

MIT Integration Bee: 2024 Regular session

Question 1

$$\int_1^{2024} \lfloor \log_{43}(x) \rfloor dx \quad (1.1)$$

Solution

$$I = \int_1^{43} 0 dx + \int_{43}^{1849} 1 dx + \int_{1849}^{2024} 2 dx = 1806 + 350 = 2156. \quad (1.2)$$

Question 2

$$\int \frac{dx}{x^{2024} - x^{4047}} \quad (2.1)$$

Solution

$$\begin{aligned} I &= \int \left(\frac{1}{x^{2024}} + \frac{1}{x} \cdot \frac{1}{1 - x^{2023}} \right) dx \\ &= \int \left(\frac{1}{x^{2024}} + \frac{1}{x} + \frac{x^{2022}}{1 - x^{2023}} \right) dx \\ &= -\frac{1}{2023 x^{2023}} + \ln x - \frac{1}{2023} \ln(1 - x^{2023}) + C. \end{aligned} \quad (2.2)$$

Question 3

$$\int_0^1 x^2(1-x)^{2024} dx \quad (3.1)$$

Solution

$$I = B(3, 2025) = \frac{2 \cdot 2024!}{2027!} = \frac{2}{2025 \times 2026 \times 2027}. \quad (3.2)$$

Question 4

$$\int \frac{2023x + 1}{x^2 + 2024} dx \quad (4.1)$$

Solution

$$\begin{aligned} I &= 2023 \int \frac{x}{x^2 + 2024} dx + \int \frac{1}{x^2 + 2024} dx \\ &= \frac{2023}{2} \ln(x^2 + 2024) + \frac{1}{\sqrt{2024}} \arctan \frac{x}{\sqrt{2024}} + C \end{aligned} \quad (4.2)$$

Question 5

$$\int_0^{\pi/2} \sec^2(x) e^{-\sec^2(x)} dx \quad (5.1)$$

Solution Notice that $\sec^2 x = \tan^2 x + 1$. With a change of variable $t = \tan x$, we have

$$I = \int_0^\infty e^{-t^2-1} dt = \frac{\sqrt{\pi}}{2e}. \quad (5.2)$$

Question 6

$$\int \cot x \cot 2x dx \quad (6.1)$$

Solution

$$\begin{aligned} I &= \int \frac{\cos 2x}{2 \sin^2 x} dx = \int \frac{1 - 2 \sin^2 x}{2 \sin^2 x} dx \\ &= -\frac{1}{2} \cot x - x + C. \end{aligned} \quad (6.2)$$

Question 7

$$\int \frac{\sinh^2 x}{\tanh 2x} dx \quad (7.1)$$

Solution

$$\begin{aligned} I &= \int \frac{\sinh x \cosh 2x}{2 \cosh x} dx = \frac{1}{2} \int \sinh 2x dx - \frac{1}{2} \int \tanh x dx \\ &= \frac{1}{4} \cosh 2x - \frac{1}{2} \ln (\cosh x) + C. \end{aligned} \quad (7.2)$$

Question 8

$$\int \arctan (\sqrt{x}) dx \quad (8.1)$$

Solution

$$\begin{aligned} I &= x \arctan (\sqrt{x}) - \frac{1}{2} \int \frac{\sqrt{x}}{1+x} dx \\ &= x \arctan (\sqrt{x}) - \int \frac{t^2}{1+t^2} dt \quad \left(t = \sqrt{x}, \quad x = t^2, \quad dx = 2t dt \right) \\ &= (x+1) \arctan (\sqrt{x}) - \sqrt{x} + C. \end{aligned} \quad (8.2)$$

Question 9

$$\int_0^\infty \frac{x \ln x}{x^4 + 1} dx \quad (9.1)$$

Solution With the change of variable $t = 1/x$, we find that

$$I = \int_0^\infty \frac{x \ln x}{x^4 + 1} dx = - \int_0^\infty \frac{t \ln t}{t^4 + 1} dt = -I. \quad (9.2)$$

Therefore, the result is $I = 0$.

Question 10

$$\int_0^{10} \lfloor x \lfloor x \rfloor \rfloor dx \quad (10.1)$$

Solution We first study the integral over a general interval $[n, n+1]$ for any $n \in \mathbb{N}$, which is

$$I_n = \int_n^{n+1} \lfloor x \lfloor x \rfloor \rfloor dx = \int_n^{n+1} \lfloor nx \rfloor dx. \quad (10.2)$$

It is trivial that $I_0 = 0$. For $n \geq 1$, the integrand is a piecewise constant function, with its value ranging from n^2 to $n^2 + n - 1$. The integral is equivalent to the average of all these values, giving

$$I_n = \frac{1}{n} [n^2 + (n^2 + 1) + \cdots + (n^2 + n - 1)] = n^2 + \frac{n-1}{2}, \quad \text{for } n \geq 1. \quad (10.3)$$

Therefore, we obtain

$$\begin{aligned} I &= \sum_{k=0}^9 I_k = (1^2 + 2^2 + \cdots + 9^2) + \frac{1+2+\cdots+8}{2} \\ &= \frac{9 \times 10 \times 19}{6} + 18 = 303. \end{aligned} \quad (10.4)$$

Question 11

$$\int_0^1 e^{-x} \sqrt{1 + \cot^2 (\arccos e^{-x})} dx \quad (11.1)$$

Solution With the change of variable $t = \arccos e^{-x}$, we have

$$t(0) = 0, \quad t(1) = \arccos e^{-1}, \quad x = -\ln |\cos t|, \quad dx = \tan t dt. \quad (11.2)$$

