Quasi-static elastic response to surface loading

1 Propagator matrix method

The propagator matrix method (Gilbert and Backus, 1966; Singh, 1970) is applied to model the quasi-static elastic response to the surface loading. The subsurface elastic structure is idealized as a plane-layered medium over a uniform half-space. The method solves the static elasticity equilibrium equation and Hooke's law in the transform domain. The two horizontal directions are Fourier transformed to wavenumbers k_x and k_y. We do not yet consider the shear traction at the surface, and thus only the P-SV system is implemented. The ordinary differential equation system is written as

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}z} = \frac{\mathrm{d}}{\mathrm{d}z}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0 & -k & \frac{1}{\mu} & 0 \\ \frac{k\lambda}{\lambda+2\mu} & 0 & 0 & \frac{1}{\lambda+2\mu} \\ \frac{4\mu k^2(\lambda+\mu)}{\lambda+2\mu} & 0 & 0 & -\frac{k\lambda}{\lambda+2\mu} \\ 0 & 0 & k & 0 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \mathbf{A}\mathbf{U}. \tag{1}$$

The components of the displacement-stress vector $\mathbf{U}$ are defined as

$$\begin{aligned} -kU_1 &= ik_x\tilde{u}_x + ik_y\tilde{u}_y, & U_2 &= \tilde{u}_z, \\ -kU_3 &= ik_x\tilde{\sigma}_{xz} + ik_y\tilde{\sigma}_{yz}, & U_4 &= \tilde{\sigma}_{zz}, \end{aligned} \tag{2}$$

where $\tilde{u}_i$ and $\tilde{\sigma}_{ij}$ are displacement and stress components, respectively, in the Fourier domain under the following convention

$$\tilde{u}(k_x, k_y, z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u(x, y, z)e^{-i(k_x x + k_y y)}\,\mathrm{d}x\mathrm{d}y, \tag{3}$$

with horizontal wavenumbers k_x, k_y and the magnitude $k = \sqrt{k_x^2 + k_y^2}$. For each elastic layer, the matrix $\mathbf{A}$ in Eq. (1) is constructed from the Lamé parameters λ and μ obtained from the density ρ and P- and S-wave seismic velocities v_P and v_S. We begin with the homogeneous solution at the top of the underlying half-space for each Fourier mode,

$$\begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = c_1\begin{bmatrix} \frac{1}{2k\mu} \\ \frac{1}{2k\mu} \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} \frac{\lambda+2\mu}{2k^2\mu(\lambda+\mu)} \\ -\frac{1}{2k^2(\lambda+\mu)} \\ \frac{1}{k} \\ 0 \end{bmatrix}, \tag{4}$$

where $c_1(k_x, k_y)$ and $c_2(k_x, k_y)$ are coefficients to be determined by the surface conditions. This solution is then propagated upward through each layer by

$$\mathbf{U}(z_{\mathrm{top}}) = e^{\mathbf{A}h}\mathbf{U}(z_{\mathrm{btm}}), \tag{5}$$

where z_{top} and z_{btm} denote the top and bottom of the layer and $h = z_{\mathrm{top}} - z_{\mathrm{btm}}$ is the layer thickness. The propagator is the matrix exponential $e^{\mathbf{A}h}$. At the surface $z = 0$, we impose the following boundary conditions

$$U_3 = 0, \qquad U_4 = -\tilde{p}(k_x, k_y), \qquad \text{at } z = 0 \tag{6}$$

to solve for the coefficients c_1 and c_2, where $\tilde{p}(k_x, k_y)$ is the Fourier transform of the input surface pressure field at a particular time. Under the quasi-static assumption, we repeat solving for coefficients c_1 and c_2 given the pressure field $\tilde{p}(k_x, k_y)$ at different times t. The final displacements and stress components are obtained from the inverse Fourier transform of **U** evaluated at the surface. The derivation can also be seen in Segall (2010).

2 Benchmark solutions

The Sorrells problem (Sorrells, 1971) and Boussinesq problem (Boussinesq, 1885) are used to benchmark the propagator matrix method (Fig. 1). As we solve the elasticity equation in the Fourier domain, periodic loading is exactly represented, and the Sorrells problem is well-suited to this method. On the other hand, for the Boussinesq problem, the domain-averaged mean of the analytical solution should be removed to compare with the numerical result. We can notice from Eq. (4 that the zero-wavenumber component ($k = 0$) needs to be separately considered since k appears in the denominator of the initial homogeneous solution. This component is not included in the solution.

