Asymptotic Analysis of Differential Equations (1): Linear ODE

We analyze the asymptotic expansion of the solution for linear ODE on the complex plane. Denote the domain $\Omega \subseteq \mathbb{C}$, the meromorphic function domain $m(\Omega)$ and holomorphic function ring $\mathbb{O}(\Omega)$, the $n \times n$ matrix $M(m(\Omega), n)$ with its elements composed of function $f \in m(\Omega)$. For a matrix $A \in M(m(\Omega), n)$, a linear ODE can be represented by

$$y'(z) = A(z)y(z), \qquad y \in \mathbb{C}^n$$

Based on variation of parameters, we only need to study the homogeneous equation. For an initial value problem $y(z_0) = y_0$, we can first solve for the matrix equation

$$Y'(z) = A(z)Y(z), \qquad Y(z_0) = I \tag{$*$}$$

With this fundamental matrix, we have $y = Yy_0$. Hence, we will focus on this matrix equation.

- Qualitative theory of solutions (6.1)

Cauchy Theorem

For $y' = f(z, y)$ with $y(z_0) = y_0$, if f is analytic then there exists a unique analytic solution.

Consider $(*)$ has a solution Y_0 around z_0. For a path γ starting from z_0, we can perform analytic continuation of Y_0 into a neighborhood of γ. When γ goes back to z_0, we can obtain another solution Y_γ. Then we state that there exists an invertible $C_\gamma \in GL(\mathbb{C}, n)$ such that $Y_\gamma = Y_0 C_\gamma$. We define a mapping $\rho_A: \gamma \mapsto C_\gamma$, and when γ_1 and γ_2 are homotopic, we have $C_{\gamma_1} = C_{\gamma_2}$. This implies that $\rho_A: \pi_1(\Omega^*, z_0) \to GL(\mathbb{C}, n)$, with Ω^* is the domain with poles removed. Moreover, if $\gamma = \gamma_1 \circ \gamma_2$, then $C_\gamma = C_{\gamma_1} C_{\gamma_2}$. So ρ_A is a group homomorphism and a representation of π_1.

For the matrix equation $(*)$, consider a transformation $P \in GL(\mathbb{O}(\Omega), n)$ and denote $Z = PY$.

$$Z' = P'Y + PY' = (P'P^{-1} + PAP^{-1})Z = BZ$$

The two mappings ρ_A and ρ_B are equivalent. $P'P^{-1} + PAP^{-1}$ is a meromorphic connection on vector bundles on a complex manifold, an example of Riemann-Hilbert correspondence.

Local problem

Let z_0 is a pole of A, with r denoted as the Poincaré rank. This implies that

$$A(z) = (z - z_0)^{-r}\tilde{A}(z), \qquad \tilde{A}(z) \in M(\mathbb{O}(z_0), n), \qquad \tilde{A}(z_0) \neq 0$$

Without loss of generality, take $z_0 = 0$ and we have

$$z^r Y'(z) = \tilde{A}(z)Y(z)$$

Since z_0 is a pole, $Y(z_0)$ may not exists, and we only focus on the equation. The solution is highly influenced by the Poincaré rank r.

When $r = 1$, consider a constant matrix A and we have

$$zY'(z) = AY(z)$$

Select a branch cut C from $z = 0$, we have $\ln z \in \mathbb{O}(\Omega \setminus C)$ and $Y(z) = e^{A \ln z}$. Consider A has the Jordan normal form $A = PJP^{-1}$ with $J = \Lambda + N$. Then we can write

$$Y(z) = P\left(e^{J \ln z}\right)P^{-1}, \qquad e^{J \ln z} = \Lambda^{z}\left(\sum_{k=0}^{n} \frac{\ln^k z}{k!} N^k\right)$$

The singularity is regular for $r = 1$. Going around $z = 0$, we obtain C_γ as follows

$$Y_0\left(ze^{2\pi i}\right) = e^{A(\ln z + 2\pi i)} = Y_0(z)e^{2\pi i A}, \qquad C_\gamma = e^{2\pi i A}$$

When $r = 2$, still consider a constant matrix A and we have

$$z^2 Y'(z) = AY(z), \qquad Y(z) = e^{-A/z}$$

Now $z = 0$ becomes an essential singularity, and the solution only exists in a sector. We cannot go around $z = 0$ as in the previous case. For $r \geq 2$, the singularity is irregular.

- Majorant series & Cauchy theorem

Let $\Omega \subseteq \mathbb{C}^{d+1}$ with coordinates $\tilde{y} = (z, y)$ with $z \in \mathbb{C}$, $y \in \mathbb{C}^d$ and a function $f \in \mathbb{O}(\Omega, \mathbb{C}^d)$.

$$y' = f(z, y), \qquad \tilde{y}_0 = (z_0, y_0) \in \Omega$$

The function f is analytic when there exists $r > 0$ such that when $\tilde{y} \in B(\tilde{y}_0, r)$, the following series is convergent

$$f(\tilde{y}) = \sum_{j \geq 0} c_j (\tilde{y} - \tilde{y}_0)^j, \qquad j = \{j_0, j_1, \cdots, j_d\}$$

The neighborhood is $B(\tilde{y}_0, r) = \{\tilde{y} \in \Omega \mid |\tilde{y} - \tilde{y}_0| < r\}$ with the L_∞ norm $|\tilde{y}| = \max|y_i|$. The above notation means

$$f(\tilde{y}) = \sum_{j_0, \cdots, j_d \geq 0} c_{j_0 \cdots j_d} (z - z_0)^{j_0} (y_1 - y_{10})^{j_1} \cdots (y_d - y_{d0})^{j_d}$$

Majorant series

Consider a formal power series $f(\tilde{y})$. If there exists another series $F(\tilde{y})$ such that $\forall j$ we have $|a_j| \leq A_j$, then $F(\tilde{y})$ is a majorant series of $f(\tilde{y})$.

$$f(\tilde{y}) = \sum_{j \geq 0} a_j (\tilde{y} - \tilde{y}_0)^j, \qquad F(\tilde{y}) = \sum_{j \geq 0} A_j (\tilde{y} - \tilde{y}_0)^j$$

If $F(\tilde{y})$ converges in $B(\tilde{y}_0, r)$, then $f(\tilde{y})$ also converges. We can then call $F(\tilde{y})$ as the majorant function of $f(\tilde{y})$.

Corollary. If $f(\tilde{y})$ is analytic around $\tilde{y}_0$, i.e., it can be expanded on $B(\tilde{y}_0, R)$ into a convergent series, then for any $r \in (0, R)$, there exists a constant $M > 0$ such that we can write down the majorant function $F(\tilde{y})$ as

$$F(\tilde{y}) = M \prod_{k=0}^{d} \left(1 - \frac{y_k - y_{k0}}{r}\right)^{-1}, \qquad \tilde{y} \in \bar{B}(\tilde{y}_0, r)$$

Proof. Since f converges in $B(\tilde{y}_0, R)$, then it absolutely converges in $\bar{B}(\tilde{y}_0, r)$. If we select a $\tilde{y} \in \partial B(\tilde{y}_0, r)$ on the boundary, we have the following convergent series

$$\sum_{j \geq 0} |a_j| |\tilde{y} - \tilde{y}_0|^j = \sum_{j \geq 0} |a_j| r^{j_0 + \cdots + j_d}$$

Then there exists $M > 0$ such that

$$|a_j| r^{|j|} \leq M, \qquad |a_j| \leq \frac{M}{r^{|j|}}, \qquad |j| = j_0 + \cdots + j_d$$

Now we can construct a majorant function

$$F(\tilde{y}) = \sum_{j \geq 0} \frac{M}{r^{|j|}} (\tilde{y} - \tilde{y}_0)^j = M \prod_{k=0}^{d} \sum_{j_k \geq 0} \left(\frac{y_k - y_{k0}}{r}\right)^{j_k} = M \prod_{k=0}^{d} \left(1 - \frac{y_k - y_{k0}}{r}\right)^{-1} \qquad \blacksquare$$

Corollary. For $F(\tilde{y})$ defined above, consider the Cauchy problem

$$y_j' = F(z, y), \qquad y_j(z_0) = y_{j0}, \qquad j = 1,2, \cdots, d$$

There exists $\rho > 0$ such that it has a unique analytic solution on $B(z_0, \rho)$.

Proof. Denote $u(z) = y_1(z) - y_{10}$. Based on the Cauchy problem, we have

$$\left(y_i - y_j\right)' = 0, \qquad u(z) = y_i(z) - y_{i0}, \qquad \forall i, j = 1,2, \cdots, d$$

The ODE for $u(z)$ can be obtained as

$$u'(z) = F(z, y) = M \left(1 - \frac{z - z_0}{r}\right)^{-1} \left(1 - \frac{u}{r}\right)^{-d}, \qquad u(z_0) = 0$$

The solution is

$$u(z) = r - r \left[1 + (d+1) M \ln \left(1 - \frac{z - z_0}{r}\right)\right]^{\frac{1}{d+1}}$$

To guarantee convergence, we can obtain the radius ρ as

$$\left|\frac{z - z_0}{r}\right| < 1, \qquad \left|(d+1) M \ln \left(1 - \frac{z - z_0}{r}\right)\right| < 1, \qquad \rho = r \left[1 - e^{-\frac{1}{(d+1)M}}\right] \qquad \blacksquare$$

Cauchy theorem

Let $\Omega \subseteq \mathbb{C}^{d+1}$ and denote analytic functions $f: \Omega \to \mathbb{C}^d$ with a point $(z_0, y_0) \in \Omega$. There exists $\rho > 0$ such that the Cauchy problem has a unique analytic solution in $B(z_0, \rho)$.

$$y_j' = f_j(z, y), \qquad y_j(z_0) = y_{j0}, \qquad j = 1,2,\cdots,d$$

Proof. Without loss of generality, assume $z_0 = 0$ and $y_0 = 0$. We consider the solution in the form of a power series

$$f_j(\tilde{y}) = f_j(z, y) = \sum_{J \geq 0} a_{jJ} \tilde{y}^J, \qquad y_j(z) = \sum_{m \geq 0} c_{jm} z^m, \qquad j = 1,2,\cdots,d$$

Then we have

$$y_j' = \sum_{k \geq 0} c_{j(k+1)}(k+1) z^k = \sum_{J \geq 0} a_{jJ} z^{J_0} y_1^{J_1} \cdots y_d^{J_d} = f_j(\tilde{y})$$

Substitute $y_j(z)$ into the RHS and compare the coefficients. We can obtain

$$c_{jm} = P_{jm}\big(a_{jJ} \mid |J| \leq m-1\big)$$

The polynomial P_{jm} has positive coefficients. To prove the solution is convergent, consider

$$\hat{y}_j' = F(z, y) = M \prod_{k=0}^{d} \left(1 - \frac{y_k}{r}\right)^{-1}$$

Here M is sufficiently large such that $F(z, y)$ is the majorant function for all $f_1, \cdots, f_d$. We have

$$\hat{y}_j(z) = \sum_{m \geq 0} \hat{c}_{jm} z^m, \qquad \hat{c}_{jm} = P_{jm}\big(A_J \mid |J| \leq m-1\big)$$

A_J is the coefficient for the majorant series $F(\tilde{y})$. Since

$$|c_{jm}| = |P_{jm}(a_{jJ})| \leq P_{jm}(|a_{jJ}|) \leq P_{jm}(A_J) = \hat{c}_{jm}$$

Therefore, the formal series $y_j(z)$ converges. ■

Corollary. For the matrix equation

$$Y'(z) = A(z)Y(z), \qquad Y(z_0) = I$$

If A is analytic near z_0, then there exists a unique analytic solution.

Theorem. Consider $F \in M(\mathbb{O}(\Omega), d)$ with the following equation and its formal solution

$$zy' = Fy, \qquad y(z) = \sum_{k \geq 0} c_k (z - z_0)^k, \qquad c_k \in \mathbb{C}^n$$

There exists $\rho > 0$ such that $y(z)$ converges in $B(z_0, \rho)$ and thus is an analytic solution.

Proof. Assume $z_0 = 0$. Consider $F(z)$ can be expanded as

$$F(z) = \sum_{k\geq 0} F_k z^k, \qquad F_k \in M(\mathbb{C}, d) = \mathbb{C}^{d\times d}$$

The equation becomes

$$\sum_{m\geq 0} m c_m z^m = \left(\sum_{k\geq 0} F_k z^k\right)\left(\sum_{l\geq 0} c_l z^l\right) = \sum_{m\geq 0}\left(\sum_{k+l=m} F_k c_l\right) z^m$$

Comparing the coefficients gives

$$m c_m = \sum_{k+l=m} F_k c_l, \qquad (F_0 - mI)c_m = -\sum_{k=1}^{m} F_k c_{m-k}$$

For $m = 0$ we have $F_0 c_0 = 0$. While for $m = 1$, we have

$$c_1 = F_0 c_1 + F_1 c_0, \qquad (F_0 - I)c_1 = -F_1 c_0$$

If F_0 does not have 1 as an eigenvalue, we can obtain the unique solution of c_1. We take $k \in \mathbb{N}$ that is sufficiently large such that for all $\lambda > k$, the matrix $F_0 - \lambda I$ is invertible. Denote

$$f(\lambda) = |(F_0 - \lambda I)^{-1}|_\infty, \qquad \lambda > k$$

Then we have $f \in C(k, +\infty)$ continuous, and when $\lambda \to +\infty$ we have $f(\lambda) \to 0$. This implies that there exists $C > 0$ such that $f(m) \leq C$ for all $m > k$. The coefficients c_m are bounded as

$$|c_m|_\infty = \left| -(F_0 - \lambda I)^{-1} \sum_{k=1}^{m} F_k c_{m-k} \right|_\infty \leq C \sum_{k=1}^{m} |F_k|_\infty |c_{m-k}|_\infty$$

Define $v_m = |c_m|_\infty$ for $m \leq k$, and

$$v_m = C \sum_{j=1}^{m} \left|F_j\right|_\infty v_{m-j}, \qquad m > k$$

This guarantees $|c_m| \leq v_m$. We want to show that $\{v_m\}$ corresponds to a convergent series.

$$u(z) = \sum_{m\geq 0} v_m z^m, \qquad \phi(z) = \sum_{m\geq 1} |F_m|_\infty z^m$$

We can show that (all norms are $|\cdot|_\infty$)

$$u(z) = [1 - C\phi(z)]^{-1}\left[|c_0| + \sum_{l=1}^{k}\left(|c_l| - C\sum_{j=1}^{l}\left|F_j\right|\left|c_{l-j}\right|\right)z^l\right]$$

This is proved by comparing the coefficients, after multiplying $1 - c\phi(z)$ to the LHS.

$$[z^m]: \quad v_m - C\sum_{j=1}^{m}\left|F_j\right| v_{m-j} = |c_m| - C\sum_{j=1}^{m}\left|F_j\right|\left|c_{m-j}\right|, \qquad m \leq k$$

$$[z^m]: \quad v_m - C\sum_{j=1}^{m}\left|F_j\right| v_{m-j} = 0, \qquad m > k$$

The numerator of $u(z)$ is a polynomial which is convergent. As $\phi(0)=0$, there exists $\delta_1>0$ such that when $|z|<\delta_1$, we have $1-C\phi(z)\neq 0$ and $(1-C\phi(z))^{-1}$ is analytic on $B(0,\delta_1)$. Therefore, we prove the majorant $u(z)$ is analytic, and thus $y(z)$. ■

- Asymptotic behavior near ordinary and regular singular points (6.2)

$$zy'(z)=F(z)y(z), \qquad F\in M(B(0,1),n)$$

Now consider the matrix equation

$$zY'(z)=A(z)Y(z), \qquad A\in M(\mathbb{O}(\Omega),n)$$

We require $A(0)\neq 0$ which implies that $z=0$ is a singular point. The domain $\Omega\colon |z|<\rho$. Our goal is to find a transform $P\in GL(\mathbb{O}(\Omega),n)$ such that $Y=PX$ and

$$zX'(z)=B(z)X(z), \qquad B=P^{-1}AP-zP^{-1}P'$$

We want to choose B to be as simple as possible. The matrix equation to be solved is

$$zP'(z)=A(z)P(z)-P(z)B(z)$$

With the formal power series, the equation becomes

$$\sum_{m\geq 0} mP_m z^m=\left(\sum_{k\geq 0} A_k z^k\right)\left(\sum_{l\geq 0} P_l z^l\right)-\left(\sum_{l\geq 0} P_l z^l\right)\left(\sum_{k\geq 0} B_k z^k\right)$$

Taking the coefficient of z^m, we obtain

$$mP_m=A_0P_m-P_mB_0+\sum_{k=1}^{m}(A_kP_{m-k}-P_{m-k}B_k)$$

$$(A_0-mI)P_m-P_mB_0=\sum_{k=1}^{m}(P_{m-k}B_k-A_kP_{m-k})$$

For $m=0$, we have $B_0=P_0^{-1}A_0P_0$. One choice is $P_0=I$ and $B_0=A_0$. Another better one is to choose P_0 such that $B_0=J_0$ is the Jordan normal form of A_0.

Corollary 1. For $A,B\in M(\mathbb{C},n)$, define the following map

$$\varphi_{AB}\colon\ M(\mathbb{C},n)\to M(\mathbb{C},n), \qquad P\mapsto AP-PB$$

Then φ_{AB} is injective if and only if A,B do not share the same eigenvalue.

Proof. When φ_{AB} is injective, assume that λ is the common eigenvalue. Then there exist non-zero $v,w\in\mathbb{C}^n$ such that $Av=\lambda v$ and $B^Tw=\lambda w$. Take $P=vw^T$, and we obtain

$$AP-PB=Avw^T-vw^TB=\lambda vv^T-\lambda vv^T=O$$

This is contradictory to φ_{AB} being injective. On the other hand, when there are no common eigenvalues between A and B, denote $V=\mathbb{C}^n$ and we can write

$$V=V_{\lambda_1}\oplus V_{\lambda_2}\oplus\cdots\oplus V_{\lambda_k}, \qquad V_{\lambda_j}=\operatorname{Ker}\left(B-\lambda_jI\right)^{m_j}$$

We can obtain a basis for V by picking from each root subspace V_λ. Let v be the basis of V_λ. If P satisfies $AP - PB = O$, we have

$$(B - \lambda I)^m v = 0, \qquad P(B - \lambda I)^m v = (A - \lambda I)^m P v = 0$$

Note that λ is not the eigenvalue of A, so $(A - \lambda I)^m$ is invertible and thus $Pv = 0$. Since v can be any vector of the basis, $P = O$ and thus φ_{AB} is injective. ■

Resonant matrix

With this corollary, we need to see if $A_0 - mI$ and B_0 share the same eigenvalue. We call a matrix A as resonant if there are two eigenvalues λ, μ such that $\lambda - \mu \in \mathbb{Z}_{>0}$.

Theorem 2. For $zY' = AY$, if A_0 is non-resonant, then there exists a transformation $Y = PX$ with $P(0) = I$ and $P(z)$ analytic around $z = 0$ such that

$$zX' = A_0 X, \qquad Y(z) = P(z) z^{A_0}$$

Proof. Since A_0 is non-resonant and $P_0 = I$, we know that $A_0 - mI$ and $B_0 = A_0$ do not share the same eigenvalue. We can then choose $B_m = O$ for $m \geq 1$, and there are corresponding P_m

$$P_m = \varphi_{A_0 - mI, B_0}^{-1}\left(-\sum_{k=1}^{m} A_k P_{m-k}\right), \qquad zX' = A_0 X$$

Therefore, we obtain a formal solution $Y(z) = P(z) z^{A_0}$. The equation for $P(z)$ is

$$zP'(z) = A(z)P(z) - P(z)A_0$$

Now take a basis $e_1, \dots, e_{n^2}$ for $M(\mathbb{C}, n)$, and we have

$$P(z) = \sum_{j=1}^{n^2} p_j e_j, \qquad zP'(z) = M(z)P(z), \qquad M(z){:}\ \varphi_{A(z), A_0}$$

From the existence of an analytic solution for the matrix equation, $P(z)$ is analytic at $z = 0$. ■

If A_0 is resonant, then $(A_0 - mI)P_m - P_m B_0$ is not an isomorphism, so we cannot ensure the existence of P_m for arbitrarily chosen B_m.

$$(A_0 - mI)P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k})$$

As an example, we can choose

$$\sum_{k=1}^{m-1} (P_{m-k} B_k - A_k P_{m-k}) + B_m - A_m = O, \qquad P_m = O$$

In this case, we can obtain the following solution.

