# Turbulence

## Lecture 1. Introduction

➢ Notations

1. $xyz$-coordinate, $uvw$-coordinate. $x_i$ and $u_i$ to denote each component.
2. Governing equation: Incompressible Navier-Stokes equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\nabla^2 u$$

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial\left(\rho u_j u_i\right)}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

$$\frac{D\boldsymbol{u}}{Dt}=\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}=-\frac{1}{\rho}\boldsymbol{\nabla}p+\frac{\mu}{\rho}\boldsymbol{\nabla}^2\boldsymbol{u}$$

3. Relation symbols
   a. Proportional: $A\propto B$, not exactly matching dimension
   b. Scale: $A\sim B$, proportional and dimensions match, only need a $O(1)$ pre-factor to turn the relation into equality
   c. Almost equal: $A\approx B$, dimensions match and their ratio is close to 1

   <u>Example</u>: Stagnation pressure $p_0$

$$p_0-p_\infty\propto v_\infty^2,\qquad p_0-p_\infty\sim\rho v_\infty^2,\qquad p_0-p_\infty\approx\frac{1}{2}\rho v_\infty^2$$

➢ Characteristics of turbulence

- ♦ Vortices / 3D vortical structures
- ♦ Irregular (chaotic, “random”)
- ♦ Wide range of scales (small and large eddies)
- ♦ Mixing of mass, momentum, heat
- ♦ Dissipation (turbulence needs energy from shear / buoyancy / body forces to sustain)
  Much faster dissipation for turbulent processes than laminar processes
- ♦ Continuum phenomenon
  Small turbulent eddies $O(10\ \mu\text{m})$ $\gg$ Mean free path of molecules $\sim 60$ nm
  Average distance of molecules in the air is 4 nm (calculated from density)
  [We do not see individual molecule within eddies, but see a continuum]

- Large Reynolds number, highly nonlinear (dominant advection term in NS eqn.)

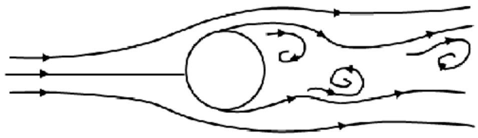
$$\mathrm{Re} = \frac{\rho U L}{\mu} = \frac{UL}{\nu} \gg 1, \qquad ul \gg \nu$$

with $u$ is eddy velocity and $l$ is eddy size

<u>Example</u>: Flow over a cylinder at $\mathrm{Re} = 10^4$ with diameter $D$

$$\frac{\rho U D}{\mu} = 10^4$$

Turbulent wake appears behind the cylinder



(d) $Re \approx 1000$

- Sources of differences in realizations

  a. Experiment: Initial conditions and environmental perturbations

  b. Simulation: Similar variability [e.g., $(a + b) + c \neq a + (b + c)$ computationally]

- Statistics are reproducible for different realizations

$$\bar{u} = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t\,, \qquad \bar{u}_{r_1} = \bar{u}_{r_2} = \cdots\,, \qquad \overline{u^2}_{r_1} = \overline{u^2}_{r_2} = \cdots$$

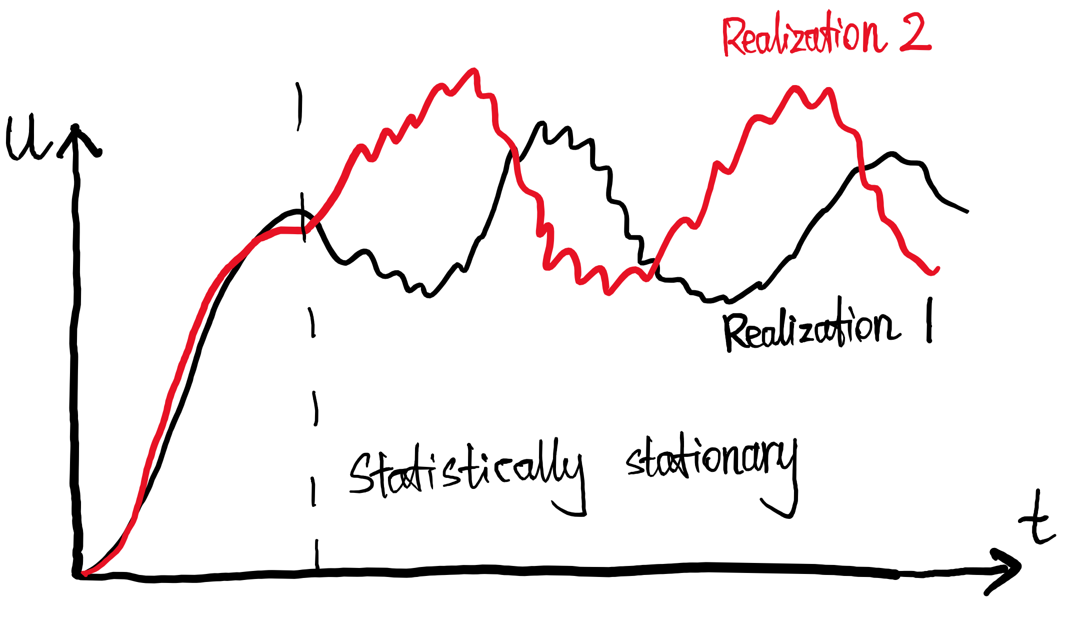
$\bar{u}$, $\overline{u^2}$, $\overline{u^3}$ are flow statistics

- SEM: Statistical error of the means, obtained from several realizations

$$\bar{u}_{estim} = \frac{\bar{u}_{r_1} + \bar{u}_{r_2} + \cdots + \bar{u}_{r_n}}{n}, \qquad \mathrm{SEM} = \frac{\mathrm{STD}(\bar{u})}{\sqrt{n}}$$

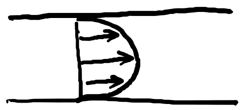
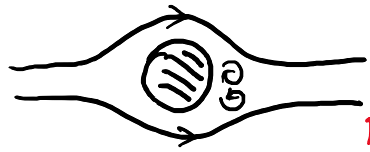
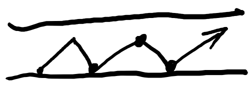
Note: $\mathrm{STD}(\bar{u}) \ll \mathrm{STD}(u_{r_i})$

95% confidence level is $1.96 \times SEM$ based on Gaussian distribution from central limit theorem



**Lecture 2.**

- Introduction to turbulence theory: Statistical theory
  - Ideas for turbulence theory is inspired by kinetic theory of gases

| | **Kinetic theory** | **Turbulence theory** |
|---|---|---|
| Target system | Averaged flow over molecular motions<br>Laminar flow<br> | Averaged flow<br> |
| Underlying physics | Chaotic molecular dynamic (M.D.)<br> | Chaotic flow, N.S. eqn. (high Re)<br> |
| Reduced model | Navier-Stokes equation<br>Obtain isotropic molecular viscosity: $\nu_M, \mu_M$ | Not available<br>RANS framework<br>Turbulent viscosity $\nu_T$ not isotropic |
| Why works / not work | - Separation of scales between M.D. (mean free path) and size of pipe<br>- Chaos is in equilibrium: Maxwell distribution of M.D. velocity | - No separation of scales, eddies are as large as cylinder diameter $D$<br>- Chaos is non-equilibrium |

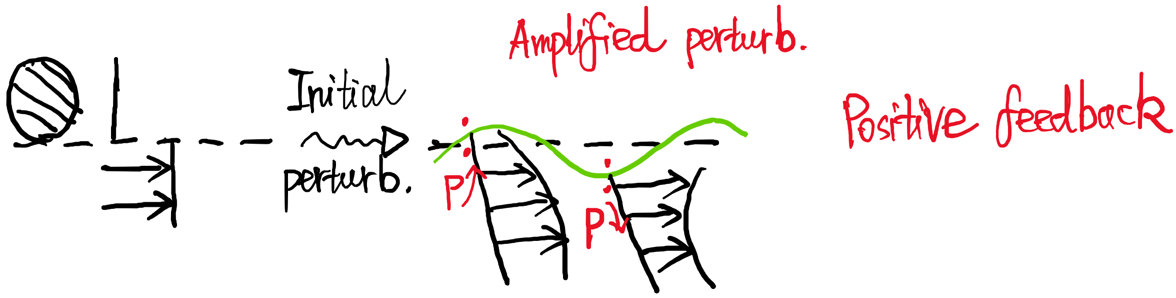
K.T.

CFD

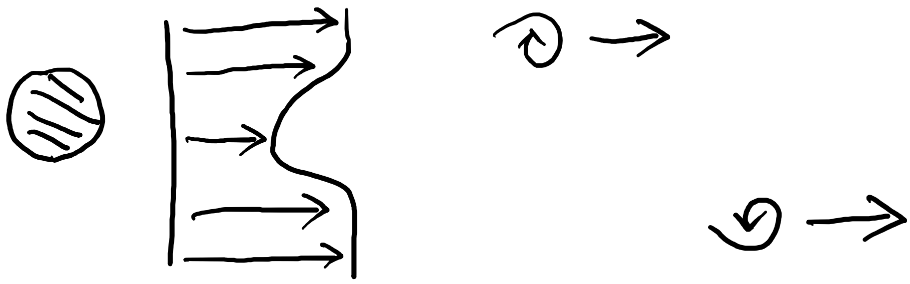
  - Useful insights obtained from analogy to kinetic theory

    Prediction of scaling laws, order of magnitude of behavior

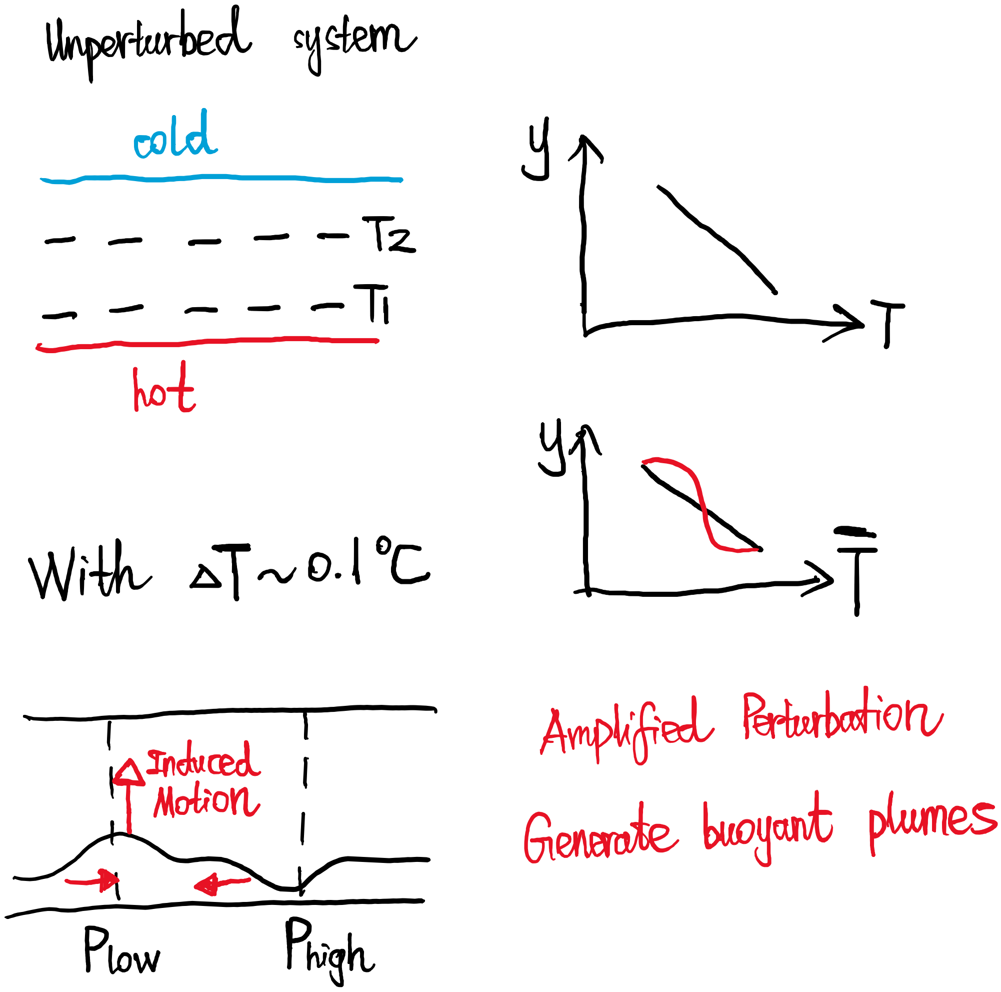
- How turbulence is triggered (Flow instability )
  - Example: Kelvin-Helmholtz instability




  - Other instability: Karmann instability (interaction between both shear layers)



- Other instability: Rayleigh-Benard instability, significant impact on heat transfer




- Picture of instability

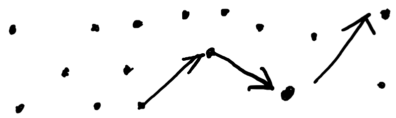
  Instability → Large eddies → Secondary instabilities → Smaller eddies → Even smaller eddies → Viscous dissipation

  2D instability → 3D instability → Turbulent spots → Fully developed turbulence

- Video: Space developing shear layer with weak inflow turbulence
  - Kelvin-Helmholtz instability
  - Vortex stretching

**Lecture 3. Random walk**

- Mixing as a laminar flow concept (random walk)



Molecules diffuse due to Brownian motion. Consider the diffusion of tracer particle.

Follow one particle released at origin, displacement $\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_2, \ldots$ between adjacent collisions

Very irregular and not repeatable process, so we study the statistics

$\langle \boldsymbol{r}(t) \rangle$ denotes average over many realizations (ensemble average)

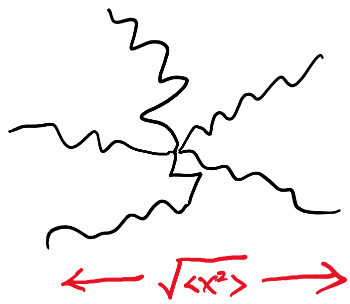
$$\langle \boldsymbol{r}(t) \rangle = \langle \boldsymbol{u}(\boldsymbol{x} = \boldsymbol{0}, t = 0) \rangle \cdot t$$

For stagnant fluid

$$\boldsymbol{u} = \boldsymbol{0} \;\rightarrow\; \langle \boldsymbol{r}(t) \rangle = \boldsymbol{0}$$

Higher-order statistics:

$$\langle \boldsymbol{r}(t) \cdot \boldsymbol{r}(t) \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = R^2$$




For isotropic fluid

$$\langle \boldsymbol{r}(t) \cdot \boldsymbol{r}(t) \rangle = 3\langle x^2 \rangle$$

Calculate this characteristic length scale:

$$\langle x^2 \rangle = \langle (\Delta x_1 + \Delta x_2 + \cdots + \Delta x_n)^2 \rangle = \langle \Delta x_1^2 \rangle + \cdots + \langle \Delta x_1 \Delta x_2 \rangle + \cdots$$

The diagonal terms become $n\langle \Delta x^2 \rangle$, and independency indicates zero off-diagonal terms

$$\langle x^2 \rangle = n\langle \Delta x^2 \rangle = n l_{RW}^2, \qquad l_{RW} \equiv \sqrt{\langle \Delta x^2 \rangle}$$

With $n\langle \Delta t \rangle = t$, and the length of random walk $l_{RW}$. Now we define the velocity of random walk, and we obtain

$$u_{RW} \equiv \frac{l_{RW}}{\langle \Delta t \rangle}, \qquad \langle x^2 \rangle = l_{RW}^2 \cdot \frac{t}{\langle \Delta t \rangle} = l_{RW} u_{RW} t, \qquad x_{rms} \propto \sqrt{t}$$

This indicates the size of the spherical cloud is proportional to $\sqrt{t}$

- Connection to continuum diffusion

$$\frac{\partial C}{\partial t} = D\nabla^2 C, \qquad C(t = 0) = \delta(\boldsymbol{x})$$

The shape at later time is a spreading Gaussian function

For this process we have

$$\langle x^2 \rangle = \frac{1}{N}\sum_{i=1}^{N} x_i(t)^2 = \frac{\iiint_V x^2 C(\boldsymbol{x})\, dV}{\iiint_V C(\boldsymbol{x})\, dV} = \iiint_V x^2 C(\boldsymbol{x})\, dV$$

The final step uses the initial $\delta(\boldsymbol{x})$ distribution of $C(\boldsymbol{x})$ and conservation of particles

Integrating the PDE over space gives

$$\frac{\partial}{\partial t}\iiint_V x^2 C\, dV = D\iiint_V x^2 \frac{\partial^2 C}{\partial x^2} dV + D\iint_{xz} x^2 \left(\int_y \frac{\partial^2 C}{\partial y^2} dy\right) dxdz + D\iint_{xz} x^2 \left(\int_z \frac{\partial^2 C}{\partial z^2} dz\right) dxdy$$

$$= D\iiint_V x^2 \frac{\partial^2 C}{\partial x^2} dV = -D\iiint_V 2x \frac{\partial C}{\partial x} dV + \iint_{yz} x^2 \frac{\partial C}{\partial x}\bigg|_{-\infty}^{+\infty} dydz$$

$$= D\iiint_V 2C\, dV = 2D$$

The first step uses zero boundary conditions at infinity, and then keeps integration by part

$$\frac{\partial}{\partial t}\langle x^2 \rangle = 2D, \qquad \langle x^2 \rangle = 2Dt, \qquad 2D = l_{RW} u_{RW} = \frac{\langle \Delta x^2 \rangle}{\langle \Delta t_{RW} \rangle}$$

For gas dynamics, $u_{RW} \sim \alpha$ (speed of sound), $l_{RW} \sim l$ (mean free path), $D \sim \alpha l$

Example: For air at atmospheric condition

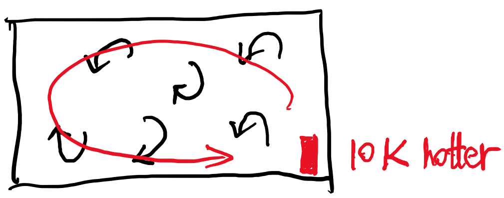
$$\alpha = 340 \text{ m/s}, \qquad l = 68 \text{ nm}, \qquad \alpha l = 2.3 \times 10^{-5} \text{ m}^2/s, \qquad \nu = 1.5 \times 10^{-5} \text{ m}^2/s$$

Example: Heat transfer in a room. Heater at one end, and the room width is $L = 5$ m

Without turbulence, the diffusion time is

$$t = \frac{L^2}{2D} = 6.6 \times 10^5 \text{ s} \sim 8 \text{ days}$$

In reality, $u \neq 0$ is chaotic




➢ Connection to turbulence

A phenomenological model for turbulent mixing

$$D_T \sim u_{eddy}\, l_{eddy}$$

Consider the heater is 10 K hotter than initial temperature, so the density is 3% less. The buoyant acceleration is $\sim 0.03g = 0.3 \text{ m/s}^2$. Over 10 cm, $u_{eddy} \sim 0.24$ m/s, and average in the room $u_{eddy} \sim 5$ cm/s.

