Set Family Decision Diagrams

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How to compute efficiently on sets ?

- represent sets in a compact way
- compute on a whole set instead on a single element
    - aka SIMD or *graphic card computing*
- respect union : set homomorphism

various approaches based on decision diagrams.

Set Family Decision Diagrams
Informal Definition

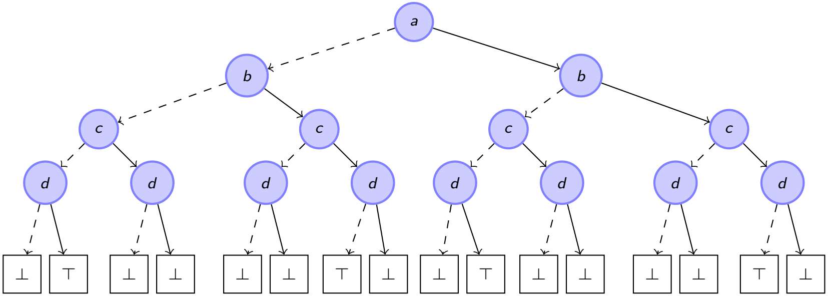
A SFDD is a directed acyclic graph where

- each node represent a term
- each node has two children, indicating whether or not the term is contained
- each path from the root to an accepting terminal represents a set of terms
- terms are totally ordered

Set Family Decision Diagrams
Example Full

Encodes with the order $a < b < c < d$ the sets:

- $\{a, b, c\}$
- $\{a, d\}$
- $\{b, c\}$
- $\{d\}$



Set Family Decision Diagrams

Example Reduction Part 1

Encodes with the order $a < b < c < d$ the sets:

- $\{a, b, c\}$
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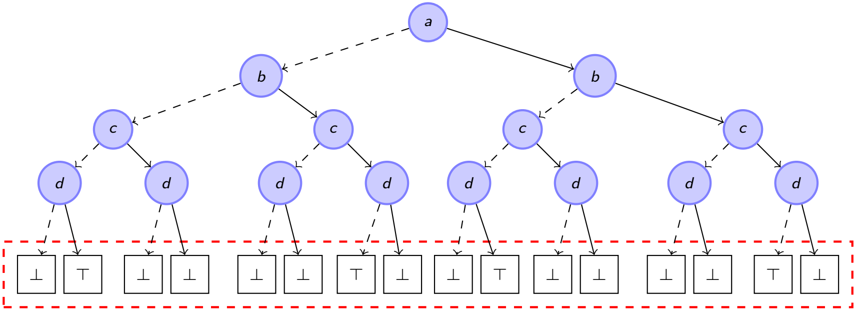

Set Family Decision Diagrams
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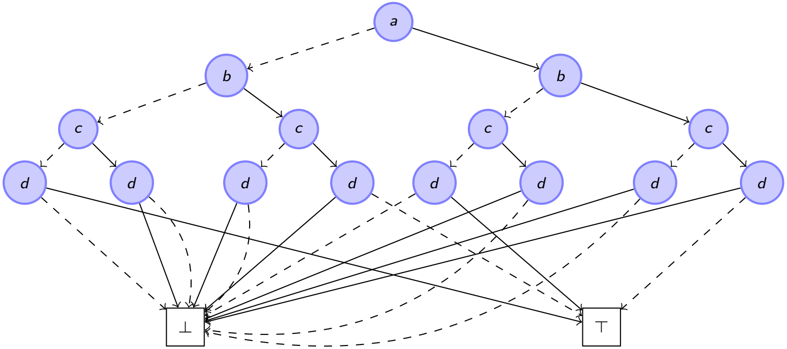
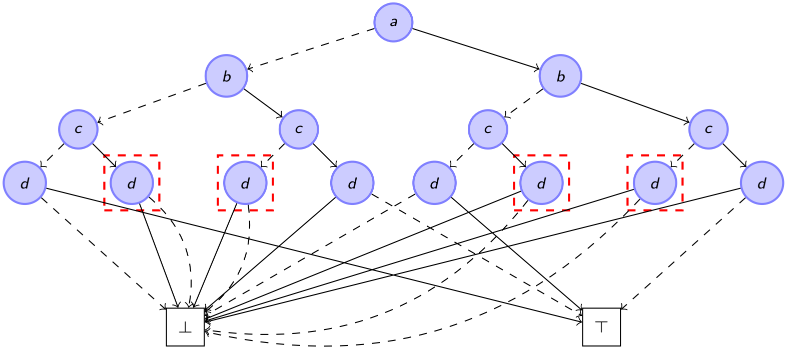


