

MIT Integration Bee: 2023 Semifinal

Semifinal #1

Question 1

$$\int e^{\cos x} \cos (2x + \sin x) \, dx \tag{1.1}$$

Solution

$$\begin{aligned} I &= \int e^{\cos x} \cos x \cos (x + \sin x) \, dx + \int \sin (x + \sin x) \, d(e^{\cos x}) \\ &= e^{\cos x} \sin (x + \sin x) - \int e^{\cos x} \cos (x + \sin x) \, dx \\ &= e^{\cos x} \sin (x + \sin x) - \int e^{\cos x} \cos x \cos (\sin x) \, dx - \int \sin (\sin x) \, d(e^{\cos x}) \\ &= e^{\cos x} \sin (x + \sin x) - e^{\cos x} \sin (\sin x) + C. \end{aligned} \tag{1.2}$$

Question 2

$$\int_0^1 \left(9x^9 - x^{90} + 9x^{99} - x^{900} + 9x^{909} - x^{990} + 9x^{999} - x^{9000} + \dots \right) dx \quad (2.1)$$

Solution The contribution from the dominant terms in the integrand is

$$I_1 = \int_0^1 \left(9x^9 + 9x^{99} + 9x^{999} + 9x^{9999} + \dots \right) dx = \sum_{k=1}^{\infty} \frac{9}{10^k} = 1. \quad (2.2)$$

The remaining part becomes

$$I_2 = \int_0^1 \left(-x^{90} - x^{900} + 9x^{909} - x^{990} - x^{9000} + 9x^{9009} - x^{9090} + 9x^{9099} - x^{9900} + 9x^{9909} - x^{9990} - \dots \right) dx. \quad (2.3)$$

This can be considered an alternating series that eventually sums to zero. Therefore, we can say

$$I = I_1 + I_2 = 1. \quad (2.4)$$

Question 3

$$\int_0^\pi \frac{2 \cos x - \cos (2021x) - 2 \cos (2022x) - \cos (2023x) + 2}{1 - \cos 2x} dx \quad (3.1)$$

Solution Using the trigonometric identity, we have

$$\cos (2021x) + \cos (2023x) = 2 \cos (2022x) \cos x. \quad (3.2)$$

Therefore, the integral becomes

$$I = \int_0^\pi \frac{2(1 + \cos x)[1 - \cos (2022x)]}{2(1 - \cos^2 x)} dx = \int_0^\pi \frac{1 - \cos (2022x)}{1 - \cos x} dx. \quad (3.3)$$

Denote the general result as

$$I_n = \int_0^\pi \frac{1 - \cos nx}{1 - \cos x} dx, \quad n \in \mathbb{N}. \quad (3.4)$$

Obviously we have $I_0 = 0$ and $I_1 = \pi$. The **reduction formula** can be obtained as

$$\begin{aligned} I_{n+1} &= \int_0^\pi \frac{1 - \cos (n+1)x}{1 - \cos x} dx = \int_0^\pi \frac{1 + \cos (n-1)x - 2 \cos nx \cos x}{1 - \cos x} dx \\ &= -I_{n-1} + \int_0^\pi \frac{2 - 2 \cos nx + 2 \cos nx - 2 \cos nx \cos x}{1 - \cos x} dx \\ &= 2I_n - I_{n-1} + \int_0^\pi 2 \cos nx dx = 2I_n - I_{n-1} \quad \text{for } n \geq 1. \end{aligned} \quad (3.5)$$

This implies that $\{I_n\}$ is an arithmetic sequence, and thus $I_n = n\pi$. Finally, we have

$$I = I_{2022} = 2022\pi. \quad (3.6)$$

Question 4

$$\int \frac{3 \ln x - 1 + 2x}{x \ln x + x^2 + 2x^4} dx \quad (4.1)$$

Solution Note that

$$\frac{d}{dx} \ln(\ln x + x + 2x^3) = \frac{1 + x + 6x^3}{x(\ln x + x + 2x^3)}, \quad \frac{1}{x} = \frac{\ln x + x + 2x^3}{x(\ln x + x + 2x^3)}. \quad (4.2)$$