Consider that large eddy contributes more important for mixing. Largest eddy can have the room size, so $l_{eddy} \sim 5$ m, $D_T \sim 0.25\ \mathrm{m^2/s}$. Turbulent mixing gives the time scale

$$t = \frac{L^2}{2D_T} = 50\ \mathrm{s}$$

Ratio of mixing rate is the Reynolds number

$$\frac{D_T}{D_M} = \frac{u_{eddy}\, l_{eddy}}{\nu} = \mathrm{Re}$$

The other interpretation is the ratio between inertial and viscous forces

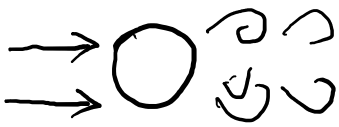
Low Re indicates that before eddy forms, Brownian motion will smear out all perturbations

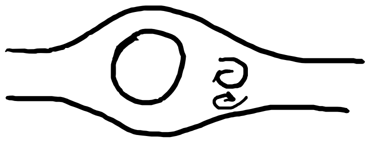
## Lecture 4. Scaling of semi-parallel flows for jets

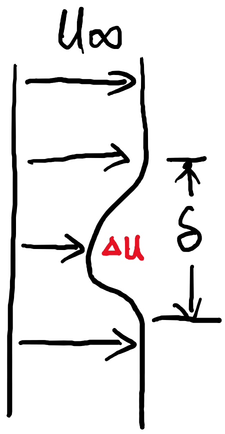
- Laminar case

  - Free shear flows: waves, jets, shear layers

    Example of wake: Instantaneous VS Average pictures



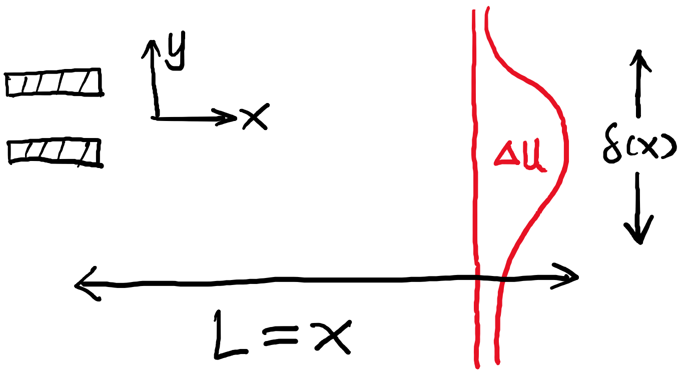





Study $\delta(x)$ and $\Delta U(x)$

  - Jets ($U_\infty = 0$), Round jet

    Semi-parallel with $\delta \ll x$. Study $\delta(x)$ and $\Delta U(x)$




  - Scaling analysis for laminar and steady flow

    $x$-momentum equation is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial z^2}$$

For scaling analysis:

$$y, z \sim \delta, \qquad v \sim w \sim \Delta U\frac{\delta}{x} \text{ (from continuity } \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0)$$

So we have

$$\frac{(\Delta U)^2}{x} + \Delta U\frac{\delta}{x}\cdot\frac{\Delta U}{\delta} \sim \frac{p}{\rho x} + \nu\frac{\Delta U}{x^2} + \nu\frac{\Delta U}{\delta^2} \sim \frac{p}{\rho x} + \nu\frac{\Delta U}{\delta^2}$$

To estimate pressure, we need $y$-momentum equation

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\frac{\partial^2 v}{\partial x^2}+\nu\frac{\partial^2 v}{\partial y^2}+\nu\frac{\partial^2 v}{\partial z^2}$$

This scales to

$$\frac{(\Delta U)^2\delta}{x^2}\sim-\frac{p}{\rho\delta}+\nu\frac{\Delta U\delta}{x^3}+\nu\frac{\Delta U}{x\delta}\sim-\frac{p}{\rho\delta}+\nu\frac{\Delta U}{x\delta},\qquad \frac{p}{\rho}\sim\max\left\{\frac{(\Delta U)^2\delta^2}{x^2},\frac{\nu\Delta U}{x}\right\}$$

So the pressure term is either much smaller than advection or the viscous term, it can be neglected. Into the $x$-momentum equation, we have

$$\frac{(\Delta U)^2}{x}\sim\nu\frac{\Delta U}{\delta^2}$$

- This scaling relation simplifies governing equation for semi-parallel flows

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\simeq\nu\nabla_\perp^2 u$$

This equation works for jets, shear layers, wakes. For plumes, the buoyancy force needs to be included

- Seek a power law solution

$$\Delta U\propto x^m,\qquad \delta\propto x^n$$

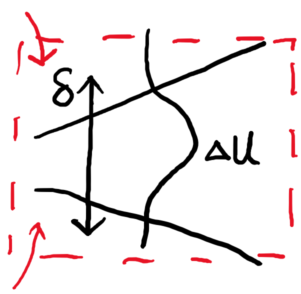
From scaling relation, we have

$$\delta^2\sim\frac{\nu x}{\Delta U},\qquad x^{2n}\propto x^{1-m},\qquad 2n=1-m$$

We still need one more constraint, and it will be from constant momentum deficit

- Mass continuity

If we assume constant axial flow rate, net flow rate is about $\delta^2\Delta U\propto x^0$, and then $2n+m=0$, but in fact this does not hold. This is due to entrainment of the radially inward flow.



- General momentum conservation equation including body force (see Handout)
- Constant momentum deficit

$$\iint_{yz} \rho u(u - U_\infty)\, dydz = \text{const} \propto x^0$$

For a round jet with $U_\infty = 0$

$$\iint_S \rho u^2 dS \sim \rho(\Delta U)^2 \delta^2 \propto x^0, \qquad m + n = 0$$

For a 2D jet with $U_\infty = 0$

$$\iint_S \rho u^2 dS \sim \rho(\Delta U)^2 \delta \propto x^0, \qquad 2m + n = 0$$

➢ Turbulent case (averaged velocity field)

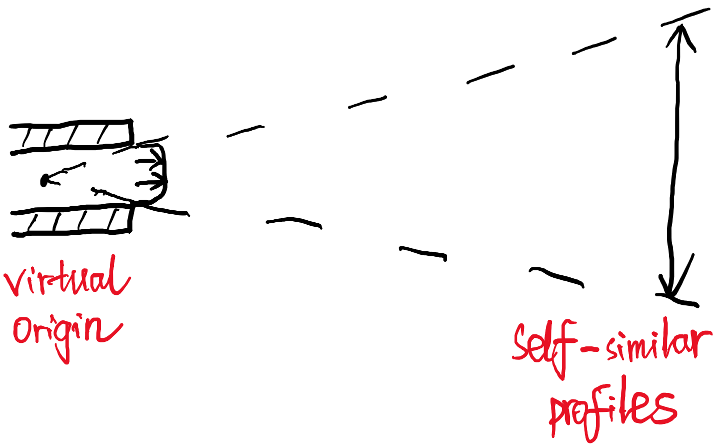
- Phenomenological governing equation

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \bar{w}\frac{\partial \bar{u}}{\partial z} \simeq \nu_T \nabla_\perp^2 \bar{u}$$

- Scaling analysis

$$\bar{u}\frac{\partial \bar{u}}{\partial x} \sim \nu_T \frac{\partial^2 \bar{u}}{\partial y^2}, \qquad \nu_T \sim u_{eddy}\, l_{eddy} \sim \Delta U \delta, \qquad \frac{(\Delta U)^2}{x} \sim \Delta U \delta \cdot \frac{\Delta U}{\delta^2}, \qquad \delta \sim x$$

| Term / Momentum balance | 2D planar jet ($n = -2m$) | 3D round jet ($n = -m$) |
|---|---|---|
| Laminar ($2n = 1 - m$) | $n = \frac{2}{3},\ m = -\frac{1}{3}$ <br> $\delta \sim x^{\frac{2}{3}},\ \Delta U \sim x^{-\frac{1}{3}}$ | $n = 1,\ m = -1$ <br> $\delta \sim x,\ \Delta U \sim \frac{1}{x}$ |
| Turbulent ($n = 1$) | $n = 1,\ m = -\frac{1}{2}$ <br> $\delta \sim x,\ \Delta U \sim \frac{1}{\sqrt{x}}$ | $n = 1,\ m = -1$ <br> $\delta \sim x,\ \Delta U \sim \frac{1}{x}$ |



**Lecture 5. Scaling of semi-parallel flows for wakes**

- Recap Lecture 4

  For semi-parallel flows, the dominant balance for a laminar flow is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\simeq \nu\nabla_\perp^2 u$$

  Phenomenologically (not rigorous governing eqn.), for a turbulent flow we have

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}+\bar{w}\frac{\partial \bar{u}}{\partial z}\simeq \nu_T\nabla_\perp^2 \bar{u}$$

  Seek the following scaling solution:

$$\Delta u\propto x^m,\qquad \delta\propto x^n$$

  (Note: If $n\geq 1$, then the semi-parallel assumption is violated)

  We need term balance:

$$\bar{u}\frac{\partial \bar{u}}{\partial x}\sim \nu_T\frac{\partial^2\bar{u}}{\partial y^2},\qquad \nu_T\sim\Delta U\delta$$

  with global conservation analysis

- Data analysis for jets

  Given velocity field in space and time, calculate

$$\Delta u(x)=\bar{u}(x,y=0,z=0)$$

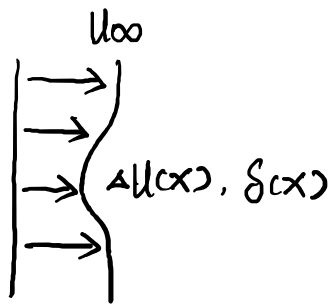
  Obtain the virtual origin based on

$$\frac{1}{\Delta u}\propto x-x_0$$

  Then check if the velocity profiles collapse by plotting relationship of these two variables

$$\frac{y}{x-x_0}\ \text{ and }\ \bar{u}(x,z=0)\cdot(x-x_0)$$

- Scaling analysis of wakes




  - Global balance (momentum deficit)

$$\iint_S u(u-U_\infty)dS=\text{const.}$$

    Since $U_\infty\propto x^0$ and $\Delta U\propto x^m$, then $U_\infty-\Delta U\propto O(U_\infty)$

    For 2D case:

$$\iint_S u(u-U_\infty)dA\sim U_\infty\Delta U\delta\propto x^0,\qquad m+n=0$$

For 3D case:

$$\iint_S u(u-U_\infty)dA \sim U_\infty \Delta U \delta^2 \propto x^0, \qquad m+2n=0$$

- Term balance (advection in $y, z$ direction can also be neglected)

  For laminar flow:

$$U_\infty \frac{\Delta U}{x} \sim \nu \frac{\Delta U}{\delta^2}, \qquad \delta^2 \propto x, \qquad 2n=1$$

  For turbulent flow:

$$U_\infty \frac{\Delta U}{x} \sim \Delta U \delta \cdot \frac{\Delta U}{\delta^2}, \qquad \frac{\Delta U}{\delta} \propto \frac{1}{x}, \qquad m=n-1$$

| Term / Momentum balance | 2D wake ($m+n=0$) | 3D wake ($m+2n=0$) |
|---|---|---|
| Laminar ($2n=1$) | $n=\frac{1}{2},\ m=-\frac{1}{2}$<br>$\delta \sim \sqrt{x},\ \Delta U \sim \frac{1}{\sqrt{x}}$ | $n=\frac{1}{2},\ m=-1$<br>$\delta \sim \sqrt{x},\ \Delta U \sim \frac{1}{x}$ |
| Turbulent ($m=n-1$) | $n=\frac{1}{2},\ m=-\frac{1}{2}$<br>$\delta \sim \sqrt{x},\ \Delta U \sim \frac{1}{\sqrt{x}}$ | $n=\frac{1}{3},\ m=-\frac{2}{3}$<br>$\delta \sim x^{\frac{1}{3}},\ \Delta U \sim x^{-\frac{2}{3}}$ |

- Data analysis for wakes

  Investigate term balance, estimate turbulent eddy viscosity (nearly a constant)

$$\bar{u}\frac{\partial \bar{u}}{\partial x} \sim \nu_T \frac{\partial^2 \bar{u}}{\partial y^2}$$

  Influence of Reynolds number (molecular viscosity), compare $\nu_T$ and $\nu_M$

  For 3D turbulent wake, at very large distance the local Reynolds number ($\mathrm{Re} \propto \Delta U \delta$) is small and becomes laminar

**Lecture 6. RANS equations**

- Recap Lecture 5

  Approximate PDE: Example for steady heat transport equation

$$\bar{u}\frac{\partial\bar{\theta}}{\partial x}+\bar{v}\frac{\partial\bar{\theta}}{\partial y}+\bar{w}\frac{\partial\bar{\theta}}{\partial z}\simeq\gamma_T\nabla_\perp^2\bar{\theta},\qquad \bar{u}\frac{\partial\bar{\theta}}{\partial x}\sim\gamma_T\frac{\partial^2\bar{\theta}}{\partial y^2}$$

- Reynolds averaged N-S equation

  - RANS: PDE governing averaged fields

    Smooth fields, often steady, sufficient for most applications

$$\overline{C_D(u)}=C_D(\bar{u})$$

Counterexamples (taking the mean cannot study these topics): Noise in aeroacoustics, unstable modes, aero-optics, combustion

  - Systematic derivation of RANS

    a. Start with NS eqn.

$$\frac{\partial u_j}{\partial x_j}=0,\qquad \frac{\partial u_i}{\partial t}+\frac{\partial(u_ju_i)}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2u_i}{\partial x_j\partial x_j}$$

Other form of momentum balance

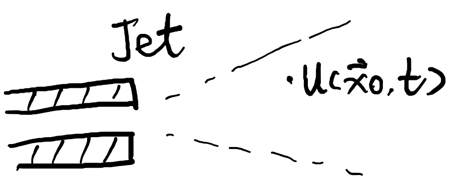
$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=\frac{1}{\rho}\frac{\partial\sigma_{ij}}{\partial x_i},\qquad \sigma_{ij}=-p\delta_{ij}+2\mu S_{ij},\qquad S_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

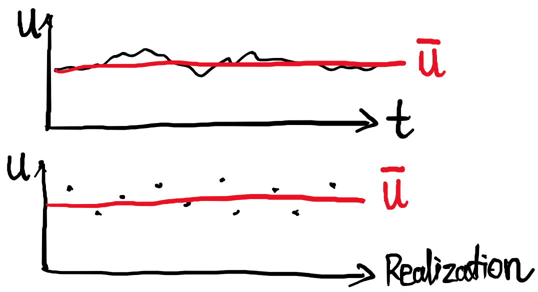
b. Reynolds decomposition: Mean + Fluctuations

$$u_i(\boldsymbol{x},t)=U_i+u_i'$$

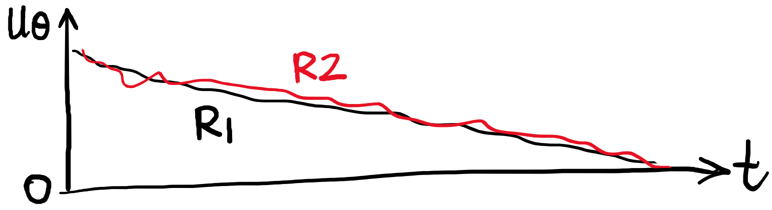
The mean can be time average or ensemble average

$$U_i=\bar{u}_i=\lim_{T\to\infty}\frac{1}{T}\int_t^{t+T}u_i\,dt\,,\qquad U_i=\langle u_i\rangle$$



Jet is "statistically time stationary" means statistics do not depend on $t_r$, and thus we have $\langle u_i\rangle = \bar{u}_i$. One counterexample is a decaying turbulence ($\bar{u}_\theta = 0$ for large $T$)




c. Apply averaging on NS eqn.

$$\frac{\partial\langle u_j\rangle}{\partial x_j} = 0, \qquad \frac{\partial\langle u_i\rangle}{\partial t} + \frac{\partial\langle u_j u_i\rangle}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial x_i} + \nu\frac{\partial^2\langle u_i\rangle}{\partial x_j \partial x_j}$$

4 equations with primary 4 unknowns: $\langle u_i\rangle, \langle p\rangle$. Additional unknowns: $\langle u_i u_j\rangle$

$$\langle u_i u_j\rangle = \langle u_i\rangle\langle u_j\rangle + \langle u_i' u_j'\rangle$$

d. Substitute into RANS eqn.

$$\frac{\partial\langle u_i\rangle}{\partial t} + \frac{\partial\left(\langle u_j\rangle\langle u_i\rangle\right)}{\partial x_j} = \frac{1}{\rho}\frac{\partial T_{ij}}{\partial x_j}$$

$$T_{ij} = -\langle p\rangle\delta_{ij} + 2\mu\langle S_{ij}\rangle - \rho\langle u_j' u_i'\rangle, \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)$$

The stress tensor includes mean pressure, viscous stress, Reynolds stress

e. Statistically stationary case

$$\frac{\partial\left(\bar{u}_j\bar{u}_i\right)}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x_i} + \nu\frac{\partial^2\bar{u}_i}{\partial x_j\partial x_j} - \frac{\partial\overline{u_j' u_i'}}{\partial x_j}$$

- Turbulence closure problem

RANS is exact but unclosed. Need turbulence models

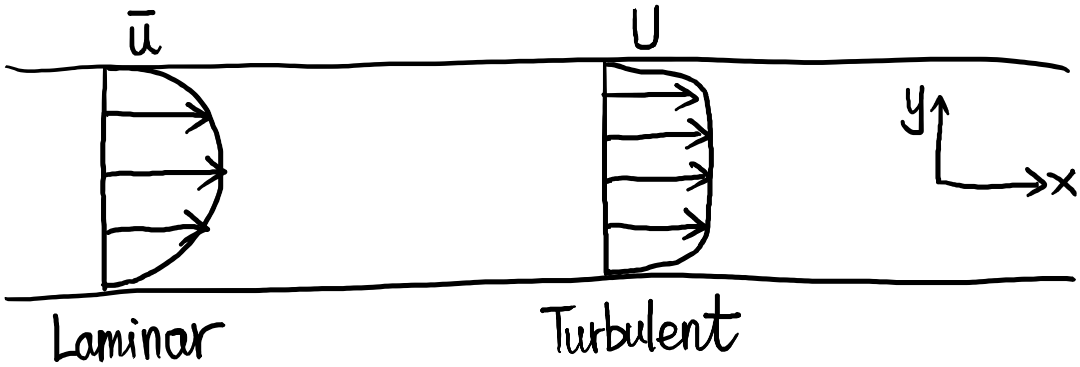
$$\langle u_i' u_j'\rangle = f_{ij}(\cdot)$$

Phenomenological model discussed in Lecture 4 & 5 is one option

$$-\langle u_j' u_i'\rangle = 2\nu_T\langle S_{ij}\rangle$$

This model is local: turbulent flux can be calculated from local gradient of mean flow. In reality the process can be non-local.