Set Family Decision Diagrams

Example Reduction Part 2: Don't belongs

Encodes with the order $a < b < c < d$ the sets:

- $\{a, b, c\}$
- $\{a, d\}$
- $\{b, c\}$
- $\{d\}$



Set Family Decision Diagrams
Example Reduction Part 2: Don't belongs

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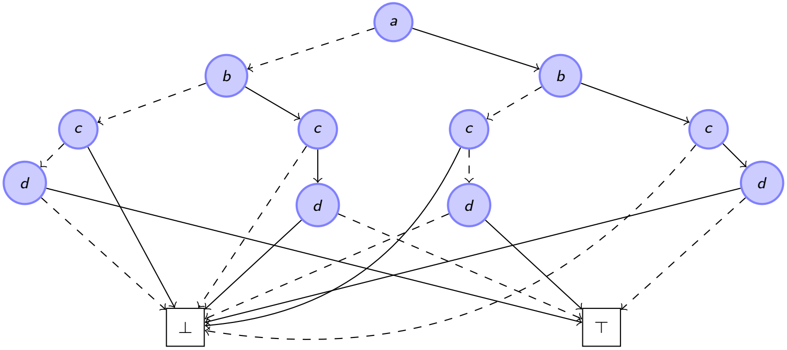

Set Family Decision Diagrams
Example Reduction Part 3: Factorization nodes

Encodes with the order $a < b < c < d$ the sets:

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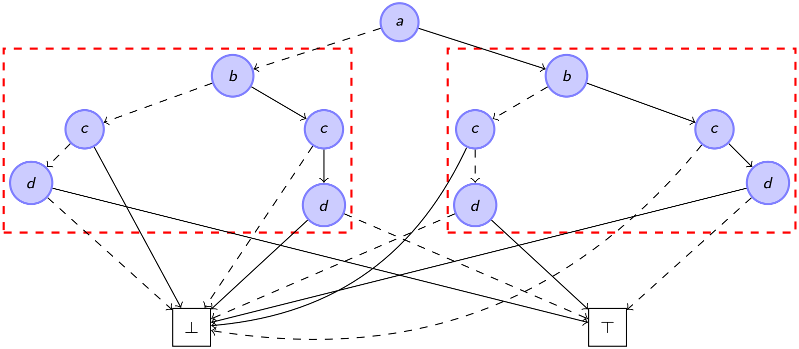
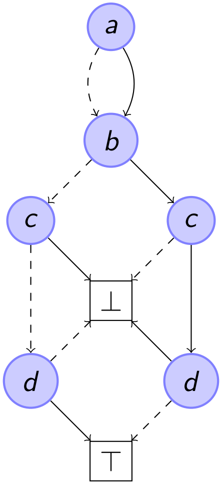
- $\{a, d\}$
- $\{b, c\}$
- $\{d\}$



Set Family Decision Diagrams

Example Reduction Part 3: Factorization nodes



Encodes with the order $a < b < c < d$ the sets:

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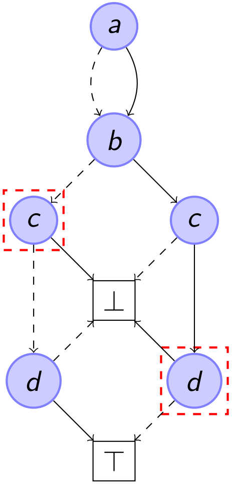
Set Family Decision Diagrams

