MIT Integration Bee: 2024 Quarterfinal

Quarterfinal #1

Question 1

$$\int \ln x \left[\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x \right] \mathrm{d}x \tag{1.1}$$

Solution Denote the following functions

$$f(x) = \left(\frac{x}{e}\right)^x, \qquad g(x) = \left(\frac{e}{x}\right)^x. \tag{1.2}$$

We notice that

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{\mathrm{d}\ln f(x)}{\mathrm{d}x} = (x\ln x - x)' = \ln x, \\ \frac{g'(x)}{g(x)} &= \frac{\mathrm{d}\ln g(x)}{\mathrm{d}x} = (x - x\ln x)' = -\ln x. \end{aligned} \tag{1.3}$$

Therefore, the result is

$$I = \int [f(x) + g(x)] \ln x \, \mathrm{d}x = f(x) - g(x) = \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + C. \tag{1.4}$$

Question 2

$$\int_0^\infty \frac{\sin^3 x}{x} \mathrm{d}x \tag{2.1}$$

Solution Using the triple-angle formula and the **Dirichlet integral**, we have

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin 3x}{x} \mathrm{d}x = \int_0^\infty \frac{3\sin x - 4\sin^3 x}{x} \mathrm{d}x = \frac{3\pi}{2} - 4I. \tag{2.2}$$

Therefore, we have

$$I = \int_0^\infty \frac{\sin^3 x}{x} \mathrm{d}x = \frac{\pi}{4}. \tag{2.3}$$

Question 3

$$\int \begin{vmatrix} x & 1 & 0 & 0 & 0 \\ 1 & x & 1 & 0 & 0 \\ 0 & 1 & x & 1 & 0 \\ 0 & 0 & 1 & x & 1 \\ 0 & 0 & 0 & 1 & x \end{vmatrix} \mathrm{d}x \tag{3.1}$$

Solution Denote the determinant as $F_5(x)$. We can first obtain the following **recurrence relation**

$$F_n(x) = xF_{n-1}(x) - F_{n-2}(x), \qquad \text{with } F_1(x) = x, \quad F_2(x) = x^2 - 1. \tag{3.2}$$

Therefore, we have

$$\begin{aligned} F_3(x) &= xF_2(x) - F_1(x) = x^3 - 2x, \\ F_4(x) &= xF_3(x) - F_2(x) = x^4 - 3x^2 + 1, \\ F_5(x) &= xF_4(x) - F_3(x) = x^5 - 4x^3 + 3x. \end{aligned} \tag{3.3}$$

The integral is thus evaluated as

$$I = \int F_5(x)\,\mathrm{d}x = \frac{1}{6}x^6 - x^4 + \frac{3}{2}x^2 + C. \tag{3.4}$$

Tiebreakers Question 1

$$\int_0^{2024} x^{2024} \log_{2024}(x)\,\mathrm{d}x \tag{4.1}$$

Solution

$$\begin{aligned} I &= \frac{1}{\ln 2024} \int_0^{2024} x^{2024} \ln x\,\mathrm{d}x \\ &= \frac{1}{2025 \ln 2024} \left(x^{2025} \ln x \Big|_0^{2024} - \int_0^{2024} x^{2024}\,\mathrm{d}x \right) \\ &= \frac{2024^{2025}}{2025} - \frac{2024^{2025}}{2025^2 \ln 2024}. \end{aligned} \tag{4.2}$$

Tiebreakers Question 2

$$\lim_{t\to\infty}\int_0^2\left[x^{-2024t}\prod_{n=1}^{2024}\sin\left(nx^t\right)\right]\mathrm{d}x \tag{5.1}$$

Solution First, we study the following function

$$f_\alpha(x)=\lim_{t\to\infty}\frac{\sin\alpha x^t}{x^t}. \tag{5.2}$$

When $x>1$, we have $\left|\sin\alpha x^t\right|\leq 1$ but $x^t\to\infty$. Hence, we have $f_\alpha(x)=0$ for $x>1$. On the other hand, when $0<x<1$ we have $x^t\to 0$ and thus $f_\alpha(x)=\alpha$. As a summary, we obtain

$$f_\alpha(x)=\lim_{t\to\infty}\frac{\sin\alpha x^t}{x^t}=\begin{cases}\alpha, & |x|<1,\\ 0, & |x|>1.\end{cases} \tag{5.3}$$

Using this result, we speculate that

$$f(x)=\lim_{t\to\infty}x^{-2024t}\prod_{n=1}^{2024}\sin\left(nx^t\right)=\prod_{n=1}^{2024}f_n(x)=\begin{cases}2024!, & |x|<1,\\ 0, & |x|>1.\end{cases} \tag{5.4}$$

Finally, the integral is evaluated as

$$I=\int_0^2 f(x)\,\mathrm{d}x=2024!. \tag{5.5}$$

Quarterfinal #2

Question 1

$$\lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} \frac{(kx)^4}{n^5}\, \mathrm{d}x \tag{6.1}$$