- Example: turbulent channel flow




RANS equation is

$$\frac{\partial \bar{u}^2}{\partial x}+\frac{\partial(\bar{u}\bar{v})}{\partial y}+\frac{\partial(\bar{u}\bar{w})}{\partial z}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\nu\nabla^2\bar{u}-\frac{\partial\overline{u'u'}}{\partial x}-\frac{\partial\overline{u'v'}}{\partial y}-\frac{\partial\overline{u'w'}}{\partial z}$$

Under homogeneous assumption, we get an ODE in $y$-direction

$$0=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x}+\nu\frac{\partial^2\bar{u}}{\partial y^2}-\frac{\partial\overline{u'v'}}{\partial y}$$

The pressure gradient force is not a function of $y$ due to homogeneity

Velocity fluctuations scale to $u'_{rms}\sim 5-10\%\ U,\ v'_{rms}\sim 1-5\%\ U$. Although this fluctuation is small, but the contribution is still larger than $\nu_m$ and mixes in the $y$-direction much more efficient than molecular diffusion

**Lecture 7. RANS-type equations for multi-physics problems**

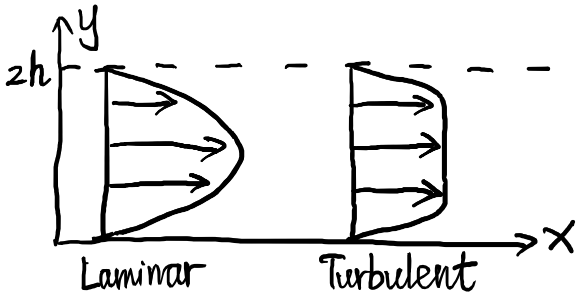
- Recap Lecture 6

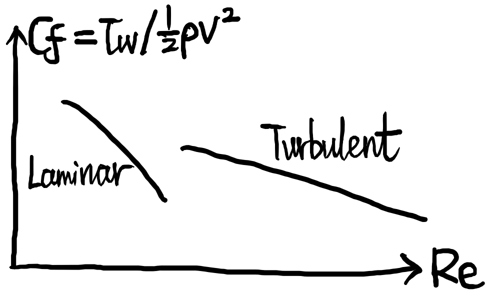
$$\frac{\partial\langle u_i\rangle}{\partial t}+\frac{\partial\left(\langle u_j\rangle\langle u_i\rangle\right)}{\partial x_j}=\frac{1}{\rho}\frac{\partial}{\partial x_i}\left[-\langle p\rangle\delta_{ij}+2\mu\langle S_{ij}\rangle-\rho\langle u_j'u_i'\rangle\right],\qquad \frac{\partial\langle u_j\rangle}{\partial x_j}=0$$

- Example: Turbulent channel flow

Fixed mean pressure gradient

$$-\frac{\partial\bar{p}}{\partial x}\cdot 2h=2\tau_w=2\mu\left.\frac{\partial\bar{u}}{\partial y}\right|_{y=0}$$




RANS equation for turbulent channel flow

$$0=-\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x}+\nu\frac{\partial^2\bar{u}}{\partial y^2}-\frac{\partial\overline{u'v'}}{\partial y}$$

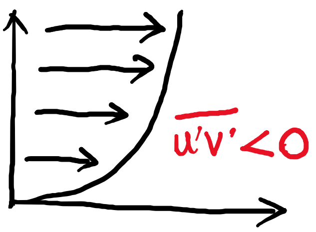
Turbulent velocity scales: $u'_{rms}$ , $v'_{rms}$ $\sim 0-10\%\ U$

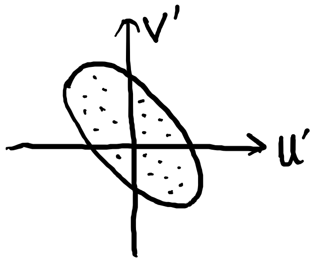
- Revisit scaling of 3D turbulent jet

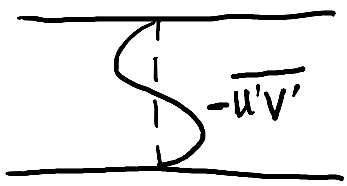
$$\bar{u}\frac{\partial\bar{u}}{\partial x}\sim\nu_T\frac{\partial^2\bar{u}}{\partial y^2},\qquad \nu_T\sim\Delta U\delta,\qquad \frac{(\Delta U)^2}{x}\sim\Delta U\delta\cdot\frac{\Delta U}{\delta^2},\qquad \delta\sim x$$

Note that $\nu_T\sim 0.1\Delta U\cdot\delta$, and $\delta\ll x$. However, the contribution from turbulent mixing is still much larger than the contribution from molecular viscous stress

The meaning of $\overline{u'v'}$ is the correlation between $u'$ and $v'$ in the channel

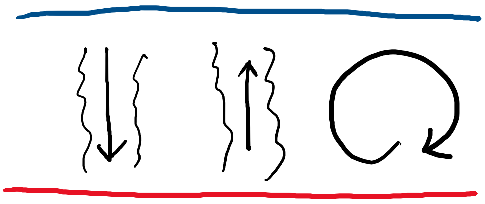



$$L_{ref}=h,\qquad U_{ref}=\sqrt{\tau_w/\rho},\qquad T_{ref}=L_{ref}/U_{ref}$$

- Rayleigh-Benard convection



$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{g}\beta T$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})T = \alpha\nabla^2 T, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

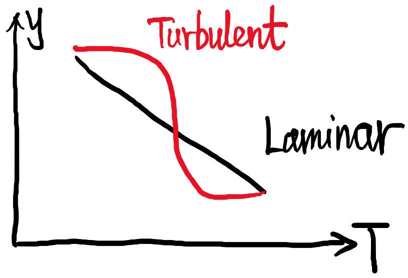
Boussinesq approximation: $\beta T \approx \rho/\rho_0$ with thermal expansion coefficient $\beta$

Special case of infinite parallel plates

$$\bar{u} = \bar{v} = \bar{w} = 0$$

$y$-momentum equation

$$-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} - \frac{\partial \overline{v'u'}}{\partial y} + g\beta\bar{T} = 0$$




Temperature evolution

$$\frac{\partial T}{\partial t} + \frac{\partial (u_j T)}{\partial x_j} = \alpha\nabla^2 T$$

- Electro convective chaos

Dimensionless governing equations

$$0 = -\nabla p + \nabla^2\boldsymbol{u} + \frac{\kappa}{\varepsilon}\rho\boldsymbol{E} \qquad \text{Small Re, LHS = 0}$$

$$-\varepsilon\,\boldsymbol{\nabla}\cdot\boldsymbol{E} = \rho, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho + \boldsymbol{\nabla}\cdot(c\boldsymbol{E}) = \nabla^2\rho, \qquad \frac{\partial c}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}c + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{E}) = \nabla^2 c$$

$$\rho = [\mathrm{Na}^+] - [\mathrm{Cl}^-], \qquad c = [\mathrm{Na}^+] + [\mathrm{Cl}^-]$$

RANS equation will lead to additional unknowns such as $\overline{u_i'c'}, \overline{\rho' E_i'}$

**Lecture 8. Statistical symmetry and homogeneity**

- Statistical symmetry

  Eqns and B.C. remain invariant to mirroring of domain along a coordinate

  Transformation: $x_n \to -x_n,\ u_n \to -u_n$

  Implication: $\langle\cdot\rangle = 0$ for quantities that change sign due to this transformation

- Statistical homogeneity

  Eqns and B.C. remain invariant to translation along a coordinate

  Transformation: $x_n \to x_n + l$

  Implication: $\partial\langle\cdot\rangle/\partial x_n = 0$

- Navier-Stokes equations
  - Symmetric and homogeneous for all 3 spatial coordinates
  - Homogeneous with time
  - Important to check boundary conditions

- Example: Fully developed turbulent channel flow

  B.C. $u = v = w = 0$ at $y = 0, 2h$

  Background pressure gradient

$$\frac{\partial p}{\partial x} = \frac{\Delta p}{\Delta L} + \frac{\partial p'}{\partial x}$$

  Governing equations

$$\frac{\partial u_j}{\partial x_j} = 0, \qquad \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\nabla^2 u_i$$

  - $z$-direction symmetry

| Physics intuition | Symmetric transform | Implication |
| --- | --- | --- |
| $\overline{w} = \overline{\widetilde{w}}$ | $\overline{w} = -\overline{\widetilde{w}}$ | $\overline{w} = 0$ |
| $\overline{u'w'} = \overline{u'\widetilde{w}'}$ | $\overline{u'w'} = -\overline{u'\widetilde{w}'}$ | $\overline{u'w'} = 0$ |

  - $x, z$-direction homogeneity: $\partial\langle\cdot\rangle/\partial x = \partial\langle\cdot\rangle/\partial z = 0$

- $x$-direction is not symmetric

$$\frac{\partial p}{\partial x} = \frac{\Delta p}{\Delta L} + \frac{\partial p'}{\partial x}, \qquad \frac{\partial p}{\partial \tilde{x}} = -\frac{\Delta p}{\Delta L} + \frac{\partial p'}{\partial \tilde{x}}$$

- $y$-direction is not homogeneous, but is symmetric at centerline

$$\bar{v}(y = h) = 0, \qquad \frac{\partial \bar{u}}{\partial y}(y = h) = 0, \qquad \overline{u'v'}(y = h) = 0, \qquad \frac{\partial \overline{u'u'}}{\partial y}(y = h) = 0$$

- Continuity: Homogeneity in $x, z$ gives $\partial v/\partial y = 0$, and with B.C. we have $\bar{v} = 0$

➢ Example: Rayleigh-Benard convection

- $x, z$-direction: Symmetric and homogeneous
- $y$-direction: Not symmetric and not homogeneous

➢ Example: Stoke's first problem

- B.C. $u_{plate} = UH(t)$. This problem is not symmetric and not homogeneous in time
- $x$-direction: Not symmetric, but homogeneous
- $y$-direction: Not symmetric and not homogeneous
- $z$-direction: Symmetric and homogeneous: $\bar{w} = \overline{w'v'} = \overline{w'u'} = 0$, $\partial\langle\cdot\rangle/\partial z = 0$
- Simplified RANS equation (with continuity giving $\bar{v} = 0$)

$$\frac{\partial \bar{u}}{\partial t} = \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'v'}}{\partial y}$$

➢ Example: Fully developed 3D round jet

$$\frac{\partial u}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r u_r u) + \frac{1}{r}\frac{\partial}{\partial \theta}(u_\theta u) + \frac{\partial u^2}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2}\right]$$

- $\theta$-direction: Symmetric and homogeneous, $\overline{u_\theta} = 0$, $\partial\langle\cdot\rangle/\partial\theta = 0$
- $r, x$-direction: Not symmetric and not homogeneous
- Simplified RANS equation (no external pressure gradient)

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{u}_r\frac{\partial \bar{u}}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{u}}{\partial r}\right) - \left[\frac{\partial \overline{u'u'}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\overline{u_r' u'}\right)\right]$$

**Lecture 9. Turbulence closure**

- Recap

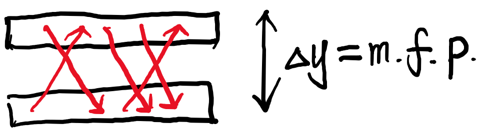
For turbulent jet, the eddy is nearly circle in $xy$-plane (i.e. $u'_{rms} \sim v'_{rms}$). But for turbulent channel flow, eddy is elongated in the stream-wise direction (i.e. $u'_{rms} \gg v'_{rms}$)

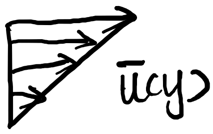
On the wall we have instantaneous zero velocity: $u = v = w = 0$. Besides, continuity equation gives $\partial v/\partial y = 0$. But $\partial u/\partial y \neq 0$ and has a large gradient.

RANS equations are unclosed. For example, Rayleigh-Benard convection has 5 eqs. and 5 standard unknowns $(\bar{u}, \bar{v}, \bar{w}, \bar{\theta}, \bar{p})$ and unclosed terms like $\overline{\theta' v'}$

- Boussinesq approximation (**locality and isotropy**)

Based on analogy between molecular mixing and turbulent mixing




Model the unclosed term with

$$-\overline{u'v'} \simeq \Delta y \, v' \, \frac{\partial \bar{u}}{\partial y}, \qquad \nu_T \sim \Delta y \, v', \qquad \mathrm{m.f.p.\ is\ mean\ free\ path}$$

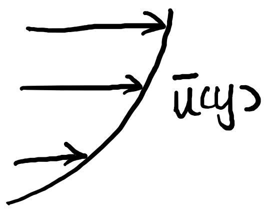
In general, with major assumptions of locality and isotropy

$$-\overline{u_i' u_j'} \simeq \nu_T \left( \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{1}{3} \overline{u_k' u_k'} \delta_{ij}$$

A problem of the first term: The trace of Reynolds stress is 2 TKE (which is positive), while the trace of velocity gradient is the divergence (which is zero). The variance (e.g. $\overline{u'u'}$) should be further captured by turbulent pressure term, which can be absorbed into $\bar{p}\delta_{ij}$.

For scaling with sound speed in the air, $\overline{u'u'} \sim \alpha^2 \sim \gamma RT$, which is related to pressure.

- Example: Mean parallel flow

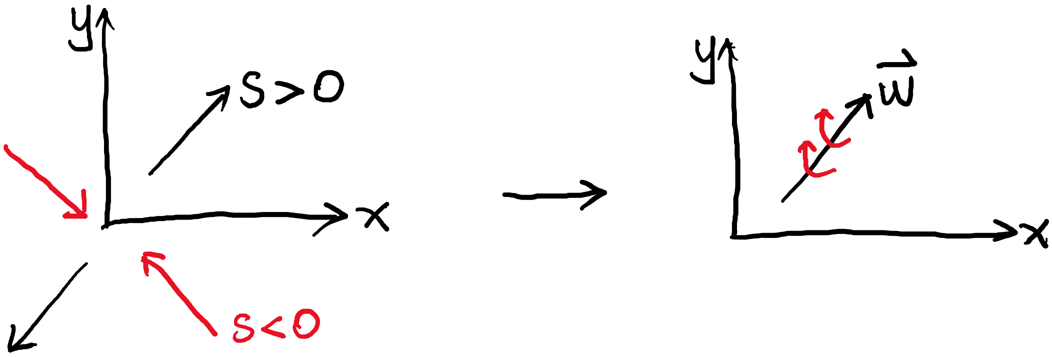



Boussinesq approximation gives

$$-\overline{u'v'} = \nu_T \frac{\partial \bar{u}}{\partial y} > 0$$

Mechanism of sustaining $\overline{u'v'} < 0$ is tilting & stretching (3D).

$$\bar{s}_{ij} = \begin{bmatrix} 0 & \bar{s}_{12} & 0 \\ \bar{s}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \bar{s}_{ij}^{P} = \mathrm{diag}(s, -s, 0)$$




Consider the stress only have non-zero $\bar{s}_{12}$, there will be stretching along 45° direction. Vortex stretching also indicates $\overline{u'v'} < 0$.

➢ Prandtl mixing length model (for parallel and semi-parallel flows)

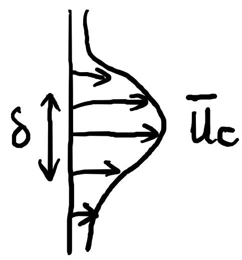
This models the eddy velocity based on mean velocity gradient

$$\nu_T \sim u_{eddy}\, l_{eddy}, \qquad u_{eddy} \sim l_{eddy} \left|\frac{\partial \bar{u}}{\partial y}\right|$$

So the turbulent viscosity is written based on the mixing length $l_m$

$$\nu_T = l_m^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

Mixing length is analogous to mean free path when interpreting viscosity, and is different for different flows




♦ Mixing length model for 2D jets

Define $\delta$ based on the half center-line velocity

$$l_m(x) = C\delta(x)$$

with dimensionless pre-specified constant $C$. Now the RANS equation for 2D planar jet is now closed

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{\partial}{\partial y}\left[(\nu + \nu_T)\frac{\partial \bar{u}}{\partial y}\right], \qquad \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \qquad \nu_T = C^2\delta^2 \left|\frac{\partial \bar{u}}{\partial y}\right|$$

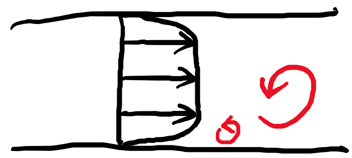
♦ Side note: General non-local model

$$-\overline{u'v'}(y) = \int_{y-\infty}^{y+\infty} \nu_T(y; y') \frac{\partial \bar{u}}{\partial y}(y')\, dy'$$

- Mixing length model for turbulent channel flow

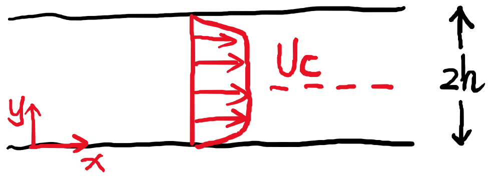
$$l_m = \kappa y$$

with von Karman constant $\kappa = 0.4$



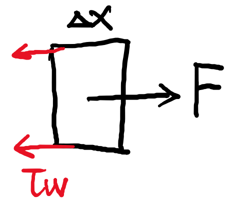
**Lecture 10. Turbulence channel flows & boundary layers**

- Turbulent channel flow




Two "boundary layers" with thickness $h$ or $\delta$

Reference parameters: $\rho_{\text{ref}} = \rho_{\text{fluid}},\ L_{\text{ref}} = h$




Derive reference velocity from pressure gradient (known)

$$F = -\frac{\partial \bar{p}}{\partial x} \cdot \Delta x \cdot 2h, \qquad \tau_w = \mu \frac{\partial \bar{u}}{\partial y}$$

C.V. analysis gives the friction velocity

$$\tau_w = \mu \frac{\partial \bar{u}}{\partial y}\bigg|_{y=0} = -h \frac{\partial \bar{p}}{\partial x}, \qquad u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\nu \frac{\partial \bar{u}}{\partial y}\bigg|_{\text{wall}}}$$

Another reference length scale is the viscous length

$$\delta_\nu = \frac{\nu}{u_\tau}$$

Inner (viscous) units:

$$u^+ = \frac{u}{u_\tau}, \qquad y^+ = \frac{y}{\delta_\nu}$$

Outer units:

$$u^+ = \frac{u}{u_\tau}, \qquad \eta = \frac{y}{\delta}$$

RANS equation for channel:

$$\frac{\partial}{\partial y}\left[\nu \frac{\partial \bar{u}}{\partial y} - \overline{u'v'}\right] = \frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} = -\frac{\tau_w}{\rho h} = -\frac{u_\tau^2}{h}$$

Non-dimensional version, and integrate it for once, we obtain

$$\frac{\partial}{\partial \eta}\left[\frac{\nu}{u_\tau h}\frac{\partial \bar{u}^+}{\partial \eta} - \overline{u^{+\prime} v^{+\prime}}\right] = -1, \qquad \frac{1}{\mathrm{Re}_\tau}\frac{\partial \bar{u}^+}{\partial \eta} - \overline{u^{+\prime} v^{+\prime}} = 1 - \eta$$

For simple notation, we will use

$$\frac{1}{\mathrm{Re}_\tau}\frac{\partial \bar{u}}{\partial y} - \overline{u'v'} = 1 - y, \qquad \mathrm{Re}_\tau = \frac{u_\tau h}{\nu} = \frac{\delta}{\delta_\nu} = h^+$$

$\mathrm{Re}_\tau$ can also be interpreted as the ratio of outer and inner length scales

What is the profile of $\bar{u}(\mathrm{Re}_\tau, y)$, the center-line velocity $U_c^+(\mathrm{Re}_\tau)$?

