MIT Integration Bee: 2026 Qualifying Round

Question 1

$$\int_{-\pi}^{\pi} \sin^{2025}(x)\cos^{2026}(x)\,\mathrm{d}x = 0. \tag{1}$$

Question 2

$$\int e^{2026e^x + x}\,\mathrm{d}x = \frac{1}{2026}e^{2026e^x} + C. \tag{2}$$

Question 3

$$\int_0^{2026} \left\{\frac{\lfloor x \rfloor}{3}\right\}\mathrm{d}x = \sum_{k=0}^{674}\left(0 + \frac{1}{3} + \frac{2}{3}\right) + 0 = 675. \tag{3}$$

Question 4

$$\int_{-1}^{1} \underbrace{|x + |x + |\cdots + |x + |x||\cdots|||}_{2026\,x}\,\mathrm{d}x \tag{4}$$

Solution Denote $f_1(x) = x$, and note that

$$f_2(x) = x + |f_1(x)| = \begin{cases} 0, & x < 0, \\ 2x, & x \geq 0, \end{cases} \qquad f_3(x) = x + |f_2(x)| = \begin{cases} x, & x < 0, \\ 3x, & x \geq 0, \end{cases}$$
$$f_4(x) = x + |f_3(x)| = \begin{cases} 0, & x < 0, \\ 4x, & x \geq 0, \end{cases} \qquad f_5(x) = x + |f_4(x)| = \begin{cases} x, & x < 0, \\ 5x, & x \geq 0, \end{cases} \tag{5}$$

Therefore, we have

$$I_{2026} = \int_{-1}^{1} |f_{2026}(x)| = \int_0^1 2026x\,\mathrm{d}x = 1013. \tag{6}$$

Question 5

$$\int \frac{\mathrm{d}x}{\sqrt{x+1} - \sqrt{x-1}} = \frac{1}{3}(x+1)^{3/2} + \frac{1}{3}(x-1)^{3/2} + C. \tag{7}$$

Question 6

$$\int \sqrt{1+\cosh x}\,\mathrm{d}x = \int \sqrt{2\cosh^2\left(\frac{x}{2}\right)}\,\mathrm{d}x = 2\sqrt{2}\sinh\left(\frac{x}{2}\right) + C. \tag{8}$$

Question 7

$$\int \frac{2^{\ln x}}{x^2}\,\mathrm{d}x \tag{9}$$

Solution Note that

$$\left(2^{\ln x}\right)' = 2^{\ln x}\cdot\frac{\ln 2}{x}, \qquad \left(\frac{2^{\ln x}}{x}\right)' = \frac{2^{\ln x}}{x^2}\left(\ln 2 - 1\right). \tag{10}$$

Therefore, we have

$$\int \frac{2^{\ln x}}{x^2}\,\mathrm{d}x = \frac{2^{\ln x}}{x\left(\ln 2 - 1\right)} + C = \frac{x^{\ln 2 - 1}}{\ln 2 - 1} + C. \tag{11}$$

This implies a better way to calculate the integral using the fact $a^{\ln b} = b^{\ln a}$, which gives

$$\int \frac{2^{\ln x}}{x^2}\,\mathrm{d}x = \int x^{\ln 2 - 2}\,\mathrm{d}x = \frac{x^{\ln 2 - 1}}{\ln 2 - 1} + C. \tag{12}$$

Question 8

$$\int_0^{1/2}\left(\sum_{n=2}^{\infty} x^n\right)\mathrm{d}x = \int_0^{1/2}\left(\frac{1}{1-x} - 1 - x\right)\mathrm{d}x = \ln 2 - \frac{5}{8}. \tag{13}$$

Question 9

$$\int x^2 \sin x\,\mathrm{d}x = \left(2 - x^2\right)\cos x + 2x\sin x + C. \tag{14}$$

Question 10

$$\int \frac{(x-1)^2}{2e^x + x^2 + 1}\,\mathrm{d}x = \int \left(1 - \frac{2e^x + 2x}{2e^x + x^2 + 1}\right)\mathrm{d}x = x - \ln\left(2e^x + x^2 + 1\right) + C. \tag{15}$$