Solution Re-organize the order of operations, we have

$$I = \lim_{n\to\infty} \frac{1}{n^5} \sum_{k=1}^{n} \int_0^1 k^4 x^4 \, \mathrm{d}x = \lim_{n\to\infty} \frac{1}{5n^5} \sum_{k=1}^{n} k^4. \tag{6.2}$$

Note that the leading order term of the sum is

$$\sum_{k=1}^{n} k^4 = \frac{1}{5} n^5 + o(n^5). \tag{6.3}$$

Therefore, after taking the limit we have

$$I = \lim_{n\to\infty} \frac{1}{5n^5} \sum_{k=1}^{n} k^4 = \frac{1}{25}. \tag{6.4}$$

Question 2

$$\int_0^1 \frac{\ln\left(1+x^2+x^3+x^4+x^5+x^6+x^7+x^9\right)}{x}\,\mathrm{d}x \tag{7.1}$$

Solution We notice that

$$1+x^2+x^3+x^4+x^5+x^6+x^7+x^9 = (1+x^2)(1+x^3)(1+x^4). \tag{7.2}$$

Therefore, we study the following integral

$$F(\alpha) = \int_0^1 \frac{\ln(1+x^\alpha)}{x}\,\mathrm{d}x, \qquad \text{with } \alpha > 0. \tag{7.3}$$

With a simple **change of variable** $t = x^n$, we have

$$F(\alpha) = \frac{1}{\alpha}\int_0^1 \frac{\ln(1+t)}{t}\,\mathrm{d}t = \frac{\pi^2}{12\alpha}. \tag{7.4}$$

Finally, the result is

$$I = F(2) + F(3) + F(4) = \frac{\pi^2}{12}\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right) = \frac{13\pi^2}{144}. \tag{7.5}$$

Note The following type of integral has appeared several times

$$\begin{aligned}\int_0^1 \frac{\ln(1+t)}{t}\,\mathrm{d}t &= \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}\int_0^1 t^{n-1}\,\mathrm{d}t \\ &= \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.\end{aligned} \tag{7.6}$$

Using integration by parts, we also have

$$\int_0^1 \frac{\ln t}{1+t}\,\mathrm{d}t = -\int_0^1 \frac{\ln(1+t)}{t}\,\mathrm{d}t = -\frac{\pi^2}{12}. \tag{7.7}$$

Question 3

$$\int_0^1 \left(1 - \sqrt[2024]{x}\right)^{2024} \mathrm{d}x \tag{8.1}$$

Solution We study the following general integral

$$F(\alpha) = \int_0^1 \left(1 - x^{\frac{1}{\alpha}}\right)^{\alpha} \mathrm{d}x. \tag{8.2}$$

With a simple **change of variable** $t = x^{1/\alpha}$, we have

$$\begin{aligned} F(\alpha) &= \alpha \int_0^1 (1-t)^{\alpha} t^{\alpha-1} \, \mathrm{d}t \\ &= \alpha B(\alpha+1, \alpha) = \frac{\Gamma^2(\alpha+1)}{\Gamma(2\alpha+1)}. \end{aligned} \tag{8.3}$$

Finally, the integral is evaluated as

$$I = F(2024) = \frac{\Gamma^2(2025)}{\Gamma(4049)} = \frac{(2024!)^2}{4048!} = \frac{1}{\binom{4048}{2024}}. \tag{8.4}$$

Quarterfinal #3

Question 1

$$\int_0^{2\pi} \operatorname{card}\left(\{\lfloor \sin x\rfloor, \lfloor \cos x\rfloor, \lfloor \tan x\rfloor, \lfloor \cot x\rfloor\}\right)\, \mathrm{d}x \tag{9.1}$$

Solution We summarize the cardinality of the set in the following table.

	$\left(0, \frac{\pi}{4}\right)$	$\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$	$\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$	$\left(\frac{3\pi}{4}, \pi\right)$	$\left(\pi, \frac{5\pi}{4}\right)$	$\left(\frac{5\pi}{4}, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$	$\left(\frac{7\pi}{4}, 2\pi\right)$
$\lfloor \sin x\rfloor$	0	0	0	0	-1	-1	-1	-1
$\lfloor \cos x\rfloor$	0	0	-1	-1	-1	-1	0	0
$\lfloor \tan x\rfloor$	0	≥ 1	< -1	-1	0	≥ 1	< -1	-1
$\lfloor \cot x\rfloor$	≥ 1	0	-1	< -1	≥ 1	0	-1	< -1
card	2	2	3	3	3	3	3	3

Therefore, the integral is evaluated as

$$I = \frac{\pi}{2}(2+3+3+3) = \frac{11\pi}{2}. \tag{9.2}$$

Question 2

$$\int_0^{+\infty} \frac{\mathrm{d}x}{(x+1)\left(\ln^2 x + \pi^2\right)} \tag{10.1}$$