Example Reduction Part 4: Remove takes nodes whose then branch is $\perp$



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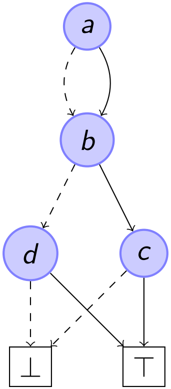
Set Family Decision Diagrams

Example Reduction Part 4: Remove takes nodes whose then branch is $\perp$



Encodes with the order $a < b < c < d$ the sets:

- $\{a, b, c\}$
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- $\{d\}$

From Set Family Decision Diagrams to BDD

Equivalence of sets with boolean functions

A set S over terms $T = \{t_1, t_2, ..., t_m\}$ is represented as a function f from T to $\mathbb{B}$, such as:

$$\begin{array}{ll} \forall s \in S, & f_S(s) = t \\ \forall s \in T - S, & f_S(s) = f \end{array}$$

A familly of sets $F = \{S_1, S_2, ..., S_n\}$, is defined as the following boolean function of arity m: $F : \mathbb{B} \times \mathbb{B} \times ... \times \mathbb{B} \rightarrow \mathbb{B}$ such as

$$\begin{array}{ll} \forall i \in 1...n, F(f_{S_i}(t_1), f_{S_i}(t_2), ..., f_{S_i}(t_m)) & = t \\ \textit{otherwise} & = f \end{array}$$

Why Set Family Decision Diagrams?
Operations

This shows the correspondance between SFDD and BDD if we provide a total order over elements of T.

Although they are structurally similar, they benefit from different operations.

Moreover SFDD can be extended to other decision diagrams such as MFDD (encoding set of <KEY,VALUE>) and ΣDD (encoding set of Σ Terms) seamlessly.

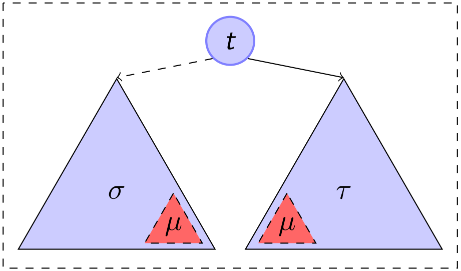
Definition (Formal definition)

Let T be a set of terms. The set of SFDDs $\mathbb{S}$ is inductively defined by:

- $\bot \in \mathbb{S}$ is the rejecting terminal
- $\top \in \mathbb{S}$ is the accepting terminal
- $\langle t, \tau, \sigma \rangle \in \mathbb{S}$ if and only if $t \in T \land \tau, \sigma \in \mathbb{S}$

