

MIT Integration Bee: 2024 Semifinal

Semifinal #1

Question 1

$$\int_{-\infty}^{+\infty} \frac{(x^3 - 4x) \sin x + (3x^2 - 4) \cos x}{(x^3 - 4x)^2 + \cos^2 x} dx \quad (1.1)$$

Solution Denote the following functions

$$f(x) = \cos x, \quad g(x) = x^3 - 4x, \quad t(x) = \frac{f(x)}{g(x)}. \quad (1.2)$$

We thus have

$$I = - \int_{-\infty}^{+\infty} \frac{f'g - fg'}{f^2 + g^2} dx = - \int_{-\infty}^{+\infty} \frac{1}{1 + t^2} dt. \quad (1.3)$$

Note that there are **singularities** $x = 0, \pm 2$ after the substitution. Because $\cos 2 < 0$, we have

$$\begin{aligned} t(-\infty) &\rightarrow 0, & t(-2^-) &\rightarrow +\infty, & t(-2^+) &\rightarrow -\infty, & t(0^-) &\rightarrow +\infty, \\ t(0^+) &\rightarrow -\infty, & t(2^-) &\rightarrow +\infty, & t(2^+) &\rightarrow -\infty, & t(\infty) &\rightarrow 0. \end{aligned}$$

Therefore, the proper way to evaluate the integral should be

$$\begin{aligned} I &= - \left(\arctan t \Big|_{x=-\infty}^{x=-2^-} + \arctan t \Big|_{x=-2^+}^{x=0^-} + \arctan t \Big|_{x=0^+}^{x=2^-} + \arctan t \Big|_{x=2^+}^{x=+\infty} \right) \\ &= - [\arctan(+\infty) - 0 + 2 \arctan(+\infty) - 2 \arctan(-\infty) + 0 - \arctan(-\infty)] \\ &= -6 \arctan(+\infty) = -3\pi. \end{aligned} \quad (1.4)$$

Question 2

$$\int_0^{\infty} \frac{x e^{-2x}}{e^{-x} + 1} dx \quad (2.1)$$

Solution With a **change of variable**, we have

$$\begin{aligned} I &= \int_0^{\infty} \frac{x e^{-2x}}{e^{-x} + 1} dx \\ &= - \int_0^1 \frac{t \ln t}{t + 1} dt \quad \left(t = e^{-x}, \quad x = -\ln t, \quad dx = -\frac{dt}{t} \right). \end{aligned} \quad (2.2)$$

Using **Taylor series**, we have

$$I = - \int_0^1 \ln t \, dt + \int_0^1 \frac{\ln t}{t + 1} dt = 1 + \sum_{n=0}^{+\infty} (-1)^n \int_0^1 t^n \ln t \, dt. \quad (2.3)$$

Based on the following results ([see 2024 Final: Question 2](#))

$$\int_0^1 t^n \ln t \, dt = -\frac{1}{(n+1)^2} \quad n \in \mathbb{N}^*, \quad (2.4)$$

and the **Basel problem**, we have

$$I = 1 + \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2} = 1 - \frac{\pi^2}{12}. \quad (2.5)$$

Question 3

$$\int_0^{\pi/2} \sin(\cot^2 x) \sec^2 x \, dx \quad (3.1)$$

Solution With a **change of variable**, we have

$$\begin{aligned} I &= \int_0^{\pi/2} \sin(\cot^2 x) \sec^2 x \, dx \\ &= \int_0^{+\infty} \frac{\sin t^2}{t^2} \, dt \quad \left(t = \cot x, \quad \tan x = \frac{1}{t}, \quad \sec^2 x \, dx = -\frac{dt}{t^2} \right). \end{aligned} \quad (3.2)$$

Integration by parts leads to the **Fresnel integral**

$$\begin{aligned} I &= \int_0^{+\infty} \frac{\sin t^2}{t^2} \, dt = -\frac{\sin t^2}{t} \Big|_0^{+\infty} + 2 \int_0^{+\infty} \cos t^2 \, dt \\ &= 2 \int_0^{+\infty} \cos t^2 \, dt = \sqrt{\frac{\pi}{2}}. \end{aligned} \quad (3.3)$$

Note One particular result for the definite Fresnel integral is

$$\int_0^{+\infty} \cos(x^\alpha) \, dx = \Gamma\left(1 + \frac{1}{\alpha}\right) \cos\left(\frac{\pi}{2\alpha}\right), \quad (3.4)$$

$$\int_0^{+\infty} \sin(x^\alpha) \, dx = \Gamma\left(1 + \frac{1}{\alpha}\right) \sin\left(\frac{\pi}{2\alpha}\right), \quad \text{for } \alpha > 1. \quad (3.5)$$

These are obtained by evaluating the integral of e^{-z^α} using the contour in Fig. 1.

$$e^{i\theta} \int_0^{+\infty} e^{-ir^\alpha} \, dr = \int_0^{+\infty} e^{-x^\alpha} \, dx = \frac{1}{\alpha} \int_0^{+\infty} t^{\frac{1}{\alpha}-1} e^{-t} \, dt = \Gamma\left(1 + \frac{1}{\alpha}\right). \quad (3.6)$$

The real and imaginary parts correspond to Eqs (3.4) and (3.5), respectively.