Therefore, the integral becomes

$$\begin{aligned} I &= \int_0^{\arccos e^{-1}} \cos t \sqrt{1 + \cot^2 t} \cdot \tan t dt \\ &= \int_0^{\arccos e^{-1}} \sqrt{\sin^2 t + \cos^2 t} dt = \arccos e^{-1}. \end{aligned} \quad (11.3)$$

Question 12

$$\int_1^3 \frac{1 + \frac{1+\dots}{x+\dots}}{x + \frac{1+\dots}{x+\dots}} dx \quad (12.1)$$

Solution Denote the integrand as $f(x)$. Therefore, we have

$$f(x) = \frac{1 + f(x)}{x + f(x)}, \quad f(x) = \frac{1 - x + \sqrt{(x-1)^2 + 4}}{2}. \quad (12.2)$$

The integral can be evaluated as

$$\begin{aligned} I &= -1 + \frac{1}{2} \int_0^2 \sqrt{x^2 + 4} dx \\ &= -1 + \frac{1}{2} \left(\frac{1}{2} x \sqrt{x^2 + 4} + 2 \ln \left| \frac{x + \sqrt{x^2 + 4}}{2} \right| \right) \Bigg|_0^2 \\ &= -1 + \sqrt{2} + \ln(1 + \sqrt{2}). \end{aligned} \quad (12.3)$$

Note A better way is to use **integration by parts**

$$x = \frac{1 + f - f^2}{f}, \quad f(1) = 1, \quad f(3) = \sqrt{2} - 1. \quad (12.4)$$

The integral can be alternatively calculated as

$$I = xf|_1^3 - \int_1^{\sqrt{2}-1} \left(\frac{1}{f} + 1 - f \right) df = \sqrt{2} - 1 - \ln(\sqrt{2} - 1). \quad (12.5)$$

Question 13

$$\int_0^1 \frac{2x(1-x)^2}{1+x^2} dx \quad (13.1)$$

Solution

$$I = \int_0^1 \left(2x - 4 + \frac{4}{1+x^2} \right) dx = \pi - 3. \quad (13.2)$$

Question 14

$$\int e^{e^x+3x} dx \quad (14.1)$$

Solution With the change of variable $t = e^x$, we have

$$\begin{aligned} I &= \int e^{e^x+3x} dx = \int e^{t+3 \ln t} \frac{dt}{t} \\ &= \int t^2 e^t dt = (t^2 - 2t + 2)e^t + C \\ &= (e^{2x} - 2e^x + 2)e^{e^x} + C. \end{aligned} \quad (14.2)$$

Question 15

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} 2 \left(1 - \frac{|x|}{\sqrt{3}} \right) dx \quad (15.1)$$

Solution

$$I = 2\sqrt{3} - \frac{4}{\sqrt{3}} \int_0^{\sqrt{3}/2} x dx = \frac{3\sqrt{3}}{2}. \quad (15.2)$$

Question 16

$$\int \frac{\ln(1+x^2)}{x^2} dx \quad (16.1)$$

Solution

$$\begin{aligned} I &= -\frac{\ln(1+x^2)}{x} + 2 \int \frac{dx}{1+x^2} \\ &= -\frac{\ln(1+x^2)}{x} + 2 \arctan x + C. \end{aligned} \quad (16.2)$$

Question 17

$$\int 2^x x^2 dx \quad (17.1)$$

Solution

$$\begin{aligned} I &= \int 2^x x^2 dx = \frac{1}{\ln 2} \left(2^x x^2 - 2 \int 2^x x dx \right) \\ &= \frac{2^x x^2}{\ln 2} - \frac{2}{\ln^2 2} \left(2^x x - \frac{2^x}{\ln 2} \right) + C \\ &= \frac{2^x}{\ln^3 2} \left(x^2 \ln^2 2 - 2x \ln 2 + 2 \right) + C. \end{aligned} \quad (17.2)$$

Question 18

$$\int_0^1 \sqrt{x^8 - x^6 + x^4} \cdot \sqrt{1+x^2} dx \quad (18.1)$$

Solution With the change of variable $t = x^3$, we have

$$\begin{aligned} I &= \int_0^1 x^2 \sqrt{1+x^6} dx = \frac{1}{3} \int_0^1 \sqrt{1+t^2} dt \\ &= \frac{1}{6} \left[t \sqrt{1+t^2} + \ln \left(t + \sqrt{1+t^2} \right) \right]_0^1 = \frac{1}{6} \left[\sqrt{2} + \ln \left(1 + \sqrt{2} \right) \right]. \end{aligned} \quad (18.2)$$

Question 19

$$\int_1^{\infty} \frac{e^x + xe^x}{x^2 e^{2x} - 1} dx \quad (19.1)$$

Solution With the change of variable $t = xe^x$, we have

$$\begin{aligned} I &= \int_1^{\infty} \frac{d(xe^x)}{x^2 e^{2x} - 1} = \int_e^{\infty} \frac{dt}{t^2 - 1} \\ &= \frac{1}{2} \ln \left(\frac{t-1}{t+1} \right) \Big|_e^{\infty} = \frac{1}{2} \ln \left(\frac{e+1}{e-1} \right). \end{aligned} \quad (19.2)$$

Question 20

$$\int_0^{\infty} (80x^3 - 60x^4 + 14x^5 - x^6) e^{-x} dx \quad (20.1)$$

Solution For each term, we can obtain

$$\begin{aligned} I_n &= \int_0^{\infty} a_n x^n e^{-x} dx = \int_0^{\infty} n a_n x^{n-1} e^{-x} dx = \dots \\ &= \int_0^{\infty} n! \cdot a_n e^{-x} dx = n! \cdot a_n. \end{aligned} \quad (20.2)$$

Therefore, we have

$$I = 80 \times 6 - 60 \times 24 + 14 \times 120 - 720 = -80 \times 12 + 120 \times 8 = 0. \quad (20.3)$$