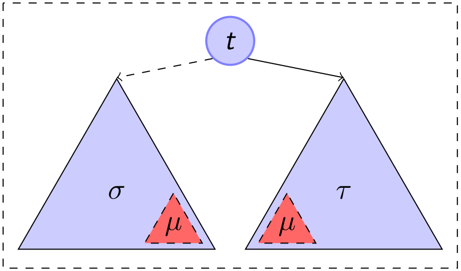
Theory VS Implementation

Theory

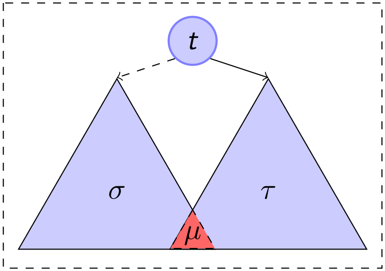


Theory VS Implementation

Theory



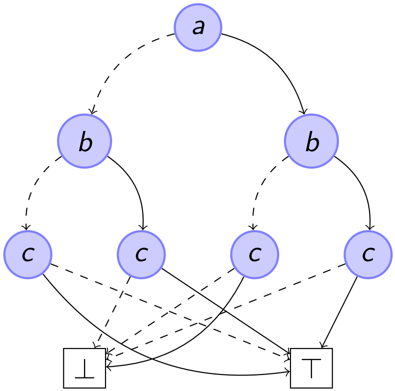
Implementation



Set Family Decision Diagrams

Brute form

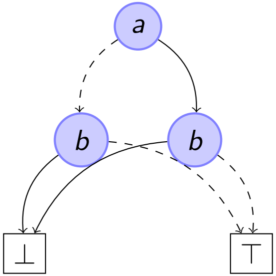
$$S = \{\emptyset, \{c\}, \{c, b\}, \{c, b, a\}\}$$



It is not optimal (neither unique, in fact depends on the constraint) as there is no common part (except the terminals) and several representation for the same set.

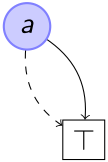
Set Family Decision Diagrams
Uniqueness

$S = \{\emptyset, \{a\}\}$



Representation uniqueness ?

$S = \{\emptyset, \{a\}\}$



Set Family Decision Diagrams
Reductions

From the brute ordered shape, we can reduce slightly the unnecessary nodes:

- remove negative nodes, i.e nodes with accept branch pointing to $\bot$, they are not providing any information.
- share common sub trees (not expressed in this formal definition)

clean : $\mathbb{S} \rightarrow \mathbb{S}$ removes a negative node from all sets that contain it:

$$\mathit{clean}(\bot) = \bot$$

$$\mathit{clean}(\top) = \top$$

$$\mathit{clean}(\langle t, \tau, \sigma \rangle) = \begin{cases} \mathit{clean}(\sigma) & \text{if } \tau = \bot \\ \langle t, \mathit{clean}(\tau), \mathit{clean}(\sigma) \rangle & \text{if } \mathit{otherwise} \end{cases}$$

NB: *clean* is an homomorphism.

Set Family Decision Diagrams
Canonical Form

Let $S \in \mathbb{S}$ be the SFDD $\langle t, \tau, \sigma \rangle$, we call τ its take node and σ its skip node.

S is canonical if for all its nodes, the skip node and take node represent greater terms or terminals, and no take node is the rejecting terminal. (sharing ?)

Definition (Canonical form)

Let T be a set of terms, and $< \in T \times T$ a total ordering on T. A SFDD $S \in \mathbb{S}$ is canonical if and only if

- S is the rejecting terminal $\bot$
- S is the accepting terminal $\top$
- $S = \langle t, \tau, \sigma \rangle$ where
    - $\tau = \langle t_\tau, \tau_\tau, \sigma_\tau \rangle \implies t < t_\tau$ and $\tau \neq \bot$
    - $\sigma = \langle t_\sigma, \tau_\sigma, \sigma_\sigma \rangle \implies t < t_\sigma$
    - τ and σ are canonical

Set Family Decision Diagrams
Implementation as graph

From the brute ordered shape, we can reduce by the *clean* operation. Shared trees are themselves describded by the fact that equivalent subtrees are collapsed by an equivalence relation. $\equiv \subseteq \mathbb{S} \times \mathbb{S}$ identify similar sets:

$$\begin{aligned} \bot &\equiv \bot \\ \top &\equiv \top \\ \langle t, \tau, \sigma \rangle &\equiv \langle t, \tau', \sigma' \rangle && \text{if } \tau \equiv \tau' \wedge \sigma \equiv \sigma' \end{aligned}$$

The structure which is implemented is then $\mathbb{S} = \mathit{clean}(\mathbb{S}_{\mathit{brute}}) / \equiv$. Implementations share same subtrees and memorization can be used due to the functional nature of operations (no side effects).