Proposition 3. For $zY' = AY$, we have a resonant A_0. Let M be the largest positive integer such that $M = \lambda - \mu$ for the eigenvalues. Then there exists $Y = PX$ with analytic $P(z)$ such that

$$zX' = \left(A_0 + \sum_{k=1}^{M} B_k z^k \right) X$$

B_k is non-zero only when there are eigenvalues such that $k = \lambda - \mu$.

A better choice is given as follows. For $zY' = AY$, consider $A_0 = P_0 J_0 P_0^{-1}$ with J_0 as the Jordan normal form. Take $Y = P_0 X$, and then we have

$$zX = (P_0^{-1} A P_0) X = (J_0 + A_1 z + \cdots + A_m z^m) X$$

Without loss of generality, assume $A_0 = \Lambda + N_0$ already the Jordan normal form ($P_0 = I$), and its eigenvalues are ordered by decreasing Re λ_α. As N_0 is strictly upper triangular, we have

$$(N_0)_{\alpha\beta} = 0, \qquad \alpha \geq \beta, \qquad (N_0)_{\alpha\beta} \neq 0, \qquad \lambda_\alpha \neq \lambda_\beta$$

When $m = 1$, the matrix equation becomes

$$(A_0 - I)P_1 - P_1 A_0 = B_1 - A_1$$

Using Einstein summation notation, the (α, β) element becomes

$$\Lambda_{\alpha\gamma}(P_1)_{\gamma\beta} + (N_0)_{\alpha\gamma}(P_1)_{\gamma\beta} - (P_1)_{\alpha\beta} - (P_1)_{\alpha\gamma}(\Lambda)_{\gamma\beta} - (P_1)_{\alpha\gamma}(N_0)_{\gamma\beta} = (B_1)_{\alpha\beta} - (A_1)_{\alpha\beta}$$

For the diagonal matrix, we have $\Lambda_{ij} = \lambda_i \delta_{ij}$, which leads to

$$(\lambda_\alpha - \lambda_\beta - 1)(P_1)_{\alpha\beta} + (N_0)_{\alpha\gamma}(P_1)_{\gamma\beta} - (P_1)_{\alpha\gamma}(N_0)_{\gamma\beta} = (B_1)_{\alpha\beta} - (A_1)_{\alpha\beta}$$

For $(n, 1)$ element, we have $(N_0)_{n\gamma} = (N_0)_{\gamma 1} = 0$, which leads to

$$(\lambda_n - \lambda_1 - 1)(P_1)_{n1} = (B_1)_{n1} - (A_1)_{n1}$$

If $\lambda_n - \lambda_1 \neq 1$, we can choose

$$(B_1)_{n1} = 0, \qquad (P_1)_{n1} = -\frac{(A_1)_{n1}}{\lambda_n - \lambda_1 - 1}$$

If $\lambda_n - \lambda_1 = 1$, we can choose $(B_1)_{n1} = (A_1)_{n1}$, and $(P_1)_{n1}$ is arbitrary. We can continue this process for $(n, 2)$ element and so on using previously determined P_1. This implies that we can find B_1 and P_1, and $(B_1)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = 1$. In general, B_m and P_m exist, and $(B_m)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = m$.

Proposition 3'. With this new choice of B_m (now denoted as N_m) and P_m, we have

$$zX' = (\Lambda + N_0 + N_1 z + \cdots + N_m z^m) X$$

$(N_k)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j = k$. This implies that non-zero elements are possible only when $i < j$ since we have ordered the eigenvalues, and N_k are strictly upper triangular.

Corollary 4. With this new choice of Λ and N_k, we have

$$z^{\Lambda} N_k = N_k z^k z^{\Lambda}$$

Proof. Note that

$$\left(\lambda_\alpha - \lambda_\beta - k\right)(N_k)_{\alpha\beta} = 0, \qquad \Lambda N_k - N_k \Lambda - k N_k = O$$

Therefore, we have

$$z^{\Lambda} N_k = \sum_{l \geq 0} \frac{(\ln z)^l}{l!} \Lambda^l N_k = \sum_{l \geq 0} \frac{(\ln z)^l}{l!} N_k (\Lambda + k)^l = N_k z^{\Lambda + k} \qquad \blacksquare$$

Corollary 5. For the following equation

$$z X' = (\Lambda + N_0 + N_1 z + \cdots + N_m z^m) X$$

Its solution is

$$\xi = z^{\Lambda} z^N, \qquad N = N_0 + N_1 + \cdots + N_m$$

Proof. Using the previous Corollary, we can directly calculate

$$z \xi' = (\Lambda z^{\Lambda}) z^N + z^{\Lambda} (N z^N) = \Lambda \xi + (N_0 + N_1 z + \cdots + N_m z^m) \xi \qquad \blacksquare$$

Theorem 6. For matrix equation $zY' = AY$, assume that A_0 has a Jordan normal form $\Lambda + N_0$, with the eigenvalues ordered by $\operatorname{Re} \lambda_\alpha$. Then there exists $P(z) \in GL(\mathbb{O}(\Omega), n)$ and a strictly upper triangular constant matrix $N \in M(\mathbb{C}, n)$ such that

$$Y(z) = P(z) z^{\Lambda} z^N$$

$(N)_{ij} \neq 0$ only possible when $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$.

To calculate the solution, since N is a nilpotent matrix with $N^{n+1} = O$, we have

$$z^{\Lambda} = \operatorname{diag}\left(z^{\lambda_\alpha}\right), \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l$$

For any $\theta_0 \in \mathbb{R}$, the solution $Y(z)$ is analytic in $S(\theta_0) = \{z \in \Omega \mid \theta_0 < \arg z < \theta_0 + 2\pi\}$.

Consider $z \to z e^{2\pi i}$, the solution becomes

$$Y\left(z e^{2\pi i}\right) = P(z) z^{\Lambda} e^{2\pi i \Lambda} z^N e^{2\pi i N}$$

From Corollary 4, with $z = e^{2\pi i}$ we have

$$e^{2\pi i \Lambda} N_k = N_k e^{2\pi i k} e^{2\pi i \Lambda} = N_k e^{2\pi i \Lambda}, \qquad e^{2\pi i \Lambda} N = N e^{2\pi i \Lambda}$$

This shows that $e^{2\pi i\Lambda}$ commutes with N, and thus $M = e^{2\pi i\Lambda}e^{2\pi iN}$. Based on this property, we call M the **monodromic matrix** of the matrix equation, and (Λ, N) the **monodromic data** that determine the multivalued properties of the solution.

Example: Bessel equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

We define the vector Y as

$$Y = \begin{bmatrix} y \\ xy' \end{bmatrix}, \qquad Y' = \begin{bmatrix} y' \\ xy'' + y' \end{bmatrix} = \begin{bmatrix} y' \\ \dfrac{\alpha^2 - x^2}{x} y \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{x} \\ \dfrac{\alpha^2 - x^2}{x} & 0 \end{bmatrix} \begin{bmatrix} y \\ xy' \end{bmatrix}$$

Then we obtain the corresponding matrix equation

$$xY' = A(x)Y, \qquad A(x) = \begin{bmatrix} 0 & 1 \\ \alpha^2 - x^2 & 0 \end{bmatrix}$$

The coefficients of the power series of $A(x)$ are

$$A_0 = \begin{bmatrix} 0 & 1 \\ \alpha^2 & 0 \end{bmatrix}, \qquad A_1 = O, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

To diagonalize A_0 (which also set $P_0 = I$), consider the following transform

$$\Phi = \begin{bmatrix} xy' + \alpha y \\ xy' - \alpha y \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ \alpha & -1 \end{bmatrix} Y, \qquad x\Phi' = \left(\begin{bmatrix} \alpha & 0 \\ 0 & -\alpha \end{bmatrix} + \frac{x^2}{2\alpha} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right) \Phi$$

When A_0 is non-resonant, we have $2\alpha \notin \mathbb{Z}$ and the solution can be obtained from Theorem 2.

When A_0 is resonant with $2\alpha \in \mathbb{Z}$:
If 2α is odd, as $A_1 = 0$ we can choose $B_1 = P_1 = O$, and then for all $m \geq 2$ we can similarly set $B_m = O$ and solve for P_m. The solution can still be written as $Y(z) = P(z)z^\Lambda$.

If 2α is even, for $m = 2$ the equation is

$$(A_0 - 2I)P_2 - P_2A_0 = B_2 - A_2$$

As an example, consider $\alpha = 1$ and we have

$$\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} P_2 - P_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = B_2 - \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

Explicitly writing out the elements for P_2, we have

$$P_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{bmatrix} -2a & 0 \\ -4c & -2d \end{bmatrix} = B_2 - \frac{1}{2} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

This constrains $(B_2)_{12}$ and a valid choice is

$$B_2 = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad P_2 = \frac{1}{8} \begin{bmatrix} -2 & 0 \\ -1 & 2 \end{bmatrix}$$

For $m \geq 3$ we can still set $B_m = O$ and solve for P_m. This implies that the final matrix N is

$$N = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l = \begin{bmatrix} 1 & \frac{1}{2}\ln z \\ 0 & 1 \end{bmatrix}, \qquad z^\Lambda = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$$

We know that for $\alpha = 1$ the solutions are $J_1(z)$ and $Y_1(z)$. The term $\ln z$ contributes to $Y_1(z)$.

In general, for a linear ODE of the form

$$x^n y^{(n)} + p_1(x) x^{n-1} y^{(n-1)} + \cdots + p_n(x) y = 0$$

We can choose vector Y as

$$Y = \left[y, xy', x^2 y'', \cdots, x^{n-1} y^{(n-1)}\right]^T$$

For each element y_j, we can obtain the recursive relation

$$y_j = x^{j-1} y^{(j-1)}, \qquad x y_j' = (j-1) y_j + y_{j+1}$$

This leads to the matrix equation

$$xY' = A(x)Y = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 \\ & & & \ddots & \ddots & \\ & & & & n-2 & 1 \\ -p_n(x) & \cdots & \cdots & \cdots & -p_2(x) & n-1-p_1(x) \end{bmatrix} Y$$

Global problem

Consider the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, and the only meromorphic functions on $\hat{\mathbb{C}}$ are the rational functions, denoted as $\mathbb{K}$. For matrix $A \in M(\mathbb{K}, n)$ and $Y'(z) = A(z)Y(z)$, we want to know when the equation only has regular singular points. For rational functions, we can write the matrix $A(z)$ as

$$A(z) = \sum_{j=1}^{k} \frac{P_j(z)}{(z - z_j)^{m_j}} + P_0(z), \qquad \deg(p_j) < m_j$$

If z_j are regular, we have $m_j = 1$ and P_j is a constant matrix. To investigate the behavior at $z = \infty$, consider $w = 1/z$ and

$$\tilde{Y}(w) = Y\left(\frac{1}{w}\right), \qquad \frac{\mathrm{d}\tilde{Y}}{\mathrm{d}w} = -\frac{1}{w^2} Y'\left(\frac{1}{w}\right) = -\frac{1}{w^2} A\left(\frac{1}{w}\right) \tilde{Y}(w)$$

If $z = \infty$ $(w = 0)$ is a regular singularity, we require $w^{-1}A(w^{-1})$ to be analytic at $w = 0$, which is equivalent to $zA(z)$ having a limit as $z \to \infty$, and this requires $P_0(z) = O$. Therefore, if the equation only has regular singularities on $\hat{\mathbb{C}}$, we have $w\tilde{Y}' = \tilde{A}\tilde{Y}$ with

$$A(z) = \sum_{j=1}^{k} \frac{P_j}{z - z_j}, \qquad \tilde{A}(w) = \sum_{j=1}^{k} \frac{P_j}{w z_j - 1}, \qquad \tilde{A}(0) = -\sum_{j=1}^{k} P_j$$

We can then use a linear fractional transformation to obtain

$$Y'(z) = A(z)Y(z), \qquad A(z) = \sum_{j=1}^{N} \frac{A_j}{z - z_j}, \qquad \sum_{j=1}^{N} A_j = 0$$

Now $z = \infty$ is regular. The singularities z_j decompose $\mathbb{C}$ into simply connected polygons U_α, and there is an analytic solution of the equation in each of them. Every side $\overline{z_j z_k}$ corresponds to a monodromic matrix M_{jk} and thus define a map $(A_j) \mapsto (M_{jk})$, which is related to the Riemann-Hilbert problem.

- Asymptotic behavior near irregular singular points (6.3)

$$z^{r+1}Y'(z) = A(z)Y(z), \qquad r \in \mathbb{N}^*, \qquad r \geq 1$$

We call r as the Poincaré rank, and recall the following classification:

$r = -1$: Ordinary point $\qquad$ $r = 0$: Regular singularity $\qquad$ $r \geq 1$: Irregular singularity

First, we can consider the scalar equation with dimension $n = 1$. We have

$$a(z) = \sum_{k\geq 0} a_k z^k, \qquad \frac{y'}{y} = \frac{a(z)}{z^{r+1}} = \sum_{k=0}^{r-1} \frac{a_k}{z^{r+1-k}} + \frac{a_r}{z} + \sum_{k\geq r+1} a_k z^{k-r-1}$$

The solution is

$$\ln y(z) = \sum_{k=0}^{r-1} \frac{a_k}{k-r} \frac{1}{z^{r-k}} + a_r \ln z + \sum_{k\geq r+1} \frac{a_k}{k-r} z^{k-r}, \qquad y(z) = P(z)\, z^{\rho}\, e^{Q(z^{-1})}$$

The exponent is $\rho = a_r$, and the analytic function $P(z)$ and polynomial $Q(w)$ are defined as

$$P(z) = \exp\left(\sum_{k\geq r+1} \frac{a_k}{k-r} z^{k-r}\right), \qquad Q(w) = \sum_{k=0}^{r-1} \frac{a_k}{k-r} w^{r-k}$$

For the matrix case, we still want to find a transformation P such that $Y = PX$ with

$$z^{r+1}X'(z) = B(z)X(z), \qquad B(z) = P^{-1}AP - z^{r+1}P^{-1}P'$$

The matrix $B(z)$ should be as simple as possible. For the equation of $B(z)$, we similarly obtain

$$z^{r+1}P' = AP - PB$$

Written in formal power series, the coefficients for z^m are

$$[z^m]\, z^{r+1}P' = [z^m] \sum_{k\geq 0} kP_k z^{k+r} = (m-r)P_{m-r}, \qquad P_j = O \text{ for } j < 0$$

$$[z^m]\,(AP - PB) = \sum_{k=0}^{m} (A_k P_{m-k} - P_{m-k} B_k)$$

Therefore, we obtain the following set of equations

$$A_0 P_m - P_m B_0 = \sum_{k=1}^{m} (P_{m-k} B_k - A_k P_{m-k}) + (m - r) P_{m-r}$$

We want to properly choose B_m to make the equations simple. Consider A_0 is already reduced to its Jordan normal form, which also gives $P_0 = I$ and $B_0 = A_0$. We need to iteratively solve the matrix equation of the form

$$A_0 P_m - P_m A_0 = B_m - A_m + \sum_{k=1}^{m-1} (P_{m-k} B_k - A_k P_{m-k}) = B_m + F_m$$

The LHS is always resonant. For simplicity, we assume that A_0 has n different eigenvalues and is already diagonalized as $A = \lambda_i \delta_{ij}$. For each element (α, β), we have

$$(\lambda_\alpha - \lambda_\beta)(P_m)_{\alpha\beta} = (B_m)_{\alpha\beta} + (F_m)_{\alpha\beta}$$

When $\alpha \neq \beta$ (off-diagonal elements), we can choose

$$(B_m)_{\alpha\beta} = 0, \qquad (P_m)_{\alpha\beta} = \frac{(F_m)_{\alpha\beta}}{\lambda_\alpha - \lambda_\beta}$$

When $\alpha = \beta$ (diagonal elements), we can choose

$$(B_m)_{\alpha\beta} = -(F_m)_{\alpha\beta}, \qquad (P_m)_{\alpha\beta} = 0$$

Theorem 1. For the matrix equation $z^{r+1} Y'(z) = A(z) Y(z)$, consider that A_0 has n different eigenvalues. There exist an invertible $P(z)$ and a diagonal $B(z)$ such that $Y = PX$ and

$$z^{r+1} X'(z) = B(z) X(z)$$

Corollary 2. With a diagonal $B(z)$, similar to the scalar case, we can define

$$Q(w) = \sum_{k=0}^{r-1} \frac{B_k}{k - r} w^{r-k}, \qquad \rho = B_r, \qquad F'(z) = \left(\sum_{k \geq r+1} B_k z^{k-r-1} \right) F(z), \qquad F(0) = I$$

Note that ρ is a constant diagonal matrix, $Q(w)$ is a diagonal matrix with each element being a polynomial of degree r. Then the solution can be written as

$$X(z) = F(z)\, z^\rho\, e^{Q(z^{-1})}, \qquad Y(z) = P(z)\, F(z)\, z^\rho\, e^{Q(z^{-1})}$$

The result uses the property that ρ and Q are commutable since they are diagonal.

Theorem 3. For an analytic $A(z)$ with rank $r \geq 1$, consider that A_0 has n different eigenvalues. The formal solution of the matrix equation $z^{r+1} Y'(z) = A(z) Y(z)$ is given as

$$Y(z) = \hat{Y}(z)\, z^\rho\, e^{Q(z^{-1})}, \qquad \hat{Y}(0) = I$$

For arbitrary $\theta_1, \theta_2 \in \mathbb{R}$ with $0 < \theta_1 - \theta_2 < \pi/r$, there exists $R > 0$ such that the equation has an analytic solution in $S(\theta_1, \theta_2) \cap B(0, R)$, where $S(\theta_1, \theta_2)$ denotes the sector

$$S(\theta_1, \theta_2) = \{z \in \mathbb{C} \mid \theta_1 < \arg z < \theta_2\}$$

Also, as $z \to 0$ within the domain $S(\theta_1, \theta_2)$, the asymptotic behavior should be interpreted as

$$Y(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \tilde{Y}(z)$$

For an irregular singularity at $z = \infty$, similarly consider $w = 1/z$ and we have

$$\tilde{Y}(w) = Y\left(\frac{1}{w}\right), \qquad w^{-r+1}\, \tilde{Y}'(w) = \tilde{A}(w) Y(w), \qquad \tilde{A}(w) = -A\left(\frac{1}{w}\right)$$

The formal solution can be written as

$$\tilde{Y}(w) = \hat{Y}(w)\, w^{\rho}\, e^{Q(w)}$$

The solution is analytic within the domain $S(\theta_1, \theta_2) \cap \{w \in \mathbb{C} \mid |w| > R\}$.

Theorem (Sibuya 1962). For $\theta_0 \in \mathbb{R}$, there exists a sufficiently small $\delta > 0$ such that there is a solution $Y(z)$ in $S(\theta_0 - \delta, \theta_0 + \pi/r) \cap B(0, R)$.

