray changes direction at the maximum rate here.

4.11 Question 11

For P-wave generated by a shear dislocation with scalar moment $M_0(t)$, the far-field and intermediate-field terms are (from eq. 4.32 in the book)

$$\mathbf{u}^P(\mathbf{x}, t) = \frac{1}{4\pi\rho\alpha^3}\mathbf{A}^{FP}\frac{1}{r}\dot{M}_0\left(t - \frac{r}{\alpha}\right) + \frac{1}{4\pi\rho\alpha^2}\mathbf{A}^{IP}\frac{1}{r^2}M_0\left(t - \frac{r}{\alpha}\right). \tag{4.66}$$

Only focused on the radial component, the radiation patterns $\mathbf{A}^{FP}$ and $\mathbf{A}^{IP}$ become (from eq. 4.33 in the book)

$$\mathbf{A}^{FP} \cdot \hat{\mathbf{r}} = \sin 2\theta \cos\phi, \qquad \mathbf{A}^{IP} \cdot \hat{\mathbf{r}} = 4\sin 2\theta \cos\phi. \tag{4.67}$$

Now with $T^P = r/\alpha$ as the travel time, the P-wave displacement pulse shape is proportional to

$$u_r^P(\mathbf{x}, t) \propto \dot{M}_0\left(t - T^P\right) + \frac{4\alpha}{r}M_0\left(t - T^P\right) = \dot{M}_0\left(t - T^P\right) + \frac{4}{T^P}M_0\left(t - T^P\right). \tag{4.68}$$

4.12 Question 12

The rays whose travel times are stationary but not minima correspond to surface-reflected phases. As an example shown in Fig. 4.4, consider the true ray path A-P_0-B with the surface reflection point P_0. Without losing generality, we assume that the radius of the sphere is $R = 1$ and A is the North pole with coordinate $(\theta, \phi) = (0, 0)$. Under this coordinate system, we can set $P_0 = (\theta, 0)$ and $B = (2\theta, 0)$. The distance of the true ray path can be calculated as

$$L_0 = \overline{AP_0} + \overline{P_0B} = 4\sin\left(\frac{\theta}{2}\right). \tag{4.69}$$

Now consider another $P = (\theta_P, 0)$ which is not the true surface reflection point. The path length can be calculated as

$$L(P) = \overline{AP} + \overline{PB} = 2\left[\sin\left(\frac{\theta_P}{2}\right) + \sin\left(\theta - \frac{\theta_P}{2}\right)\right] = 4\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta - \theta_P}{2}\right). \tag{4.70}$$

Therefore, we always have $L(P) < L_0$ as long as $P \neq P_0$. By taking the derivative, we can also show that

$$\frac{\partial L}{\partial \theta_P} = 2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta - \theta_P}{2}\right) = 0 \qquad \text{at } \theta_P = \theta. \tag{4.71}$$

This demonstrates that the true ray path corresponds to a travel time that is stationary, and moreover, is maximal along the θ-direction. On the other hand, along the ϕ-direction, the travel time should be minimal.

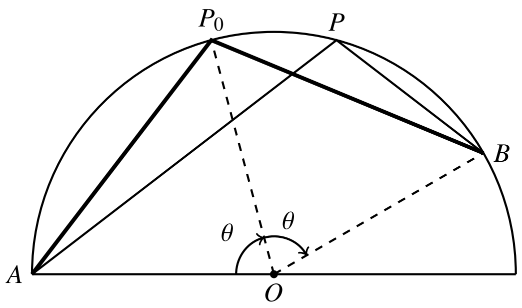

Fig. 4.4 Illustration of the surface-reflected phase in a homogeneous sphere. The true ray path is A-P_0-B with P_0 being the midpoint of the arc, while P denotes the perturbed surface reflection point.

4.13 Question 13

Consider a Cartesian coordinate system with x-, y-, and z-directions in the North, East, and Down directions, respectively. The Cartesian components of moment tensor $\mathbf{M}$ are related to the components of fault slip $\bar{\mathbf{u}}$ and fault normal $\boldsymbol{\nu}$ as

$$M_{pq} = \mu A \left(\bar{u}_p \nu_q + \bar{u}_q \nu_p\right). \tag{4.72}$$

a) If we have $M_{xz} = M_{zx} \neq 0$ as the only non-zero components, the shear faulting can have two potential setups.

1. A horizontal fault plane ($\boldsymbol{\nu} = \hat{\mathbf{e}}_z$) with slip in the North direction ($\bar{\mathbf{u}} = \bar{u}\,\hat{\mathbf{e}}_x$).
2. A vertical fault plane following East-West strike direction ($\boldsymbol{\nu} = \hat{\mathbf{e}}_x$) with slip in the vertical direction ($\bar{\mathbf{u}} = \bar{u}\,\hat{\mathbf{e}}_z$).

b) As already shown in **a)**, there are two scenarios corresponding to the same moment tensor.

c) If we have $M_{xz} = M_{zx} \neq 0$ and $M_{yz} = M_{zy} \neq 0$ as the only non-zero components, we can first consider a horizontal fault plane ($\boldsymbol{\nu} = \hat{\mathbf{e}}_z$) and then sum up the fault slips. The total fault slip can be obtained as

$$\bar{\mathbf{u}} = \frac{M_{xz}}{\mu A}\,\hat{\mathbf{e}}_x + \frac{M_{yz}}{\mu A}\,\hat{\mathbf{e}}_y, \qquad \text{in the direction } \phi = \arctan\left(\frac{M_{yz}}{M_{xz}}\right). \tag{4.73}$$

Note that the direction ϕ is with respect to the x-axis, pointing toward the y-axis. Therefore, this moment tensor can have two potential setups.