Set Family Decision Diagrams
Examples

We give some basic examples of SFDD for a given set of sets from S and a total order $a < b < c$:

$$S = \{a, b, c\}$$

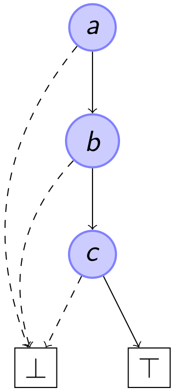
$$\wp(S) = \{\{a, b, c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \varnothing\}$$

$$\wp(S) - S = \{\{a, b\}, \{b, c\}, \{a, c\}, \{a\}, \{b\}, \{c\}, \varnothing\}$$

$$\wp(S) - \varnothing = \{\{a, b, c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}$$

Set Family Decision Diagrams
Example

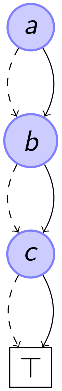
$$enc(\{\{a, b, c\}\})$$



Set Family Decision Diagrams

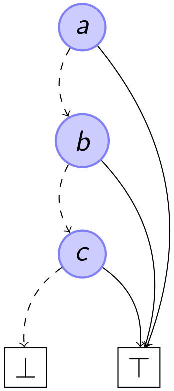
Example

$enc(\wp(\{a, b, c\}))$



Set Family Decision Diagrams
Example

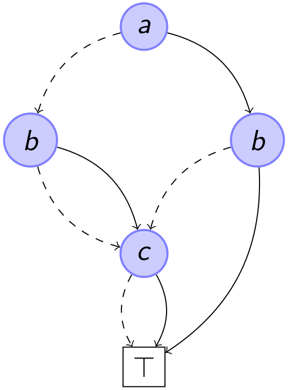
$enc(\{\{a\},\{b\},\{c\}\})$



Set Family Decision Diagrams
Example

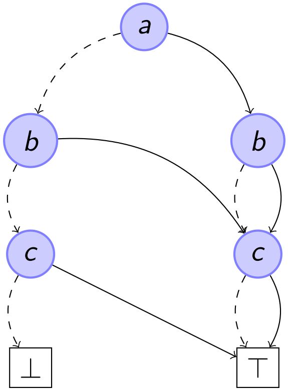
$enc(\wp(\{a,b,c\}) - \{a,b,c\})$

It is not a good case of encoding.



Set Family Decision Diagrams
Example

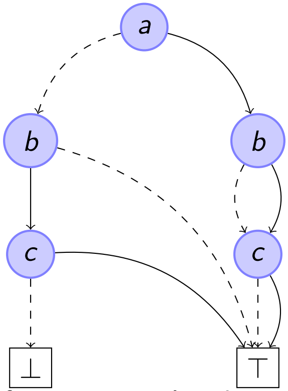
$enc(\wp(\{a, b, c\}) - \emptyset)$



It is one of the bad case we can expect, comparable to the previous one. But we can do worse.

Set Family Decision Diagrams
Example

$enc(\{\{a,b,c\},\{a,b\},\{a,c\},\{b,c\},\{a\},\emptyset\})$



It is one of the worst case we can expect if we remove also the non singleton sets, comparable to the previous one, we need $2^{|S|} - k$ nodes to encode $\wp(S) - \emptyset$.

Set Family Decision Diagrams
Example

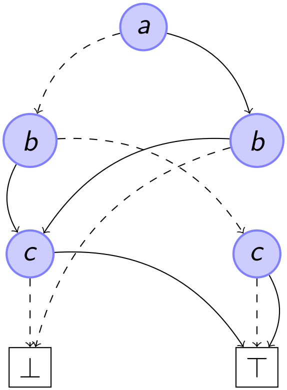
$$S_{i+1} = \{\emptyset\} \cup (S_i \oplus \{e_{i+1}\}), 0 \leq i \leq n-1$$

$$S_0 = \{\emptyset\}$$

$$S1 = \{\emptyset, \{a\}\}$$

$$S2 = \{\emptyset, \{b\}, \{a, b\}\}$$

$$S3 = \{\emptyset, \{c\}, \{c, b\}, \{c, b, a\}\}$$



It is one of the worst case we can expect if we remove also the non singleton sets, comparable to the previous one, we need $2^{|S|} - k$ nodes to encode $\wp(S) - \emptyset$.