Given center-line velocity, the Reynolds number is thus

$$\mathrm{Re} = \frac{U_c h}{\nu} = U_c^+ \frac{u_\tau h}{\nu} = U_c^+ \, \mathrm{Re}_\tau$$

**Derivation of the velocity profile**

1. For the outer zone, we have

$$-\overline{u^{+\prime} v^{+\prime}} = 1 - \frac{y}{\delta}$$

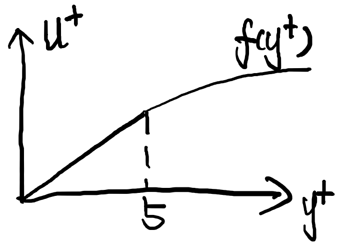
For wall-bounded flows, near the wall the molecular mixing dominates. So for the entire region, we cannot say turbulent stress always dominates, which is different from jets.

However, sufficiently away from the wall ($y^+ \gg 1$, a condition related to $\mathrm{Re}_\tau$), we can safely neglect the viscous stress, and consider the following expression independent of $\mathrm{Re}_\tau$

$$\frac{\bar{u} - u_c}{u_\tau} = g\left(\frac{y}{\delta}\right)$$

2. For the inner zone ($y \ll \delta$), we have $u^+$ independent of Reynolds number

$$u^+ = \frac{\bar{u}}{u_\tau} = f(y^+)$$




The velocity gradient satisfies

$$\left.\frac{du^+}{dy^+}\right|_{y=0} = \left.\frac{\delta_\nu}{u_\tau}\frac{d\bar{U}}{dy}\right|_{y=0} = \frac{\delta_\nu}{\nu u_\tau} \cdot \nu \left.\frac{d\bar{U}}{dy}\right|_{y=0} = \frac{\delta_\nu}{\nu u_\tau} \cdot u_\tau^2 = 1$$

So the simplified momentum balance is

$$\frac{\partial \bar{u}^+}{\partial y^+} - \overline{u^{+\prime} v^{+\prime}} \simeq 1$$

3. For overlap zone ($\delta_\nu \ll y \ll \delta$), typical values are

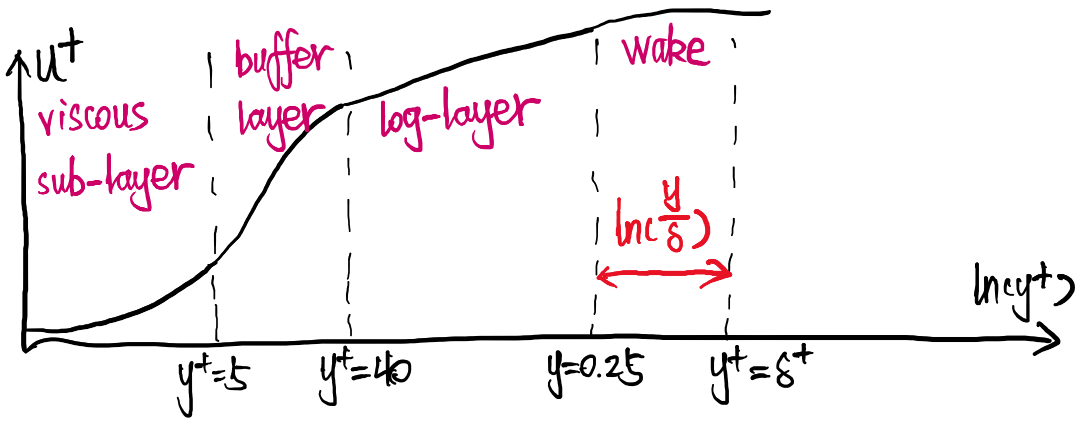
$$y^+ > 40, \qquad \frac{y}{\delta} < 0.2$$

In this case, both inner and outer scaling relations should be valid

$$\frac{\partial u^+}{\partial \eta} = \frac{dg}{d\eta}, \qquad \frac{\partial u^+}{\partial \eta} = \frac{y^+}{\eta}\frac{df}{dy^+}$$

Therefore, both derivatives should be constant

$$\eta \frac{dg}{d\eta} = y^+ \frac{df}{dy^+} = \text{const.}, \qquad g(\eta) = A \ln \eta + B$$




The log-layer has the expression

$$u^+ = \frac{1}{\kappa}\ln y^+ + A, \qquad \kappa = 0.4, \qquad A = 5.5$$

- Prandtl mixing length model & log-layer

  In the overlap zone, we have

$$-\overline{u^{+\prime}v^{+\prime}} = 1$$

Mixing length model indicates

$$\nu_T \frac{\partial \bar{u}}{\partial y} = l_m^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \frac{\partial \bar{u}}{\partial y} = 1, \qquad l_m = \kappa y$$

Therefore, the log-layer can be obtained

$$\kappa y \frac{\partial \bar{u}}{\partial y} = 1, \qquad \frac{\partial \bar{u}}{\partial y} = \frac{1}{\kappa y}, \qquad \bar{u} = \frac{1}{\kappa}\ln y + B$$

A strong weakness of mixing length model: At the centerline, velocity gradient is 0. But in reality, $\nu_T \neq 0$ as there is strong mixing around the centerline.

**Lecture 11. Reynolds stress transport equation**

How to model Reynolds stress $\overline{u_i' u_j'}$, and how it is influenced by velocity gradient $\partial \bar{u}_l / \partial x_k$

- Transport equation for Reynolds stress

1. Start with NS equation

$$\frac{\partial u_i}{\partial t} + \frac{\partial (u_k u_i)}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i$$

2. RANS equation

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial (\overline{u_k u_i})}{\partial x_k} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \nabla^2 \bar{u}_i$$

3. Subtraction between these two equations, evolution of perturbation

$$\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{u}_k u_i' + u_k' \bar{u}_i + u_k' u_i' - \overline{u_k' u_i'}\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial x_i} + \nu \nabla^2 u_i'$$

$$\frac{\partial u_j'}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{u}_k u_j' + u_k' \bar{u}_j + u_k' u_j' - \overline{u_k' u_j'}\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial x_j} + \nu \nabla^2 u_j'$$

4. Cross multiplication and Reynolds-averaging

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} + \frac{\partial}{\partial x_k}\left(\bar{u}_k \overline{u_i' u_j'}\right) = -\frac{1}{\rho}\left(\frac{\partial \overline{p' u_j'}}{\partial x_i} + \frac{\partial \overline{p' u_i'}}{\partial x_j} - 2\,\overline{p' S_{ij}'}\right)$$

$$+\nu \frac{\partial^2 \overline{u_i' u_j'}}{\partial x_k \partial x_k} - 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}$$

$$-\overline{u_j' u_k'}\frac{\partial \bar{u}_i}{\partial x_k} - \overline{u_i' u_k'}\frac{\partial \bar{u}_j}{\partial x_k} - \frac{\partial \overline{u_k' u_i' u_j'}}{\partial x_k}$$

Reynolds stress transport equation give 6 addition equations

$\overline{u_i' u_j'}$ is now primary unknowns, but we have new unclosed terms: pressure correlation terms (energy re-distribution into isotropy), velocity gradient correlation terms (dissipation), triple correlation terms (turbulent transport of Reynolds stress)

a. Compared with DNS, we can reduce the time and spatial resolution needed for modeling, but we have more equations to solve
b. Same advection, diffusion, production terms appear for Reynolds stress, more physics interpretation and potentially better to capture non-local effects

- Turbulent kinetic energy equation

$$\mathrm{TKE} = e = \frac{\overline{u_i'u_i'}}{2} = \frac{1}{2}(\overline{u'u'} + \overline{v'v'} + \overline{w'w'}), \qquad \mathrm{MKE} = \frac{1}{2}\bar{u}_i\bar{u}_i$$

Take $i = j$ for the Reynolds stress transport equation

$$\frac{D\mathrm{TKE}}{Dt} = \frac{\partial \mathrm{TKE}}{\partial t} + \bar{u}_k \frac{\partial \mathrm{TKE}}{\partial x_k} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[-\overline{p'u_j'} - \frac{\rho}{2}\overline{u_j'u_i'u_i'} + 2\mu\,\overline{u_i'S_{ij}'}\right] - 2\nu\overline{S_{ij}'S_{ij}'} - \overline{u_i'u_j'} \cdot \overline{S_{ij}}$$

The change of TKE, when observed following a moving eddy blob, is contributed by:

Dissipation $-\varepsilon$, Production (overall positive) $P$, Pressure work, Triple correlation, Transport by viscous stress

Sign of the production term: $-\overline{u_i'u_j'} \cdot \overline{S_{ij}} \sim \nu_T\, \overline{S_{ij}} \cdot \overline{S_{ij}}$, positive in an overall sense

If doing Boussinesq approximation for the triple correlation term, we now commit less errors compared with modeling Reynolds stress

- Kinetic energy of mean flow

1. Start with RANS equation

$$\frac{D\bar{u}_i}{Dt} = \frac{1}{\rho}\frac{\partial T_{ij}}{\partial x_j}, \qquad T_{ij} = -\bar{p}\delta_{ij} + 2\mu\bar{S}_{ij} - \rho\overline{u_i'u_j'}$$

2. Cross multiplication

$$\frac{D\mathrm{MKE}}{Dt} = \frac{\partial \mathrm{MKE}}{\partial t} + \bar{u}_k\frac{\partial \mathrm{MKE}}{\partial x_k} = \frac{1}{\rho}\frac{\partial \bar{u}_i T_{ij}}{\partial x_j} + \frac{\partial \bar{u}_i}{\partial x_j}\delta_{ij}\bar{p} - 2\nu\frac{\partial \bar{u}_i}{\partial x_j}\bar{S}_{ij} + \overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}$$

3. Simplification and manipulation

$$\frac{D\mathrm{MKE}}{Dt} = \frac{\partial \mathrm{MKE}}{\partial t} + \bar{u}_k\frac{\partial \mathrm{MKE}}{\partial x_k} = \frac{1}{\rho}\frac{\partial \bar{u}_i T_{ij}}{\partial x_j} - 2\nu\bar{S}_{ij}\bar{S}_{ij} + \overline{u_i'u_j'} \cdot \bar{S}_{ij}$$

The change of MKE, when observed following a moving fluid, is contributed by:

Divergence of flux, Small viscous term (negative, often small except near the wall), Minus Production

$$\text{Boundaries} \rightarrow \text{MKE} \xrightarrow{\text{Production}} \text{TKE} \xrightarrow{\varepsilon} \text{Heat}$$

**Lecture 12. TKE equation for canonical problems (0D or 1D problems)**

➢ Recap Lecture 11: TKE equation

$$\mathrm{TKE} = \frac{1}{2}\overline{u_i' u_i'}, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}_j \frac{\partial}{\partial x_j}$$

$$\frac{D\mathrm{TKE}}{Dt} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left[-\overline{p' u_j'} - \frac{\rho}{2}\,\overline{u_j' u_i' u_i'} + 2\mu\,\overline{u_i' S_{ij}'}\right] - 2\nu\overline{S_{ij}' S_{ij}'} - \overline{u_i' u_j'}\cdot\overline{S_{ij}}$$

$$P = -\overline{u_i' u_j'}\cdot\overline{S_{ij}}, \qquad \varepsilon = 2\nu\overline{S_{ij}' S_{ij}'}$$

➢ Turbulent channel flow

1. TKE equation

$$0 = \frac{1}{\rho}\frac{\partial}{\partial y}\left[-\overline{p' v'} - \rho\overline{v'\frac{u_j' u_j'}{2}} + 2\mu\,\overline{u_i' S_{i2}'}\right] - 2\nu\overline{S_{ij}' S_{ij}'} - \overline{u' v'}\frac{\partial \bar{u}}{\partial y}$$

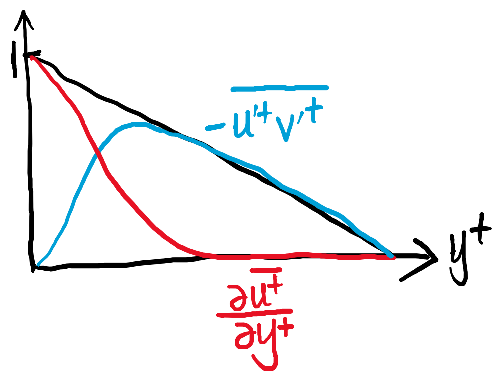
$$P = -\overline{u' v'}\frac{\partial \bar{u}}{\partial y}, \qquad \varepsilon = 2\nu\overline{S_{ij}' S_{ij}'}$$

2. MKE equation

$$0 = \underbrace{-\frac{\bar{u}}{\rho}\frac{\partial \bar{p}}{\partial x}}_{+} + \frac{1}{\rho}\frac{\partial}{\partial y}\left[\bar{u}\cdot\mu\frac{\partial \bar{u}}{\partial y} - \bar{u}\cdot\overline{u' v'}\right] \underbrace{-P}_{-} - 2\nu\bar{S}_{ij}\bar{S}_{ij}$$

3. Production term

Recall RANS equation




$$\frac{\partial \overline{u^+}}{\partial y^+} - \overline{u'^+ v'^+} = 1 - \eta$$

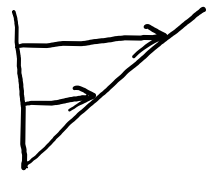
$$P^+_{max} = 0.25, \text{ at } y^+ \simeq 12$$

4. In the log layer, the non-dimensional quantities are

$$\frac{\langle u'^2\rangle}{\mathrm{TKE}} = 1, \qquad \frac{\langle v'^2\rangle}{\mathrm{TKE}} = 0.4, \qquad \frac{\langle w'^2\rangle}{\mathrm{TKE}} = 0.6, \qquad \frac{\langle u' v'\rangle}{\mathrm{TKE}} = -0.28$$

$$\frac{\kappa}{\varepsilon}\frac{\partial \bar{u}}{\partial y} = S\frac{\kappa}{\varepsilon} = 3.2, \qquad 3.2 = -\frac{S\langle u' v'\rangle}{0.28\varepsilon} = \frac{P}{0.28\varepsilon}, \qquad \frac{P}{\varepsilon} = 0.9$$

➢ Homogeneous shear flow



$$\bar{u} = sy, \qquad \bar{v} = \bar{w} = 0$$

1. TKE equation

$$\frac{\partial \mathrm{TKE}}{\partial t} = -\overline{u'v'}\frac{\partial \bar{u}}{\partial y} - \varepsilon$$

The statistics are homogeneous in $y$-direction, and RANS become an ODE

2. Non-dimensional quantities (for tuning and testing RANS models)

$$\frac{\langle u'^2 \rangle}{\mathrm{TKE}} = 1, \qquad \frac{\langle v'^2 \rangle}{\mathrm{TKE}} = 0.4, \qquad \frac{\langle w'^2 \rangle}{\mathrm{TKE}} = 0.6, \qquad \frac{\langle u'v' \rangle}{\mathrm{TKE}} = -0.28, \qquad \frac{P}{\varepsilon} = 1.7$$

3. Analytic solution of TKE

$$\frac{\partial \mathrm{TKE}}{\partial t} = 0.28 \cdot \mathrm{TKE} \cdot \mathrm{S} \cdot \left(1 - \frac{1}{1.7}\right) = 0.12 \cdot S \cdot \mathrm{TKE}$$

TKE in this flow exponentially grows

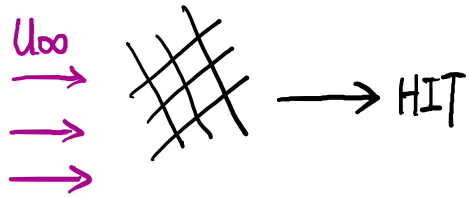
$$\mathrm{TKE} = \mathrm{TKE}_0 \cdot \exp(0.12\, St)$$

4. Empirical relationship for dissipation

$$\epsilon \sim \frac{u^3}{l} \sim \frac{(\mathrm{TKE})^{\frac{3}{2}}}{l}, \qquad l \sim \frac{(\mathrm{TKE})^{\frac{3}{2}}}{\epsilon} \sim \exp(0.06\, St)$$

➢ Homogeneous & Isotropic turbulence (grid turbulence)

The length scale of the decay is much larger than the turbulence itself (~ grid size), so locally it is a homogeneous & isotropic turbulence




1. TKE equation

$$\frac{d\mathrm{TKE}}{dt} = -\epsilon$$

2. Power law solution from observation

$$\mathrm{TKE} = \mathrm{TKE}_0 \cdot \left(\frac{t}{t_0}\right)^{-n}, \qquad \epsilon \propto t^{-n-1}$$

Length scale of the turbulence grows with time, but turbulent viscosity decreases with time.