1. A horizontal fault plane ($\boldsymbol{\nu} = \hat{\mathbf{e}}_z$) with slip in the ϕ direction.
2. A vertical fault plane whose strike is orthogonal to the ϕ direction, with slip in the vertical direction ($\bar{\mathbf{u}} = \bar{u}\,\hat{\mathbf{e}}_z$).

d) Based on the Hooke's Law for an isotropic solid elastic medium, we have

$$e_{zx} = \frac{\tau_{zx}}{2\mu}, \qquad e_{zy} = \frac{\tau_{zy}}{2\mu}. \tag{4.74}$$

At the traction-free surface of the Earth, there are no shear strains e_{zx} and e_{zy}.

4.14 Question 14

For a **shear dislocation** of arbitrary orientation, its moment tensor $\mathbf{M}$ can be decomposed into four elementary moment tensors (Box 4.4 in the book)

$$\mathbf{M} = \mathbf{M}^{(1)} \cos\delta \cos\lambda + \mathbf{M}^{(2)} \sin\delta \cos\lambda - \mathbf{M}^{(3)} \cos 2\delta \sin\lambda + \mathbf{M}^{(4)} \sin 2\delta \sin\lambda. \tag{4.75}$$

The elementary matrices $\mathbf{M}^{(2)}$ and $\mathbf{M}^{(4)}$ are given as

$$\mathbf{M}^{(2)}(\phi_s) = M_0 \begin{bmatrix} -\sin 2\phi_s & \cos 2\phi_s & 0 \\ \cos 2\phi_s & \sin 2\phi_s & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{M}^{(4)}(\phi_s) = M_0 \begin{bmatrix} -\sin^2\phi_s & \frac{1}{2}\sin 2\phi_s & 0 \\ \frac{1}{2}\sin 2\phi_s & -\cos^2\phi_s & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.76}$$

For this problem, we denote

$$\tilde{\mathbf{M}}^{(2)} = \mathbf{M}^{(2)}(\phi_s), \qquad \tilde{\mathbf{M}}^{(4)} = \mathbf{M}^{(4)}\left(\phi_s^{(4)}\right), \qquad \text{with } \phi_s^{(4)} = \phi_s + \frac{\pi}{4}. \tag{4.77}$$

Since we have

$$\sin^2\phi_s^{(4)} = \frac{1+\sin 2\phi_s}{2}, \qquad \cos^2\phi_s^{(4)} = \frac{1-\sin 2\phi_s}{2}, \qquad \sin 2\phi_s^{(4)} = \cos 2\phi_s, \tag{4.78}$$

when the strike of the $\tilde{\mathbf{M}}^{(4)}$ dislocation is $\pi/4$ greater than that of the $\tilde{\mathbf{M}}^{(2)}$ dislocation, we have

$$\frac{1}{2}\tilde{\mathbf{M}}^{(2)} = \tilde{\mathbf{M}}^{(4)} + M_0 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{4.79}$$

Note that $\tilde{\mathbf{M}}^{(2)}$ represents a vertical strike-slip fault, while $\tilde{\mathbf{M}}^{(4)}$ represents a thrust fault dipping at 45° with pure up-dip slip, but having a strike angle that is 45° greater.

We now focus on the radiation pattern of the far-field SH waves. Note that the second term on the right-hand side of Eq. (4.79) represents a CLVD source. For an arbitrary moment tensor, the far-field S wave is (eq. 4.29 in the book)

$$u_n^{FS} = -\left(\frac{\gamma_n\gamma_p - \delta_{np}}{4\pi\rho\beta^3}\right)\gamma_q \frac{1}{r}\dot{M}_{pq}\left(t - \frac{r}{\beta}\right). \tag{4.80}$$

For a CLVD source, the displacement becomes

$$u_n^{FS} \propto -\frac{1}{2}\left(\gamma_n\gamma_1 - \delta_{n1}\right)\gamma_1 - \frac{1}{2}\left(\gamma_n\gamma_2 - \delta_{n2}\right)\gamma_2 + \left(\gamma_n\gamma_3 - \delta_{n3}\right)\gamma_3, \tag{4.81}$$

where $\hat{\gamma}$ is the same as the radial direction $\hat{\mathbf{r}}$ in the spherical coordinate. The SH direction then corresponds to $\hat{\phi}$, which has $\phi_3 = 0$ and is orthogonal to $\hat{\gamma}$, stated as below

$$\hat{\phi} = \left(-\sin\phi, \cos\phi, 0\right), \qquad \hat{\gamma}\cdot\hat{\phi} = \gamma_n\phi_n = \gamma_1\phi_1 + \gamma_2\phi_2 = 0. \tag{4.82}$$

Therefore, we can show that

$$\mathbf{u}^{FS}\cdot\hat{\phi} = u_n^{FS}\phi_n \propto -\frac{1}{2}\gamma_n\phi_n\left(\gamma_1^2 + \gamma_2^2 - 2\gamma_3^2\right) + \frac{1}{2}\left(\gamma_1\phi_1 + \gamma_2\phi_2 - 2\gamma_3\phi_3\right) = 0, \tag{4.83}$$

which indicates that a CLVD source does not contribute to far-field SH waves. Hence, the two elementary moment tensors $\tilde{\mathbf{M}}^{(2)}/2$ and $\tilde{\mathbf{M}}^{(4)}$ have the same radiation pattern, and thus generate same SH waves in a spherically symmetric Earth.

However, if the sources are in an isotropic but laterally inhomogeneous Earth, the heterogeneity can couple the P-SV and SH systems together, so a CLVD source can also contribute to SH waves and the above conclusion will be changed.

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