Corollary 4. There exists $\delta > 0, R > 0$ such that there is a solution $Y(z)$ satisfying Theorem 3 in the following domain

$$S_l = \left\{z \in \mathbb{C}^* \mid \frac{\pi}{r}(l-1) - \delta < \arg z < \frac{\pi}{r} l\right\} \cap B(0, R), \qquad l = 1,2,\cdots,2r$$

Stokes phenomenon

Now consider the intersection

$$S(l, l+1) = \left\{z \in \mathbb{C}^* \mid \frac{\pi}{r} l - \delta < \arg z < \frac{\pi}{r} l\right\} \cap B(0, R)$$

Corollary 4 indicates that there are solutions Y_l and Y_{l+1} in this domain $S(l, l+1)$. Hence, there is a constant matrix C_l, the **Stokes multiplier**, such that $Y_{l+1}(z) = Y_l(z) C_l$.

In S_l and S_{l+1} respectively, we have

$$Y_l(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \hat{Y}(z), \qquad Y_{l+1}(z)\, z^{-\rho}\, e^{-Q(z^{-1})} \sim \hat{Y}(z), \qquad z \to 0$$

Multiply the second equation with the inverse of the first one, and we have

$$e^{Q(z^{-1})} z^{\rho} C_l\, z^{-\rho} e^{-Q(z^{-1})} \sim I, \qquad z \to 0, \qquad z \in S(l, l+1)$$

For each element (α, β), we have

$$(C_l)_{\alpha\beta}\, e^{q_\alpha(z^{-1}) - q_\beta(z^{-1})}\, z^{\rho_\alpha - \rho_\beta} \sim \delta_{\alpha\beta}, \qquad z \to 0, \qquad z \in S(l, l+1)$$

When $\alpha = \beta$ (diagonal), we have $(C_l)_{\alpha\alpha} = 1$. When $\alpha \neq \beta$ (off-diagonal), note that

$$q_\alpha(z^{-1}) - q_\beta(z^{-1}) = \frac{\lambda_\beta - \lambda_\alpha}{r} z^{-r} + o(z^{-r})$$

Consider a ray $\gamma \in S(l, l+1)$. If there exists a ray γ such that as $z \to 0$ along γ, we have

$$\mathrm{Re}\{(\lambda_\beta - \lambda_\alpha) z^{-r}\} > 0, \qquad \text{then } \ (C_l)_{\alpha\beta} = 0$$

If there does not exist such a ray γ for the exponent, then nothing can be said about $(C_l)_{ij}$. If the eigenvalues λ_n are sorted by lexicographic order (λ_R, λ_I), then C_l must be an upper or lower triangular matrix, dependent on l being odd or even.

We define the **Stokes ray** as those that lead to

$$\mathrm{Re}\{(\lambda_\beta - \lambda_\alpha) z^{-r}\} = 0$$

There are $M = n(n-1)r$ Stokes rays in total. Each ray corresponds to a Stokes factor.

Example: Airy equation

$$y'' = zy$$

The corresponding matrix equation is

$$Y = \begin{bmatrix} y \\ y' \end{bmatrix}, \qquad Y' = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ zy \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ z & 0 \end{bmatrix} Y$$

To analyze the behavior at $z = \infty$, we rewrite it as

$$z^{-1} Y'(z) = \begin{bmatrix} 0 & 1/z \\ 1 & 0 \end{bmatrix} Y, \qquad r = 2$$

- Exercise

Regular singular point

$$Y(z) = \begin{bmatrix} y_1(z) \\ y_2(z) \end{bmatrix}, \qquad zY'(z) = \begin{bmatrix} -1/2 + z & z \\ z & 1/2 + z \end{bmatrix} Y(z) = A(z) Y(z)$$

$z = 0$ is a regular singularity. The coefficients of the power series of $A(z)$ are

$$A_0 = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad A_k = O, \qquad k \geq 2$$

We already have a diagonal A_0, which implies $P_0 = I$ and $B_0 = A_0$. For $m = 1$ we have

$$(A_0 - I) P_1 - P_1 A_0 = B_1 - A_1$$

Explicitly writing out the elements for P_1, we have

$$P_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \begin{bmatrix} -a & -2b \\ 0 & -d \end{bmatrix} = B_1 - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

We can obtain a lower triangular B_1, as well as the corresponding P_1 as

$$P_1 = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

For $m \geq 2$, there is no resonance and we can choose $B_m = O$. As an example, for $m = 2$

$$(A_0 - 2I)P_2 - P_2 A_0 = B_2 + P_1 B_1 - A_1 P_1$$

We can solve for P_2 as

$$P_2 = \begin{bmatrix} 1/4 & 1/2 \\ 0 & 3/4 \end{bmatrix}, \qquad B_2 = O$$

Repeat this process and we can also obtain

$$P_3 = \begin{bmatrix} -1/12 & 5/16 \\ -1/4 & 5/12 \end{bmatrix}, \qquad B_3 = O$$

The monodromic data (Λ, N) are then given as

$$\Lambda = \operatorname{diag}\left(-\frac{1}{2}, \frac{1}{2}\right), \qquad N = B_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The transformation $P(z)$ is given as

$$P(z) = \sum_{k \geq 0} P_k z^k = \begin{bmatrix} 1 + z + \frac{1}{4}z^2 - \frac{1}{12}z^3 + \cdots & \frac{1}{2}z + \frac{1}{2}z^2 + \frac{5}{16}z^3 + \cdots \\ -\frac{1}{4}z^3 + \cdots & 1 + z + \frac{3}{4}z^2 + \frac{5}{12}z^3 + \cdots \end{bmatrix}$$

The fundamental solution matrix becomes

$$Y(z) = P(z) z^{\Lambda} z^{N}, \qquad z^{\Lambda} = \operatorname{diag}\left(\frac{1}{\sqrt{z}}, \sqrt{z}\right), \qquad z^N = \sum_{l=0}^{n} \frac{(\ln z)^l}{l!} N^l = \begin{bmatrix} 1 & 0 \\ \ln z & 1 \end{bmatrix}$$

Asymptotic Analysis of DEs (2): Linear ODE with Parameters

For $x \in I \subseteq \mathbb{R}$ and $y \in \Omega \subseteq \mathbb{R}^n$, consider the following ODE with respect to a small parameter $\varepsilon \in B^*(0,\delta)$ given as

$$F(x,y,y',\varepsilon) = 0, \qquad y(x_0) = y_0$$

We want to study the asymptotic behavior of its solution $y(x;\varepsilon)$ as $\varepsilon \to 0^{\pm}$.

- Formal power series expansion (7.1)

Assume that the solution can be written as

$$y(x;\varepsilon) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots$$

The ODE now becomes

$$F(x, y_0 + \varepsilon y_1 + \cdots, y_0' + \varepsilon y_1' + \cdots; \varepsilon) = 0$$

The Taylor expansion with respect to ε around $p_0 = (x, y_0, y_0'; 0)$ is

$$F = F(p_0) + \varepsilon\left[\frac{\partial F}{\partial y}(p_0) y_1 + \frac{\partial F}{\partial y'}(p_0)\, y_1' + \frac{\partial F}{\partial \varepsilon}(p_0)\right]$$

$$+\varepsilon^2\left[\frac{\partial F}{\partial y} y_2 + \frac{\partial F}{\partial y'} y_2' + \frac{1}{2} y_1^T \frac{\partial^2 F}{\partial y \partial y} y_1 + \frac{1}{2} y_1'^T \frac{\partial^2 F}{\partial y' \partial y'} y_1' + \frac{1}{2}\frac{\partial^2 F}{\partial \varepsilon^2}\right] = 0$$

For each order of ε, we have

$$\varepsilon^0\!: \; F(x, y_0, y_0'; 0) = 0, \qquad y_0 = y_0(x)$$

$$\varepsilon^1\!: \; y_1' = A(x) y_1 + B_1(x), \qquad A = -\left(\frac{\partial F}{\partial y'}\right)^{-1}\frac{\partial F}{\partial y}, \qquad B_1 = -\left(\frac{\partial F}{\partial y'}\right)^{-1}\frac{\partial F}{\partial \varepsilon}$$

Note that for $[\varepsilon^1]$, the derivatives are evaluated at $(x, y_0(x), y_0'(x), 0)$. For $[\varepsilon^2]$ we have

$$\varepsilon^2\!: \; y_2' = A(x) y_2 + B_2(x), \qquad B_2(x) = -\left(\frac{\partial F}{\partial y'}\right)^{-1}(\cdots)$$

As long as the fundamental matrix of $y' = A(x)y$ is known, we can recursively solve $y_n(x)$.

There are several issues arising

- F may not be defined at $\varepsilon = 0$.
- $F(x, y_0, y_0', 0)$ may not have a solution (e.g., boundary layer equation).
- The Jacobian $\partial F/\partial y'$ is not invertible at $p_0 = (x, y_0, y_0'; 0)$.
- The properties of the formal power series are bad.

Now simply consider a function $y(x;\varepsilon)$ with its formal power series

$$y(x;\varepsilon) = \sum_{n\geq 0} y_n(x)\varepsilon^n, \qquad \varepsilon \to 0$$

The equivalent statement is that for $\forall N \in \mathbb{N}$ we have

$$\lim_{\varepsilon \to 0} \frac{y(x;\varepsilon) - \sum_{n=0}^{N} y_n(x)\varepsilon^n}{y_N(x)\varepsilon^N} = 0$$

If the function series has pointwise but not uniform convergence, then the remainder depends on x and is unbounded at some points. The partial sum is thus not practical to use.

Example: Duffing equation

$$y'' + y + \varepsilon y^3 = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

Multiplying y' gives

$$\left(\frac{1}{2}y'^2 + \frac{1}{2}y^2 + \frac{\varepsilon}{4}y^4\right)' = 0, \qquad (y')^2 + y^2 + \frac{\varepsilon}{2}y^4 = 1 + \frac{\varepsilon}{2}$$

The constant is determined from the initial conditions. This leads to an elliptical integral

$$x = \pm \int_1^y \frac{\mathrm{d}y}{\sqrt{\left(1 + \frac{\varepsilon}{2}\right) - y^2 - \frac{\varepsilon}{2}y^4}}$$

We notice that $y_{\min} = -1$ and $y_{\max} = 1$. The period of the oscillator is

$$T = 2\int_{y_{\min}}^{y_{\max}} \frac{\mathrm{d}y}{\sqrt{\left(1 + \frac{\varepsilon}{2}\right) - y^2 - \frac{\varepsilon}{2}y^4}}$$

If we directly expand it into a formal power series, we have

$$(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots) + \varepsilon(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots)^3 = 0$$

The initial conditions are

$$y_0(0) = 1, \qquad y_0'(0) = 0, \qquad y_k(0) = 0, \qquad y_k'(0) = 0, \qquad k \geq 1$$

For each order of ε, we have

$$\varepsilon^0\!: \; y_0'' + y_0 = 0, \qquad y_0 = \cos x$$

$$\varepsilon^1\!: \; y_1'' + y_1 + y_0^3 = 0, \qquad y_1 = \frac{1}{32}(\cos 3x - \cos x) - \frac{3}{8}x \sin x$$

The $x \sin x$ term gives an increasing amplitude with x. We can similarly obtain

$$y_2 = -\frac{9}{128}x^2 \cos x + \frac{3}{32}x \sin x - \frac{9}{256}x \sin 3x + \cdots$$

The **secular terms** such as $x^n \cos x$ make the partial sum useless for computation. The reason for this behavior is the resonance with the forcing term involving y_0 to y_{n-1}. Now we consider a simpler version of the Duffing equation

$$y'' + y + \varepsilon y = 0, \qquad y(x;\varepsilon) = \cos\left(\sqrt{1+\varepsilon}\, x\right)$$

The period deviates slightly from 2π, and the Taylor expansion will lead to secular terms. This shows the limitation of the method of direct expansion.

➢ Poincaré-Lindstedt, Poincaré-Lighthill-Kuo (PLK), Strained coordinate method (9.3)

Consider the following example (Tsien, 1956)

$$(x+\varepsilon u)u' + u = 0, \qquad u(1) = 1$$

First we try using the formal power series

$$u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

For each order of ε, we have

$$\varepsilon^0\colon\ x u_0' + u_0 = 0, \qquad u_0(1) = 1, \qquad u_0 = \frac{1}{x}$$

$$\varepsilon^1\colon\ x u_1' + u_0 u_0' + u_1 = 0, \qquad u_1(1) = 0, \qquad u_1 = \frac{1}{2x}\left(1 - \frac{1}{x^2}\right)$$

We can similarly obtain

$$\varepsilon^2\colon\ x u_2' + u_2 + u_0 u_1' + u_1 u_0' = 0, \qquad u_2 = -\frac{1}{2x^3}\left(1 - \frac{1}{x^2}\right)$$

The solution is ordinary around $x = 1$, but is singular at $x = 0$. In other words, the solution is uniformly convergent in $[a, +\infty)$ for any $a > 0$, but not in $(0, +\infty)$.

Strained coordinate (9.3.3)

We introduce the strained coordinate $x = x(\xi)$ with the formal power series

$$u(x;\varepsilon) = u_0(\xi) + \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + \cdots$$
$$x(x;\varepsilon) = \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \cdots$$

Now we denote u' and x' as the derivatives with respect to ξ. The operator becomes

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}\xi}\frac{\mathrm{d}}{\mathrm{d}\xi} = \frac{1}{x'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}$$

Then the ODE becomes

$$(x+\varepsilon u)u' + x'u = 0$$

For each order of ε, we have

$$\varepsilon^0\colon\ \xi u_0' + u_0 = 0, \qquad u_0 = \frac{1}{\xi}$$

$$\varepsilon^1\colon\ \xi u_1' + u_1 = -x_1 u_0' - x_1' u_0 - u_0 u_0', \qquad x_1(1) = u_1(1) = 0$$

Here both x_1 and u_1 are unknowns. We require that the singularity of u_1 at $\xi = 0$ is not higher than the singularity of u_0. We want to find x_1 such that the RHS is ordinary at $\xi = 0$. A simple choice is to let the RHS be zero, which gives

$$(\xi u_1)' = -\left(x_1 u_0 + \frac{1}{2}u_0^2\right)' = 0, \qquad x_1 = \frac{1}{2}\left(\xi - \frac{1}{\xi}\right), \qquad u_1 = 0$$

We can similarly obtain

$$\varepsilon^2\colon\ (x_2 + u_1)u_0' + (x_1 + u_0)u_1' + \xi u_2' + u_2 + u_0 x_2' + u_1 x_1' = 0$$

$$\xi u_2' + u_2 = \frac{x_2}{\xi^2} - \frac{x_2'}{\xi}, \qquad x_2(1) = u_2(1) = 0$$

The choice $x_2 = u_2 = 0$ is valid. For $n \geq 2$, the equation is homogeneous with respect to x_n, and we can always choose $x_n = u_n = 0$. Hence, we obtain an exact solution

$$u(\xi;\varepsilon) = \frac{1}{\xi}, \qquad x(\xi;\varepsilon) = \xi + \frac{\varepsilon}{2}\left(\xi - \frac{1}{\xi}\right)$$

Writing as $u = u(x;\varepsilon)$, we have

$$u = -\frac{x}{\varepsilon} + \sqrt{\left(\frac{x}{\varepsilon}\right)^2 + \frac{2}{\varepsilon} + 1}$$

Example: Duffing equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y + \varepsilon y^3 = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

Now we consider the solution

$$y(x;\varepsilon) = y_0(\xi) + \varepsilon y_1(\xi) + \varepsilon^2 y_2(\xi) + \cdots$$
$$x(x;\varepsilon) = \xi + \varepsilon x_1(\xi) + \varepsilon^2 x_2(\xi) + \cdots$$

The second-order derivative operator becomes

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{x'(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\frac{y'(\xi)}{x'(\xi)}\right) = \frac{y''x' - y'x''}{(x')^3}$$

The equation then becomes

$$y''x' - y'x'' + (x')^3(y + \varepsilon y^3) = 0$$

The initial conditions are

$$y_0(0) = 1, \qquad y_0'(0) = 0, \qquad y_k(0) = y_k'(0) = 0, \qquad x_k(0) = 0, \qquad k \geq 1$$

There is no constraint on $x_k'(0)$ and we can set $x_k'(0) = 0$. For each order of ε, we have

$$\varepsilon^0\!: \; y_0'' + y_0 = 0, \qquad y_0 = \cos\xi$$
$$\varepsilon^1\!: \; y_1'' + y_1 = y_0'x_1'' + 2y_0''x_1' - y_0^3$$

Using the solution y_0, we have

$$y_1'' + y_1 = -\sin\xi\, x_1'' - 2\cos\xi\, x_1' - \frac{3}{4}\cos\xi - \frac{1}{4}\cos 3\xi$$

The forcing term $\cos\xi$ leads to resonance, and we want to remove the secular term by setting

$$\sin\xi\, x_1'' + 2\cos\xi\, x_1' + \frac{3}{4}\cos\xi = 0, \qquad x_1 = -\frac{3}{8}\xi$$

Then the equation for y_1 becomes

$$y_1'' + y_1 = -\frac{1}{4}\cos 3\xi\,, \qquad y_1 = \frac{1}{32}(\cos 3\xi - \cos\xi)$$

We can further obtain (e.g., Fourier series expansion)

$$\begin{aligned}\varepsilon^2\!: \; y_2'' + y_2 &= y_1'x_1'' + y_0'x_2'' + 2y_0''x_2' + 2y_1''x_1' - 3y_0''(x_1')^2 - 3y_0'x_0'x_1'' - 3y_0^2 y_1 \\ &= -\sin\xi\, x_2'' - 2\cos\xi\, x_2' + \frac{57}{128}\cos\xi + \frac{1}{16}\cos 3\xi - \frac{3}{128}\cos 5\xi\end{aligned}$$

To remove the secular term, we need to set

$$-\sin\xi\, x_2'' - 2\cos\xi\, x_2' + \frac{57}{128}\cos\xi = 0, \qquad x_2 = \frac{57}{256}\xi$$

With this choice of x_2, we can solve for y_2. Eventually, the solution is

$$x = \xi\left(1 - \frac{3}{8}\varepsilon + \frac{57}{256}\varepsilon^2 - \cdots\right) = \xi\left(1 + \sum_{k\geq 1}\omega_k\varepsilon^k\right)$$

This is the typical form of the strained coordinate for weakly nonlinear oscillations.

- Method of multiple scales (9.3.4)

$$y'' + 2\varepsilon y' + y = 0, \qquad y(0) = 1, \qquad y'(0) = 0$$

The exact solution is obtained from the characteristic equation

$$\lambda^2 + 2\varepsilon\lambda + 1 = 0, \qquad \lambda_{1,2} = -\varepsilon \pm i\sqrt{1-\varepsilon^2}$$

$$y(x) = e^{-\varepsilon x}\cos\left(\sqrt{1-\varepsilon^2}\, x + \theta_0\right), \qquad \theta_0 = -\arctan\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}$$

First we try using the strained coordinate method

$$y''x' - y'x'' + 2\varepsilon(x')^2y' + (x')^3y = 0$$

For each order of ε, we have

$$\varepsilon^0:\ y_0'' + y_0 = 0, \qquad y_0 = \cos\xi$$

$$\varepsilon^1:\ y_1'' + y_1 = -\sin\xi\, x_1'' - 2\cos\xi\, x_1' - 2\sin\xi$$

We still want to remove the secular term, but now $x_1(\xi)$ becomes singular at $\xi = \pi$

$$x_1 = 1 - \xi\cot\xi$$

The issue is due to the lack of amplitude information ($e^{-\varepsilon x}$) in the strained coordinate method.