Set Family Decision Diagrams
Union

The union of two SFDDs is given by:

$$
\begin{aligned}
A \cup B &= B \cup A \\
A \cup A &= A \\
\bot \cup A &= A \\
\top \cup \langle t, \tau, \sigma \rangle &= \langle t, \top \cup \tau, \top \cup \sigma \rangle \\
\langle t, \tau, \sigma \rangle \cup \langle t', \tau', \sigma' \rangle &= \begin{cases} \langle t, \tau, \sigma \cup \langle t', \tau', \sigma' \rangle \rangle & \text{if } t < t' \\ \langle t, \tau \cup \tau', \sigma \cup \sigma' \rangle & \text{if } t = t' \\ \langle t', \tau', \sigma' \cup \langle t, \tau, \sigma \rangle \rangle & \text{if } t > t' \end{cases}
\end{aligned}
$$

Set Family Decision Diagrams
Intersection

The intersection of two SFDDs is given by:

$$
\begin{aligned}
A \cap B &= B \cap A \\
A \cap A &= A \\
\bot \cap A &= \bot \\
\top \cap \langle t, \tau, \sigma \rangle &= \top \cap \sigma \\
\langle t, \tau, \sigma \rangle \cap \langle t', \tau', \sigma' \rangle &= \begin{cases} \sigma \cap \langle t', \tau', \sigma' \rangle & \text{if } t < t' \\ \langle t, \tau \cap \tau', \sigma \cap \sigma' \rangle & \text{if } t = t' \\ \langle t, \tau, \sigma \rangle \cap \sigma' & \text{if } t > t' \end{cases}
\end{aligned}
$$

Set Family Decision Diagrams
Encoding

The encoding of a set into a SFDD is given by:

$$\begin{aligned}
\mathsf{enc}(\varnothing) &= \bot \\
\mathsf{enc}(\{\varnothing\}) &= \top \\
\mathsf{enc}(S \cup \{s\}) &= \mathsf{enc}(S) \cup \mathsf{enc}(\{s\}) \\
t < \min(s) \implies \mathsf{enc}(\{s \cup \{t\}\}) &= \langle t, \mathsf{enc}(\{s\}), \bot \rangle
\end{aligned}$$

Set Family Decision Diagrams
Decoding

The decoding of one SFDD is given by:

$$\begin{aligned}\mathrm{dec}(\bot) &= \varnothing \\ \mathrm{dec}(\top) &= \{\varnothing\} \\ \mathrm{dec}(\langle t, \tau, \sigma\rangle) &= (\mathrm{dec}(\tau) \oplus t) \cup \mathrm{dec}(\sigma)\end{aligned}$$

Where $\oplus$ is defined as follows:

$$\bigcup_{s \in S} \{s\} \oplus t = \bigcup_{s \in S} \{s \cup \{t\}\}$$

The decoding/encoding of one set is the identity (and the reverse):

$$\forall S \subseteq \mathcal{P}(T), \mathrm{dec}(\mathrm{enc}(S)) = S$$
$$\forall S \in \mathbb{S}, \mathrm{enc}(\mathrm{dec}(S)) = S$$

Set Family Decision Diagrams
Plunging

We write as index the reference set T for the encoding : enc_T

- Extending the reference set from T to T' ($T \subseteq T'$) does not imply changing the representation:
  $\forall S \subseteq \mathcal{P}(T) \Rightarrow \mathrm{enc}_T(S) = \mathrm{enc}_{T'}(S)$
- Under some constraint we can reduce the reference set $T' \subseteq T$, with or without change,$\forall S \subseteq \mathcal{P}(T)$:
  - case 1: $S \cap (T - T') = \emptyset \Rightarrow$
    $\mathrm{enc}_T(S) = \mathrm{enc}_{T'}(S)$
  - case 2: $S \cap (T - T') \neq \emptyset \Rightarrow$
    $\mathrm{enc}_T(S \cap T') = \mathrm{enc}_{T'}(S \cap T') = \mathrm{enc}_T(S) \ominus (T - T')$

Where $\cup, -$ and $\cap$ are defined as extension of set operation on family of sets, $\ominus$ is defined later on SFDD.