Question 11

$$\int_{-1}^{1} \max\left\{0, \sqrt{1-x^2}-\frac{1}{2}\right\} \mathrm{d}x = 2\int_{0}^{\sqrt{3}/2} \left(\sqrt{1-x^2}-\frac{1}{2}\right) \mathrm{d}x = \frac{\pi}{3}-\frac{\sqrt{3}}{4}. \tag{16}$$

Question 12

$$\int_0^1 \sqrt{x^2+x+\sqrt{x^2+x+\sqrt{\cdots}}}\, \mathrm{d}x \tag{17}$$

Solution The integrand can be solved as

$$f^2 = x^2+x+f, \qquad f(x) = x+1. \tag{18}$$

We discard the other solution $f(x) = -x$ since we need $f(x) \geq 0$. Therefore, we have

$$I = \int_0^1 (x+1)\, \mathrm{d}x = \frac{3}{2}. \tag{19}$$

Question 13

$$\int \left(\cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x\right) \mathrm{d}x \tag{20}$$

Solution The integrand is equivalent to

$$f(x) = \cos x \left[\left(1-\sin^2 x\right)^2 - 10\left(1-\sin^2 x\right)\sin^2 x + 5\sin^4 x\right]. \tag{21}$$

With a **change of variable** $t = \sin x$, we have

$$\begin{aligned} I &= \int \left[\left(1-t^2\right)^2 - 10\left(1-t^2\right)t^2 + 5t^4\right] \mathrm{d}t && (22) \\ &= \frac{1}{5}\left(16\sin^5 x - 20\sin^3 x + 5\sin x\right) + C = \frac{1}{5}\sin 5x + C. && (23) \end{aligned}$$

Note One can also directly use the following results

$$\begin{aligned} \cos 5x + i\sin 5x = \left(e^{ix}\right)^5 &= (\cos x + i\sin x)^5 \\ &= \left(\cos^5 x - 10\cos^3 x\sin^2 x + 5\cos x\sin^4 x\right) \\ &+ i\left(5\cos^4 x\sin x - 10\cos^2 x\sin^3 x + \sin^5 x\right). \end{aligned} \tag{24}$$

Question 14

$$\int \arctan \sqrt{x}\, \mathrm{d}x \tag{25}$$

Solution Denote $t(x) = \arctan \sqrt{x}$, which gives $x(t) = \tan^2 t$. **Integration by parts** leads to

$$\begin{aligned} I = \int t(x)\, \mathrm{d}x = xt - \int x(t)\, \mathrm{d}t = xt - \int \tan^2 t\, \mathrm{d}t = xt - \int \left(\sec^2 t - 1\right) \mathrm{d}t \quad (26)\\ = xt - \tan t + t + C = (x+1) \arctan \sqrt{x} - \sqrt{x} + C. \quad (27) \end{aligned}$$

Question 15

$$\int_0^{1000} \left(\lfloor\lceil x\rceil\rfloor + \lceil\lfloor x\rfloor\rceil + \lfloor\{x\}\rfloor + \{\lfloor x\rfloor\} + \lceil\{x\}\rceil + \{\lceil x\rceil\}\right)\, \mathrm{d}x \tag{28}$$

Solution When $x \in [k, k+1]$, we can write $x = k + t$ with $t = \{x\} \in [0, 1]$. This gives

$$\int_k^{k+1} f(x)\, \mathrm{d}x = \int_0^1 \left[(k+1) + k + 0 + 0 + 1 + 0\right]\, \mathrm{d}t = 2\,(k+1)\,. \tag{29}$$

Therefore, the integral is evaluated as

$$I = 2 \sum_{k=1}^{1000} k = 1001000. \tag{30}$$

Question 16

$$\int \sqrt{\frac{\cos x \cot x \csc x}{\sin x \tan x \sec x}}\, \mathrm{d}x = \int \cot^2 x\, \mathrm{d}x = -x - \cot x + C. \tag{31}$$

Question 17

$$\int_{-\infty}^{+\infty} \frac{e^{-x^2}}{1 + e^{2x}}\, \mathrm{d}x \tag{32}$$

Solution With a **change of variable** $t = -x$, we have

$$I = \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{1+e^{2x}}\, \mathrm{d}x = \int_{-\infty}^{+\infty} \frac{e^{-t^2}}{1+e^{-2t}}\, \mathrm{d}t = \int_{-\infty}^{+\infty} \frac{e^{2t}e^{-t^2}}{1+e^{2t}}\, \mathrm{d}t. \tag{33}$$