In this damped oscillation, there are two (time) scales for the fast oscillation and slow damping, respectively. We introduce a number of scales

$$T_0 = x, \qquad T_1 = \varepsilon x, \qquad T_k = \varepsilon^k x$$

Consider the solution of the form

$$y(x;\varepsilon) = Y(T_0, T_1, \cdots, T_k; \varepsilon) = Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots$$

The goal is to convert the original ODE into PDEs with more degrees of freedom introduced. With the notations D_x and ∂_k, the derivative operator becomes

$$D_x y = \frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{k\geq 0}\frac{\partial Y}{\partial T_k}\frac{\partial T_k}{\partial x} = \sum_{k\geq 0}\varepsilon^k\frac{\partial Y}{\partial T_k} = \sum_{k\geq 0}\varepsilon^k\partial_k Y$$

The damped oscillation equation then becomes

$$[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2) + 2\varepsilon(\partial_0 + \varepsilon\partial_1) + 1](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2) = o(\varepsilon^2)$$

The initial conditions are analyzed as

$$y(0) = Y(T_0(0), T_1(0), \cdots; \varepsilon) = Y(\mathbf{0}; \varepsilon) = Y_0(\mathbf{0}) + \varepsilon Y_1(\mathbf{0}) + \cdots = 1$$

$$y'(0) = \left(\sum_{k\geq 0} \varepsilon^k \partial_k\right)\left(\sum_{l\geq 0} \varepsilon^l Y_l(\mathbf{0})\right) = 0$$

This leads to the initial conditions for $Y_k(\mathbf{0})$ given as

$$Y_0(\mathbf{0}) = 1, \qquad Y_k(\mathbf{0}) = 0, \qquad k \geq 1$$

For the derivatives, the first several orders give the initial conditions

$$\partial_0 Y_0(\mathbf{0}) = 0, \qquad \partial_1 Y_0(\mathbf{0}) + \partial_0 Y_1(\mathbf{0}) = 0, \qquad \partial_0 Y_2(\mathbf{0}) + \partial_1 Y_1(\mathbf{0}) + \partial_2 Y_0(\mathbf{0}) = 0$$

For ε^0 term, we have

$$\varepsilon^0\!: \ \partial_0^2 Y_0 + Y_0 = 0, \qquad Y_0 = A_0(T_1, \cdots) \cos T_0 + B_0(T_1, \cdots) \sin T_0$$

For ε^1 term, we have

$$\varepsilon^1\!: \ \partial_0^2 Y_1 + Y_1 = -2\partial_1\partial_0 Y_0 - 2\partial_0 Y_0 = 2(\partial_1 A_0 + A_0) \sin T_0 - 2(\partial_1 B_0 + B_0) \cos T_0$$

To remove the secular terms, we set the coefficients of the resonant forcing as zero

$$\partial_1 A_0 + A_0 = 0, \qquad \partial_1 B_0 + B_0 = 0$$

We can update the general solutions for A_0 and B_0 as

$$A_0(T_1, \cdots) = e^{-T_1} A_0(T_2, \cdots), \qquad B_0(T_1, \cdots) = e^{-T_1} B_0(T_2, \cdots)$$

The equation of Y_1 then gives

$$\partial_0^2 Y_1 + Y_1 = 0, \qquad Y_1 = A_1(T_1, \cdots) \cos T_0 + B_1(T_1, \cdots) \sin T_0$$

For ε^2 terms, we can further obtain

$$\partial_0^2 Y_2 + Y_2 + 2(\partial_0\partial_1 + \partial_0) Y_1 + (2\partial_0\partial_2 + \partial_1^2 + 2\partial_1) Y_0 = 0$$

Note that from previous results, we already have

$$Y_0 = e^{-T_1} A_0(T_2, \cdots) \cos T_0 + e^{-T_1} B_0(T_2, \cdots) \sin T_0$$

$$\partial_2\partial_0 Y_0 = -e^{-T_1}(\partial_2 A_0) \sin T_0 + e^{-T_1}(\partial_2 B_0) \cos T_0\,, \qquad (\partial_1^2 + 2\partial_1) Y_0 = -Y_0$$

$$(\partial_0\partial_1 + \partial_0) Y_1 = -(\partial_1 A_1 + A_1) \sin T_0 + (\partial_1 B_1 + B_1) \cos T_0$$

The equation for Y_2 is thus obtained as

$$\partial_0^2 Y_2 + Y_2 = (2\partial_2 A_0 + B_0) e^{-T_1} \sin T_0 + (-2\partial_2 B_0 + A_0) e^{-T_1} \cos T_0$$
$$-(\partial_1 A_1 + A_1) \sin T_0 + (\partial_1 B_1 + B_1) \cos T_0$$

To remove these secular terms, we require

$$\partial_2 A_0 = -\frac{1}{2} B_0, \qquad \partial_2 B_0 = \frac{1}{2} A_0, \qquad \partial_2^2 A_0 + \frac{1}{4} A_0 = 0$$
$$\partial_1 A_1 + A_1 = 0, \qquad \partial_1 B_1 + B_1 = 0$$

We can update the general solutions of the coefficients as

$$A_0(T_1, \cdots) = A_0(T_3, \cdots) e^{-T_1} \cos\left(\frac{1}{2} T_2\right) + B_0(T_3, \cdots) e^{-T_1} \sin\left(\frac{1}{2} T_2\right)$$

$$B_0(T_1, \cdots) = A_0(T_3, \cdots) e^{-T_1} \sin\left(\frac{1}{2} T_2\right) - B_0(T_3, \cdots) e^{-T_1} \cos\left(\frac{1}{2} T_2\right)$$

$$A_1(T_1,\cdots)=e^{-T_1}A_1(T_2,\cdots),\qquad B_1(T_1,\cdots)=e^{-T_1}B_1(T_2,\cdots)$$

The equation of Y_2 then gives

$$\partial_0^2 Y_2+Y_2=0,\qquad Y_2=A_2(T_1,\cdots)\cos T_0+B_2(T_1,\cdots)\sin T_0$$

As a summary, now we obtain

$$\begin{aligned}Y_0&=\left[A_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right)+B_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right)\right]\cos T_0\\&\quad+\left[A_0(T_3,\cdots)e^{-T_1}\sin\left(\frac{1}{2}T_2\right)-B_0(T_3,\cdots)e^{-T_1}\cos\left(\frac{1}{2}T_2\right)\right]\sin T_0\\Y_1&=e^{-T_1}A_1(T_2,\cdots)\cos T_0+e^{-T_1}B_1(T_2,\cdots)\sin T_0\end{aligned}$$

At the initial $x=0$, we have

$$Y_0(\mathbf{0})=A_0(T_3,\cdots)=1,\qquad \partial_0Y_0(\mathbf{0})=-B_0(T_3,\cdots)=0$$

$$Y_1(\mathbf{0})=A_1(T_2,\cdots)=0,\qquad \partial_1Y_0(\mathbf{0})+\partial_0Y_1(\mathbf{0})=-A_0(T_3,\cdots)+B_1(T_2,\cdots)=0$$

With these coefficients, we have

$$Y_0=e^{-T_1}\cos\left(T_0-\frac{1}{2}T_2\right),\qquad Y_1=e^{-T_1}\sin T_0$$

The summary of the current solution is

$$y(x;\varepsilon)=Y_0+\varepsilon Y_1+\cdots=e^{-\varepsilon x}\left[\cos\left(x-\frac{1}{2}\varepsilon^2x+\cdots\right)+\varepsilon\sin(x+\cdots)\right]+\cdots$$

Example 1: Van der Pol oscillator (p397)

$$y''+\varepsilon(y^2-1)y'+y=0$$

We want to obtain a general solution. The equation becomes

$$\begin{aligned}&[\partial_0^2+2\varepsilon\partial_0\partial_1+\varepsilon^2(2\partial_0\partial_2+\partial_1^2)+\cdots](Y_0+\varepsilon Y_1+\varepsilon^2Y_2+\cdots)\\&\qquad+\varepsilon(Y_0^2+2\varepsilon Y_0Y_1-1)(\partial_0+\varepsilon\partial_1)(Y_0+\varepsilon Y_1)+(Y_0+\varepsilon Y_1+\varepsilon^2Y_2+\cdots)=0\end{aligned}$$

For ε^0 term, we still have

$$\varepsilon^0\!:\ \partial_0^2Y_0+Y_0=0,\qquad Y_0=A_1(T_1,T_2,\cdots)\cos\big(T_0+B_1(T_1,T_2,\cdots)\big)$$

For ε^1 term, denote $\theta=T_0+B_1$ and we have

$$\begin{aligned}\varepsilon^1\!:\ \partial_0^2Y_1+Y_1&=-2\partial_0\partial_1Y_0-(Y_0^2-1)\partial_0Y_0\\&=2(\partial_1A_1)\sin\theta+2A_1\cos\theta\,(\partial_1B_1)+(A_1^2\cos^2\theta-1)A_1\sin\theta\\&=\left(2\partial_1A_1-A_1+\frac{1}{4}A_1^3\right)\sin\theta+2A_1(\partial_1B_1)\cos\theta+\frac{1}{4}A_1^3\sin3\theta\end{aligned}$$

To remove the secular terms, we require

$$2\partial_1A_1-A_1+\frac{1}{4}A_1^3=0,\qquad \partial_1B_1=0$$

We can then solve for A_1 and B_1 as

$$\frac{1}{A_1^2}=\frac{1}{4}(C_1e^{-T_1}+1),\qquad A_1=\frac{2}{\sqrt{1+C_1(T_2,\cdots)e^{-T_1}}},\qquad B_1=B_2(T_2,\cdots)$$

Now the equation for Y_1 can be solved as

$$\partial_0^2 Y_1 + Y_1 = \frac{1}{4} A_1^3 \sin 3\theta\,, \qquad Y_1 = -\frac{A_1^3}{32}\sin(3\theta) + C_2\cos\theta$$

The summary of the current solution is

$$y(x;\varepsilon) = \frac{2}{\sqrt{1 + C_1(\varepsilon^2 x, \cdots)e^{-\varepsilon x}}}\cos\big(x + B_2(\varepsilon^2 x, \cdots)\big) + \varepsilon Y_1 + o(\varepsilon)$$

Example 2: Mathieu equation (9.2)

$$y'' + (\delta(\varepsilon) + \varepsilon\cos x)y = 0$$

We want to properly choose $\delta(\varepsilon)$ such that the solution still has a period of 2π. Consider

$$\delta(\varepsilon) = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 + \cdots$$

Directly expanding $y(x;\varepsilon)$ into the formal power series, we have

$$(y_0'' + \varepsilon y_1'' + \cdots) + (\delta_0 + \varepsilon\delta_1 + \cdots + \varepsilon\cos x)(y_0 + \varepsilon y_1 + \cdots) = 0$$

For ε^0 term, to keep the 2π-periodicity we have

$$\varepsilon^0\!: \ y_0'' + \delta_0 y_0 = 0, \qquad y_0 = A_0\cos\big(\sqrt{\delta_0}x\big) + B_0\sin\big(\sqrt{\delta_0}x\big)\,, \qquad \delta_0 = n^2, \quad n \in \mathbb{N}^*$$

Take $\delta_0 = 1$, and for ε^1 term we have

$$\varepsilon^1\!: \ y_1'' + \delta_0 y_1 = -\delta_1 y_0 - y_0\cos x$$

$$y_1'' + y_1 = -(\delta_1 + \cos x)(A_0\cos x + B_0\sin x)$$

To remove the secular terms, we require $\delta_1 = 0$ and then y_1 is solved as

$$y_1 = -\frac{A_0}{2} + \frac{A_0}{6}\cos 2x + \frac{B_0}{6}\sin 2x + A_1\cos x + B_1\sin x$$

For ε^2 term, we have

$$\begin{aligned}\varepsilon^2\!: \ y_2'' + y_2 = &-\delta_2(A_0\cos x + B_0\sin x)\\ &- \cos x\left(-\frac{A_0}{2} + \frac{A_0}{6}\cos 2x + \frac{B_0}{6}\sin 2x + A_1\cos x + B_1\sin x\right)\end{aligned}$$

To remove the secular terms, we have

$$A_0\left(-\delta_2 + \frac{5}{12}\right) = 0, \qquad -B_0\left(\delta_2 + \frac{1}{12}\right) = 0$$

Therefore, since the initial conditions determine A_0 and B_0, not all conditions will lead to the same period of 2π. When A_0 or B_0 is zero, it is possible to keep the same period.

Now we study the Mathieu equation using the method of multiple scales. Directly set $\delta_0 = 1$ and $\delta_1 = 0$, and we have

$$\begin{aligned}&[\partial_0^2 + 2\varepsilon\partial_0\partial_1 + \varepsilon^2(2\partial_0\partial_2 + \partial_1^2)](Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2)\\ &\qquad + (1 + \varepsilon\cos T_0 + \varepsilon^2\delta_2)(Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2) = o(\varepsilon^2)\end{aligned}$$

For ε^0 and ε^1 terms, we have

$$\varepsilon^0:\ \partial_0^2 Y_0 + Y_0 = 0, \qquad Y_0 = A_0(T_1,\cdots)\cos T_0 + B_0(T_1,\cdots)\sin T_0$$

$$\varepsilon^1:\ \partial_0^2 Y_1 + Y_1 = -2(-\partial_1 A_0 \sin T_0 + \partial_1 B_0 \cos T_0) - A_0 \frac{1+\cos 2T_0}{2} - \frac{B_0}{2}\sin 2T_0$$

To remove the secular terms, we require

$$\partial_1 A_0 = 0, \qquad \partial_1 B_0 = 0, \qquad A_0 = A_0(T_2,\cdots), \qquad B_0 = B_0(T_2,\cdots)$$

The general solution to Y_1 is the same as previous

$$Y_1 = -\frac{A_0}{2} + \frac{A_0}{6}\cos 2T_0 + \frac{B_0}{6}\sin 2T_0 + A_1(T_1,\cdots)\cos T_0 + B_1(T_1,\cdots)\sin T_0$$

For ε^2 term, we have

$$\varepsilon^2:\ \partial_0^2 Y_2 + Y_2 = -2(-\partial_1 A_1 \sin T_0 + \partial_1 B_1 \cos T_0) - 2(-\partial_2 A_0 \sin T_0 + \partial_2 B_0 \cos T_0)$$
$$+\frac{1}{2}A_0\cos T_0 - \frac{1}{12}A_0\cos T_0 - \frac{B_0}{12}\sin T_0 - \delta_2 A_0 \cos T_0 - \delta_2 B_0 \sin T_0 + \cdots$$

The non-resonant forcing terms are neglected. To remove the secular terms, we require

$$2\partial_1 A_1 + 2\partial_2 A_0 = \left(\frac{1}{12} + \delta_2\right)B_0, \qquad -2\partial_1 B_1 - 2\partial_2 B_0 = \left(-\frac{5}{12} + \delta_2\right)A_0$$

Note that from ε^1 term, we have A_0 and B_0 depending on T_2 and further. Consider a simpler case with $A_1 = B_1 = 0$, which correspond to specific initial conditions. This gives

$$\partial_2 A_0 = \frac{1}{2}\left(\frac{1}{12} + \delta_2\right)B_0, \qquad \partial_2 B_0 = \frac{1}{2}\left(\frac{5}{12} - \delta_2\right)A_0$$

This leads to a second-order equation for A_0 as

$$\partial_2^2 A_0 + K_2 A_0 = 0, \qquad K_2 = \frac{1}{4}\left(\delta_2 + \frac{1}{12}\right)\left(\delta_2 - \frac{5}{12}\right)$$

Depending on the sign of K_2, we have

$$A_0 = C_1 \cos\sqrt{K_2}T_2 + C_2 \sin\sqrt{K_2}T_2\,, \qquad \delta_2 < -\frac{1}{12} \text{ or } \delta_2 > \frac{5}{12}$$
$$A_0 = C_1 e^{\sqrt{-K_2}T_2} + C_2 e^{-\sqrt{-K_2}T_2}, \qquad -\frac{1}{12} < \delta_2 < \frac{5}{12}$$

The summary of the current solution is

$$y(x;\varepsilon) = A_0 \cos T_0 + B_0 \sin T_0 + \varepsilon Y_1 + \cdots$$

For the exponential case, the finite energy of the system implies $C_1 = 0$, while the exponential decay cannot be observed. This corresponds to the band gap.

Asymptotic Analysis of Differential Equations (3): WKBJ method

For $x \in I = [\alpha, \beta] \subseteq \mathbb{R}$ and $y \in \Omega \subseteq \mathbb{R}^n$, consider the following ODE with respect to a large parameter λ given as

$$y''(x) + f(x;\lambda)\, y(x) = 0, \qquad \lambda \to +\infty$$

The function $f(x;\lambda)$ has the asymptotic expansion

$$f(x;\lambda) \sim \lambda^2 \sum_{n \geq 0} f_n(x) a_n(\lambda), \qquad \lambda \to +\infty, \qquad x \in I$$

This method originates from the analysis of Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi, \qquad \psi'' + \frac{2m(V-E)}{\hbar^2}\psi = 0$$

The classical limit corresponds to $\hbar \to 0^+$ ($\lambda = \hbar^{-1} \to +\infty$). The issue of this problem lies in the function $f(x;\lambda) \to \infty$ as $\lambda \to +\infty$. Note that

$$\frac{y''}{f} + y = 0 \qquad \Longrightarrow \qquad y = 0 \quad \text{when} \quad \lambda \to +\infty$$

We cannot obtain a useful solution from ε^0 term, since $\varepsilon = \lambda^{-1} \to 0^+$ is in the highest-order derivative term, unlike the ODE with parameters analyzed by previous methods.