These observations indicate $1 < n < 2$, and empirical values are

$$n = 1.3,\ \epsilon \propto t^{-2.3},\ l \propto t^{0.35},\ \mathrm{Re}_l \propto t^{-0.3}$$

**Lecture 13. Introduction to correlations**

- Spatial correlation

Correlations in homogeneous flow: Only dependent on separation vector $\boldsymbol{r}$

$$R_{uu}(\boldsymbol{r}) \equiv \overline{u'(\boldsymbol{x})u'(\boldsymbol{x}+\boldsymbol{r})}$$

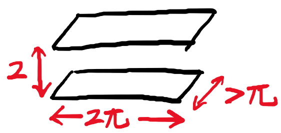
More general, the second order correlation tensor

$$R_{ij}(\boldsymbol{r}) = \langle u_i'(\boldsymbol{x}+\boldsymbol{r},t)\, u_j{}'(\boldsymbol{x},t)\rangle$$

Special case (average in all homogeneous directions)

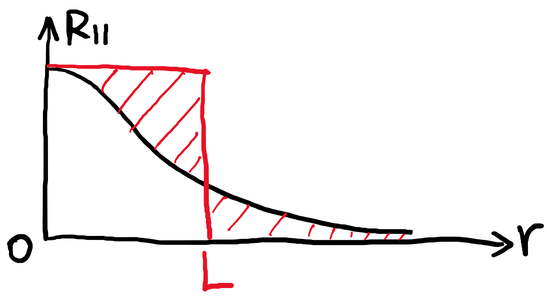
$$R_{11}(r\hat{\boldsymbol{i}}) = \langle u'(x+r,y,z,t)\, u'(x,y,z,t)\rangle, \qquad R_{11}(0) = \langle u'u'\rangle$$

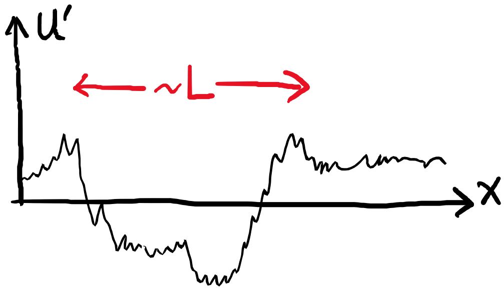
- Integral length scale (**size of large structures ∝ *L***)




$$L = \frac{\int_0^\infty R_{11}(r)\, dr}{\langle u'u'\rangle}$$

For homogeneous directions, we need **simulation domain size ≫ *L*** (several times of $L$)




- Temporal correlations

$$R_{11}(\tau) = \langle u'(x,y,z,t+\tau)\, u'(x,y,z,t)\rangle$$

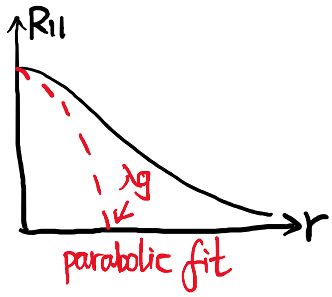
$$\text{Integral Time} = \frac{\int_0^\infty R_{11}(\tau)\, d\tau}{\langle u'u'\rangle}$$

For homogeneous directions, we need **simulation time ≫ integral time scale**

In SEM formula, each segment should be independent. **The window size should be larger than the integral time**

- Taylor micro-scale

$$R_{11}(r) \simeq \overline{u'u'} + \frac{1}{2}\frac{\partial^2 R}{\partial r^2}\bigg|_{r=0} r^2, \qquad \lambda_g = \left(\frac{2\overline{u'u'}}{-\dfrac{\partial^2 R}{\partial r^2}\bigg|_{r=0}}\right)^{1/2}$$



Defined based on the leading orders of Taylor series. Taylor Reynolds number is

$$\mathrm{Re}_\lambda = \frac{U_{rms}\lambda}{\nu}$$

This micro-scale $\lambda$ characterizes turbulent dissipation, but **not the smallest eddy size**

$$\epsilon \sim \nu \frac{U_{rms}^2}{\lambda^2}$$

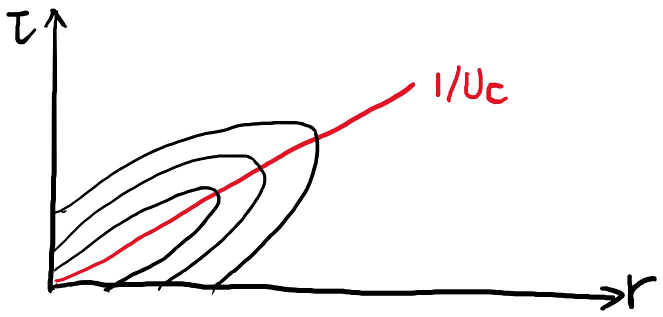
- Space-time correlations

The flow is homogeneous in $x, z$ and time (e.g. turbulent channel flow)

$$R_{uu}(r,\tau) = \langle u'(x+r,y,z,t+\tau)\, u'(x,y,z,t)\rangle$$

Convective velocity (at which the features are moving across the probe) is usually the same with the local mean velocity.

But in some scenarios, these two velocities can be very different. On the wall there are pressure or shear spots moving, but the local mean velocity on the wall is zero. These shear spots are related to moving vortices a bit away from the wall posing footprints on the wall.




- How to compute correlations

1. Data should be defined on uniform mesh for the directions to compute correlations (e.g. $\Delta x$ and $\Delta t$ of the data should be uniform)
2. Choice $r$ or $\tau$ when performing shift-multiply-average

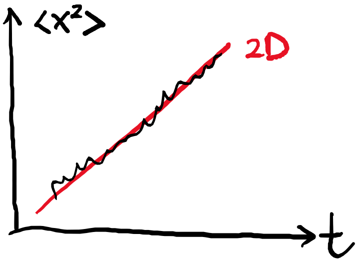
$$r = 0, \Delta x, 2\Delta x, 3\Delta x, \dots, \qquad \tau = 0, \Delta t, 2\Delta t, 3\Delta t, \dots$$

3. Do not use for loops in MATLAB when doing average
4. Periodic extension in spatial domain, but not for time

$$\tau_{max} \leq \frac{T_{max}}{2}$$

i.e. at least 50% of overlap

## Lecture 14. Measurement of eddy diffusivity




- Lagrangian methods

Consider a statistically homogeneous process

$$\langle x^2 \rangle = 2Dt$$

with $x$ denotes the position of **Lagrangian particles.** However, this formula indicates that $r = \sqrt{\langle x^2 \rangle}$, with a singular velocity at $x = 0$.

We need to take the limit of $t \to \infty$ to estimate $D$

$$D = \frac{1}{2}\frac{d}{dt}\langle x^2 \rangle = \frac{1}{2}\langle \frac{dx^2}{dt} \rangle = \langle x(t)\, u(t) \rangle = \langle u(t) \int_0^t u(t')\, dt' \rangle$$

$$= \langle \int_0^t u(t)\, u(t')\, dt' \rangle = \int_0^t \langle u(t)\, u(t') \rangle\, dt'$$

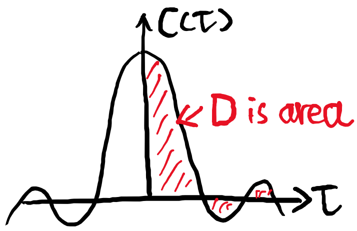
Define the time difference as $\tau = t' - t$

$$D = \int_{-t}^{0} \langle u(t)\, u(t+\tau) \rangle\, d\tau = \int_0^t \langle u(t)\, u(t+\tau) \rangle\, d\tau$$

where we use time symmetry at the last step. At the limit of $t \to \infty$, we have

$$D = \int_0^\infty \langle u(t)\, u(t+\tau) \rangle\, d\tau = \int_0^\infty C(\tau)\, d\tau$$

Note that $C(\tau)$ is for a moving Lagrangian particle.




- Generalization & application in transport of a scalar quantity

$$\frac{\partial c}{\partial t} + \frac{\partial (u_j c)}{\partial x_j} = D_M \nabla^2 c, \qquad \frac{\partial \bar{c}}{\partial t} + \frac{\partial (\bar{u}_j \bar{c})}{\partial x_j} = \frac{\partial}{\partial x_j}\left[ D_M \frac{\partial \bar{c}}{\partial x_j} - \overline{u_j' c'} \right]$$

Boussinesq approximation (local approximation, gradient diffusion model), as well as its anisotropic extension, gives

$$-\overline{u_j' c'} \simeq D_T \frac{\partial \bar{c}}{\partial x_j}, \qquad -\overline{u_i' c'} \simeq D_{ij} \frac{\partial \bar{c}}{\partial x_j}$$

The goal is to determine $D_{ij}$. We can estimate it from correlations

$$D_{ij} = \int_0^\infty C_{ij}(\tau)\, d\tau, \qquad C_{ij}(\tau) = \langle \frac{1}{2} u_i(t)\, u_j(t+\tau) + \frac{1}{2} u_j(t) u_i(t+\tau) \rangle$$

Note: $u_i(t)$ obtained from particle tracking is a term in the following expression

$$\boldsymbol{X}^{NH} = \boldsymbol{X}^{N} + \Delta t\, \boldsymbol{u}(\boldsymbol{X}_n) + \text{Random } \Delta \boldsymbol{x}$$

where the random walk should satisfy that its diffusivity is $D_M$

- Measurement of eddy diffusivity for inhomogeneous systems

For inhomogeneous systems (e.g. Rayleigh-Benard convection), simplified RANS equation for a passive scalar is

$$0 = \frac{\partial}{\partial y}\left[D_M \frac{\partial \bar{c}}{\partial y} - \overline{v'c'}\right]$$

with Boussinesq model and $y$-dependent eddy diffusivity

$$-\overline{v'c'} = D_T(y)\frac{\partial \bar{c}}{\partial y}$$

General closure eddy diffusivity operator (input-output relation) is

$$-\overline{u_i'c'}(\boldsymbol{x}) = \int_{V(\boldsymbol{y})} D_{ij}(\boldsymbol{x},\boldsymbol{y})\frac{\partial \bar{c}}{\partial y_j}\, d^3\boldsymbol{y}$$

This relation is exact, even though it is linear. Because for scalar quantity the governing equation is linear.

Hamba (2004, 2005, physics of fluids) proposed the idea of solving DNS with the following condition

$$\frac{\partial \bar{c}}{\partial y_j} = \delta(\boldsymbol{y} - \boldsymbol{y}_0)$$

Post-processing of simulation data will give

$$-\overline{u_i'c'} = D_{ij}(\boldsymbol{x},\boldsymbol{y}_0)$$

and then we can repeat for all $\boldsymbol{y}_0$

- Introduction to macroscopic forcing method (Mani & Park, 2021, PR-Fluids)

$$\frac{\partial c}{\partial t} + \frac{\partial (u_j c)}{\partial x_j} = D_M \nabla^2 c + S(\boldsymbol{x})$$

For 1D limit RANS-space in $x_2$-direction

$$-\overline{u_2'c'}(x_2) = \int_{y_2} D_{ij}(x_2,y_2)\frac{\partial \bar{c}}{\partial y_2}\, dy_2$$

with Taylor expansion around $x_2$

$$\frac{\partial \bar{c}}{\partial y_2} = \frac{\partial \bar{c}}{\partial x_2} + (y_2 - x_2)\frac{\partial^2 \bar{c}}{\partial x_2^2} + \frac{(y_2 - x_2)^2}{2}\frac{\partial^3 \bar{c}}{\partial x_2^3} + \cdots$$

we have

$$-\overline{u_2' c'}(x_2) = D^{(0)}(x_2)\frac{\partial \bar{c}}{\partial x_2} + D^{(1)}(x_2)\frac{\partial^2 \bar{c}}{\partial x_2^2} + D^{(2)}(x_2)\frac{\partial^3 \bar{c}}{\partial x_2^3} + \cdots$$

with each coefficient (moment of different orders) calculated as

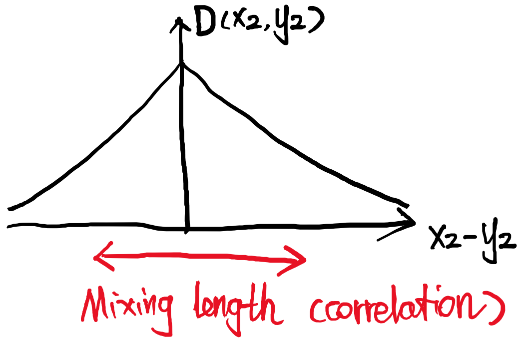
$$D^{(0)} = \int_y D_{22}(x_2, y_2)\, dy_2\,, \qquad D^{(1)} = \int_y (y_2 - x_2) D_{22}(x_2, y_2)\, dy_2$$

$D^{(0)}$ is the Boussinesq term, and higher order terms capture non-Boussinesq effects.

Calculate $D^{(0)}(x_2)$: Choose $S(x_2)$ such that $\bar{c}(x_2) = x_2$, and this can be done by adding a nudging term for the scalar evolution equation

$$-\overline{u_2' c'}(x_2) = D^{(0)}(x_2)$$

Calculate $D^{(1)}(x_2)$: Choose $S(x_2)$ such that $\bar{c}(x_2) = x_2^2/2$. Similarly choose polynomial form of $S(x_2)$




However, the previous expansion is not convergent.

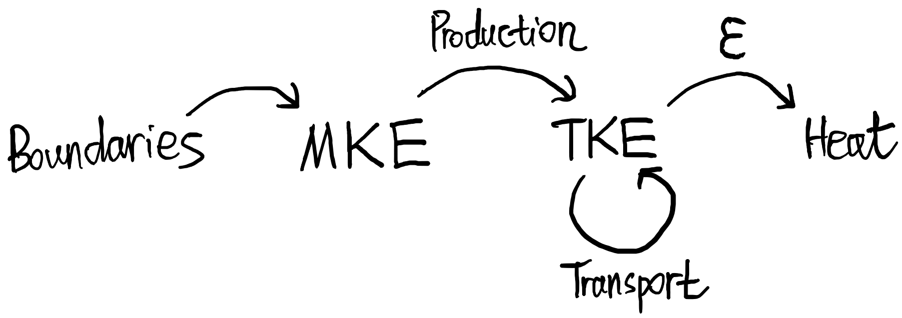
A general converging closure operator is

$$\left[1 + a_1(x_2)\frac{\partial}{\partial x_2} + a_2\frac{\partial^2}{\partial x_2^2} + \cdots\right]\left(-\overline{u_2' c'}\right)(x_2) = a_0\frac{\partial \bar{c}}{\partial x_2}$$

The macro-scale (RANS space) denotes $\bar{q}$, while micro-scale (fluctuation space) denotes $q'$

**Lecture 15. Kolmogorov scale**

➢ Energy budget



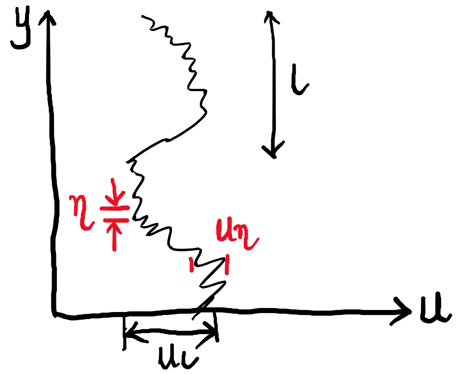

Production term

$$P = -\langle u_i' u_j' \rangle \langle S_{ij} \rangle$$

Dissipation term

$$\epsilon = 2\nu \langle S_{ij}' S_{ij}' \rangle$$

In the integral sense, $P \simeq \epsilon$ (most cases away from the wall). They are in the same order.




Instantaneous picture of velocity profile:

Large eddy scale: $l$

Smallest eddy size (Kolmogorov scale): $\eta$

We need to find $\eta$ and $u_\eta$

➢ Scaling analysis of smallest eddy size

For large eddy scale $l$ and velocity scale $u_l$, they are related to the geometry and background flow velocity

Scaling of production and dissipation terms gives

$$P = -\langle u_i' u_j' \rangle \langle s_{ij} \rangle \sim \frac{u_l^3}{l}, \qquad \epsilon \sim \nu \frac{u_\eta^2}{\eta^2}, \qquad \frac{u_l^3}{l} \sim \nu \frac{u_\eta^2}{\eta^2}$$

Note the difference in scaling of derivative in RANS and fluctuation spaces

- Example: Jet with $l \sim \delta \sim 0.5\ \mathrm{m}$, $u_l \sim 100\ \mathrm{m/s}$, $\nu \sim 10^{-5}\ \mathrm{m^2/s}$

$$P \sim 2 \times 10^6\ \mathrm{m^2/s^3}, \qquad \frac{u_\eta}{\eta} \sim \sqrt{P/\nu} \sim 4 \times 10^5\ \mathrm{s^{-1}}$$

Consider $\eta = 0.1\ \mathrm{mm}$, then we have $u_\eta = 40\ \mathrm{m/s}$. To see if this is correct, we need another constraint, which is the Reynolds number

$$\mathrm{Re}_\eta = \frac{\eta u_\eta}{\nu} \sim 1$$

This indicates that these scales are dominated by viscous stress. Two constraints give

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}, \qquad u_\eta = \frac{\nu}{\eta} = (\nu\epsilon)^{1/4}, \qquad t_\eta = \frac{\eta}{u_\eta} = \left(\frac{\nu}{\epsilon}\right)^{1/2}$$

For the above jet, the scaling analysis indicates

$$\boldsymbol{\eta = 5\ \mu m}, \qquad u_\eta = 2\ \mathrm{m/s}$$

- Example: A mixer with power 500 W, mixing 2 L of maple sytup with

$$\nu = 1.2 \times 10^{-4}\ \mathrm{m^2/s}, \qquad \rho = 1.3 \times 10^3\ \mathrm{kg/m^3}$$

Since all power goes into dissipation, we have

$$\epsilon = \frac{\text{Power}}{\text{Mass}} = \frac{500}{1.3 \times 10^3 \times 2 \times 10^{-3}}\ \mathrm{m^2/s^3} = 190\ \ \mathrm{m^2/s^3}$$

The corresponding Kolmogorov scale is $\eta = 0.3$ mm

- Estimation of DNS computational cost

Number of mesh points in one direction

$$\frac{l}{\eta} = \left(\frac{l^4 \epsilon}{\nu^3}\right)^{\frac{1}{4}} = \left(\frac{l u_l}{\nu}\right)^{\frac{3}{4}} = \mathrm{Re}_l^{\frac{3}{4}}$$

Example: For the previous jet example, number of mesh points in 3D

$$\mathrm{Re}_l \sim 5 \times 10^6, \qquad N_{3D} \sim 10^{15}$$

Typically people choose $\Delta > \eta$ with $\Delta = 1.5\ \eta$, which is related to the prefactor. $\eta$ is only the scale of eddy, and the eddy size in reality is larger

For DNS of turbulent channel flow, people use

$$\Delta x^+ < 10, \qquad \Delta z^+ < 5, \qquad \Delta y^+ < 0.5$$

Near the wall there are hairpin vortices, and the features are elongated in flow direction

**Lecture 16. Different scales of eddies**

- Recap Lecture 15

$$\eta \equiv \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}, \qquad u_\eta \equiv \frac{\nu}{\eta} = (\nu\epsilon)^{\frac{1}{4}}, \qquad \epsilon = 2\nu\,\overline{S'_{ij}S'_{ij}} = 15\nu\,\overline{\left(\frac{\partial u}{\partial x}\right)^2} \quad \text{for HIT}$$

For large scale quantities

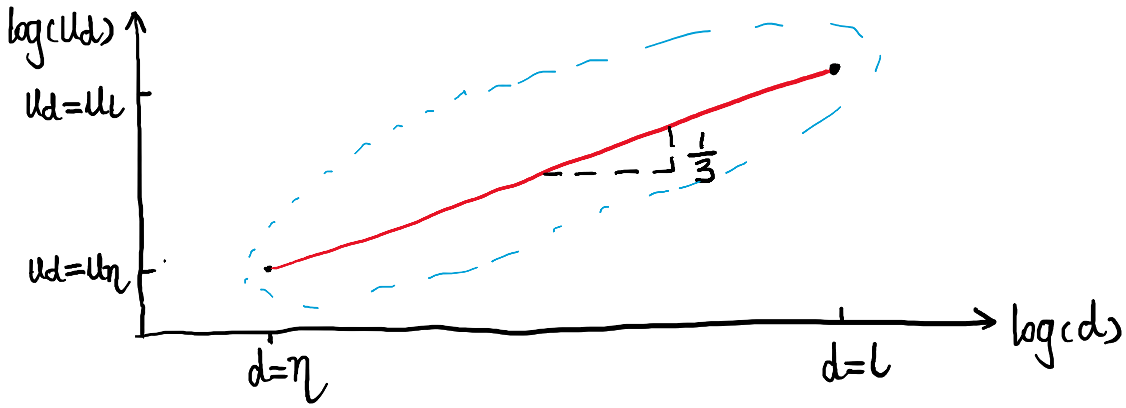
$$u_l \equiv \sqrt{\text{TKE}}, \qquad l \equiv \frac{u_l^3}{\epsilon}$$

We only need to measure TKE and dissipation to calculate these scales

- Intermediate eddies

$$\epsilon = \frac{u_l^3}{l} = \nu\frac{u_\eta^2}{\eta^2} = \frac{u_\eta^3}{\eta}, \qquad \frac{u_\eta}{u_l} = \left(\frac{\eta}{l}\right)^{1/3}$$

The expectation relationship is sketched below. The true scenario will be a cloud.




