

MIT Integration Bee: 2024 Quarterfinal

Quarterfinal #1

Question 1

$$\int \ln x \left[\left(\frac{x}{e} \right)^x + \left(\frac{e}{x} \right)^x \right] dx \quad (1.1)$$

Solution Denote the following functions

$$f(x) = \left(\frac{x}{e} \right)^x, \quad g(x) = \left(\frac{e}{x} \right)^x. \quad (1.2)$$

We notice that

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{d \ln f(x)}{dx} = (x \ln x - x)' = \ln x, \\ \frac{g'(x)}{g(x)} &= \frac{d \ln g(x)}{dx} = (x - x \ln x)' = -\ln x. \end{aligned} \quad (1.3)$$

Therefore, the result is

$$I = \int [f(x) + g(x)] \ln x \, dx = f(x) - g(x) = \left(\frac{x}{e} \right)^x - \left(\frac{e}{x} \right)^x + C. \quad (1.4)$$

Question 2

$$\int_0^\infty \frac{\sin^3 x}{x} dx \quad (2.1)$$

Solution Using the triple-angle formula and the **Dirichlet integral**, we have

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin 3x}{x} dx = \int_0^\infty \frac{3 \sin x - 4 \sin^3 x}{x} dx = \frac{3\pi}{2} - 4I. \quad (2.2)$$

Therefore, we have

$$I = \int_0^\infty \frac{\sin^3 x}{x} dx = \frac{\pi}{4}. \quad (2.3)$$

Question 3

$$\int \begin{vmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{vmatrix} dx \quad (3.1)$$

Solution Denote the determinant as $F_5(x)$. We can first obtain the following **recurrence relation**

$$F_n(x) = xF_{n-1}(x) - F_{n-2}(x), \quad \text{with } F_1(x) = x, \quad F_2(x) = x^2 - 1. \quad (3.2)$$

Therefore, we have

$$\begin{aligned} F_3(x) &= xF_2(x) - F_1(x) = x^3 - 2x, \\ F_4(x) &= xF_3(x) - F_2(x) = x^4 - 3x^2 + 1, \\ F_5(x) &= xF_4(x) - F_3(x) = x^5 - 4x^3 + 3x. \end{aligned} \quad (3.3)$$

The integral is thus evaluated as

$$I = \int F_5(x) dx = \frac{1}{6}x^6 - x^4 + \frac{3}{2}x^2 + C. \quad (3.4)$$

Tiebreakers Question 1

$$\int_0^{2024} x^{2024} \log_{2024}(x) dx \quad (4.1)$$

Solution

$$\begin{aligned} I &= \frac{1}{\ln 2024} \int_0^{2024} x^{2024} \ln x dx \\ &= \frac{1}{2025 \ln 2024} \left(x^{2025} \ln x \Big|_0^{2024} - \int_0^{2024} x^{2024} dx \right) \\ &= \frac{2024^{2025}}{2025} - \frac{2024^{2025}}{2025^2 \ln 2024}. \end{aligned} \quad (4.2)$$

Tiebreakers Question 2

$$\lim_{t \rightarrow \infty} \int_0^2 \left[x^{-2024t} \prod_{n=1}^{2024} \sin(nx^t) \right] dx \quad (5.1)$$

Solution First, we study the following function

$$f_\alpha(x) = \lim_{t \rightarrow \infty} \frac{\sin \alpha x^t}{x^t}. \quad (5.2)$$

When $x > 1$, we have $|\sin \alpha x^t| \leq 1$ but $x^t \rightarrow \infty$. Hence, we have $f_\alpha(x) = 0$ for $x > 1$. On the other hand, when $0 < x < 1$ we have $x^t \rightarrow 0$ and thus $f_\alpha(x) = \alpha$. As a summary, we obtain

$$f_\alpha(x) = \lim_{t \rightarrow \infty} \frac{\sin \alpha x^t}{x^t} = \begin{cases} \alpha, & |x| < 1, \\ 0, & |x| > 1. \end{cases} \quad (5.3)$$

Using this result, we speculate that

$$f(x) = \lim_{t \rightarrow \infty} x^{-2024t} \prod_{n=1}^{2024} \sin(nx^t) = \prod_{n=1}^{2024} f_n(x) = \begin{cases} 2024!, & |x| < 1, \\ 0, & |x| > 1. \end{cases} \quad (5.4)$$

Finally, the integral is evaluated as

$$I = \int_0^2 f(x) dx = 2024!. \quad (5.5)$$

Quarterfinal #2

Question 1

$$\lim_{n \rightarrow \infty} \int_0^1 \sum_{k=1}^n \frac{(kx)^4}{n^5} dx \quad (6.1)$$