- WKBJ method (7.2)

We first transform the ODE into the **Riccati equation**

$$u = (\ln y)' = \frac{y'}{y}, \qquad u' + u^2 + f = 0$$

From the solution of the Riccati equation, the solution of the original equation is

$$y = \exp\left(\int_{x_0}^{x} u(s;\lambda)\, \mathrm{d}s\right)$$

Consider the asymptotic series

$$u(x;\lambda) \sim \sum_{n \geq 0} u_n(x) b_n(\lambda), \qquad f(x;\lambda) \sim \lambda^2 \sum_{n \geq 0} f_n(x) a_n(\lambda), \qquad \lambda \to +\infty$$

This implies the following constraints

$$a_0(\lambda) = 1, \qquad a_{n+1}(\lambda) = o\big(a_n(\lambda)\big), \qquad b_{n+1}(\lambda) = o\big(b_n(\lambda)\big), \qquad \lambda \to +\infty$$

The Riccati equation becomes

$$[u_0(x)b_0(\lambda) + u_1(x)b_1(\lambda) + \cdots]' + [u_0(x)b_0(\lambda) + u_1(x)b_1(\lambda) + \cdots]^2 \\ + \lambda^2[f_0(x)a_0(\lambda) + f_1(x)a_1(\lambda) + \cdots] = 0$$

The leading order terms are

$$u_0'(x)b_0(\lambda) + u_0^2(x)b_0^2(\lambda) + \lambda^2 f_0(x) = o\big(b_0(\lambda)\big) + o\big(b_0^2(\lambda)\big) + o(\lambda^2)$$

We analyze the dominant balance among these three terms, and we need to set

$$b_0(\lambda) = \lambda, \qquad u_0^2(x) + f_0(x) = o(\lambda^2), \qquad u_0(x) = \pm\sqrt{-f_0(x)}$$

The **turning points** at which $f_0(x) = 0$ govern the behavior of the solution in different regimes. The next order terms give

$$u_0'(x)\lambda + 2u_0(x)u_1(x)\lambda b_1(\lambda) + \lambda^2 f_1(x)a_1(\lambda) = o(\lambda) + o\big(\lambda b_1(\lambda)\big) + o\big(\lambda^2 a_1(\lambda)\big)$$

There are several cases depending on the order of $a_1(\lambda)$:

- Case 1: Dominant balance of term I and II (special $C = 0$ of Case 3)

$$\lambda^2 a_1(\lambda) = o(\lambda), \qquad a_1(\lambda) = o\left(\frac{1}{\lambda}\right), \qquad b_1(\lambda) = 1, \qquad u_1(x) = -\frac{u_0'}{2u_0}$$

- Case 2: Dominant balance of term II and III

$$\lambda = o\big(\lambda^2 a_1(\lambda)\big), \qquad b_1(\lambda) = \lambda a_1(\lambda), \qquad u_1(x) = -\frac{f_1}{2u_0}$$

- Case 3: Dominant balance of all three terms

$$\lim_{\lambda\to+\infty} \lambda a_1(\lambda) = C, \qquad b_1(\lambda) = 1, \qquad u_1(x) = -\frac{u_0' + Cf_1}{2u_0}$$

This process ends when we reach $b_N(\lambda) = O(1)$, and this gives

$$u(x;\lambda) \sim \sum_{n=0}^{N} u_n(x)b_n(\lambda) + o(1), \qquad \lambda \to +\infty$$

$$y(x;\lambda) \sim \exp\left(\sum_{n=0}^{N} b_n(\lambda)\int_{x_0}^{x} u_n(s)\,\mathrm{d}s\right)\big(1 + o(1)\big), \qquad \lambda \to +\infty$$

Usually, we study the case with $a_n(\lambda) = \lambda^{-n}$, from which we choose $u(x;\lambda)$ as

$$u(x;\lambda) \sim \lambda\sum_{n\ge 0} u_n(x)\lambda^{-n}$$

The Riccati equation becomes

$$\sum_{n\ge0} u_n'(x)\lambda^{-n-1} + \left(\sum_{k\ge0} u_k(x)\lambda^{-k}\right)\left(\sum_{l\ge0} u_l(x)\lambda^{-l}\right) + \sum_{n\ge0} f_n(x)\lambda^{-n} = 0$$

For each order of λ, we have

$$\lambda^0\colon\ u_0^2 + f_0 = 0, \qquad u_0 = \pm\sqrt{-f_0}$$

$$\lambda^{-n}\colon\ u_{n-1}' + \sum_{k=0}^{n} u_k u_{n-k} + f_n = 0, \qquad u_n = -\frac{1}{2u_0}\left(u_{n-1}' + \sum_{k=1}^{n-1} u_k u_{n-k} + f_n\right)$$

Consider $x_0 \in (\alpha,\beta)$ such that $f(x_0) \neq 0$, and assume that we can further find $\delta > 0$ such that $f_0(x) > M$ or $f_0(x) < -M$ when $x_0 \in B(x_0,\delta)$. We also assume $f_n \in C^\infty$, and we have

$$u(x;\lambda) \sim \lambda u_0(x) + u_1(x) + O(\lambda^{-1}), \qquad \lambda \to +\infty$$

The first two terms are given as

$$u_0(x) = \pm\sqrt{-f_0}, \qquad u_1(x) = -\frac{u_0' + f_1}{2u_0} = -\frac{1}{4}\frac{f_0'}{f_0} - \frac{f_1}{2u_0}$$

Therefore, we have the solutions

$$u^{\pm}(x;\lambda) = \pm\lambda\sqrt{-f_0} - \frac{1}{4}\frac{f_0'}{f_0} \pm \frac{f_1}{2\sqrt{-f_0}}$$

$$y^{\pm}(x;\lambda) = f_0^{-\frac{1}{4}} \exp\left(\pm\int_{x_0}^{x} \lambda\sqrt{-f_0}\,\mathrm{d}s \pm \frac{1}{2}\int_{x_0}^{x} \frac{f_1}{\sqrt{-f_0}}\,\mathrm{d}s\right) \cdot \big(1 + o(1)\big)$$

More specifically, depending on the sign of $f_0(x)$ on $B(x_0,\delta)$, we have

$$y^{\pm}(x;\lambda) = f_0^{-\frac{1}{4}} \exp\left(\pm i\lambda\int_{x_0}^{x} \sqrt{f_0}\,\mathrm{d}s \pm \frac{i}{2}\int_{x_0}^{x} \frac{f_1}{\sqrt{f_0}}\,\mathrm{d}s\right) \cdot \big(1 + o(1)\big), \qquad f_0(x_0) > 0$$

$$y^{\pm}(x;\lambda) = |f_0|^{-\frac{1}{4}} \exp\left(\pm\lambda\int_{x_0}^{x} \sqrt{|f_0|}\,\mathrm{d}s \mp \frac{1}{2}\int_{x_0}^{x} \frac{f_1}{\sqrt{|f_0|}}\,\mathrm{d}s\right) \cdot \big(1 + o(1)\big), \qquad f_0(x_0) < 0$$

Example

$$y''(x) + [\lambda^2 + \varepsilon\mu(x)]y(x) = 0, \qquad \lambda \to +\infty$$

This describes the propagation of light in a medium with spatial variation in the refractive index given by $\varepsilon\mu(x)$. For this equation, we have

$$f_0 = 1, \qquad f_1 = 0, \qquad f_2 = \varepsilon\mu(x), \qquad f_n = 0, \qquad n \geq 3$$

Directly from the WKBJ method, we can obtain

$$u_0^{\pm}(x) = \pm i, \qquad u_1^{\pm}(x) = 0, \qquad u_2^{\pm}(x) = \pm\frac{i\varepsilon}{2}\mu(x)$$

Since $f_0 > 0$, the solution is

$$y^{\pm}(x;\lambda) = \exp\left(\pm i\lambda(x - x_0) \pm \frac{i\varepsilon}{2\lambda}\int_{x_0}^{x} \mu(s)\,\mathrm{d}s\right) \cdot \big(1 + o(\lambda^{-1})\big), \qquad \lambda \to +\infty$$

The dominant term is a high-frequency oscillation.

Consistency of WKBJ asymptotic series

We first study the case with $f_0(x_0) > 0$, which gives

$$u_0^{\pm}(x) = \pm i\sqrt{f_0(x)}, \qquad u_n^{\pm}(x) = -\frac{1}{2u_0^{\pm}}\left({u_{n-1}^{\pm}}' + \sum_{k=1}^{n-1} u_k^{\pm}u_{n-k}^{\pm} + f_n\right)$$

Denote the exact solution as $u(x;\lambda)$. We want to know if there exists $\delta > 0$ such that

$$\frac{u(x;\lambda) - \lambda \sum_{n=0}^{N} u_n(x)\lambda^{-n}}{\lambda^{-N-1}} \rightrightarrows 0, \qquad N \to \infty, \qquad x \in B(x_0, \delta)$$

Denote the partial sum as $u_N(x;\lambda)$ and the difference as $\Delta_N(x;\lambda) = u(x;\lambda) - u_N(x;\lambda)$. The initial value of the error at $x = x_0$ satisfies

$$U_N(\lambda) = \Delta_N(x_0;\lambda) = O(\lambda^{-N}), \qquad \lambda \to +\infty$$

With $u = u_N + \Delta_N$, substituting it into the Riccati equation leads to

$$\Delta_N' + u_N' + \Delta_N^2 + 2u_N\Delta_N + u_N^2 + f = 0, \qquad \Delta_N' + \Delta_N^2 + 2\lambda u_0 \Delta_N + A_N\Delta_N = B_N$$

We organize some terms into A_N and B_N with the following behaviors

$$A_N(x;\lambda) = 2\sum_{n=1}^{N} u_n \lambda^{-n+1} = O(1), \qquad B_N(x;\lambda) = u_N' + u_N^2 + f = O(\lambda^{-N+1})$$

Introduce an exponential integrating factor

$$E(x,y;\lambda) = \exp\left(2\lambda \int_y^x u_0(s)\,\mathrm{d}s\right) = \exp\left(2i\lambda \int_y^x \sqrt{f_0(s)}\,\mathrm{d}s\right), \qquad \frac{\partial E}{\partial x} = 2\lambda u_0(x)E$$

Apply it to the Riccati equation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}[E(x,x_0;\lambda)\Delta_N(x;\lambda)] = 2\lambda u_0 \Delta_N E + \Delta_N' E = (B_N - A_N\Delta_N - \Delta_N^2)E$$

Integrate both sides, and with the initial values we have

$$E(x,x_0;\lambda)\Delta_N(x;\lambda) - U_N(\lambda) = \int_{x_0}^{x} E(s,x_0;\lambda)(B_N - A_N\Delta_N - \Delta_N^2)\,\mathrm{d}s$$

Since $E(y,x;\lambda) = E^{-1}(x,y;\lambda)$, this leads to

$$\Delta_N(x;\lambda) = T_N[\Delta_N](x;\lambda) = U_N(\lambda)E(x_0,x;\lambda) + \int_{x_0}^{x} E(s,x;\lambda)(B_N - A_N\Delta_N - \Delta_N^2)\,\mathrm{d}s$$

The nonlinear integral operator is denoted as T_N. The above expression implies that Δ_N is a fixed point of T_N. For two functions $w_1, w_2 \in \{f \in C^0 \,|\, \|f\| = \sup|f(x)| \le M\}$, we have

$$\|T_N[w_1] - T_N[w_2]\| \le \left|\int_{x_0}^{x} (|A_N| + |w_1| + |w_2|)\,\mathrm{d}s\right| \|w_2 - w_1\| \le C_1\delta\|w_2 - w_1\|$$

We can choose δ such that $\|T_N[w_1] - T_N[w_2]\| \le \|w_2 - w_1\|$, which is a contraction mapping. Eventually, we have

$$\|\Delta_N\| = \|T_N[\Delta_N]\| \le \frac{C_2}{\lambda^N} = O(\lambda^{-N})$$

This proves that if initially we have $O(\lambda^{-N})$ error at $x = x_0$, then it holds uniformly over (α, β).

In terms of the case with $f_0(x_0) < 0$, for the $+$ solution we have

$$u_0^+(x) = \sqrt{|f_0(x)|}$$

The integrating factor now becomes

$$E^{+}(x,y;\lambda)=\exp\left(2\lambda\int_{y}^{x}\sqrt{|f_0(s)|}\,\mathrm{d}s\right)$$

When $\lambda\to+\infty$, it is an exponential growth ($x>y$) or decay ($x<y$). In this case, $y^{+}(x;\lambda)$ is consistent only in $[x_0,x_0+\delta)$. Similarly, y^{-} is consistent only in $(x_0-\delta,x_0]$. The validity of WKBJ asymptotics is in the direction of exponential growth.

Turning points

Consider the following example

$$f(x;\lambda)=\lambda^2 x,\qquad y''+\lambda^2 xy=0$$

The ODE can be transformed into

$$z=-\lambda^{2/3}x,\qquad Y'(z)-zY(z)=0$$

This is the Airy equation, from which we can write down the general solution as

$$Y(z)=C_1\mathrm{Ai}(z)+C_2\mathrm{Bi}(z)$$

For $x<0$, $f(x)<0$, when $\lambda\to+\infty$ we have $z\to+\infty$

$$\mathrm{Ai}(z)=\frac{1}{2\sqrt{\pi}}\frac{1}{z^{1/4}}e^{-\frac{2}{3}z^{3/2}}\left(1+O\left(z^{-3/2}\right)\right)$$

$$\mathrm{Bi}(z)=\frac{1}{\sqrt{\pi}}\frac{1}{z^{1/4}}e^{\frac{2}{3}z^{3/2}}\left(1+O\left(z^{-3/2}\right)\right),\qquad z\to+\infty$$

For $x>0$, $f(x)>0$, when $\lambda\to+\infty$ we have $z\to-\infty$

$$\mathrm{Ai}(z)=\frac{1}{\sqrt{\pi}}\frac{1}{|z|^{1/4}}\left[\sin\left(\frac{2}{3}|z|^{3/2}+\frac{\pi}{4}\right)+O\left(|z|^{-3/2}\right)\right]$$

$$\mathrm{Bi}(z)=\frac{1}{\sqrt{\pi}}\frac{1}{|z|^{1/4}}\left[\cos\left(\frac{2}{3}|z|^{3/2}+\frac{\pi}{4}\right)+O\left(|z|^{-3/2}\right)\right],\qquad z\to-\infty$$

Near the turning point, the transition from exponential to oscillatory behaviors is connected by the Airy function. We can define three regimes: left (y_L, exponential), middle (y_M, Airy) and right (y_R, oscillatory). The connection problem is to find a proper y_R given the solution y_L, such that there exists y_M to form a smooth solution near the turning point x_*.

Consider another example

$$f(x;\lambda)=\lambda^2\mathrm{sgn}(x),\qquad y''(x)+f(x;\lambda)\,y(x)=0$$

We can obtain the exact general solution as

$$y=Ae^{\lambda x}+Be^{-\lambda x},\qquad x<0,\qquad y=Ce^{i\lambda x}+De^{-i\lambda x},\qquad x>0$$

The connection conditions can be chosen as $y(0^-)=y(0^+)$ and $y'(0^-)=y'(0^+)$. This gives

$$C=\frac{1-i}{2}A+\frac{1+i}{2}B,\qquad D=\frac{1+i}{2}A+\frac{1-i}{2}B$$

For asymptotic analysis, we first need to choose a solution in $x<0$ such that

$$y_L^+ = e^{\lambda x}\big(1 + o(1)\big), \qquad x < 0, \qquad \lambda \to +\infty$$

In this case, the connection problem is well-posed and we obtain

$$A = 1, \qquad B = 0, \qquad C = \frac{1-i}{2}, \qquad D = \frac{1+i}{2}$$

However, if we choose the other solution in $x < 0$ such that

$$y_L^- = e^{-\lambda x}\big(1 + o(1)\big), \qquad x < 0, \qquad \lambda \to +\infty$$

Since $e^{\lambda x} = e^{-\lambda x} o(1)$ for $x < 0$, the coefficient A is arbitrary and thus we cannot determine a unique oscillatory solution in $x > 0$ to form the connection. Vice versa, if we choose a solution in $x > 0$ such that

$$y_R = Ce^{i\lambda x} + De^{-i\lambda x} + o(1), \qquad x > 0, \qquad \lambda \to +\infty$$

The connection conditions give

$$A = \frac{1+i}{2}C + \frac{1-i}{2}D, \qquad B = \frac{1-i}{2}C + \frac{1+i}{2}D$$

The asymptotic behavior for the solution in $x < 0$ is

$$y_L = Ae^{\lambda x} + Be^{-\lambda x} = Be^{-\lambda x}\big(1 + o(1)\big), \qquad x < 0, \qquad \lambda \to +\infty$$

If $B = 0$, then we need to refer to higher order terms to pose the connection problem.

We assume that $I = (\alpha, \beta)$ only has one turning point x_* such that with $f_0'(x_*) = \nu^2 > 0$. The functions are smooth with $f_n \in C^\infty(I)$. For $f_0(x)$, near the turning point we have

$$f_0(x) = \nu^2(x - x_*) + o(x - x_*)$$

We want to find $\delta(\lambda)$ such that there exists a consistent asymptotic solution within the region

$$x \in \big(x_1, x_* - \delta(\lambda)\big) \cup (x_* + \delta(\lambda), x_2)$$

Based on WKBJ results, we first obtain

$$u_0(x) = \pm\sqrt{-\nu^2(x - x_*) + o(x - x_*)} = O\left(\sqrt{|x - x_*|}\right)$$

Specifically, we have

$$u_0 = \pm\nu\sqrt{|x - x_*|} + o\left(\sqrt{|x - x_*|}\right), \qquad x < x_*$$

$$u_0 = \pm i\nu\sqrt{x - x_*} + o\big(\sqrt{x - x_*}\big), \qquad x > x_*$$

Since f_n is finite, by induction we can further estimate

$$u_1(x) = -\frac{u_0' + f_1}{2u_0} = O(|x - x_*|^{-1})$$

$$u_n(x) = -\frac{1}{2u_0}\left(u_{n-1}' + \sum_{k=1}^{n-1} u_k u_{n-k} + f_n\right) = O\left(|x - x_*|^{\frac{1-3n}{2}}\right)$$

To obtain a valid asymptotic series, we require

$$\lim_{\lambda\to+\infty}\frac{u_{n+1}\lambda^{-n}}{u_n\lambda^{-(n-1)}}=\frac{|x-x_*|^{\frac{1}{2}-\frac{3}{2}(n+1)}\lambda^{-n}}{|x-x_*|^{\frac{1}{2}-\frac{3}{2}n}\lambda^{-n+1}}=\lambda^{-1}|x-x_*|^{-\frac{3}{2}}=0$$

This leads to the condition on $\delta(\lambda)$ as

$$|x-x_*|^{-\frac{3}{2}}=o(\lambda),\qquad \delta(\lambda)=\lambda^{-p},\qquad 0<p<\frac{2}{3}$$

In the outer region, it can be shown that the consistent solution exists for $0<p<2/3$.

For $x\in(x_1,x_*-\lambda^{-p})$, we have

$$y_L^+(x;\lambda)=|f_0(x)|^{-\frac{1}{4}}\exp\left(\lambda\int_{x_*}^{x}\sqrt{|f_0(s)|}\,\mathrm{d}s-\frac{1}{2}\int_{x_*}^{x}\frac{f_1(s)}{\sqrt{|f_0(s)|}}\mathrm{d}s\right)\left(1+O\left(\lambda^{\frac{3p}{2}-1}\right)\right)$$

For $x\in(x_*+\lambda^{-p},x_2)$, we have

$$y_R^\pm(x;\lambda)=f_0(x)^{-\frac{1}{4}}\exp\left(\pm i\lambda\int_{x_*}^{x}\sqrt{f_0(s)}\,\mathrm{d}s\pm\frac{i}{2}\int_{x_*}^{x}\frac{f_1(s)}{\sqrt{f_0(s)}}\mathrm{d}s\right)\left(1+O\left(\lambda^{\frac{3p}{2}-1}\right)\right)$$

Across the turning point for $x\in(x_*-\delta(\lambda),x_*+\delta(\lambda))$, we need to directly solve the equation. We assume that for sufficiently large λ, $f(x;\lambda)$ only have one zero point $x_*(\lambda)$. As $\lambda\to+\infty$, $x_*(\lambda)$ becomes the turning point x_*. For sufficiently small $|x-x_*|$, we uniformly have

$$f_0(x;\lambda)=\nu^2\big(x-x_*(\lambda)\big)+o\big(x-x_*(\lambda)\big)$$

$$f(x;\lambda)=\lambda^2\nu^2\big(x-x_*(\lambda)\big)\Big(1+\big(x-x_*(\lambda)\big)h(x;\lambda)\Big),\qquad h=O(1)$$

With the same transformation

$$x=x_*(\lambda)-\alpha z,\qquad \alpha=(\lambda\nu)^{-2/3},\qquad Y(z;\lambda)=y(x_*(\lambda)-\alpha z;\lambda)$$

The original ODE is converted into the Airy equation

$$Y''-zY=\lambda^{-2/3}z^2g(z;\lambda)Y,\qquad g=O(1),\qquad z\to 0$$

Consider the general solution satisfying

$$Y(z;\lambda)=a(z;\lambda)\,\mathrm{Ai}(z)+b(z;\lambda)\,\mathrm{Bi}(z),\qquad Y'(z;\lambda)=a(z;\lambda)\,\mathrm{Ai}'(z)+b(z;\lambda)\,\mathrm{Bi}'(z)$$

This implies that we choose the coefficients under the following constraint

$$a'(z;\lambda)\,\mathrm{Ai}(z)+b'(z;\lambda)\,\mathrm{Bi}(z)=0$$

Now the Airy equation becomes

$$a'(z;\lambda)\,\mathrm{Ai}'(z)+b'(z;\lambda)\,\mathrm{Bi}'(z)=\lambda^{-2/3}z^2g(z;\lambda)\big(a(z;\lambda)\,\mathrm{Ai}(z)+b(z;\lambda)\,\mathrm{Bi}(z)\big)$$

Using the asymptotic behaviors of the Airy functions and their derivatives, the determinant of the Wronskian and its inverse are obtained as

$$\det W=\mathrm{Ai}(z)\,\mathrm{Bi}'(z)-\mathrm{Ai}'(z)\,\mathrm{Bi}(z)=\frac{1}{\pi},\qquad W^{-1}=\frac{1}{\det W}\begin{pmatrix}\mathrm{Bi}'(z) & -\mathrm{Bi}(z)\\ -\mathrm{Ai}'(z) & \mathrm{Ai}(z)\end{pmatrix}$$