- Energy transfer

$$P = -\langle u_i'u_j'\rangle\langle S_{ij}\rangle \sim \nu_l\langle S_{ij}\rangle\langle S_{ij}\rangle, \qquad \epsilon \sim \nu_\eta\langle S'_{ij}S'_{ij}\rangle$$

$$\text{MKE} \xrightarrow{P} \text{TKE} \xrightarrow{\epsilon} \text{Heat}$$

$$\xrightarrow{P} l, u_l \rightarrow d_1 \rightarrow d_2 \rightarrow d_3 \rightarrow \eta, u_\eta \xrightarrow{\epsilon}$$

Mechanism of energizing eddies: Vortex stretching

$$\omega = \omega_0 e^{At}, \qquad A \sim \frac{u_d}{d}$$

For eddies with length scale $d_3$, the 'best' eddies that can efficiently stretch vortices of this scale are those with size $d_2$. This is because: larger eddies have smaller strain rate $A$, smaller eddies are dimensionally incompatible (within the structures of current eddy)

Energy flowing in per unit mass (using exponential grow of $u_d$):

$$\frac{du_d^2}{dt} \sim u_d A u_d \sim \frac{u_d^3}{d}$$

Energy flowing out per unit mass:

$$\nu_d \tilde{S}_{ij} \tilde{S}_{ij} \sim d u_d \cdot \left(\frac{u_d}{d}\right)^2 \sim \frac{u_d^3}{d}$$

Energy balance indicates that the above quantity is constant: $\boldsymbol{u_d \propto d^{1/3}}$. This analysis is not following a single eddy under stretching, but is considering eddies of different sizes that have already been mixed (statistically quasi-steady), i.e. reaching a balance between vortex stretching that tends to reduce eddy size and mixing with smaller eddies that tends to increase eddy size.

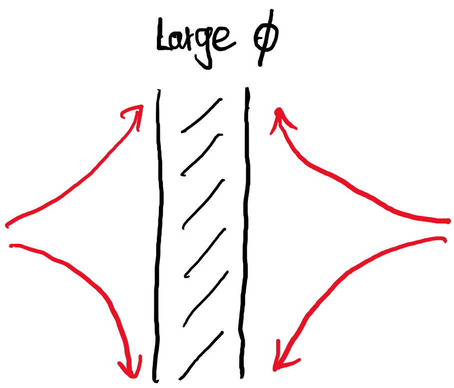
➢ Small scales for transport of a scalar field

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x_j}(u_j \phi) = \gamma \nabla^2 \phi$$

1. If $\gamma = \nu$, then we have smallest structure $\eta_\phi = \eta$.

2. Now consider we have smaller $\gamma < \nu$, qualitatively we would have $\eta_\phi < \eta$

$$\tau_{\text{stretch}} \sim \tau_\eta \sim \frac{\eta}{u_\eta} \sim \frac{\eta^2}{\nu}, \qquad \tau_{\text{diff}} \sim \frac{\eta_\phi^2}{\gamma}$$




Efficient thinning requires $\tau_{\text{stretch}} < \tau_{\text{diff}}$ indicating that diffusion (Brownian motion) will not smooth the features out. Critical point gives the Batchelor scale

$$\frac{\eta^2}{\nu} \sim \frac{\eta_\phi^2}{\gamma}, \qquad \eta_\phi = \eta \left(\frac{\gamma}{\nu}\right)^{\frac{1}{2}}, \qquad S_c \equiv \frac{\nu}{\gamma}$$

In this case, the stretching is dominated by $\eta$. For water we have Schmidt number $S_c =$ 1000, so scalar transport in water needs 30 times finer mesh

3. Now consider we have larger $\gamma > \nu$, qualitatively we would have $\eta_\phi = d > \eta$

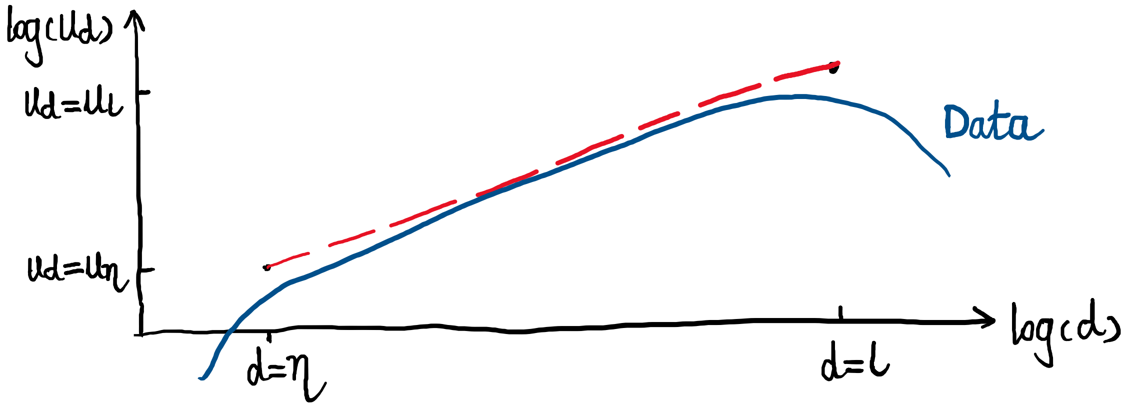
$$\tau_{\text{stretch}} \sim \frac{d}{u_d} \sim \tau_{\text{diff}} \sim \frac{d^2}{\gamma}, \qquad u_d = u_\eta \left(\frac{d}{\eta}\right)^{\frac{1}{3}}, \qquad u_\eta = \frac{\nu}{\eta}$$

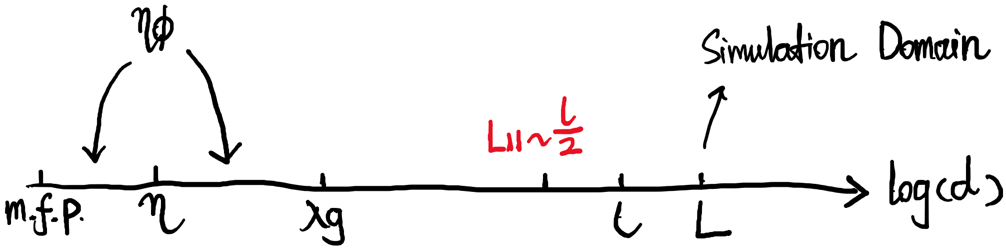
which gives the Obukov-Corsin scale

$$\eta_\phi = d = \eta \left(\frac{\gamma}{\nu}\right)^{3/4}$$

In this case, the stretching is dominated by $\eta_\phi$, and the smallest eddy to stretch the feature is the length scale of that feature. In the other case, Kolmogorov scale is the best stretcher, since we want the largest velocity gradient

**Lecture 17. Spectral analysis of homogeneous turbulence**



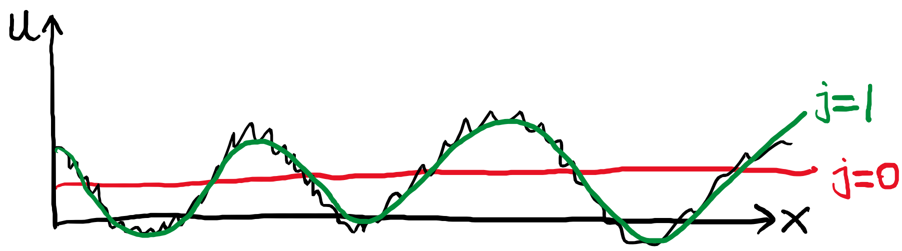




When $\eta$ approaches the mean free path, $u_l$ and thus $u_\eta$ will be comparable to sound speed, there for supersonic effects appear and the entire picture needs to be revisited

- 1D Fourier transform

Quantification of velocity in terms of scales (wavenumbers) with inverse FT

$$u(x) = \int_{-\infty}^{+\infty} \hat{u}(k)e^{ikx}dk \simeq \sum_{j=-\infty}^{+\infty} \hat{u}_j \Delta k e^{ik_j x}, \qquad k_j = j\Delta k$$




For real-valued signal, $\hat{u}(-k) = \hat{u}^*(k)$. The statistical quantity of interest is $\langle|\hat{u}(k)|^2\rangle$. This is because $\langle\hat{u}(k)\rangle = 0$ for HIT

Fourier transform gives the spectral amplitude

$$\hat{u}(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x)e^{-ikx}dx$$

But in practice, we only have finite signals. If we use the following convention

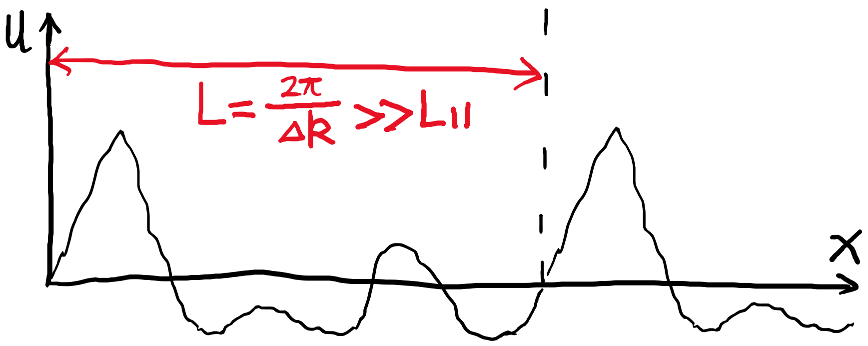
$$\hat{u}_j\Delta k = \frac{1}{L}\int_0^L u(x)e^{-ik_jx}dx$$

the reported value will still depend on $L$. We want a definition independent of box size

- Continuous & discrete transforms

$$u(x) = \sum_{j=-\infty}^{+\infty} \hat{u}_j\Delta k e^{i(j\Delta k)x}$$

$\Delta k$ is the resolution in the $k$ space. Once we select one $\Delta k$, the signal $u(x)$ will be periodic with period of $2\pi/\Delta k$, which should be consistent with box size of simulation, and should be much larger than the integral length $L_{11}$




- Connection between FT and correlation

$$\langle|\hat{u}_j\Delta k|^2\rangle = \langle\hat{u}_j\hat{u}_j^*\rangle(\Delta k)^2 = \frac{1}{L^2}\left\langle\left[\int_0^L u(x)e^{-ik_jx}dx\right]\left[\int_0^L u(x')e^{ik_jx'}dx'\right]\right\rangle$$

$$= \frac{1}{L^2}\int_0^L\int_0^L\langle u(x)u(x')\rangle e^{-ik_j(x-x')}dx'\,dx = \frac{1}{L^2}\int_0^L\int_{-x}^{L-x} R_{uu}(r)e^{-ik_jr}dr\,dx$$

$$= \frac{2\pi}{L}\cdot\frac{1}{2\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} R_{uu}(r)e^{-ik_jr}dr = \frac{2\pi}{L}\cdot\frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{uu}(r)e^{-ik_jr}dr$$

$$= \frac{2\pi}{L}\cdot\mathcal{F}\{R_{uu}(r)\}$$

Therefore, our original convention is box-size dependent with factor $2\pi/L$

- Definition of 1D spectrum

$$E_{uu}(k_x) = \mathcal{F}\{R_{uu}(r)\}$$

This is well defined, box-size independent, but very expensive to compute. It can be manipulated into the following form

$$E_{uu}(k_x) = \frac{L}{2\pi}\langle\left|\hat{u}_j \Delta k\right|^2\rangle = \frac{1}{2\pi L}\langle\left|\int_0^L u(x)e^{-ik_j x}dx\right|^2\rangle$$

In practice, we compute 1D spectrum based on FFT

$$E_{uu}(k_x) = \frac{(\Delta x)^2}{2\pi L}\langle\left|\sum_{j=1}^{N} u(x_j)e^{-ik_x x_j}\right|^2\rangle, \qquad k_x = 0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \dots$$

The factor guarantees that the quantity is independent of mesh and box-size, given that $\Delta x$ resolves small eddy and $L$ is larger than $L_{11}$. FFT parameters are

$$\Delta k = \frac{2\pi}{L}, \qquad \Delta x = \frac{L}{N}, \qquad k_j = j\Delta k, \qquad x_j = j\Delta x, \qquad x_1 = 0, \qquad x_N = L - \Delta x$$

**Lecture 18. Fourier transform in practice**

- Recap Lecture 17: 1D (power) spectrum of $u$, with mesh $\Delta x$ and box $L$

$$E_{uu}(k_x) = \mathcal{F}\{R_{uu}(r)\} = \frac{(\Delta x)^2}{2\pi L}\langle|\mathrm{FFT}\{u'(x)\}|^2\rangle, \qquad k_x = 0, \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \ldots$$

The unit of $E_{uu}$ is $U^2L$

- Properties of Fourier transform
  - Parseval's theorem

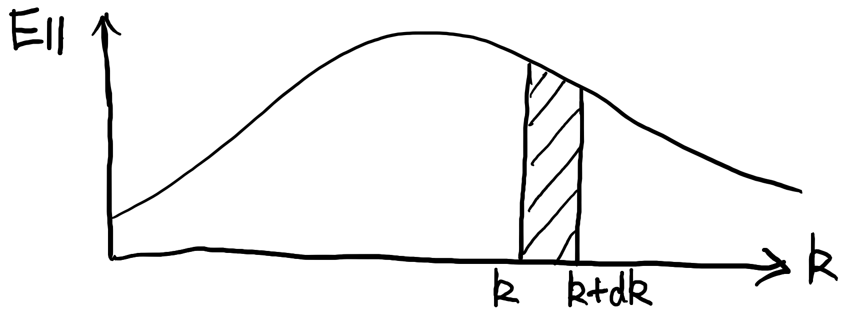
$$\int_{-\infty}^{+\infty} E_{11}(k_1)\, dk_1 = \overline{u'^2}$$

The proof is based on the cross-correlation $R_{11}$ evaluated at $r_1 = 0$

$$R_{11}(r_1) = \mathcal{F}^{-1}\{E_{11}(k_1)\} = \int_{-\infty}^{+\infty} E_{11}(k_1)e^{ik_1r_1}\, dk_1$$

$$R_{11}(0) = \overline{u'^2} = \int_{-\infty}^{+\infty} E_{11}(k_1)\, dk_1 = \frac{2\pi}{L}\sum_{j=-\frac{N}{2}}^{\frac{N}{2}-1} E_{11}(k_j)$$

The spectrum represents how kinetic energy is distributed in wavenumber domain




The area below $E_{11}(k)$ is related to

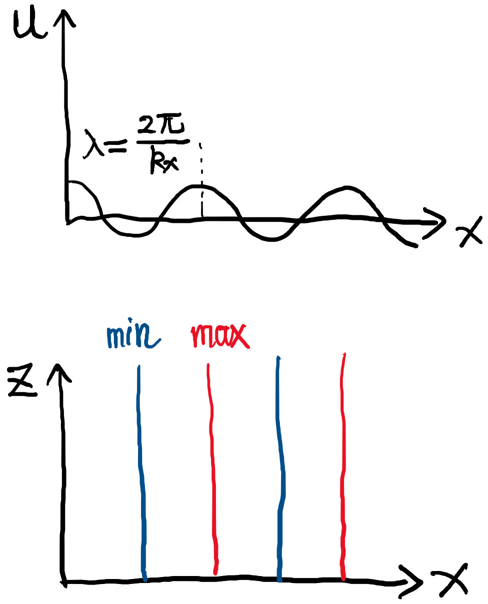
$$\frac{\text{Area}}{2} = \text{Kinetic energy due to scales } [k, k+dk]$$

  - Fourier transform of derivatives

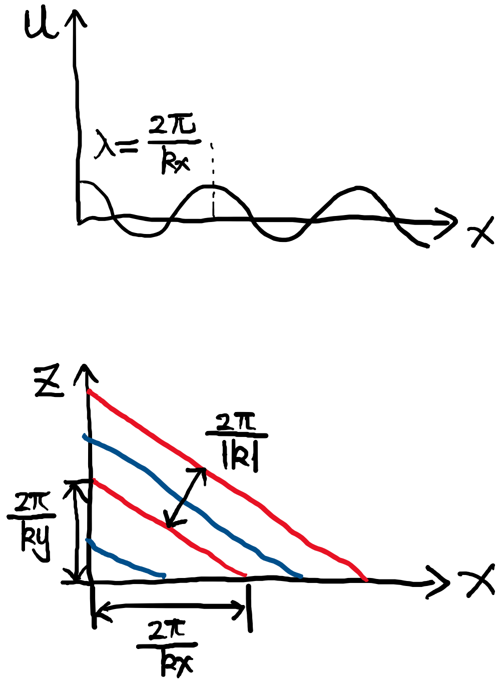
$$u(x) \leftrightarrow \hat{u}(k), \qquad \frac{du}{dx} \leftrightarrow ik\hat{u}$$

- Extension to multi-dimension (vector wavenumber $\boldsymbol{k}$)
  - Example: Channel flow at $y^{+} = 15$ with $u(t, x, z)$

Pure 1D with $\boldsymbol{k} = (k_x, 0)$




2D with $\boldsymbol{k} = (k_x, k_y)$




Direction of the vector wavenumber is normal to the wavefront

- Inverse FT in 2D is

$$u(x, z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{u}(k_x, k_z)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, dk_x\, dk_z$$

- 2D spectrum is calculated by

$$E_{uu}(k_x, k_z) = \mathrm{FT2D}\{R_{uu}(r_x, r_z)\} = \frac{(\Delta x)^2}{2\pi L_x} \cdot \frac{(\Delta z)^2}{2\pi L_z} \langle |\mathrm{FFT2}\{u'(x, z)\}|^2 \rangle$$