Solution Using several **changes of variables**, we have

$$\begin{aligned}
I &= \int_0^{\infty} \frac{\mathrm{d}x}{(x+1)\left(\ln^2 x + \pi^2\right)} \\
&= \int_{-\infty}^{+\infty} \frac{e^t\,\mathrm{d}t}{(e^t+1)\left(t^2+\pi^2\right)} \qquad \left(t = \ln x, \quad x = e^t, \quad \mathrm{d}x = e^t\,\mathrm{d}t\right) \\
&= \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{t^2+\pi^2} - \int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{(e^t+1)\left(t^2+\pi^2\right)} \\
&= \frac{1}{\pi}\arctan\left(\frac{t}{\pi}\right)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \frac{e^t\,\mathrm{d}t}{(e^t+1)\left(t^2+\pi^2\right)} \qquad (t \to -t)
\end{aligned} \tag{10.2}$$

Therefore, we have

$$2I = \frac{1}{\pi}\arctan\left(\frac{t}{\pi}\right)\Big|_{-\infty}^{+\infty} = 1, \qquad I = \frac{1}{2}. \tag{10.3}$$

Question 3

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \max\left(\{x\}, \left\{\sqrt{2}x\right\}, \left\{\sqrt{3}x\right\}\right) \mathrm{d}x \tag{11.1}$$

Solution This integral can be translated into the calculation of an **expectation**. Denote X_1, X_2, X_3 as three independent variables drawn from the uniform distribution $U[0,1]$. Denote the new random variable $Y = \max(X_1, X_2, X_3)$. Therefore, the integral is equivalent to

$$I = \mathbb{E}(Y) = \mathbb{E}\left[\max(X_1, X_2, X_3)\right]. \tag{11.2}$$

The CDF $f_Y(y)$ is computed as follows, based on the Independence among X_i

$$f_Y(y) = \mathbb{P}(Y < y) = \prod_{i=1}^{3} \mathbb{P}(X_i < y) = y^3, \qquad \text{for } y \in [0,1]. \tag{11.3}$$

Therefore, the PDF $p_Y(y)$ and the expectation are obtained as

$$p_Y(y) = \frac{\mathrm{d}f_Y}{\mathrm{d}y} = 3y^2, \qquad I = \mathbb{E}(Y) = \int_0^1 y p_Y(y)\,\mathrm{d}y = \frac{3}{4}. \tag{11.4}$$

Quarterfinal #4

Question 1

$$\int \frac{e^{2x}}{(1-e^x)^{2024}}\,\mathrm{d}x \tag{12.1}$$

Solution With a simple **change of variable** $t = e^x$, we have

$$\begin{aligned} I &= \int \frac{t\,\mathrm{d}t}{(1-t)^{2024}} = \int \frac{\mathrm{d}t}{(1-t)^{2024}} - \int \frac{\mathrm{d}t}{(1-t)^{2023}} \\ &= \frac{1}{2023\,(1-e^x)^{2023}} - \frac{1}{2022\,(1-e^x)^{2022}} + C \end{aligned} \tag{12.2}$$

Question 2

$$\lim_{n\to\infty} \log_n \left(\int_0^1 \left(1-x^3\right)^n \,\mathrm{d}x \right) \tag{13.1}$$

Solution We first study the following general integral

$$F(\alpha) = \int_0^1 (1-x^\alpha)^n \,\mathrm{d}x. \tag{13.2}$$

With the standard **change of variable** $t = x^\alpha$, we have

$$F(\alpha) = \frac{1}{\alpha}\int_0^1 (1-t)^n\, t^{\frac{1}{\alpha}-1}\,\mathrm{d}t = \frac{1}{\alpha} B\left(n+1, \frac{1}{\alpha}\right). \tag{13.3}$$

Based on the **Stirling's approximation** for the gamma function

$$\ln\Gamma(z) \sim z\ln z, \tag{13.4}$$

the limit can be obtained as

$$\begin{aligned} I(\alpha) = \lim_{n\to\infty} \frac{\ln F(\alpha)}{\ln n} &= \lim_{n\to\infty} \frac{\ln\Gamma(n+1) - \ln\Gamma(n+1+\alpha^{-1})}{\ln n} \\ &= \lim_{n\to\infty} \frac{-\alpha^{-1}\ln n}{\ln n} = -\frac{1}{\alpha}. \end{aligned} \tag{13.5}$$

Therefore, the result is

$$I = I(3) = -\frac{1}{3}. \tag{13.6}$$

Question 3

$$\int \frac{\sin x}{1+\sin x} \cdot \frac{\cos x}{1+\cos x} \, \mathrm{d}x \tag{14.1}$$

Solution Note that

$$\begin{aligned} \frac{\sin x}{1+\sin x} \cdot \frac{\cos x}{1+\cos x} &= 1 - \frac{\sin^2 x + \cos^2 x + \sin x + \cos x}{(1+\sin x)(1+\cos x)} \\ &= 1 - \left(\frac{\cos x}{1+\sin x} + \frac{\sin x}{1+\cos x} \right). \end{aligned} \tag{14.2}$$

Therefore, the integral becomes

$$\begin{aligned} I &= x - \ln(1+\sin x) + \ln(1+\cos x) + C \\ &= x + \ln\left(\frac{1+\cos x}{1+\sin x} \right) + C. \end{aligned} \tag{14.3}$$