These lead to an ODE system for $a(z;\lambda)$ and $b(z;\lambda)$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = -\pi\lambda^{-\frac{2}{3}} z^2 g \begin{pmatrix} \mathrm{Ai}(z)\,\mathrm{Bi}(z) & [\mathrm{Bi}(z)]^2 \\ -[\mathrm{Ai}(z)]^2 & -\mathrm{Ai}(z)\,\mathrm{Bi}(z) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

With initial conditions at $x - x_*(\lambda) = \pm\lambda^{-p_1}$ given by the outer solution, we want to show that the coefficients a, b are nearly constant when $\lambda \to +\infty$, as indicated by the ODE system.

$$x - x_*(\lambda) = -\alpha z = \mp\lambda^{-p_1}, \qquad z = \pm\nu^{2/3}\lambda^{2/3-p_1} = \pm C\lambda^q$$

The initial values at the left endpoint can be written as

$$a(C\lambda^q;\lambda) = 1 + \delta a, \qquad b(C\lambda^q;\lambda) = 0 + \tilde{C}\delta b, \qquad \delta a, \delta b = o(1)$$

Integrate the ODE from $C\lambda^q$ to an arbitrary z with $|z| < C\lambda^q$

$$\begin{pmatrix} a(z;\lambda) - 1 - \delta a \\ b(z;\lambda) - \tilde{C}\delta b \end{pmatrix} = -\pi\lambda^{-\frac{2}{3}} \int_{C\lambda^q}^{z} \zeta^2 g(\zeta;\lambda) \begin{pmatrix} \mathrm{Ai}(\zeta)\,\mathrm{Bi}(\zeta) & [\mathrm{Bi}(\zeta)]^2 \\ -[\mathrm{Ai}(\zeta)]^2 & -\mathrm{Ai}(\zeta)\,\mathrm{Bi}(\zeta) \end{pmatrix} \begin{pmatrix} a(\zeta) \\ b(\zeta) \end{pmatrix} \mathrm{d}\zeta$$

From this integral equation, it can be shown that

$$\|\tilde{a}\| = \|a(z;\lambda) - 1\| \le |\delta a| + C_1\lambda^{\frac{5q}{2}-\frac{2}{3}}\left(\|\tilde{a}\| + \|\tilde{b}\| + 1\right)$$

$$\|\tilde{b}\| = \|W(z)b(z;\lambda)\| \le |\delta b| + C_1\lambda^{\frac{5q}{2}-\frac{2}{3}}\left(\|\tilde{a}\| + \|\tilde{b}\| + 1\right)$$

The function $W(z)$, which describes the order of $[\mathrm{Bi}(z)]^2$, is defined by

$$W(z) = e^{\frac{4}{3}z^{3/2}}, \qquad z > 0, \qquad W(z) = 1, \qquad z \le 0$$

For the norm to be bounded by a finite value when $\lambda \to +\infty$, we require

$$\frac{5q}{2} - \frac{2}{3} < 0, \qquad q = \frac{2}{3} - p_1 < \frac{4}{15}, \qquad p_1 > \frac{2}{5}$$

Therefore, the **overlap domain** between the inner and outer regimes is

$$\lambda^{-2/3} < |x - x_*| < \lambda^{-2/5}$$

Suppose that $x_L = x_*(\lambda) - \lambda^{-p}$ with $2/5 < p < 2/3$, as $\lambda \to +\infty$ it can be shown that

$$|f_0(x)|^{-\frac{1}{4}} = \nu^{-\frac{1}{2}}\lambda^{\frac{p}{4}}\left(1 + O(\lambda^{-p}) + O(\lambda^{p-1})\right), \qquad \int_{x_*}^{x} \frac{f_1(s)}{\sqrt{|f_0(s)|}}\mathrm{d}s = O\left(\lambda^{-\frac{p}{2}}\right)$$

$$\lambda\int_{x_*}^{x}\sqrt{|f_0(s)|}\,\mathrm{d}s = -\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}}\left(1 + O(\lambda^{-p}) + O(\lambda^{p-1})\right)$$

With these results, we have

$$y_L^+(x_L;\lambda) = \nu^{-\frac{1}{2}}\lambda^{\frac{p}{4}}\exp\left(-\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + O\left(\lambda^{-\frac{p}{2}}\right) + O\left(\lambda^{\frac{3p}{2}-1}\right) + O\left(\lambda^{1-\frac{5p}{2}}\right)\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}y_L^+(x_L;\lambda) = \nu^{\frac{1}{2}}\lambda^{1-\frac{p}{4}}\exp\left(-\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + O\left(\lambda^{-\frac{p}{2}}\right) + O\left(\lambda^{\frac{3p}{2}-1}\right) + O\left(\lambda^{1-\frac{5p}{2}}\right)\right)$$

In the overlap domain, y_L^+ can be represented by a linear combination of the Airy functions.

From previous analysis, the coefficients are given as

$$a(z;\lambda) = \pi \mathrm{Bi}'(z) Y(z;\lambda) - \pi \mathrm{Bi}(z) \frac{\mathrm{d}Y}{\mathrm{d}z}(z;\lambda)$$

$$b(z;\lambda) = \pi \mathrm{Ai}(z) \frac{\mathrm{d}Y}{\mathrm{d}z}(z;\lambda) - \pi \mathrm{Ai}'(z) Y(z;\lambda)$$

For the connection problem, we have

$$Y(z;\lambda) = y_L^+(C\lambda^q;\lambda), \qquad x_L - x_*(\lambda) = -\alpha z_L = -\lambda^{-p_1}, \qquad z_L = \nu^{2/3}\lambda^{2/3-p_1} = C\lambda^q$$

At the left endpoint of the overlap domain, we obtain

$$a(C\lambda^q;\lambda) = 2\sqrt{\pi}\,\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\,(1+\delta a), \qquad b(C\lambda^q;\lambda) = \frac{\sqrt{\pi}}{2}\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\exp\left(-\frac{4}{3}C^{\frac{3}{2}}\lambda^{\frac{3q}{2}}\right)(1+\delta b)$$

Hence, in the neighborhood $|x - x_*(\lambda)| \le \lambda^{-p_1}$, or equivalently $|z| \le C\lambda^q$, we have

$$y_L^+(x;\lambda) = Y_L^+(z;\lambda) = 2\sqrt{\pi}\,\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\left[1 + E_p(\lambda)\right]\mathrm{Ai}(z)$$

$$+2\sqrt{\pi}\,\nu^{-\frac{1}{3}}\,\lambda^{\frac{1}{6}}\,W^{-1}(z)\,E_p(\lambda)\,\mathrm{Bi}(z)$$

For the asymptotic expression of $\mathrm{d}Y_L^+/\mathrm{d}z$, just add a derivative to the Airy functions. The error term $E_p(\lambda)$ represents a possibly different function satisfying

$$E_p(\lambda) = O\left(\lambda^{-p/2}\right) + O\left(\lambda^{3p/2-1}\right) + O\left(\lambda^{1-5p/2}\right), \qquad \lambda \to +\infty$$

The optimal error is obtained when $p = 1/2$, which leads to an estimate of $O(\lambda^{-1/4})$.

We also need to represent y_L^+ as a linear combination of the oscillatory solutions at the right endpoint x_R. Specifically, take $x_R = x_*(\lambda) + \lambda^{-p_1}$ (equivalently $z_R = -C\lambda^q$) and we have

$$y_L^+(x_R;\lambda) = 2\nu^{-\frac{1}{2}}\,\lambda^{\frac{p}{4}}\left[\sin\left(\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}} + \frac{\pi}{4}\right) + E_p(\lambda)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}x} y_L^+(x_R;\lambda) = 2\nu^{\frac{1}{2}}\,\lambda^{1-\frac{p}{4}}\left[\cos\left(\frac{2}{3}\nu\lambda^{1-\frac{3p}{2}} + \frac{\pi}{4}\right) + E_p(\lambda)\right]$$

The oscillatory solutions are given as

$$y_R^\pm(x_R;\lambda) = \nu^{-\frac{1}{2}}\,\lambda^{\frac{p}{4}}\exp\left(\pm\frac{2i}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + E_p(\lambda)\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} y_R^\pm(x_R;\lambda) = \pm i\nu^{\frac{1}{2}}\,\lambda^{1-\frac{p}{4}}\exp\left(\pm\frac{2i}{3}\nu\lambda^{1-\frac{3p}{2}}\right)\left(1 + E_p(\lambda)\right)$$

Hence, we have

$$y_L^+(x;\lambda) = \left(e^{-\pi i/4} + E_p(\lambda)\right) y_R^+(x;\lambda) + \left(e^{\pi i/4} + E_p(\lambda)\right) y_R^-(x;\lambda)$$

This is similar to the result of a previous example where $f(x;\lambda) = \lambda^2 \mathrm{sgn}(x)$.

For $f_0(x)$ with the following form at the turning point x_*

$$f_0(x) = (x - x_*)^m g(x)$$

The procedure remains similar, but instead of Airy functions, we need others for connection.

- Langer transformation (7.2.5)

For simplicity, consider $f(x;\lambda) = \lambda^2 f_0(x)$ that only contains the leading order term

$$y''(x) + \lambda^2 f_0(x) y(x) = 0, \qquad \lambda \to +\infty$$

We consider the nonlinear transformation

$$y(x;\lambda) = a(x) v(x;\lambda), \qquad x = g(\xi), \qquad V(\xi;\lambda) = v(x;\lambda)$$

Eventually, we obtain an ODE of the form

$$V''(\xi;\lambda) - \lambda^2 \xi V(\xi;\lambda) = F(\xi) V(\xi;\lambda), \qquad F(\xi) = -\left.\frac{a''(x)}{a(x)[g'(x)]^2}\right|_{x=g^{-1}(\xi)}$$

Langer's method provides a uniformly small error of $O(\lambda^{-1})$ over a fixed-size interval around the turning point, and can give more accurate information in the neighborhoods than techniques based on rescaling and matching to WKBJ formulae.

- Exercise

Ray equation

Asymptotic Analysis of Differential Equations (4): BV Problems

For $x \in I = [\alpha, \beta] \subseteq \mathbb{R}$ and $a(x), b(x) \in C[\alpha, \beta]$, consider the boundary-value problem with respect to a small parameter ε given as

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(\alpha) = A, \qquad y(\beta) = B, \qquad \varepsilon \to 0^+$$

- Existence of solutions for BVP (8.1)

For a fixed $I = [\alpha, \beta]$ with $y(\alpha) = A$ and $y(\beta) = B$, denote this problem as $\mathrm{BVP}(A, B)$.

Theorem. $\mathrm{BVP}(A, B)$ has a unique solution is equivalent to $\mathrm{BVP}(0,0)$ has a unique solution, which is the trivial solution $y(x; \lambda) = 0$.

Proof. When $\mathrm{BVP}(A, B)$ has a unique solution, it is obvious that $y = 0$ is the unique solution for $\mathrm{BVP}(0,0)$. Now consider $\mathrm{BVP}(0,0)$ has a unique trivial solution. We start from the solutions of the following initial-value problems (IVP)

$$\varepsilon y_1'' + a y_1' + b y_1 = 0, \qquad y_1(\alpha) = 0, \qquad y_1'(\alpha) = 1$$
$$\varepsilon y_2'' + a y_2' + b y_2 = 0, \qquad y_2(\beta) = 0, \qquad y_2'(\beta) = 1$$

We say that $y_1(\beta) \neq 0$ and $y_2(\alpha) \neq 0$. If not, then y_i becomes the solution of $\mathrm{BVP}(0,0)$ but with non-zero y_i' at the boundary, which contradicts the uniqueness of the trivial solution. Then we can construct the solution of $\mathrm{BVP}(A, B)$ as

$$y(x) = \frac{B}{y_1(\beta)} y_1(x) + \frac{A}{y_2(\alpha)} y_2(x)$$

This solution is unique since $\mathrm{BVP}(0,0)$ only has the trivial solution. ■

Now we only need to study when $\mathrm{BVP}(0,0)$ has a unique solution

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(\alpha) = y(\beta) = 0, \qquad \varepsilon \to 0^+$$

First we introduce a transformation $y = g(x)w$ to remove the y' term, which gives

$$2\varepsilon g' + a g = 0, \qquad g(x) = \exp\left(-\frac{1}{2\varepsilon}\int_\alpha^x a(s)\,\mathrm{d}s\right)$$

The ODE becomes

$$\varepsilon w'' + f w = 0, \qquad f(x) = b(x) - \frac{1}{2}a'(x) - \frac{1}{4\varepsilon}a^2(x)$$

The boundary values are still zero with $w(\alpha) = w(\beta) = 0$. Multiply by $w(x)$ and integrate the equation. Using the boundary conditions, we obtain

$$\varepsilon \int_\alpha^\beta [w'(x)]^2\,\mathrm{d}x = \int_\alpha^\beta f(x) w^2(x)\,\mathrm{d}x$$

If $f(x) \leq 0$, then $w = 0$ is the unique trivial solution of BVP(0,0). To satisfy this requirement, we notice two different cases.

$$|a(x)| \geq m > 0, \qquad f(x) \sim -\frac{1}{4\varepsilon}a^2(x) \leq 0, \qquad \varepsilon \to 0^+$$
$$b(x) - \frac{1}{2}a'(x) \leq 0, \qquad f(x) = b(x) - \frac{1}{2}a'(x) - \frac{1}{4\varepsilon}a^2(x) \leq 0$$

- Boundary layers (8.2)

We start with the following example

$$\varepsilon y'' + (1-\varepsilon)y' - (1-\varepsilon)y = 0, \qquad y(0) = y(1) = 1$$

The characteristic roots are

$$m_\pm(\varepsilon) = \frac{\varepsilon - 1 \pm \sqrt{1 + 2\varepsilon - 3\varepsilon^2}}{2\varepsilon}$$

As $\varepsilon \to 0^+$, only one of the two roots remains finite

$$m_+(\varepsilon) = 1 - \varepsilon + \varepsilon^2 - 2\varepsilon^3 + O(\varepsilon^4), \qquad m_-(\varepsilon) = -\frac{1}{\varepsilon} + \varepsilon - \varepsilon^2 + 2\varepsilon^3 - O(\varepsilon^4)$$

The general solution is

$$y(x;\varepsilon) = C_+ e^{m_+(\varepsilon)x} + C_- e^{m_-(\varepsilon)x}$$

The coefficients are solved from the boundary conditions as

$$C_+ = \frac{1 - e^{m_-(\varepsilon)}}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}}, \qquad C_- = \frac{e^{m_+(\varepsilon)} - 1}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}}$$

Outer solution

When $0 < x \leq 1$ and let $\varepsilon \to 0^+$, we can decompose the solution as

$$\begin{aligned} y(x;\varepsilon) &= e^{m_+(\varepsilon)(x-1)} + e^{m_-(\varepsilon)x}\frac{e^{m_+(\varepsilon)} - 1}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}} + e^{m_-(\varepsilon)}\frac{e^{m_+(\varepsilon)(x-1)} - e^{m_+(\varepsilon)}}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}} \\ &= e^{m_+(\varepsilon)(x-1)} + o(\varepsilon^p), \qquad \forall p \in \mathbb{N}, \qquad \varepsilon \to 0^+ \end{aligned}$$

If we consider the asymptotic series by $\{\varepsilon^n\}$, we only need to expand the first term in its Taylor series. The **outer expansion** is obtained as

$$y_{\text{out}}(x;\varepsilon) = e^{x-1} - \varepsilon(x-1)e^{x-1} + \frac{\varepsilon^2}{2}(x^2-1)e^{x-1} + O(\varepsilon^3), \qquad x > 0, \qquad \varepsilon \to 0^+$$

If we do not know the exact solution, we can still obtain the outer solution by considering the formal power series of $y(x;\varepsilon)$ as

$$y(x;\varepsilon) = \sum_{n\geq 0} y_n(x)\varepsilon^n, \qquad y_0(1) = 1, \qquad y_n(1) = 0, \qquad n \geq 1$$

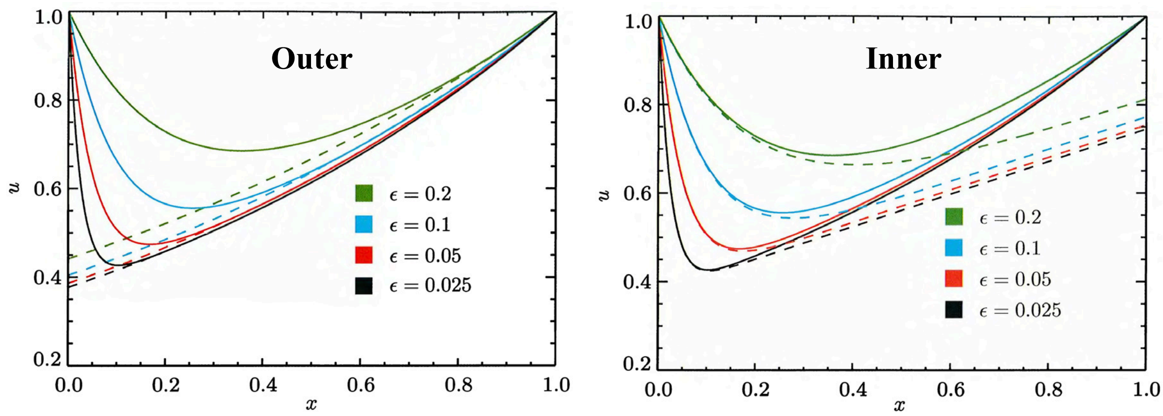
Note that we can only consider one of the two boundary conditions, since

$$\varepsilon^0\colon\ y_0' - y_0 = 0, \qquad y_0(1) = 1, \qquad y_0(x) = e^{x-1}$$

It is a first-order ODE. In general, we cannot obtain a solution that satisfies both boundary conditions for this type of problem. However, using the condition at $x = 1$ successfully gives the outer solution.

$$\varepsilon^1:\ y_0'' + y_1' - y_1 = 0, \qquad y_1(1) = 0, \qquad y_1(x) = -(x-1)e^{x-1}$$

Using the condition at $x = 0$ will not lead to a meaningful result. The comparison of the exact solution (solid) and the outer expansion (dashed) is shown below.



Inner solution

The outer expansion fails near $x = 0$ because the term $e^{-x/\varepsilon}$ that arises from $m_-(\varepsilon)$ does not converge uniformly for $x = O(\varepsilon)$ and for a fixed $0 < x \le 1$. To enlarge the thin boundary layer, we introduce an **inner variable** z, and the ODE becomes

$$z = \frac{x}{\varepsilon}, \qquad Y(z;\varepsilon) = y(\varepsilon z;\varepsilon), \qquad Y'' + (1-\varepsilon)Y' - \varepsilon(1-\varepsilon)Y = 0$$

For our example, given the exact solution, we have

$$Y(z;\varepsilon) = e^{m_+(\varepsilon)(\varepsilon z-1)} + e^{m_-(\varepsilon)\varepsilon z}\frac{e^{m_+(\varepsilon)} - 1}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}} + e^{m_-(\varepsilon)}\frac{e^{m_+(\varepsilon)(\varepsilon z-1)} - e^{m_+(\varepsilon)}}{e^{m_+(\varepsilon)} - e^{m_-(\varepsilon)}}$$

The term $e^{m_-(\varepsilon)\varepsilon z}$ is no longer singular in the exponent as $\varepsilon \to 0^+$. For a fixed $z \ge 0$, we have

$$Y(z;\varepsilon) = e^{m_+(\varepsilon)(\varepsilon z-1)} + e^{m_-(\varepsilon)\varepsilon z}\left(1 - e^{-m_+(\varepsilon)}\right) + o(\varepsilon^p), \qquad \forall p \in \mathbb{N}$$

Again, consider the asymptotic series by $\{\varepsilon^n\}$, the **inner expansion** is

$$Y(z;\varepsilon) = \left[\left(1 - \frac{1}{e}\right)e^{-z} + \frac{1}{e}\right] + \frac{z + 1 - e^{-z}}{e}\varepsilon + O(\varepsilon^2)$$

$$y_{\text{in}}(x;\varepsilon) = \left[\left(1 - \frac{1}{e}\right)e^{-\frac{x}{\varepsilon}} + \frac{1}{e}\right] + \frac{\varepsilon}{e}\left(\frac{x}{\varepsilon} + 1 - e^{-\frac{x}{\varepsilon}}\right) + O(\varepsilon^2), \qquad \varepsilon \to 0^+$$

- Outer asymptotics (8.3)

We use the formal power series to study the general BVP(A, B) given as

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \qquad y(x;\varepsilon) \sim S(x) = \sum_{n\ge0} y_n(x)\varepsilon^n$$

For each order of ε, we have

$$\varepsilon^0:\ ay_0' + by_0 = 0, \qquad y_0 = C_0 \exp\left(-\int_\alpha^x \frac{b(s)}{a(s)}\,\mathrm{d}s\right)$$

$$\varepsilon^n:\ y_{n-1}'' + ay_n' + by_n = 0$$

Let D be the domain of convergence of $S(x)$. If $\alpha \in D$, the condition $y(\alpha) = A$ can be used to determine the coefficients $\{C_n\}$, with $y_0(\alpha) = A$ and $y_n(\alpha) = 0$ for $n \geq 1$. Hence we obtain an outer solution around α. Similarly, we can solve the case with $\beta \in D$.