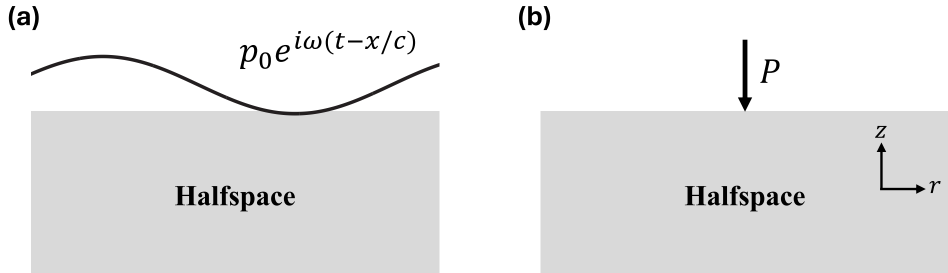

Fig. 1 The configuration of benchmark problems. **(a)** Sorrells problem. The elastic halfspace is subject to a monochromatic pressure wave loading with phase speed c much smaller than the elastic wave speed. **(b)** Boussinesq problem. The elastic halfspace is subject to a concentrated normal load P. The cylindrical coordinates are defined with the positive z-axis pointing upward.

2.1 Sorrells problem

With the quasi-static assumption $c \ll v_{P,S}$ and $\omega z c \ll 2v_{P,S}^2$, the displacement field of the Sorrells problem is given as

$$\begin{aligned} u_x &= \frac{icp_0}{2\mu\omega}\left(\frac{v_S^2}{v_P^2 - v_S^2} - \frac{\omega|z|}{c}\right) e^{-\omega|z|/c}\, e^{i\omega(t-x/c)}, \\ u_z &= -\frac{cp_0}{2\mu\omega}\left(\frac{v_P^2}{v_P^2 - v_S^2} + \frac{\omega|z|}{c}\right) e^{-\omega|z|/c}\, e^{i\omega(t-x/c)}. \end{aligned} \tag{7}$$

Eq. (7) shows an exponential decay in z-direction, and in fact it only depends on the wavenumber $k = \omega/c$. At the surface $z = 0$, the displacement amplitude ratio is $|u_x/u_z| = (v_S/v_P)^2$, which is much smaller than 1 for compliant sediments. The comparison of analytical and numerical results is shown in Fig. 2

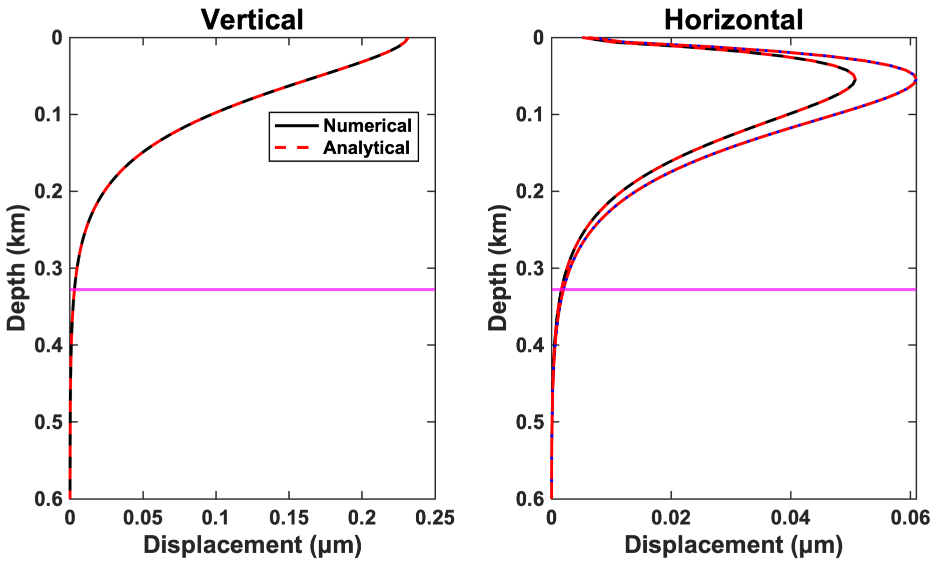

Fig. 2 Benchmark of Sorrells problem with $p_0 = 1$ Pa, $\lambda = 0.328$ km and wave azimuth $\theta = 50°$. The magenta line indicates the depth $|z| = \lambda$ as a reference to indicate the exponential decay rate.

2.2 Boussinesq problem

The Boussinesq solution describes the static elastic response of a homogeneous half-space subject to a concentrated normal load P. This is the first benchmark for the propagator matrix method. In the polar coordinate, we have $u_\theta = 0$ and

$$\begin{aligned} u_r &= \frac{Pr}{4\pi\mu}\left[\frac{|z|}{R^3} - \frac{1-2\nu}{R(|z|+R)}\right], \\ u_z &= -\frac{P}{4\pi\mu}\left[\frac{2(1-\nu)}{R} + \frac{|z|^2}{R^3}\right], \end{aligned} \tag{8}$$

where $r = \sqrt{x^2+y^2}$ and $R = \sqrt{r^2+z^2}$ (Slaughter, 2002). Here, the solution is written in terms of $|z|$, the depth from the surface. For the stress components, we have $\sigma_{r\theta} = \sigma_{\theta z} = 0$ and

$$\begin{aligned} \sigma_{rr} &= \frac{P}{2\pi}\left[\frac{1-2\nu}{R(|z|+R)} - \frac{3r^2|z|}{R^5}\right], \\ \sigma_{\theta\theta} &= \frac{P(1-2\nu)}{2\pi}\left[\frac{|z|}{R^3} - \frac{1}{R(|z|+R)}\right], \\ \sigma_{zz} &= -\frac{3P|z|^3}{2\pi R^5}, \qquad \sigma_{rz} = \frac{3Pr|z|^2}{2\pi R^5}. \end{aligned} \tag{9}$$

The comparison of analytical and numerical results is shown in Fig. 3.