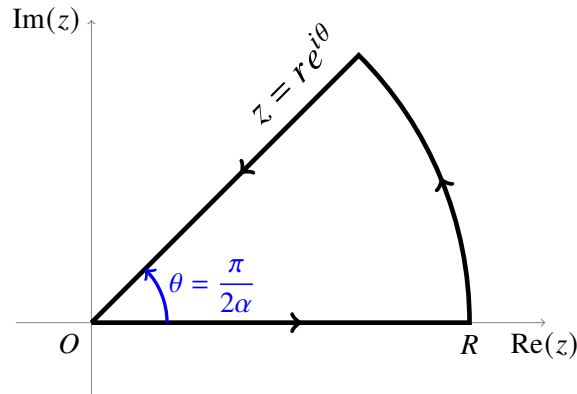


Fig. 1 Sector-shaped contour in the upper plane to evaluate the Fresnel integral.

Question 4

$$\int \cosh^2 (3x) \tanh (2x) \, dx \quad (4.1)$$

Solution Using the following identities for the hyperbolic functions

$$\cosh^2 x = \frac{1 + \cosh (2x)}{2}, \quad \cosh (3x) = 4 \cosh^3 x - 3 \cosh x \quad (4.2)$$

we have

$$\begin{aligned} I &= \frac{1}{2} \int \tanh (2x) \, dx + \frac{1}{2} \int \cosh (6x) \tanh (2x) \, dx \\ &= \frac{1}{4} \ln |\cosh (2x)| + \int \left[2 \cosh^2 (2x) - \frac{3}{2} \right] \sinh (2x) \, dx \\ &= \frac{1}{4} \ln |\cosh (2x)| + \frac{1}{3} \cosh^3 (2x) - \frac{3}{4} \cosh (2x) + C \\ &= \frac{1}{4} \ln |\cosh (2x)| + \frac{1}{12} \cosh (6x) - \frac{1}{2} \cosh (2x) + C. \end{aligned} \quad (4.3)$$

Tiebreakers Question 1

$$\int \sec^5 x \, dx \quad (5.1)$$

Solution Note that

$$(\tan x)' = \sec^2 x, \quad (\sec x)' = \tan x \sec x, \quad \tan^2 x = \sec^2 x - 1. \quad (5.2)$$

We can first obtain the **reduction formula** for the general integral. Since we have

$$\begin{aligned} I_n &= \int \sec^n x \, dx = \sec^{n-2} x \tan x - (n-2) \int \tan^2 x \sec^{n-2} x \, dx \\ &= \sec^{n-2} x \tan x - (n-2) (I_n - I_{n-2}), \end{aligned} \quad (5.3)$$

the **recurrence relation** is

$$I_n = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}, \quad n \geq 3. \quad (5.4)$$

When $n = 1, 2$, we have

$$I_1 = \int \sec x \, dx = \ln |\sec x + \tan x| + C, \quad I_2 = \int \sec^2 x \, dx = \tan x + C. \quad (5.5)$$

Therefore, we can obtain

$$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln |\sec x + \tan x| + C. \quad (5.6)$$

Tiebreakers Question 2

$$\int_{-\infty}^{+\infty} \operatorname{sech} \left(2x + 1 - \frac{1}{x-1} - \frac{2}{x+1} \right) dx \quad (6.1)$$

Solution The **Glasser's master theorem** states that the identity

$$\operatorname{PV} \int_{-\infty}^{+\infty} F(\phi(x)) dx = \operatorname{PV} \int_{-\infty}^{+\infty} F(x) dx \quad (6.2)$$

holds for any integrable function $F(x)$ and $\phi(x)$ of the form

$$\phi(x) = x - a - \sum_{n=1}^N \frac{|\alpha_n|}{x - \beta_n}, \quad (6.3)$$

with a, α_n and β_n being arbitrary real constants. Consequently, the integral is equivalent to

$$I = \int_{-\infty}^{+\infty} \operatorname{sech}(2x) dx = \int_{-\infty}^{+\infty} \frac{2e^{2x}}{e^{4x} + 1} dx = \left(\arctan e^{2x} \right) \Big|_{-\infty}^{+\infty} = \frac{\pi}{2}. \quad (6.4)$$