Depending on the domain D, we have two simple scenarios. First, if $D = [\alpha, \beta)$ or $D = (\alpha, \beta]$, then a boundary layer is present at the endpoint. If $D = [\alpha, x_0) \cup (x_0, \beta]$, then an internal layer is present at the transition point x_0.

- Inner asymptotics for boundary and internal layers (8.4)

Denote the layer thickness around x_0 as $\delta(\varepsilon) = o(1)$ as $\varepsilon \to 0^+$. The inner variable is

$$z = \frac{x - x_0}{\delta(\varepsilon)}, \qquad |z| \leq 1$$

With this rescaling, the ODE becomes

$$Y(z;\varepsilon) = y(x_0 + z\delta(\varepsilon);\varepsilon), \qquad \frac{\varepsilon}{\delta^2(\varepsilon)}Y'' + \frac{a}{\delta(\varepsilon)}Y' + bY = 0$$

Note that $a(z)$ and $b(z)$ are given as

$$a\big(x_0 + z\delta(\varepsilon)\big) = a(x_0) + a'(x_0)z\delta(\varepsilon) + O\big(\delta^2(\varepsilon)\big)$$

If $a(x_0) \neq 0$, the dominant balance gives

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim \frac{a(x_0)}{\delta(\varepsilon)}, \qquad \delta(\varepsilon) \sim \varepsilon$$

If $a(x_0) = 0$ but $a'(x_0) \neq 0$, the dominant balance gives

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim a'(x_0)z \sim b(x_0), \qquad \delta(\varepsilon) \sim \sqrt{\varepsilon}$$

The scaling relation of the layer thickness $\delta(\varepsilon)$ can be analyzed in general from the Taylor series of $a(z)$ and $b(z)$.

Assume $a(x_0) \neq 0$ for simplicity, we take $\delta(\varepsilon) = \varepsilon$ and the ODE becomes

$$Y'' + aY' + \varepsilon bY = 0, \qquad Y(z;\varepsilon) = \sum_{n\geq 0} Y_n(z)\varepsilon^n$$

The leading-order inner equation is

$$\varepsilon^0:\ Y_0'' + a(x_0)Y_0' = 0, \qquad Y_0(z) = c_1 + c_2 e^{-a(x_0)z}$$

In order to match with a reasonable outer solution, we need to choose the exponential decay solution within the inner layer. The sign of $a(x_0)$ then governs the existence of possible boundary or internal layers:

- Boundary layer at the left endpoint $x = \alpha$ with $z > 0$ can exist when $a(\alpha) > 0$
- Boundary layer at the right endpoint $x = \beta$ with $z < 0$ can exist when $a(\beta) < 0$
- Internal layer at an interior point $x = x_0$ can exists when $a(x_0) = 0$, but a different scaling may be required to achieve the dominant balance

➢ Matching of inner and outer asymptotic expansions (8.5)

Consider a possible boundary point $x_0 \in [\alpha, \beta]$ with thickness $\delta(\varepsilon)$. The inner solution $Y(z; \varepsilon)$ and the outer solution $y(x; \varepsilon)$ are given as

$$Y(z;\varepsilon) = \sum_{n\geq 0} Y_n(z)\mu_n(\varepsilon), \qquad z = \frac{x - x_0}{\delta(\varepsilon)}, \qquad y(x;\varepsilon) = \sum_{n\geq 0} y_n(x)\varepsilon^n$$

We introduce an intermediate variable w defined as

$$w = \frac{x - x_0}{\chi(\varepsilon)} = \frac{\delta(\varepsilon)}{\chi(\varepsilon)} z, \qquad \chi(\varepsilon) \to 0, \qquad \frac{\delta(\varepsilon)}{\chi(\varepsilon)} \to 0, \qquad \varepsilon \to 0^+$$

The **intermediate scale** $\chi(\varepsilon)$ is limited by $\delta(\varepsilon) \ll \chi(\varepsilon) \ll 1$, and it can define an overlap domain to connect the inner and outer expansions. Now we truncate both solutions as

$$Y_M(z;\varepsilon) = \sum_{m=0}^{M} Y_m(z)\mu_m(\varepsilon), \qquad y_N(x;\varepsilon) = \sum_{n=0}^{N} y_n(x)\varepsilon^n$$

We want to find a matched expansion $y_{\text{match}}^{NM}(w;\varepsilon)$ such that

$$y_N(x(w);\varepsilon) = y_{\text{match}}^{NM}(w;\varepsilon) + o\big(\mu_M(\varepsilon)\big), \qquad Y_M(z(w);\varepsilon) = y_{\text{match}}^{NM}(w;\varepsilon) + o(\varepsilon^N)$$

Specifically, if there is only one boundary point x_0, we can construct a single formula uniformly valid for the whole interval $[\alpha, \beta]$ as

$$y_{\text{unif}}^{NM}(x;\varepsilon) = y_N(x;\varepsilon) + Y_M\left(\frac{x - x_0}{\delta(\varepsilon)};\varepsilon\right) - y_{\text{match}}^{NM}\left(\frac{x - x_0}{\chi(\varepsilon)};\varepsilon\right)$$

➢ Examples (8.6)

Example 1. Matching of asymptotics

$$\varepsilon y'' + (1 + x^2)y' + xy = 0, \qquad y(-1) = 0, \qquad y(1) = 2$$

Since $a(x) = 1 + x^2 > 0$, we have the left endpoint $x_0 = -1$ as a boundary point. The outer expansion can be solved as

$$\varepsilon^0\colon\ (1 + x^2)y_0' + xy_0 = 0, \qquad y_0(1) = 2, \qquad y_0(x) = 2\sqrt{\frac{2}{1 + x^2}}$$

$$\varepsilon^n:\ y''_{n-1} + (1+x^2)y'_n + xy_n = 0, \qquad y_n(1) = 0$$

The solution of $y_1(x)$ can be obtained as

$$y_1(x) = \frac{1}{16}\sqrt{\frac{2}{1+x^2}}\left[\frac{24x}{(1+x^2)^2} + \frac{4x}{1+x^2} + 4\arctan x - \pi - 8\right]$$

For the inner expansion, with $a(-1) = 2 \neq 0$, we take $\delta(\varepsilon) = \varepsilon$ and the inner equation is

$$z = \frac{x+1}{\varepsilon}, \qquad Y'' + (2 - 2\varepsilon z + \varepsilon^2 z^2)Y' + \varepsilon(\varepsilon z - 1)Y = 0$$

With the choice $\mu_n(\varepsilon) = \varepsilon^n$, the inner expansion can be solved as

$$\varepsilon^0:\ Y_0'' + 2Y_0' = 0, \qquad Y_0(0) = 0, \qquad Y_0(z) = c_1(1 - e^{-2z})$$

$$\varepsilon^1:\ Y_1'' + 2Y_1' - 2zY_0' - Y_0 = 0, \qquad Y_1(0) = 0$$

The solution of $Y_1(z)$ can be obtained as

$$Y_1(z) = \frac{c_1}{4}(2z-1) + d_1 - \left[\frac{c_1}{4}(4z^2 + 2z + 1) + d_2\right]e^{-2z}, \qquad d_1 - d_2 = \frac{c_1}{2}$$

Now we need to find the intermediate scale $\chi(\varepsilon)$ to match the two expansions.

$$w = \frac{x+1}{\chi(\varepsilon)} = \frac{\varepsilon z}{\chi(\varepsilon)}, \qquad \varepsilon \ll \chi(\varepsilon) \ll 1$$

We assume a fixed $w > 0$ and let $\varepsilon \to 0^+$. The outer expansion becomes

$$y_0(w) = 2\sqrt{\frac{2}{1+(\chi w - 1)^2}} = 2 + \chi(\varepsilon)w + O(\chi(\varepsilon)^2), \qquad y_1(w) = -1 - \frac{\pi}{8} + O\big(\chi(\varepsilon)\big)$$

$$y_{\text{out}}(x(w);\varepsilon) = 2 + \chi(\varepsilon)w - \frac{8+\pi}{8}\varepsilon + O\big(\varepsilon\chi(\varepsilon)\big) + O(\chi(\varepsilon)^2)$$

The inner expansion becomes

$$Y_0(w) = c_1\left(1 - e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right) = c_1 + O\left(e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_1(w) = \frac{c_1 w}{2}\frac{\chi(\varepsilon)}{\varepsilon} + d_1 - \frac{c_1}{4} + O\left(\frac{\chi^2(\varepsilon)}{\varepsilon^2}e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

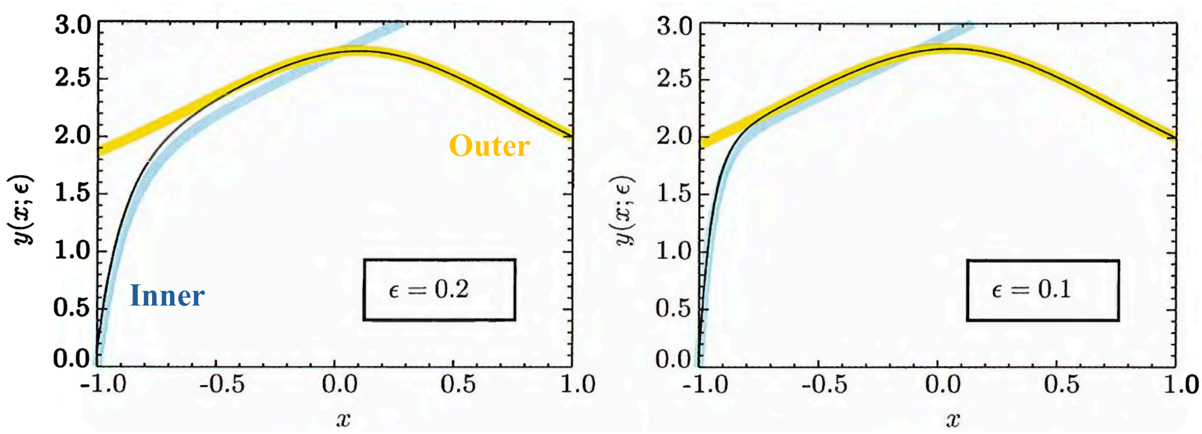
$$Y_{\text{in}}(z(w);\varepsilon) = c_1 + \frac{c_1 w}{2}\chi(\varepsilon) + \left(d_1 - \frac{c_1}{4}\right)\varepsilon + O\left(e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right) + O\left(\frac{\chi^2(\varepsilon)}{\varepsilon}e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

We need to properly choose $\chi(\varepsilon)$ so that all $O(\cdot)$ terms can be controlled. Note that they first should be $o(\varepsilon)$, which gives

$$\chi^2(\varepsilon) \ll \varepsilon, \qquad e^{-2w\frac{\chi(\varepsilon)}{\varepsilon}} \ll \varepsilon, \qquad \varepsilon\ln\varepsilon^{-1} \ll \chi(\varepsilon) \ll \sqrt{\varepsilon}$$

Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$c_1 = 2, \qquad d_1 = -\left(\frac{1}{2} + \frac{\pi}{8}\right)$$



Therefore, the approximation in the overlap domain is

$$y_{\text{match}}^{1,1}(x;\varepsilon) = 2 + \chi(\varepsilon)w - \frac{8+\pi}{8}\varepsilon = x + 3 - \frac{8+\pi}{8}\varepsilon$$

The uniformly valid approximation is

$$y_{\text{unif}}^{1,1}(x;\varepsilon) = y_{\text{out}}(x;\varepsilon) + Y_{\text{in}}\left(\frac{x+1}{\varepsilon};\varepsilon\right) - y_{\text{match}}^{1,1}(x;\varepsilon)$$

This result does not satisfy the ODE and boundary conditions, but the error is very small.

Example 2. Different scaling of layer thickness

$$\varepsilon y'' + 12x^{1/3}y' + y = 0, \qquad y(0) = y(1) = 1$$

In this case, we have

$$b(x) - \frac{1}{2}a'(x) = 1 - 2x^{-\frac{2}{3}} \le -1$$

This shows that the BVP has a unique solution. Since $a(x) > 0$ for $x \in (0,1]$, there will be no possible boundary point in this interval, hence the only possible boundary point is $x_0 = 0$. The outer expansion can be solved as

$$\varepsilon^0\colon\ 12x^{1/3}y_0' + y_0 = 0, \qquad y_0(1) = 1, \qquad y_0(x) = \exp\left(\frac{1 - x^{2/3}}{8}\right)$$

For the inner expansion, we need to use $\delta(\varepsilon)$ and find its proper scaling.

$$z = \frac{x}{\delta(\varepsilon)}, \qquad \frac{\varepsilon}{\delta^2(\varepsilon)}Y'' + \frac{12\big(z\delta(\varepsilon)\big)^{1/3}}{\delta(\varepsilon)}Y' + Y = 0$$

The dominant balance gives

$$\frac{\varepsilon}{\delta^2(\varepsilon)} \sim [\delta(\varepsilon)]^{-2/3}, \qquad \delta(\varepsilon) \sim \varepsilon^{3/4}$$

The inner equation then becomes

$$Y_0'' + 12z^{\frac{1}{3}}Y' + \sqrt{\varepsilon}Y = 0$$

$$\varepsilon^0\colon\ Y_0'' + 12z^{1/3}Y_0' = 0, \qquad Y_0(0) = 1$$

The solution can be obtained as

$$Y_0(z) = 1 + C\int_0^z e^{-9s^{4/3}}\mathrm{d}s$$

The intermediate scale can be denoted as $\chi(\varepsilon) = \varepsilon^p$, and we have

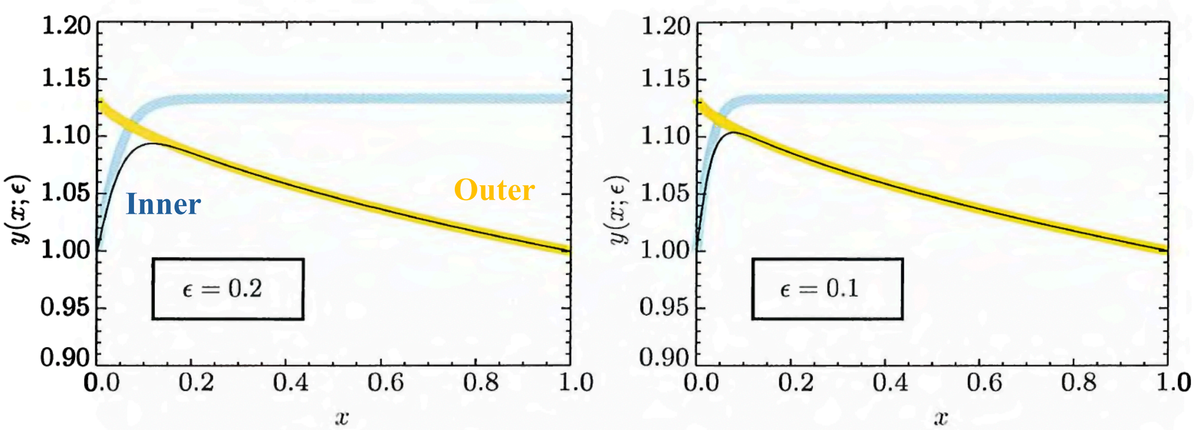
$$w = \frac{x}{\varepsilon^p} = \frac{\varepsilon^{3/4}z}{\varepsilon^p}, \qquad \varepsilon^{3/4} \ll \varepsilon^p \ll 1, \qquad 0 < p < \frac{3}{4}$$

We assume a fixed $w > 0$ and let $\varepsilon \to 0^+$. Since we keep the leading-order term, the matching condition can be simply written as

$$\lim_{x\to 0^+} y_0(x) = \lim_{z\to +\infty} Y_0(z), \qquad e^{1/8} = 1 + \frac{C}{4\sqrt{3}}\Gamma\left(\frac{3}{4}\right)$$

The uniformly valid approximation is

$$y_{\mathrm{unif}}^{0,0}(x;\varepsilon) = \exp\left(\frac{1-x^{2/3}}{8}\right) + 1 + C\int_0^{x\varepsilon^{-3/4}} e^{-9s^{4/3}}\mathrm{d}s - e^{1/8}$$



Example 3. Internal layer

$$\varepsilon y'' + xy' - \left(1 + \frac{x}{4}\right)y = 0, \qquad y(-1) = 3, \qquad y(1) = 1$$

In this case, we have

$$b(x) - \frac{1}{2}a'(x) = -\frac{3}{2} - \frac{x}{4} \leq -\frac{5}{4}$$

The only possible boundary point is $x_0 = 0$ where $a(0) = 0$. This corresponds to an internal layer point. As $a'(0) \neq 0$, the layer thickness scales as $\delta(\varepsilon) \sim \sqrt{\varepsilon}$. The outer expansions need to be solved for both regions to the left and right of $x_0 = 0$.

$$\varepsilon^0:\ xy_0' - \left(1 + \frac{x}{4}\right)y_0 = 0, \qquad y_L(-1) = 3, \qquad y_R(1) = 1$$

$$y_{L0}(x) = -3xe^{\frac{x+1}{4}}, \qquad y_{R0}(x) = xe^{\frac{x-1}{4}}$$

For the inner expansion, we take $\delta(\varepsilon) = \sqrt{\varepsilon}$, $\mu_n(\varepsilon) = \sqrt{\varepsilon}$ and the inner equation is

$$z = \frac{x}{\sqrt{\varepsilon}}, \qquad Y'' + zY' - \left(1 + \frac{z\sqrt{\varepsilon}}{4}\right)Y = 0$$

$$\varepsilon^0\!: \; Y_0'' + zY_0' - Y_0 = 0, \qquad Y_0(z) = C_1 z + C_2\left(e^{-\frac{z^2}{2}} + z\int_{-\infty}^{z} e^{-\frac{s^2}{2}}\mathrm{d}s\right)$$

The two coefficients are to be determined from the matching conditions. The intermediate scale can be chosen as $\chi(\varepsilon) = \varepsilon^{1/4}$, and for a fixed w we have

$$w = \frac{x}{\varepsilon^{1/4}} = \frac{\sqrt{\varepsilon}z}{\varepsilon^{1/4}}, \qquad x \to 0, \qquad z \to \operatorname{sgn}(w)\cdot\infty, \qquad \varepsilon \to 0^+$$