Therefore, we have

$$2I = \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{1+e^{2x}}\, \mathrm{d}x + \int_{-\infty}^{+\infty} \frac{e^{2x}e^{-x^2}}{1+e^{2x}}\, \mathrm{d}x = \int_{-\infty}^{+\infty} e^{-x^2}\, \mathrm{d}x = \sqrt{\pi}, \qquad I = \frac{\sqrt{\pi}}{2}. \tag{34}$$

Question 18

$$\int \left(\frac{\sin^2 x}{x^2} - \frac{\sin 2x}{x} \right) \mathrm{d}x \tag{35}$$

Solution Note that

$$\left(\frac{\sin x}{x} \right)' = \frac{x \cos x - \sin x}{x^2}. \tag{36}$$

The integrand can be written as

$$\frac{\sin^2 x}{x^2} - \frac{\sin 2x}{x} = \frac{\sin^2 x - 2x \sin x \cos x}{x^2} = - \sin x \left(\frac{\sin x}{x} \right)' - \cos x \left(\frac{\sin x}{x} \right). \tag{37}$$

This directly leads to

$$\int \left(\frac{\sin^2 x}{x^2} - \frac{\sin 2x}{x} \right) \mathrm{d}x = - \frac{\sin^2 x}{x} + C. \tag{38}$$

Question 19

$$\int \frac{(\ln \ln x)\,(\ln \ln \ln x)}{x \ln x} \mathrm{d}x \tag{39}$$

Solution With the following notations

$$l_2(x) = \ln \ln x, \qquad l_3(x) = \ln \ln \ln x, \tag{40}$$

we have

$$\frac{\mathrm{d}l_2(x)}{\mathrm{d}x} = \frac{1}{x \ln x}, \qquad \frac{\mathrm{d}l_3(x)}{\mathrm{d}x} = \frac{1}{x \ln x \, l_2(x)}. \tag{41}$$

Therefore, the integral is evaluated as

$$I = \int l_2 \, l_3 \, \mathrm{d}l_2 = \frac{1}{2} l_2^2 \, l_3 - \frac{1}{2} \int l_2^2 \, \mathrm{d}l_3 = \frac{1}{2} l_2^2 \, l_3 - \frac{1}{2} \int \frac{l_2}{x \ln x} \mathrm{d}x \tag{42}$$

$$= \frac{1}{2} l_2^2 \, l_3 - \frac{1}{4} l_2^2 + C = \frac{1}{4} (\ln \ln x)^2 \left[2 \, (\ln \ln \ln x) - 1 \right] + C. \tag{43}$$

Question 20

$$\int_0^{\pi/2} \cos^2\left(\frac{\pi}{2}\cos^2\left(\frac{\pi}{2}\cos^2 x\right)\right) \mathrm{d}x \tag{44}$$

Solution Consider the **reflection** $x \to \pi/2 - x$, which gives

$$\begin{aligned} \cos^2\left(\frac{\pi}{2} - x\right) &= \sin^2 x = 1 - \cos^2 x, \\ \cos^2\left(\frac{\pi}{2}\cos^2\left(\frac{\pi}{2} - x\right)\right) &= \sin^2\left(\frac{\pi}{2}\cos^2 x\right) = 1 - \cos^2\left(\frac{\pi}{2}\cos^2 x\right). \end{aligned} \tag{45}$$

Similarly, for the integrand, we also have

$$f\left(\frac{\pi}{2} - x\right) = 1 - f(x). \tag{46}$$

Based on the reflection, we have

$$I = \int_0^{\pi/2} f(x)\,\mathrm{d}x = \int_0^{\pi/2} f\left(\frac{\pi}{2} - x\right) \mathrm{d}x. \tag{47}$$

Therefore, the integral is evaluated as

$$I = \frac{1}{2}\int_0^{\pi/2} \left[f(x) + f\left(\frac{\pi}{2} - x\right)\right] \mathrm{d}x = \frac{\pi}{4}. \tag{48}$$