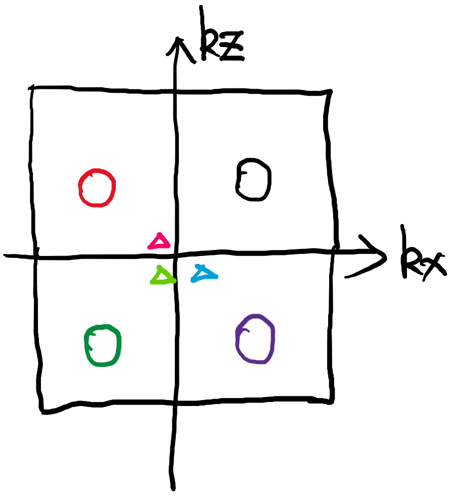
- Fourier transform with MATLAB

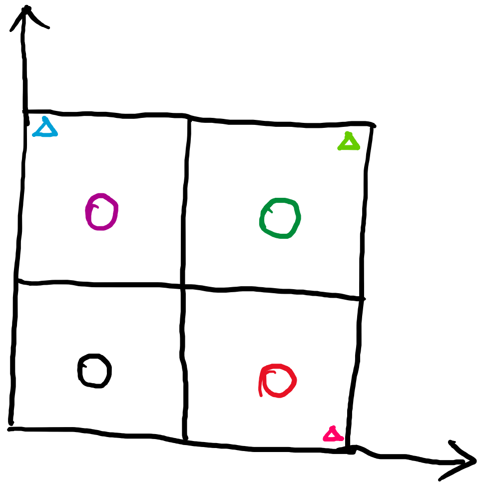
1D array representing $u(x)$, with data $u(0), u(\Delta x), \ldots, u(L - \Delta x)$ and $\Delta x = L/N$

FFT of this array gives

$$\hat{u}(0), \hat{u}(\Delta k), \ldots, \hat{u}\left(\left(\frac{N}{2} - 1\right)\Delta k\right), \hat{u}\left(-\frac{N}{2}\Delta k\right), \ldots, \hat{u}(-\Delta k), \qquad \Delta k = \frac{2\pi}{L}$$

2D FFT storage in MATLAB



**Lecture 19. Kolmogorov hypothesis**

- Recap: Quantification of turbulence in terms of scales

  Resolution requirements for DNS

  Spectral analysis: Validation & detailed comparison between experiments

- Correlation & spectrum tensor

  1D spectrum:

$$E_{11}(k_1) = \mathcal{F}\{R_{11}(r_1)\}$$

  Generalization to 3D homogeneous flows (spectrum tensor):

$$\phi_{ij}(\boldsymbol{k}) = \mathcal{F}^{(3)}\{R_{ij}(\boldsymbol{r})\} = \frac{(\Delta x)^2}{2\pi L_x}\frac{(\Delta y)^2}{2\pi L_y}\frac{(\Delta z)^2}{2\pi L_z}\langle \hat{u}_i' \hat{u}_j'^{*} \rangle$$

  Connection to TKE

$$\text{TKE} = \frac{1}{2}\langle u_i' u_i' \rangle = \frac{1}{2}\iiint_{\boldsymbol{k}} \phi_{ii}(\boldsymbol{k})\, d^3\boldsymbol{k}$$

  Kinetic energy of all structures with wavenumber $[k_i, k_i + dk_i]$ is calculated as

$$\frac{1}{2}\phi_{ii} dk_1 dk_2 dk_3$$

  In isotropic turbulence, spectrum tensor is only function of $|\boldsymbol{k}| = k$

$$\text{TKE} = \frac{1}{2}\langle u_i u_i \rangle = \int_0^{\infty} \phi_{ii}(k)\, 2\pi k^2 dk$$

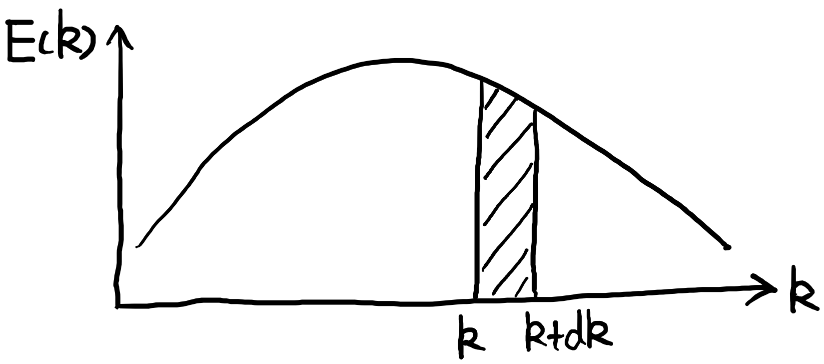
  In general, the Reynolds stress component can be expressed as

$$\langle u_i' u_j' \rangle = \iiint_{\boldsymbol{k}} \phi_{ij}(\boldsymbol{k})\, d^3\boldsymbol{k}$$

- 3D energy spectrum

  For an isotropic flow, the 3D energy spectrum $E(k)$ is

$$E(k) = 2\pi k^2 \phi_{ii}(k), \qquad \text{TKE} = \int_0^{+\infty} E(k) dk$$



Extension to non-isotropic flow: Integrate over the spherical shells in k-space

$$E(k)=\iiint_{\boldsymbol{k}}\frac{1}{2}\phi_{ii}(\boldsymbol{k}')\,\delta(|\boldsymbol{k}'|-k)\,d^3\boldsymbol{k}'$$

$$\simeq\frac{1}{\Delta k}\iiint_{V(k)}\frac{1}{2}\phi_{ii}(\boldsymbol{k}')d^3\boldsymbol{k}',\qquad V(k)=\left\{\boldsymbol{k}'\middle|k-\frac{\Delta k}{2}<|\boldsymbol{k}'|<k+\frac{\Delta k}{2}\right\}$$

Numerically, in the normalized domain ($\Delta k=1$), we compute it as

$$E(k)=\sum_{V(k)}\frac{1}{2}\phi_{ii}(\boldsymbol{k}')$$

- Kolmogorov hypothesis

$$E=E(k,\epsilon,l,\eta)$$

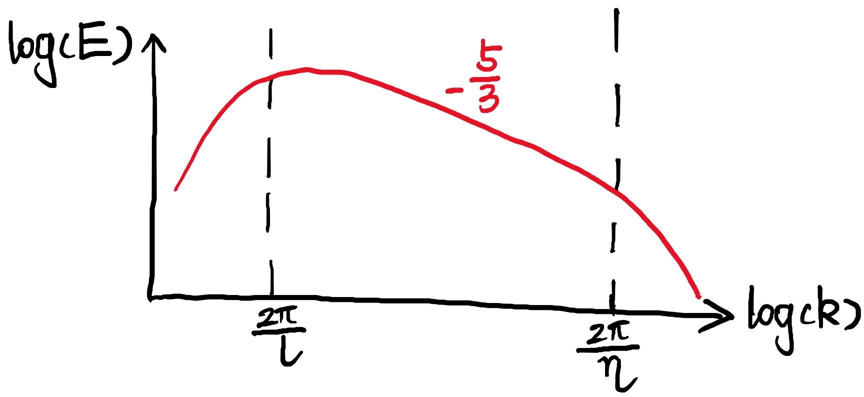
When $k\gg 2\pi/l$, $E=E(k,\epsilon,\eta)$. Similarly, when $k\ll 2\pi/\eta$, $E=E(k,\epsilon,l)$

In high Reynolds number Re $\gg 1$, there exists an overlap zone

$$E=E(k,\epsilon),\qquad \frac{2\pi}{l}\ll k\ll\frac{2\pi}{\eta}$$

Dimensional analysis gives ($E=[L^3T^{-2}], k=[L^{-1}], \epsilon=[L^2T^{-3}]$):

$$E=C\epsilon^{2/3}k^{-5/3},\qquad C\simeq 1.5$$




Connection with velocity scaling relationship

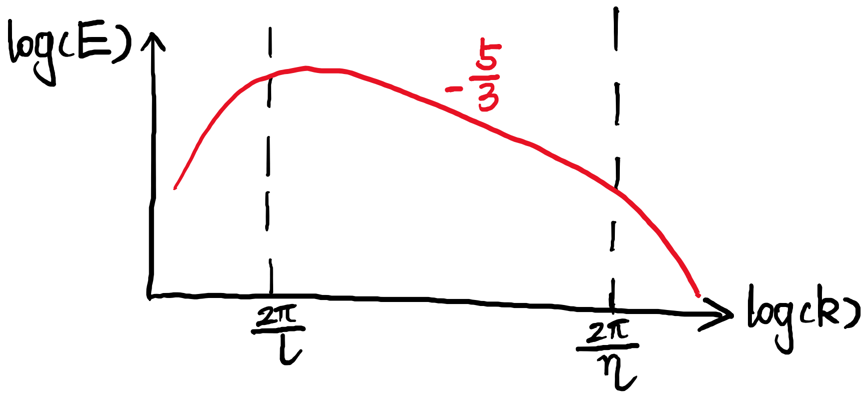
$$\text{TKE}\sim u_d^2\sim\int_{\frac{2\pi}{d}-dk}^{\frac{2\pi}{d}+dk}E(k)dk\sim\int E(k)k\,d(\log k)$$

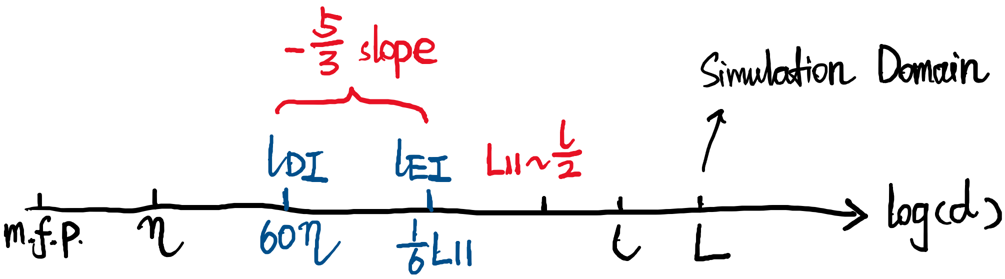
$$u_d^2\sim\epsilon^{2/3}k^{-2/3},\qquad u_d\sim(\epsilon d)^{1/3},\qquad \epsilon\sim\frac{u_d^3}{d}$$

**Lecture 20. Taylor hypothesis**

➢ Recap: 3D energy spectrum & inertial range

Observation of inertial range requires $\eta \ll \lambda \ll l$




The Reynolds number to observe Kolmogorov spectrum should satisfy

$$\frac{1}{6}L_{11} > 60\eta$$

➢ Connection between $E_{11}(k_1)$ and $E(k)$

Experimental quantification of $E(k)$

$$\boldsymbol{u}(x,y,z,t) \rightarrow R_{ii}(\boldsymbol{r}) \rightarrow \phi_{ii}(\boldsymbol{k}) \rightarrow E(k)$$

But this can be hard to calculate. We hope to quantify $E(k)$ only using $u(x,t)$. Start from the definition of energy spectrum and correlation function

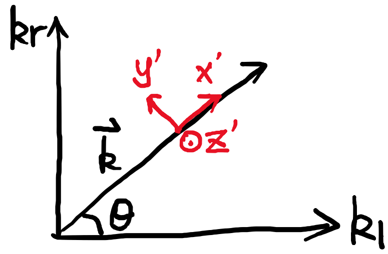
$$E_{11}(k_1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{11}(r_1)\, e^{-ik_1 r_1}\, dr_1\,, \qquad R_{11}(\boldsymbol{r}) = \iiint_V \phi_{11}(\boldsymbol{k})\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, d^3\boldsymbol{k}$$

We can obtain $E_{11}(k_1)$ by integrating $\phi_{11}(\boldsymbol{k})$ over the other two wavenumber components. The exponent does not show up because we select $\boldsymbol{k} = k_1\hat{\boldsymbol{e}}_x$.

With the assumption of isotropic turbulence, we have

$$E_{11}(k_1) = \iint_S \phi_{11}(k_1, k_y, k_z)\, dk_y dk_z = \int_0^{+\infty} \phi_{11}(k_1, k_r, 0)\, 2\pi k_r\, dk_r$$

Therefore, with the pre-factor denoted as $A$, we have




$$E_{11}(k_1) = A \int_0^{+\infty} \langle |\hat{u}_1|^2 \rangle\, 2\pi k_r\, dk_r$$

From continuity equation

$$\frac{\partial u_i}{\partial x_i} = 0, \qquad i k_i \hat{u}_i = i\boldsymbol{k} \cdot \hat{\boldsymbol{u}} = 0$$

We can define a local coordinate system, and $\hat{\boldsymbol{u}}$ is always within $y'$-$z'$ plane. Again based on isotropy, we obtain

$$\phi_{ii}(\boldsymbol{k}) = 2A \left\langle \left|\hat{u}_{y'}\right|^2 \right\rangle$$

$$A\langle |\hat{u}_1|^2 \rangle = A \sin^2\theta \left\langle \left|\hat{u}_{y'}\right|^2 \right\rangle = \frac{\sin^2\theta}{2} \phi_{ii}(\boldsymbol{k}) = \frac{1}{2} \frac{k_r^2}{|\boldsymbol{k}|^2} \phi_{ii}(\boldsymbol{k}) = \left(1 - \frac{k_1^2}{k^2}\right) \frac{E(k)}{4\pi k^2}$$

Therefore, the final expression is

$$E_{11}(k_1) = \int_0^{+\infty} \frac{E(k)}{2k^2} \left(1 - \frac{k_1^2}{k^2}\right) k_r dk_r = \int_{k_1}^{+\infty} \frac{E(k)}{2k} \left(1 - \frac{k_1^2}{k^2}\right) kdk$$

where we use the following relations

$$k^2 = k_r^2 + k_1^2, \qquad kdk = k_r dk_r$$

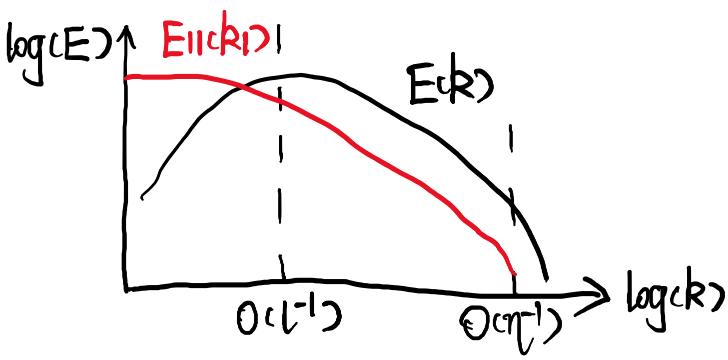
The inverse relation is

$$E(k) = k^3 \frac{d}{dk} \left[\frac{1}{k} \frac{dE_{11}(k)}{dk}\right]$$

Measuring $E_{11}(k_1)$ can predict $E(k)$ for **incompressible and isotropic** turbulence. In the inertial range, we have

$$E(k) \propto k^{-5/3} \quad \Leftrightarrow \quad E_{11}(k_1) \propto k_1^{-5/3}$$

For low wavenumbers, $E_{11}(k_1)$ is higher as it integrates over $k_r$ (higher $k$ components)



- Taylor's hypothesis

Turbulence can be approximately viewed as **frozen structure** convected past a sensor

$$R_{11}(\tau) = R_{11}(\tau V_C)$$

where $V_C$ is the convective velocity ("mean flow")

Spectrum in time can be directly obtained from $E_{11}(k_1)$

$$E_{11}(\omega) = \frac{(\Delta t)^2}{2\pi T}\langle|\hat{u}(\omega)|^2\rangle = \mathcal{F}\{R_{11}(\tau)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{11}(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{11}(r_1 = \tau V_C)\, e^{-i\omega\tau}\, d\tau = \frac{1}{V_C}E_{11}\left(k_1 = \frac{\omega}{V_C}\right)$$

Vice versa, we have

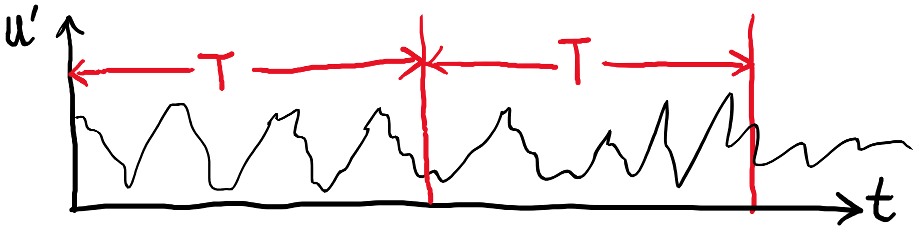
$$E_{11}(k_1) = V_C\, E_{11}(\omega = k_1 V_C)$$

- Spectral analysis for non-homogeneous flows

With isotropy for small scales and local homogeneity, we can compute $E_{11}(\omega)$ and thus $E_{11}(k_1)$ using Taylor's hypothesis. Then we can obtain $E(k)$ based on the relation between the two spectra. In the inertial range, we can then compute $\epsilon, \nu$ and $\eta$.

The computation of $E_{11}(\omega)$ uses raw data, and does not involve any assumption.