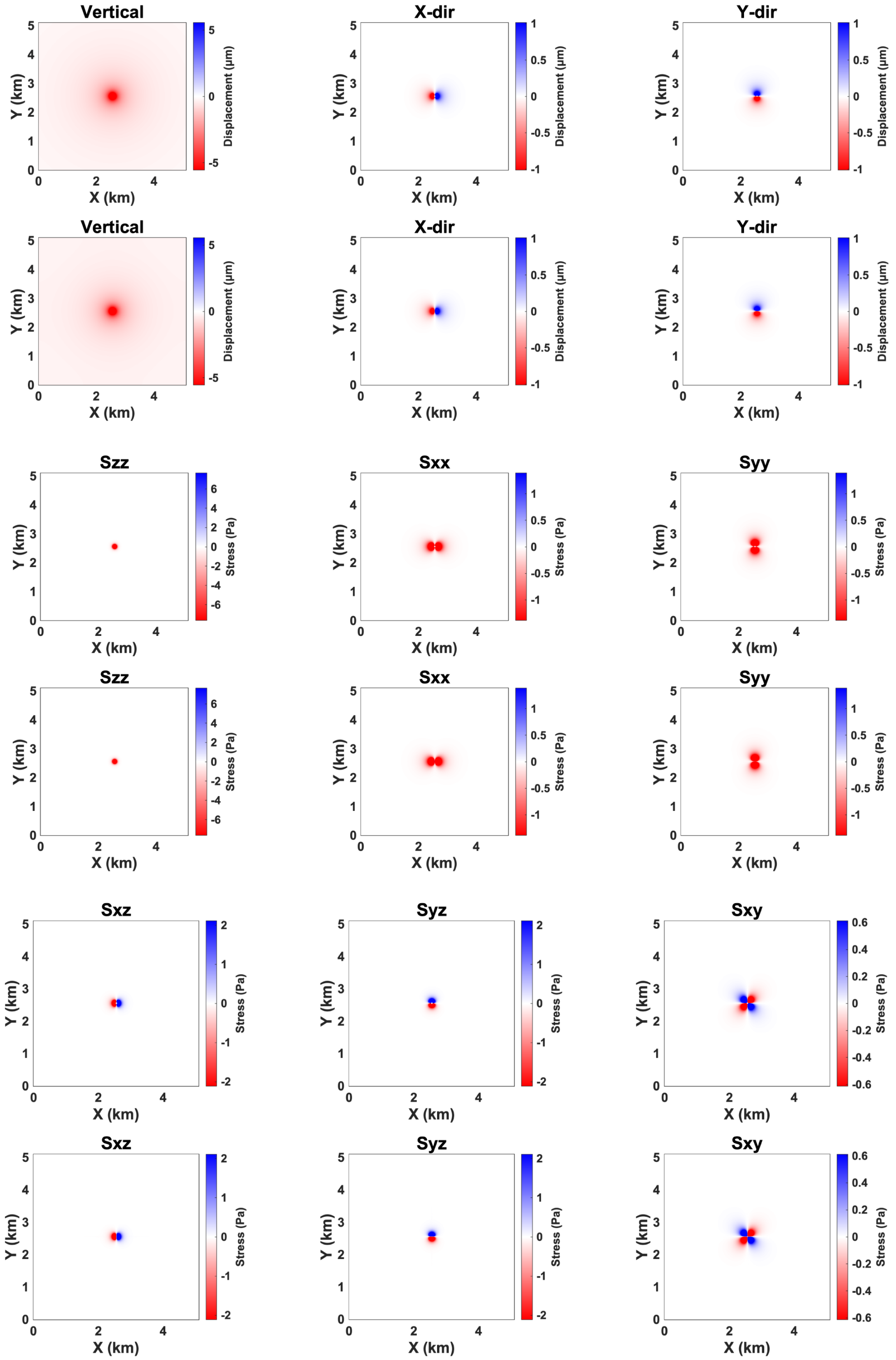

Fig. 3 Benchmark of Boussinesq problem with $P = 10^6$ N at depth $z = 0.05$ km. The properties of the elastic halfspace are $\rho = 1600\ \mathrm{kg/m^3}$, $v_P = 1.45$ km/s and $v_S = 0.27$ km/s. The analytical solutions from Eqs (8) and (9) are plotted on the top, while the numerical solutions are plotted on the bottom for each displacement and stress component.

References

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