The inner and outer expansions become

$$y_{L0}(w) = -3\varepsilon^{1/4}we^{1/4} + O\left(\sqrt{\varepsilon}\right), \qquad y_{R0}(w) = \varepsilon^{1/4}we^{-1/4} + O\left(\sqrt{\varepsilon}\right)$$

$$Y_0(w^-) = C_1\varepsilon^{-1/4}w + O\left(e^{-w^2/\sqrt{\varepsilon}}\right), \qquad Y_0(w^+) = \left(C_1 + C_2\sqrt{2\pi}\right)\varepsilon^{-1/4}w + O\left(e^{-w^2/\sqrt{\varepsilon}}\right)$$

Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$C_1 = -3e^{\frac{1}{4}}\sqrt{\varepsilon}, \qquad C_2 = \frac{e^{-1/4} + 3e^{1/4}}{\sqrt{2\pi}}\sqrt{\varepsilon}$$

The coefficients depend on ε. The approximations in the overlap domain are

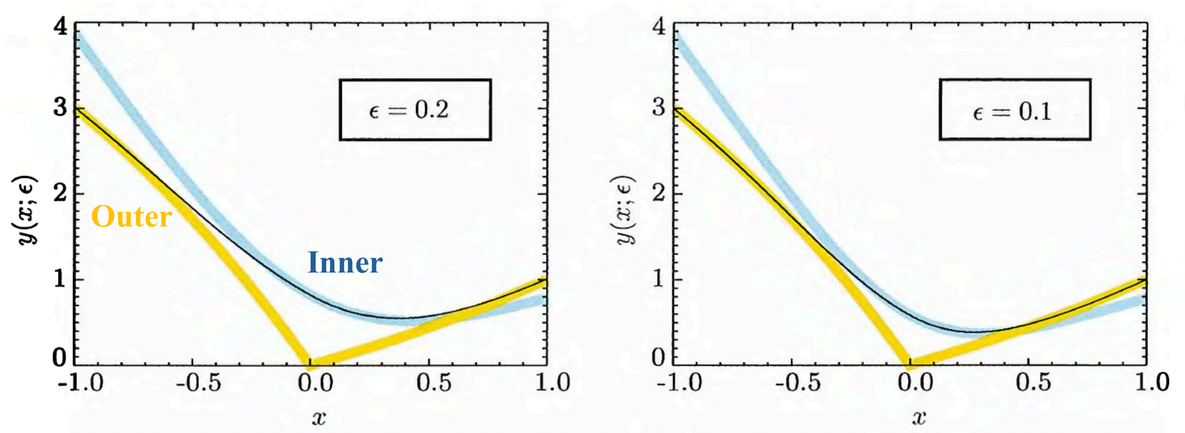
$$y_{L,\mathrm{match}}^{0,0}(x;\varepsilon) = -3e^{1/4}x, \qquad y_{R,\mathrm{match}}^{0,0}(x;\varepsilon) = e^{-1/4}x$$

The uniformly valid approximation is

$$y_{\mathrm{unif}}^{0,0}(x;\varepsilon) = y_{L0}(x) + Y_0\left(\frac{x}{\sqrt{\varepsilon}}\right) - y_{L,\mathrm{match}}^{0,0}(x;\varepsilon), \qquad x < 0$$

$$y_{\mathrm{unif}}^{0,0}(x;\varepsilon) = y_{R0}(x) + Y_0\left(\frac{x}{\sqrt{\varepsilon}}\right) - y_{R,\mathrm{match}}^{0,0}(x;\varepsilon), \qquad x > 0$$

An internal layer like this is also called a corner layer.



- Exercise

Right boundary layer

$$\varepsilon y'' - y' + x^4 y = 0, \qquad y(-1) = y(1) = 1$$

Since $a(x) = -1 < 0$, we have a right boundary point at $x = 1$. The outer expansion is

$$\varepsilon^0 : -y_0' + x^4 y_0 = 0, \qquad y_0(-1) = 1, \qquad y_0(x) = e^{\frac{x^5+1}{5}}$$

$$\varepsilon^1 : y_0'' - y_1' + x^4 y_1 = 0, \qquad y_1(-1) = 0$$

The solution of $y_1(x)$ can be obtained as-

$$y_1(x) = \frac{1}{9} e^{\frac{x^5+1}{5}} (-8 + 9x^4 + x^9)$$

For the inner expansion, with $a(1) \neq 0$, we take $\delta(\varepsilon) = \varepsilon$ and the inner equation is

$$z = \frac{x-1}{\varepsilon} < 0, \qquad Y'' - Y' + \varepsilon(\varepsilon z + 1)^4 Y = 0$$

With the choice $\mu_n(\varepsilon) = \varepsilon^n$, the inner expansion can be solved as

$$\varepsilon^0 : Y_0'' - Y_0' = 0, \qquad Y_0(0) = 1, \qquad Y_0(z) = 1 + c_1(e^z - 1)$$

$$\varepsilon^1 : Y_1'' - Y_1' + Y_0 = 0, \qquad Y_1(0) = 0$$

The solution of $Y_1(z)$ can be obtained as

$$Y_1(z) = c_1(e^z - z - 1 - ze^z) + z + c_2(e^z - 1)$$

Now we need to find the intermediate scale $\chi(\varepsilon)$ to match the two expansions.

$$w = \frac{x-1}{\chi(\varepsilon)} = \frac{\varepsilon z}{\chi(\varepsilon)}, \qquad \varepsilon \ll \chi(\varepsilon) \ll 1$$

We assume a fixed $w < 0$ and let $\varepsilon \to 0^+$. The outer expansion becomes

$$y_0(w) = e^{2/5} + e^{2/5}\chi(\varepsilon) w + O(\chi(\varepsilon)^2), \qquad y_1(w) = \frac{2}{9} e^{2/5} + O\big(\chi(\varepsilon)\big)$$

$$y_{\text{out}}(x(w); \varepsilon) = e^{2/5} + e^{2/5}\chi(\varepsilon) w + \frac{2}{9} e^{2/5} \varepsilon + O\big(\varepsilon\chi(\varepsilon)\big) + O(\chi(\varepsilon)^2)$$

The inner expansion becomes

$$Y_0(w) = 1 - c_1 + O\left(e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_1(w) = (1 - c_1) w \frac{\chi(\varepsilon)}{\varepsilon} - (c_1 + c_2) + O\left(\frac{\chi(\varepsilon)}{\varepsilon} e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_{\text{in}}(z(w); \varepsilon) = 1 - c_1 + (1 - c_1) w \chi(\varepsilon) - (c_1 + c_2)\varepsilon + O\left(e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right) + O\left(\chi(\varepsilon) e^{w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

We need to properly choose $\chi(\varepsilon)$ so that all $O(\cdot)$ terms can be controlled. Note that they first should be $o(\varepsilon)$, which gives

$$\chi^2(\varepsilon) \ll \varepsilon, \qquad e^{w\frac{\chi(\varepsilon)}{\varepsilon}} \ll \varepsilon, \qquad \varepsilon \ln \varepsilon^{-1} \ll \chi(\varepsilon) \ll \sqrt{\varepsilon}$$

Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$c_1 = 1 - e^{2/5}, \qquad c_2 = \frac{7}{9}e^{2/5} - 1$$

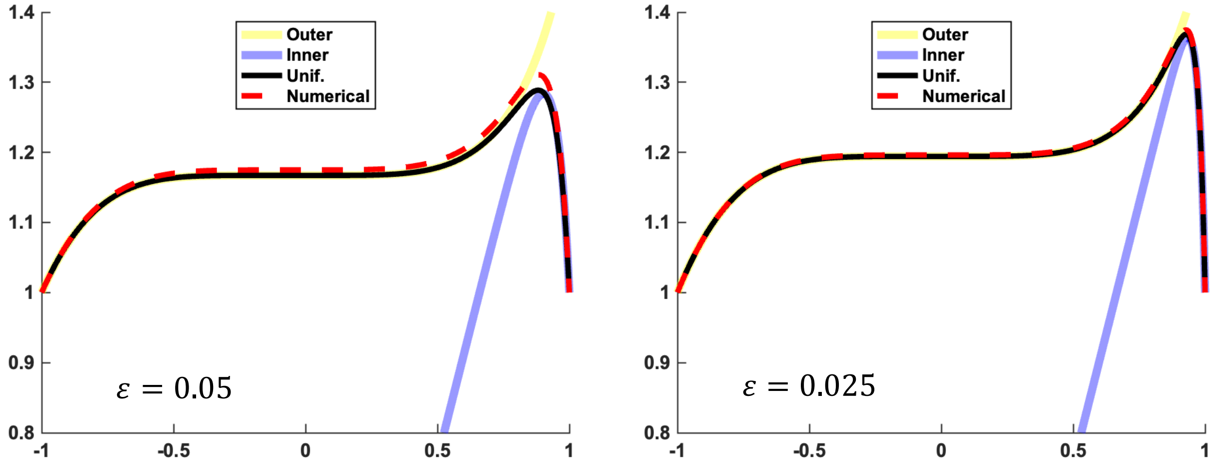
Therefore, the approximation in the overlap domain is

$$y_{\text{match}}^{1,1}(x;\varepsilon) = xe^{2/5} + \frac{2}{9}e^{2/5}\varepsilon$$

The uniformly valid approximation is

$$y_{\text{unif}}^{1,1}(x;\varepsilon) = y_{\text{out}}(x;\varepsilon) + Y_{\text{in}}\left(\frac{x-1}{\varepsilon};\varepsilon\right) - y_{\text{match}}^{1,1}(x;\varepsilon)$$

As ε becomes smaller, the approximation is closer to the numerically solved result.



Left boundary layer

$$\varepsilon y'' + y' - xy = 0, \qquad y(0) = 0, \qquad y(1) = e^{1/2}$$

Since $a(x) = 1 > 0$, we have a left boundary point at $x = 0$. The outer expansion is

$$\varepsilon^0\colon y_0' - xy_0 = 0, \qquad y_0(1) = e^{\frac{1}{2}}, \qquad y_0(x) = e^{\frac{x^2}{2}}$$

$$\varepsilon^1\colon\ y_0'' + y_1' - xy_1 = 0, \qquad y_1(1) = 0, \qquad y_1(x) = -\frac{1}{3}e^{\frac{x^2}{2}}(-4 + 3x + x^3)$$

For the inner expansion, with $a(1) \neq 0$, we take $\delta(\varepsilon) = \varepsilon$ and the inner equation is

$$z = \frac{x}{\varepsilon} > 0, \qquad Y'' + Y' - \varepsilon(\varepsilon z + 1)Y = 0$$

With the choice $\mu_n(\varepsilon) = \varepsilon^n$, the inner expansion can be solved as

$$\varepsilon^0\colon\ Y_0'' + Y_0' = 0, \qquad Y_0(0) = 0, \qquad Y_0(z) = c_1(1 - e^{-z})$$

$$\varepsilon^1\colon\ Y_1'' + Y_1' - Y_0 = 0, \qquad Y_1(0) = 0$$

The solution of $Y_1(z)$ can be obtained as

$$Y_1(z) = c_1(e^{-z} + z - 1 + ze^{-z}) + z + c_2(1 - e^{-z})$$

Now we need to find the intermediate scale $\chi(\varepsilon)$ to match the two expansions.

$$w = \frac{x}{\chi(\varepsilon)} = \frac{\varepsilon z}{\chi(\varepsilon)}, \qquad \varepsilon \ll \chi(\varepsilon) \ll 1$$

We assume a fixed $w > 0$ and let $\varepsilon \to 0^+$. The outer expansion becomes

$$y_0(w) = 1 + O(\chi(\varepsilon)^2), \qquad y_1(w) = \frac{4}{3} - \chi(\varepsilon)w + O\big(\chi^2(\varepsilon)\big)$$

$$y_{\text{out}}(x(w);\varepsilon) = 1 + \frac{4}{3}\varepsilon - \chi(\varepsilon)w\varepsilon + O\big(\varepsilon\chi^2(\varepsilon)\big) + O(\chi(\varepsilon)^2)$$

The inner expansion becomes

$$Y_0(w) = c_1 + O\left(e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_1(w) = (1 + c_1)w\frac{\chi(\varepsilon)}{\varepsilon} + (c_2 - c_1) + O\left(\frac{\chi(\varepsilon)}{\varepsilon}e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

$$Y_{\text{in}}(z(w);\varepsilon) = c_1 + (1 + c_1)w\chi(\varepsilon) + (c_2 - c_1)\varepsilon + O\left(e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right) + O\left(\chi(\varepsilon)e^{-w\frac{\chi(\varepsilon)}{\varepsilon}}\right)$$

We can similarly choose $\chi(\varepsilon)$ so that all $O(\cdot)$ terms are $o(\varepsilon)$. Now comparing the leading-order terms in the outer and inner expansion, we obtain

$$c_1 = -1 - \varepsilon, \qquad c_2 = \frac{2}{\varepsilon} - \varepsilon + \frac{4}{3}$$

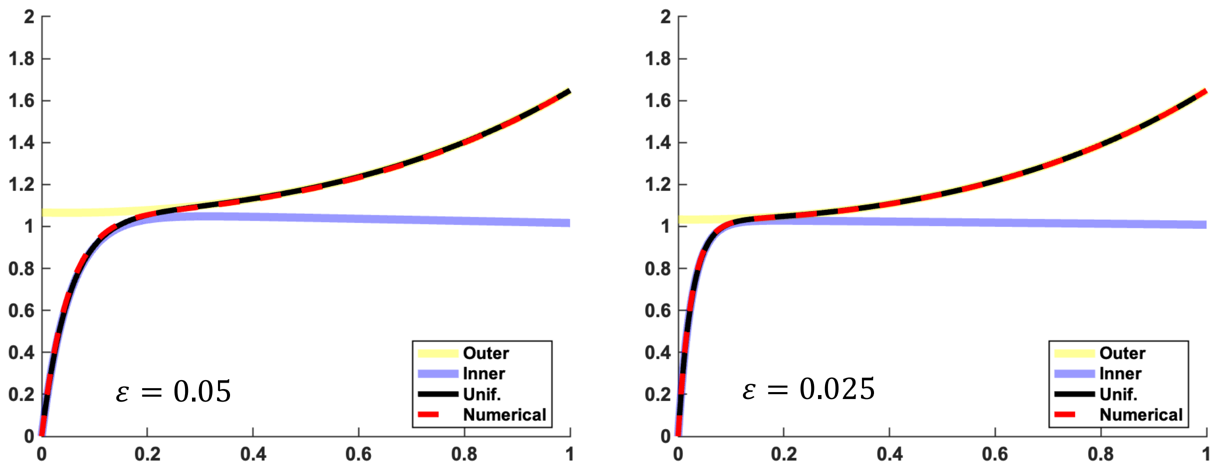
Therefore, the approximation in the overlap domain is

$$y_{\text{match}}^{1,1}(x;\varepsilon) = 1 + \frac{4}{3}\varepsilon - \varepsilon x$$

The uniformly valid approximation is

$$y_{\text{unif}}^{1,1}(x;\varepsilon) = y_{\text{out}}(x;\varepsilon) + Y_{\text{in}}\left(\frac{x}{\varepsilon};\varepsilon\right) - y_{\text{match}}^{1,1}(x;\varepsilon)$$

As ε becomes smaller, the approximation is closer to the numerically solved result.



For this problem we can obtain an exact solution. To facilitate comparison, we first write down

$$y_0(x) = e^{x^2/2}, \qquad Y_0(z) = 1 - e^{-z}, \qquad y_{\text{unif}}^{0,0}(x;\varepsilon) = e^{x^2/2} - e^{-x/\varepsilon}$$

We first introduce the transform to remove the y' term as

$$\varepsilon\xi'' - x\xi - \frac{\xi}{4\varepsilon} = 0, \qquad y(x) = \xi(x)\exp\left(-\frac{x}{2\varepsilon}\right)$$

Then with a new independent variable, the ODE becomes

$$t = \varepsilon^{-\frac{1}{3}}\left(x + \frac{1}{4\varepsilon}\right), \qquad \frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} - t\xi = 0$$

This is the Airy equation. The boundary conditions are modified to

$$\xi(t_0) = \xi\left(\frac{1}{4}\varepsilon^{-\frac{4}{3}}\right) = 0, \qquad \xi(t_1) = \xi\left(\varepsilon^{-\frac{1}{3}} + \frac{1}{4}\varepsilon^{-\frac{4}{3}}\right) = \exp\left(\frac{1+\varepsilon}{2\varepsilon}\right)$$

We can obtain the exact solution of $y(x)$ as follows

$$y(x) = \exp\left(\frac{\varepsilon + 1 - x}{2\varepsilon}\right) \cdot \frac{\mathrm{Ai}(t_0)\mathrm{Bi}\big(t(x)\big) - \mathrm{Ai}\big(t(x)\big)\mathrm{Bi}(t_0)}{\mathrm{Ai}(t_0)\mathrm{Bi}(t_1) - \mathrm{Ai}(t_1)\mathrm{Bi}(t_0)}$$

The asymptotic expansions for Airy functions are

$$\mathrm{Ai}(t) \sim \frac{t^{-1/4}}{2\sqrt{\pi}}e^{-\zeta}, \qquad \mathrm{Bi}(t) \sim \frac{t^{-1/4}}{\sqrt{\pi}}e^{\zeta}, \qquad \zeta(t) = \frac{2}{3}t^{3/2}, \qquad t \to +\infty$$

For the outer expansion, assumed a fixed x and let $\varepsilon \to 0^+$. Denote $\delta = t(x) - t_0$ and we have

$$\zeta\big(t(x)\big) - \zeta(t_0) = t_0^{1/2}\delta + \frac{1}{4}t_0^{-1/2}\delta^2 + O\big(t_0^{-3/2}\delta^3\big) = \frac{x}{2\varepsilon} + \frac{x^2}{2} + O(\varepsilon x^3)$$

$$\zeta(t_1) - \zeta(t_0) = \zeta\big(t(1)\big) - \zeta(t_0) = \frac{1}{2\varepsilon} + \frac{1}{2} + O(\varepsilon)$$

Therefore, we have

$$\begin{aligned} y(x) &\sim \exp\left(\frac{\varepsilon + 1 - x}{2\varepsilon}\right) \cdot \left(\frac{t(x)}{t_1}\right)^{\frac{1}{4}} \cdot \frac{\sinh\big[\zeta\big(t(x)\big) - \zeta(t_0)\big]}{\sinh[\zeta(t_1) - \zeta(t_0)]} \\ &\sim \exp\left(\frac{\varepsilon + 1 - x}{2\varepsilon}\right) \cdot \exp\left(\frac{x}{2\varepsilon} + \frac{x^2}{2} - \frac{1}{2\varepsilon} - \frac{1}{2}\right) \sim e^{\frac{x^2}{2}}, \qquad \varepsilon \to 0^+ \end{aligned}$$

For the inner expansion, assumed as fixed $z = x/\varepsilon$ and let $\varepsilon \to 0^+$. Now we have

$$\zeta\big(t(z)\big) - \zeta(t_0) = t_0^{1/2}\delta + \frac{1}{4}t_0^{-1/2}\delta^2 + O\big(t_0^{-3/2}\delta^3\big) = \frac{z}{2} + \frac{\varepsilon^2 z^2}{2} + O(\varepsilon^4 z^3)$$

The asymptotic behavior becomes

$$\begin{aligned} y(z) &\sim \exp\left(\frac{\varepsilon + 1 - \varepsilon z}{2\varepsilon}\right) \cdot \left(\frac{t(z)}{t_1}\right)^{\frac{1}{4}} \cdot \frac{\sinh\big[\zeta\big(t(z)\big) - \zeta(t_0)\big]}{\sinh[\zeta(t_1) - \zeta(t_0)]} \\ &\sim 2e^{-\frac{z}{2}}\sinh\left(\frac{z}{2}\right) = 1 - e^{-z}, \qquad \varepsilon \to 0^+ \end{aligned}$$

The asymptotic behavior of the exact solution is consistent with the outer and inner expansions.