The integrand can be written as

$$\frac{3 \ln x - 1 + 2x}{x \ln x + x^2 + 2x^4} = \frac{3}{x} - \frac{1 + x + 6x^3}{x(\ln x + x + 2x^3)}. \quad (4.3)$$

Therefore, the integral is calculated as

$$I = 3 \ln x - \ln(\ln x + x + 2x^3) + C. \quad (4.4)$$

Tiebreakers Question 1

$$\int_0^{\ln 2} \left\{ \frac{1}{e^x - 1} \right\} dx \quad (5.1)$$

Solution With the **change of variable**, we have

$$t = \frac{1}{e^x - 1}, \quad x = \ln \left(\frac{t+1}{t} \right), \quad dx = -\frac{dt}{t(t+1)}. \quad (5.2)$$

Therefore, the integral becomes

$$I = \int_1^{+\infty} \frac{\{t\} dt}{t(t+1)} = \sum_{n=1}^{+\infty} \int_0^1 \frac{u du}{(u+n)(u+n+1)}. \quad (5.3)$$

Each single term can be evaluated as

$$\begin{aligned} \int_0^1 \frac{u du}{(u+n)(u+n+1)} &= \int_0^1 \left(\frac{n+1}{u+n+1} - \frac{n}{u+n} \right) du \\ &= (n+1) [\ln(n+2) - \ln(n+1)] - n [\ln(n+1) - \ln n] \\ &= (n+1) \ln(n+2) - n \ln(n+1) + n \ln n - (n+1) \ln(n+1). \end{aligned} \quad (5.4)$$

Now denote the following two series

$$a_n = n \ln(n+1), \quad b_n = n \ln n, \quad n = 1, 2, 3, \dots. \quad (5.5)$$

The integral can thus be calculated as

$$\begin{aligned} I &= \sum_{n=1}^{+\infty} (a_{n+1} - a_n + b_n - b_{n+1}) = b_1 - a_1 + \lim_{n \rightarrow \infty} (a_n - b_n) \\ &= -\ln 2 + \lim_{n \rightarrow \infty} \ln \left(1 + \frac{1}{n} \right)^n = 1 - \ln 2. \end{aligned} \quad (5.6)$$

Tiebreakers Question 2

$$\int_1^9 \left(\sqrt[3]{x + \sqrt{x^2 - 1}} + \sqrt[3]{x - \sqrt{x^2 - 1}} \right) dx \quad (6.1)$$

Solution Similar approach to [2024 Final: Question 4](#). Denote

$$a(x) = \sqrt[3]{x + \sqrt{x^2 - 1}}, \quad b(x) = \sqrt[3]{x - \sqrt{x^2 - 1}}, \quad y(x) = a(x) + b(x). \quad (6.2)$$

Now we can show that

$$a^3 + b^3 = 2x, \quad ab = 1 \quad \implies \quad y^3 = a^3 + b^3 + 3ab(a + b) = 2x + 3y. \quad (6.3)$$

Using **integration by parts**, we obtain

$$I = \int_1^9 y(x) dx = xy \Big|_{x=1}^{x=9} - \int_{y(1)}^{y(9)} x(y) dy. \quad (6.4)$$

Since we have

$$y^3 = 2x + 3y \quad \implies \quad y(1) = 2, \quad y(9) = 3, \quad x(y) = \frac{1}{2}y^3 - \frac{3}{2}y, \quad (6.5)$$

we finally obtain

$$I = 25 - \frac{1}{2} \int_2^3 (y^3 - 3y) dy = 25 - \frac{35}{8} = \frac{165}{8}. \quad (6.6)$$

Semifinal #2

Question 1

$$\int \left(\sqrt{x+1} - \sqrt{x} \right)^{\pi} dx \tag{7.1}$$

Question 2

$$\int_{-2}^2 (((x^2 - 2)^2 - 2)^2 - 2) \, dx \tag{8.1}$$

Question 3

$$\int_0^{+\infty} \frac{\tanh x}{x \cosh 2x} dx \tag{9.1}$$

Question 4

$$\int \sin (4 \arctan x) \, dx \tag{10.1}$$