Semifinal #2

Question 1

$$\int_0^{+\infty} \frac{\sin(x) \sin(2x) \sin(3x)}{x^3} dx \quad (7.1)$$

Solution Using the trigonometric identities, we have

$$I = \frac{1}{2} \int_0^{+\infty} \frac{(\cos x - \cos 3x) \sin 3x}{x^3} dx = \frac{1}{4} \int_0^{+\infty} \frac{\sin 2x + \sin 4x - \sin 6x}{x^3} dx. \quad (7.2)$$

We define the following integral

$$F(\alpha) = \int_0^{+\infty} \frac{\alpha x - \sin \alpha x}{x^3} dx. \quad (7.3)$$

Using **integration by parts**, we have

$$\begin{aligned} F(\alpha) &= -\frac{\alpha - \sin \alpha x}{2x^2} \Big|_0^{+\infty} + \frac{\alpha}{2} \int_0^{+\infty} \frac{1 - \cos \alpha x}{x^2} dx \\ &= -\frac{\alpha}{2} \frac{1 - \cos \alpha x}{x} \Big|_0^{+\infty} + \frac{\alpha^2}{2} \int_0^{+\infty} \frac{\sin \alpha x}{x} dx = \frac{\pi \alpha^2}{4}. \end{aligned} \quad (7.4)$$

Finally, we obtain

$$I = \frac{1}{4} [-F(2) - F(4) + F(6)] = \pi. \quad (7.5)$$

Question 2

$$\int (1 + \ln x)(1 + \ln \ln x) \, dx \quad (8.1)$$

Solution Note that

$$(x \ln x)' = 1 + \ln x, \quad (1 + \ln \ln x)' = \frac{1}{x \ln x}. \quad (8.2)$$

We have

$$\begin{aligned} I &= \int (1 + \ln \ln x) \, d(x \ln x) \\ &= x \ln x (1 + \ln \ln x) - \int 1 \, dx \\ &= -x + x \ln x + (x \ln x) \ln \ln x + C. \end{aligned} \quad (8.3)$$

Question 3

$$\int_0^{+\infty} \frac{e^{-x^2}}{\left(x^2 + \frac{1}{2}\right)^2} dx \quad (9.1)$$

Solution It can be shown that

$$f(x) = \left(x + \frac{1}{2x}\right)^{-1} = \frac{x\left(x^2 + \frac{1}{2}\right)}{\left(x^2 + \frac{1}{2}\right)^2}, \quad f'(x) = \frac{\frac{1}{2} - x^2}{\left(x^2 + \frac{1}{2}\right)^2}. \quad (9.2)$$

Therefore, we have

$$\frac{d}{dx} \left[e^{-x^2} f(x) \right] = e^{-x^2} [f'(x) - 2xf(x)] = e^{-x^2} \cdot \frac{1 - 2\left(x^2 + \frac{1}{2}\right)^2}{\left(x^2 + \frac{1}{2}\right)^2}, \quad (9.3)$$

which implies that

$$\frac{e^{-x^2}}{\left(x^2 + \frac{1}{2}\right)^2} = \frac{d}{dx} \left[e^{-x^2} f(x) \right] + 2e^{-x^2}. \quad (9.4)$$

The integral can thus be evaluated as

$$I = \left[e^{-x^2} f(x) \right]_0^{\infty} + 2 \int_0^{+\infty} e^{-x^2} dx = 2 \int_0^{+\infty} e^{-x^2} dx = \sqrt{\pi}. \quad (9.5)$$

Question 4

$$\int \tan x \sec^2 x \cos(2x) e^{2\cos x} dx \quad (10.1)$$

Solution With a **change of variable** $t = \cos x$, we have

$$I = - \int \frac{2t^2 - 1}{t^3} e^{2t} dt \quad (10.2)$$

We can also obtain the **reduction formula** for the following integral

$$I_n = \int \frac{e^{2t}}{t^n} dt = -\frac{1}{n-1} \frac{e^{2t}}{t^{n-1}} + \frac{2}{n-1} I_{n-1}. \quad (10.3)$$

Finally, we have

$$\begin{aligned} I &= I_3 - 2I_1 = -\frac{1}{2} \frac{e^{2t}}{t^2} + I_2 - 2I_1 \\ &= -\frac{1}{2} \frac{e^{2t}}{t^2} - \frac{e^{2t}}{t} + 2I_1 - 2I_1 = -\left(\frac{1}{2t^2} + \frac{1}{t}\right) e^{2t} + C \\ &= -\left(\frac{\sec^2 x}{2} + \sec x\right) e^{2\cos x} + C. \end{aligned} \quad (10.4)$$