Solution Re-organize the order of operations, we have

$$I = \lim_{n \rightarrow \infty} \frac{1}{n^5} \sum_{k=1}^n \int_0^1 k^4 x^4 dx = \lim_{n \rightarrow \infty} \frac{1}{5n^5} \sum_{k=1}^n k^4. \quad (6.2)$$

Note that the leading order term of the sum is

$$\sum_{k=1}^n k^4 = \frac{1}{5}n^5 + o(n^5). \quad (6.3)$$

Therefore, after taking the limit we have

$$I = \lim_{n \rightarrow \infty} \frac{1}{5n^5} \sum_{k=1}^n k^4 = \frac{1}{25}. \quad (6.4)$$

Question 2

$$\int_0^1 \frac{\ln(1+x^2+x^3+x^4+x^5+x^6+x^7+x^9)}{x} dx \quad (7.1)$$

Solution We notice that

$$1+x^2+x^3+x^4+x^5+x^6+x^7+x^9 = (1+x^2)(1+x^3)(1+x^4). \quad (7.2)$$

Therefore, we study the following integral

$$F(\alpha) = \int_0^1 \frac{\ln(1+x^\alpha)}{x} dx, \quad \text{with } \alpha > 0. \quad (7.3)$$

With a simple **change of variable** $t = x^\alpha$, we have

$$F(\alpha) = \frac{1}{\alpha} \int_0^1 \frac{\ln(1+t)}{t} dt = \frac{\pi^2}{12\alpha}. \quad (7.4)$$

Finally, the result is

$$I = F(2) + F(3) + F(4) = \frac{\pi^2}{12} \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) = \frac{13\pi^2}{144}. \quad (7.5)$$

Note The following type of integral has appeared several times

$$\begin{aligned} \int_0^1 \frac{\ln(1+t)}{t} dt &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \int_0^1 t^{n-1} dt \\ &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}. \end{aligned} \quad (7.6)$$

Using integration by parts, we also have

$$\int_0^1 \frac{\ln t}{1+t} dt = - \int_0^1 \frac{\ln(1+t)}{t} dt = -\frac{\pi^2}{12}. \quad (7.7)$$

Question 3

$$\int_0^1 \left(1 - \sqrt[2024]{x}\right)^{2024} dx \quad (8.1)$$

Solution We study the following general integral

$$F(\alpha) = \int_0^1 \left(1 - x^{\frac{1}{\alpha}}\right)^{\alpha} dx. \quad (8.2)$$

With a simple **change of variable** $t = x^{1/\alpha}$, we have

$$\begin{aligned} F(\alpha) &= \alpha \int_0^1 (1-t)^{\alpha} t^{\alpha-1} dt \\ &= \alpha B(\alpha+1, \alpha) = \frac{\Gamma^2(\alpha+1)}{\Gamma(2\alpha+1)}. \end{aligned} \quad (8.3)$$

Finally, the integral is evaluated as

$$I = F(2024) = \frac{\Gamma^2(2025)}{\Gamma(4049)} = \frac{(2024!)^2}{4048!} = \frac{1}{\binom{4048}{2024}}. \quad (8.4)$$

Quarterfinal #3

Question 1

$$\int_0^{2\pi} \text{card}(\{\lfloor \sin x \rfloor, \lfloor \cos x \rfloor, \lfloor \tan x \rfloor, \lfloor \cot x \rfloor\}) \, dx \quad (9.1)$$

Solution We summarize the cardinality of the set in the following table.

	$(0, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{3\pi}{4})$	$(\frac{3\pi}{4}, \pi)$	$(\pi, \frac{5\pi}{4})$	$(\frac{5\pi}{4}, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, \frac{7\pi}{4})$	$(\frac{7\pi}{4}, 2\pi)$
$\lfloor \sin x \rfloor$	0	0	0	0	-1	-1	-1	-1
$\lfloor \cos x \rfloor$	0	0	-1	-1	-1	-1	0	0
$\lfloor \tan x \rfloor$	0	≥ 1	< -1	-1	0	≥ 1	< -1	-1
$\lfloor \cot x \rfloor$	≥ 1	0	-1	< -1	≥ 1	0	-1	< -1
card	2	2	3	3	3	3	3	3

Therefore, the integral is evaluated as

$$I = \frac{\pi}{2} (2 + 3 + 3 + 3) = \frac{11\pi}{2}. \quad (9.2)$$

Question 2

$$\int_0^{+\infty} \frac{dx}{(x+1)(\ln^2 x + \pi^2)} \quad (10.1)$$

Solution Using several **changes of variables**, we have

$$\begin{aligned}
 I &= \int_0^{+\infty} \frac{dx}{(x+1)(\ln^2 x + \pi^2)} \\
 &= \int_{-\infty}^{+\infty} \frac{e^t dt}{(e^t + 1)(t^2 + \pi^2)} \quad (t = \ln x, \quad x = e^t, \quad dx = e^t dt) \\
 &= \int_{-\infty}^{+\infty} \frac{dt}{t^2 + \pi^2} - \int_{-\infty}^{+\infty} \frac{dt}{(e^t + 1)(t^2 + \pi^2)} \\
 &= \frac{1}{\pi} \arctan\left(\frac{t}{\pi}\right) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{e^t dt}{(e^t + 1)(t^2 + \pi^2)} \quad (t \rightarrow -t)
 \end{aligned} \quad (10.2)$$