- Spectral analysis for finite-length signal




Use windows to perform averaging, with window much longer than correlation time, and time step smaller to Kolmogorov time

$$T \gg T_{11}, \qquad \Delta t \le t_\eta$$

Use window functions (e.g. Hanning window) before FFT to taper the signal

$$u'_{\text{new}}(t) = u'_{\text{raw}}(t) \cdot \frac{1}{2}\left[1 - \cos\left(\frac{2\pi t}{T}\right)\right] \cdot \sqrt{\frac{8}{3}}$$

- Energy spectrum for pressure

$$p_d \propto u_d^2 \propto d^{2/3}, \qquad E(p) \propto \frac{p_d^2}{k} \propto k^{-7/3}$$

Similar dimensional analysis can give the same result. Consider the following relation

$$E(p) \sim \rho^2 \epsilon^\alpha k^\beta$$

Given the unit of relevant physical quantities

$$E(p) = [\rho^2 U^4 L] = [\rho^2 L^5 T^{-4}], \qquad k = [L^{-1}], \qquad \epsilon = [L^2 T^{-3}]$$

we can obtain

$$-3\alpha = -4, \qquad 2\alpha - \beta = 5, \qquad \alpha = \frac{4}{3}, \qquad \beta = -\frac{7}{3}$$

Therefore, the energy spectrum for pressure scales as

$$E(p) \sim \rho^2 \epsilon^{4/3} k^{-7/3}$$

**Lecture 21. Dynamics in spectral space**

➢ Navier-Stokes equation in wavenumber domain

$$\frac{\partial u_i}{\partial t}+\frac{\partial(u_j u_i)}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\nabla^2 u_i, \qquad \frac{\partial}{\partial t}\hat{u}_i+ik_j(\hat{u}_i*\hat{u}_j)=-\frac{ik_i}{\rho}\hat{p}-\nu|\boldsymbol{k}|^2\hat{u}_i$$

With the convolution

$$\widehat{u_i u_j}=\hat{u}_i*\hat{u}_j=\int_{V'}\hat{u}_i(\boldsymbol{k}')\,\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}')\,d^3\boldsymbol{k}'$$

The proof is

$$u_i u_j=\left(\int_{V'}\hat{u}_i\,e^{i\boldsymbol{k}'\cdot\boldsymbol{r}}d^3\boldsymbol{k}'\right)\left(\int_{V''}\hat{u}_j\,e^{i\boldsymbol{k}''\cdot\boldsymbol{r}}d^3\boldsymbol{k}''\right)=\int_{V'}\int_{V''}\hat{u}_i\hat{u}_j\,e^{i(\boldsymbol{k}'+\boldsymbol{k}'')\cdot\boldsymbol{r}}d^3\boldsymbol{k}'d^3\boldsymbol{k}''$$

$$=\int_V\int_{V'}\hat{u}_i(\boldsymbol{k}')\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}')\,e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\,d^3\boldsymbol{k}'d^3\boldsymbol{k}=\mathcal{F}^{-1}\left\{\int_{V'}\hat{u}_i(\boldsymbol{k}')\hat{u}_j(\boldsymbol{k}-\boldsymbol{k}')d^3\boldsymbol{k}'\right\}$$

Together with the governing equation for $\hat{u}_i^*$

$$\frac{\partial}{\partial t}\hat{u}_i+\nu|\boldsymbol{k}|^2\hat{u}_i=-ik_j\left(\hat{u}_i*\hat{u}_j\right)-\frac{ik_i}{\rho}\hat{p}$$
$$\frac{\partial}{\partial t}\hat{u}_i^*+\nu|\boldsymbol{k}|^2\hat{u}_i^*=ik_j\left(\hat{u}_i*\hat{u}_j\right)^*+\frac{ik_i}{\rho}\hat{p}^*$$

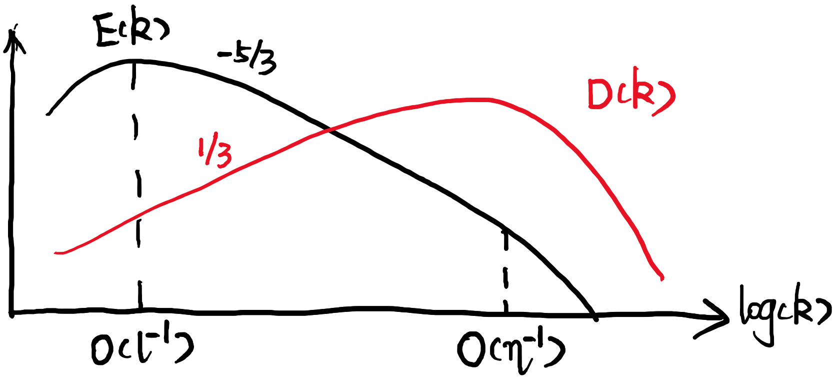
Cross-multiplication, summation, and Reynolds-averaging gives (with continuity $k_i\hat{u}_i=0$)

$$\frac{\partial}{\partial t}\langle\hat{u}_i\hat{u}_i^*\rangle+2\nu|\boldsymbol{k}|^2\langle\hat{u}_i\hat{u}_i^*\rangle=ik_j\langle\hat{u}_i\left(\hat{u}_i*\hat{u}_j\right)^*-\hat{u}_i^*\left(\hat{u}_i*\hat{u}_j\right)\rangle$$

Multiply by the pre-factor for spectrum and integrate over spherical shell gives

$$\frac{\partial}{\partial t}E(k)+2\nu|\boldsymbol{k}|^2E(k)=T(k)$$

The dissipation spectrum is $D(k)=2\nu|\boldsymbol{k}|^2E(k)$, and $T(k)$ is the energy transfer term




Connection with physical space

$$\int E(k)dk=\text{TKE},\qquad \int D(k)dk=\epsilon$$

For HIT, we can prove it as

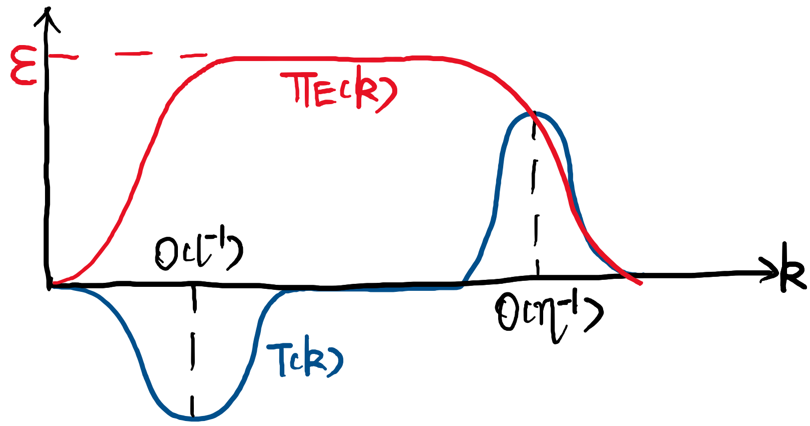
$$\epsilon = \nu\langle\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\rangle = A\iiint_{\boldsymbol{k}} \nu|\boldsymbol{k}|^2\langle\hat{u}_i\hat{u}_i^*\rangle d^3\boldsymbol{k} = \iiint_{\boldsymbol{k}} 2\nu|\boldsymbol{k}|^2\frac{\phi_{ii}(\boldsymbol{k})}{2}d^3\boldsymbol{k} = \int 2\nu|\boldsymbol{k}|^2 E(k)dk$$

Integrate the governing equation of $E(k)$ over the wavenumber domain gives

$$\frac{\partial \mathrm{TKE}}{\partial t} = -\epsilon$$

The integral of $T(k)$ over wavenumber domain is zero. It has a strong sink at large scale and a strong gain at Kolmogorov scale. The transfer $T(k)$ can be written as a divergence of flux

$$\Pi_E(k) = -\int_0^k T(k')dk', \qquad T(k) = -\frac{\partial \Pi_E(k)}{\partial k}$$



**Lecture 22. k-ε model**

➢ Recap Lecture 9

RANS equation with Boussinesq approximation

$$\frac{\bar{D}\bar{u}_i}{Dt} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[2(\nu + \nu_T)\bar{S}_{ij}\right], \qquad -\overline{u_i' u_j'} = 2\nu_T \bar{S}_{ij} - \frac{1}{3}\overline{u_k' u_k'}\delta_{ij}$$

Turbulence models try to give an expression of $\nu_T = \nu_T(\boldsymbol{x}, t)$

➢ k-ε model (Jones & Launder 1972, Launder & Sharma 1974)

From mixing length model $\nu_T \sim u' l$, express these quantities with observations

$$k = \text{TKE} \sim u'^2, \qquad \epsilon \sim \frac{u'^3}{l}, \qquad \nu_T = C_\mu \frac{k^2}{\epsilon}, \qquad C_\mu = 0.09$$

◆ TKE equation (k-equation)

$$\frac{\bar{D}k}{Dt} = \frac{\partial}{\partial x_j}\langle -\frac{p' u_j'}{\rho} - \frac{u_j' u_i' u_i'}{2} + 2\nu\, u_i' S_{ij}'\rangle + P - \epsilon, \qquad P = -\overline{u_i' u_j'} \cdot \bar{S}_{ij}$$

The diffusion flux can be written as

$$2\nu\, u_i' S_{ij}' = \nu \frac{\partial}{\partial x_j}\left(\frac{u_i' u_i'}{2}\right) + \nu\, u_i' \frac{\partial u_j'}{\partial x_i}$$

The second term is small compared with $\epsilon$, so the model neglect it. Final k-equation is

$$\frac{\bar{D}k}{Dt} = \boldsymbol{\nabla} \cdot \left(\left[\nu + \frac{\nu_T}{\sigma_k}\right] \boldsymbol{\nabla} k\right) + P - \epsilon, \qquad \sigma_k = 1$$

$\sigma_k$ denotes ratio between turbulent momentum and TKE mixing, which is suggested as 1

◆ Dissipation equation (ε-equation)

$$\frac{\bar{D}\epsilon}{Dt} = \text{Turbulent transport} + \text{Production of } \epsilon - \text{Dissipation of } \epsilon$$

From empirical comparison with k-equation, and dimensional analysis

$$\frac{\bar{D}\epsilon}{Dt} = \boldsymbol{\nabla} \cdot \left(\left[\nu + \frac{\nu_T}{\sigma_\epsilon}\right] \boldsymbol{\nabla} \epsilon\right) + C_1 \frac{P\epsilon}{k} - C_2 \frac{\epsilon^2}{k}$$

With the following constants

$$C_\mu = 0.09, \qquad \sigma_k = 1, \qquad \sigma_\epsilon = 1.3, \qquad C_1 = 1.44, \qquad C_2 = 1.92$$

- Homogeneous & isotropic turbulence

$$k = A \cdot t^{-n}, \qquad 1.15 < n < 1.45, \qquad n \sim 1.3$$

k-ε model for HIT is (using zero mean quantities)

$$\frac{dk}{dt} = -\epsilon, \qquad \frac{d\epsilon}{dt} = -C_2\frac{\epsilon^2}{k}$$

Model predictions are thus

$$\epsilon = An \cdot t^{-n-1}, \qquad C_2 = \frac{n+1}{n}, \qquad n = \frac{1}{C_2 - 1} = 1.08$$

k-ε model results in a slightly slower decaying HIT than typical observation

- Homogeneous shear flow

$$\frac{P}{\epsilon} = 1.7, \qquad \langle u'v' \rangle = -0.28k, \qquad S = \frac{\partial \bar{u}}{\partial y} = \text{const.}, \qquad P \sim \exp{(0.12St)}$$

k-ε model for homogeneous shear flow is

$$P = C_\mu \frac{k^2}{\epsilon}S^2, \qquad \frac{dk}{dt} = C_\mu \frac{k^2 S^2}{\epsilon} - \epsilon, \qquad \frac{d\epsilon}{dt} = C_1 C_\mu k S^2 - C_2 \frac{\epsilon^2}{k}$$

Consider a test solution $k = k_0 e^{\alpha t}$ and $\epsilon = \epsilon_0 e^{\alpha t}$

$$\frac{\alpha}{S} = C_\mu \frac{k_0 S}{\epsilon_0} - \frac{\epsilon_0}{k_0 S}, \qquad \frac{\alpha}{S} = C_1 C_\mu \frac{k_0 S}{\epsilon_0} - C_2 \frac{\epsilon_0}{k_0 S}$$

Model predictions are thus

$$C_\mu \left(\frac{k_0 S}{\epsilon_0}\right)^2 (C_1 - 1) = C_2 - 1, \qquad \frac{P}{\epsilon} = \frac{C_2 - 1}{C_1 - 1} = 2.1 \neq 1.7, \qquad \frac{\alpha}{S} = 0.23 \neq 0.12$$

- Log-law in wall-bounded flows

$$\frac{\overline{u'v'}}{k} = -0.28, \qquad \frac{P}{\epsilon} = 0.9$$

For k-ε model to satisfy the reported values

$$\overline{u'v'} = C_\mu \frac{k^2}{\epsilon} S, \qquad \left(\frac{\overline{u'v'}}{k}\right)^2 = C_\mu \frac{P}{\epsilon}, \qquad C_\mu = 0.09$$

This is the same as the suggested value of $C_\mu$. The implied Karman's constant is 0.43

For experiment and DNS data, we have Karman's constant $\kappa$ (might not be constant in reality) and the following scaling

$$S \propto y^{-1}, \qquad \epsilon \propto y^{-1}, \qquad P \propto y^{-1}$$

k-ε model for log-layer

$$P = \epsilon, \qquad \frac{\partial}{\partial y}\left(\frac{\nu_T}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial y}\right) + C_1\frac{P\epsilon}{k} - C_2\frac{\epsilon^2}{k} = 0$$

These equations satisfy all the reported scaling relations above

- Major issues of k-ε model

1. Realizability issue

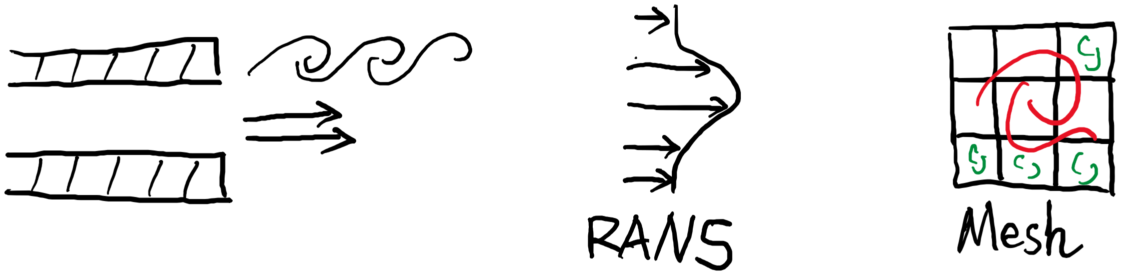
$$\frac{\langle u_1' u_2' \rangle^2}{\langle u_1' u_1' \rangle \langle u_2' u_2' \rangle} < 1$$

2. Problematic prediction of turbulent B.L. separation over smooth surfaces
3. Need damping of diffusion and production for buffer layer

- Boundary conditions of k-ε model

On the wall, $k$ should be 0. Near the wall surface, special treatment (damping) is needed to obtain finite $\epsilon$ without divergence of dissipation term $C_2\epsilon^2/k$

**Lecture 23. Large eddy simulation (LES)**




➢ Idea of LES

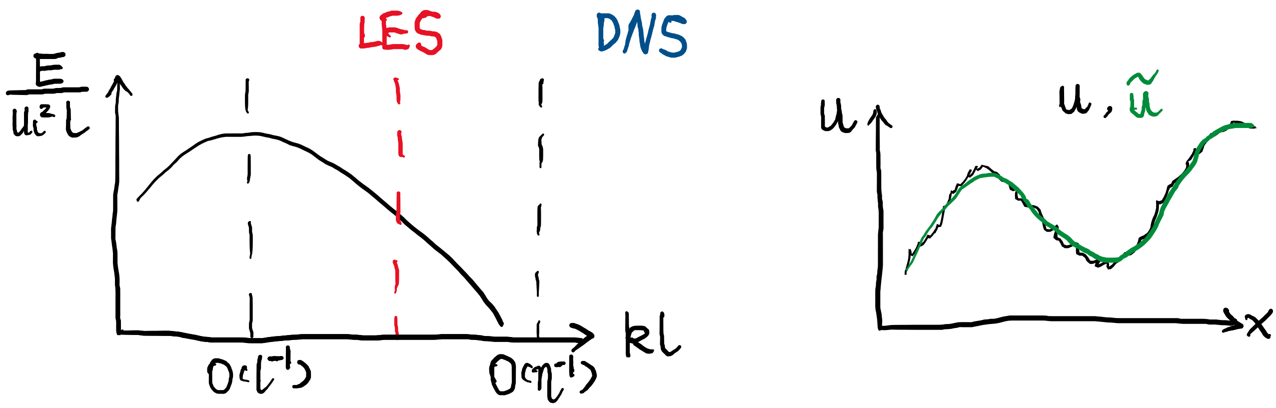
Resolve large eddies directly, and model the effects of unresolved eddies (subgrid-scale model, SGS model)

In wavenumber space, we have DNS mesh $\sim \eta$, while for LES, we will choose

$$\eta < \Delta x_{LES} \ll l$$

Energy containing eddies are resolved by LES, dissipation is not resolved.

When increasing Reynolds number by 1000 times, DNS needs to have 1000 times finer mesh, but for LES to obtain same percentage of resolved TKE, we barely need to change the mesh size, as the contribution of TKE at Kolmogorov scales is diminishingly small. However, we lose the process of dissipation and cannot resolve $D(k)$




➢ Filter operator

$$\tilde{u}(x) = \int_V u(\boldsymbol{x}')\, G(\boldsymbol{x} - \boldsymbol{x}')\, d^3\boldsymbol{x}', \qquad \hat{\tilde{u}}(k) = \hat{u}(k) \cdot \hat{G}(k)$$

Kernel $G(\boldsymbol{x}' - \boldsymbol{x})$ can be a Gaussian, and the width is the filter width $\sim \Delta$

LES criterion: To capture 80% of TKE, the mesh size should be

$$\Delta(\text{LES}) \simeq \frac{1}{12} l = \frac{1}{6} L_{11}$$

This criterion is independent of $\eta$ or Re. For example, with $l = 2$ cm and $\eta = 100\ \mu$m

$$\frac{l}{\eta} = 200, \qquad \Delta(\text{LES}) \sim \frac{l}{10} = 2\ \text{mm}, \qquad \Delta(\text{DNS}) = 100\ \mu\text{m}$$

The mesh saving is $20^3 = 8000$

- LES equations

Start by filtering Navier-Stokes equations with $u_i = \tilde{u}_i + u_i'$

$$\frac{\partial}{\partial t}\tilde{u}_i + \frac{\partial}{\partial x_j}\tilde{u}_j\tilde{u}_i = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu\nabla^2\tilde{u}_i - \frac{\partial}{\partial x_j}\left(\tilde{u}_j\tilde{u}_i - \widetilde{u_j u_i}\right), \qquad \frac{\partial \tilde{u}_i}{\partial x_i} = 0$$

Note that unlike Reynolds-averaging, we now have

$$\tilde{u}_j\tilde{u}_i - \widetilde{u_j u_i} \neq \widetilde{u_j' u_i'}$$

The final divergence term physically represents the effects of unresolved eddies on filtered momentum transport (mixing)

- Smagorinsky model (1963, MWR)

Effect of small eddies → Mixing → Diffusion of momentum

LES viscosity should scale as

$$u_\Delta \sim u_l \left(\frac{\Delta}{l}\right)^{\frac{1}{3}}, \qquad \nu_{LES} \sim \Delta \cdot u_\Delta \sim \Delta^2\left|\tilde{S}\right|, \qquad \nu_{LES} = C_S^2\Delta^2\left|\tilde{S}\right|$$

This expression is analogous to Prandtl mixing length model, but applied at LES grid scale

- Advantage of LES over RANS

1. Most of TKE is directly resolved by LES. For RANS none is resolved, but modelled
2. Effect of model error is smaller, confined to smallest scales. Smallest scale eddies are more isotropic, and it is more likely to have isotropic mixing at small scales. Boussinesq type models are thus appropriate

- Parameter $C_S$ for LES

  For free shear flows, $C_S = 0.1 \sim 1$

  Near the wall, $C_S = 0$ as $u' \to 0$. Damping is applied to $C_S$ near the wall

- Dynamic Smagorinsky model (Germano et al., Physics of Fluids, 1991)

  The quantity we want to model is

$$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j = -2\nu_{LES} \tilde{S}_{ij}$$

Introduce a filter of filter (coarser filter) with $\widetilde{\Delta} > \Delta$ (which is often $\widetilde{\Delta} = 2\Delta$)

$$T_{ij} = \widetilde{\widetilde{u_i u_j}} - \tilde{\tilde{u}}_i \tilde{\tilde{u}}_j = -2C_S^2 \, \widetilde{\Delta}^2 \left| \tilde{\tilde{S}} \right| \tilde{\tilde{S}}_{ij}$$

Criterion to choose $C_S$ is to satisfy

$$L_{ij} = T_{ij} - \tilde{\tau}_{ij} = \widetilde{\tilde{u}_i \tilde{u}_j} - \tilde{\tilde{u}}_i \tilde{\tilde{u}}_j = -2C_S^2 \left[ \widetilde{\Delta}^2 \left| \tilde{\tilde{S}} \right| \tilde{\tilde{S}}_{ij} - \Delta^2 \widetilde{\left| \tilde{S} \right| \tilde{S}_{ij}} \right]$$

Here we have 6 equations and 1 unknown, Lilly (1992) use least squares to choose $C_S$ which is spatially dependent. Ensemble averaging or selection based on homogeneous directions can be applied further.