Therefore, we have

$$2I = \frac{1}{\pi} \arctan\left(\frac{t}{\pi}\right) \Big|_{-\infty}^{+\infty} = 1, \quad I = \frac{1}{2}. \quad (10.3)$$

Question 3

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^n \max \left(\{x\}, \{\sqrt{2}x\}, \{\sqrt{3}x\} \right) dx \quad (11.1)$$

Solution This integral can be translated into the calculation of an **expectation**. Denote X_1, X_2, X_3 as three independent variables drawn from the uniform distribution $U[0, 1]$. Denote the new random variable $Y = \max(X_1, X_2, X_3)$. Therefore, the integral is equivalent to

$$I = \mathbb{E}(Y) = \mathbb{E} [\max(X_1, X_2, X_3)] . \quad (11.2)$$

The CDF $f_Y(y)$ is computed as follows, based on the independence among X_i

$$f_Y(y) = \mathbb{P}(Y < y) = \prod_{i=1}^3 \mathbb{P}(X_i < y) = y^3, \quad \text{for } y \in [0, 1]. \quad (11.3)$$

Therefore, the PDF $p_Y(y)$ and the expectation are obtained as

$$p_Y(y) = \frac{df_Y}{dy} = 3y^2, \quad I = \mathbb{E}(Y) = \int_0^1 y p_Y(y) dy = \frac{3}{4}. \quad (11.4)$$

Quarterfinal #4

Question 1

$$\int \frac{e^{2x}}{(1 - e^x)^{2024}} dx \quad (12.1)$$

Solution With a simple **change of variable** $t = e^x$, we have

$$\begin{aligned} I &= \int \frac{t dt}{(1 - t)^{2024}} = \int \frac{dt}{(1 - t)^{2024}} - \int \frac{dt}{(1 - t)^{2023}} \\ &= \frac{1}{2023 (1 - e^x)^{2023}} - \frac{1}{2022 (1 - e^x)^{2022}} + C \end{aligned} \quad (12.2)$$

Question 2

$$\lim_{n \rightarrow \infty} \log_n \left(\int_0^1 (1 - x^3)^n dx \right) \quad (13.1)$$

Solution We first study the following general integral

$$F(\alpha) = \int_0^1 (1 - x^\alpha)^n dx. \quad (13.2)$$

With the standard **change of variable** $t = x^\alpha$, we have

$$F(\alpha) = \frac{1}{\alpha} \int_0^1 (1 - t)^n t^{\frac{1}{\alpha}-1} dt = \frac{1}{\alpha} B \left(n + 1, \frac{1}{\alpha} \right). \quad (13.3)$$

Based on the **Stirling's approximation** for the gamma function

$$\ln \Gamma(z) \sim z \ln z, \quad (13.4)$$

the limit can be obtained as

$$\begin{aligned} I(\alpha) &= \lim_{n \rightarrow \infty} \frac{\ln F(\alpha)}{\ln n} = \lim_{n \rightarrow \infty} \frac{\ln \Gamma(n + 1) - \ln \Gamma(n + 1 + \alpha^{-1})}{\ln n} \\ &= \lim_{n \rightarrow \infty} \frac{-\alpha^{-1} \ln n}{\ln n} = -\frac{1}{\alpha}. \end{aligned} \quad (13.5)$$

Therefore, the result is

$$I = I(3) = -\frac{1}{3}. \quad (13.6)$$

Question 3

$$\int \frac{\sin x}{1 + \sin x} \cdot \frac{\cos x}{1 + \cos x} dx \quad (14.1)$$

Solution Note that

$$\begin{aligned} \frac{\sin x}{1 + \sin x} \cdot \frac{\cos x}{1 + \cos x} &= 1 - \frac{\sin^2 x + \cos^2 x + \sin x + \cos x}{(1 + \sin x)(1 + \cos x)} \\ &= 1 - \left(\frac{\cos x}{1 + \sin x} + \frac{\sin x}{1 + \cos x} \right). \end{aligned} \quad (14.2)$$

Therefore, the integral becomes

$$\begin{aligned} I &= x - \ln(1 + \sin x) + \ln(1 + \cos x) + C \\ &= x + \ln\left(\frac{1 + \cos x}{1 + \sin x}\right) + C. \end{aligned} \quad (14.3)$$