Homomorphisms are operations that preserve union:

$$\phi(S \cup S') = \phi(S) \cup \phi(S')$$

They also support operations that are themselves homomorphisms:

$$\forall S, (\phi_1 + \phi_2)(S) = \phi_1(S) \cup \phi_2(S)$$
$$\forall S, (\phi_1 \times \phi_2)(S) = \phi_1(S) \cap \phi_2(S)$$
$$\forall S, (\phi_1 \circ \phi_2)(S) = \phi_1(\phi_2(S))$$

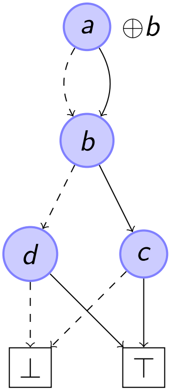
$\oplus : \mathbb{S}, T \to \mathbb{S}$ inserts a term $t \in T$ into all sets of a SFDD:

$$\begin{aligned}
\bot \oplus a &= \bot \\
\top \oplus a &= \langle a, \top, \bot \rangle \\
\langle t, \tau, \sigma \rangle \oplus a &= \begin{cases} \langle t, \tau \oplus a, \sigma \oplus a \rangle & \text{if } t < a \\ \langle t, \tau \cup \sigma, \bot \rangle & \text{if } t = a \\ \langle a, \langle t, \tau, \sigma \rangle, \bot \rangle & \text{if } t > a \end{cases}
\end{aligned}$$

NB: $\oplus$ is an homomorphism.

Set Family Decision Diagrams
Insertion

Example: $\mathrm{enc}(\{\{a,b,c\},\{a,d\},\{b,c\},\{d\}\}) \oplus b$



Set Family Decision Diagrams

Insertion

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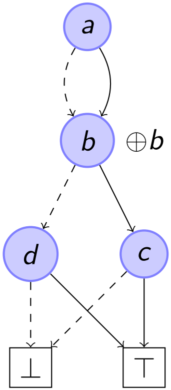

Set Family Decision Diagrams

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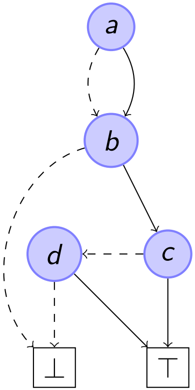


Encodes the sets:

- $\{a,b,c\}$
- $\{a,b,d\}$
- $\{b,c\}$
- $\{b,d\}$

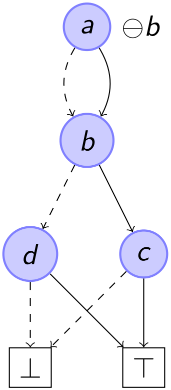
$\ominus : \mathbb{S}, T \to \mathbb{S}$ removes a term $t \in T$ from all sets that contain it:

$$\begin{aligned} \bot \ominus a &= \bot \\ \top \ominus a &= \top \\ \langle t, \tau, \sigma \rangle \ominus a &= \begin{cases} \langle t, \tau \ominus a, \sigma \ominus a \rangle & \text{if } t < a \\ \sigma \cup \tau & \text{if } t = a \\ \langle t, \tau, \sigma \rangle & \text{if } t > a \end{cases} \end{aligned}$$

NB: $\ominus$ is an homomorphism.

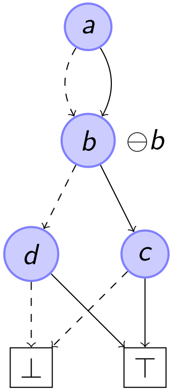
Set Family Decision Diagrams
Removal

Example: $\text{enc}(\{\{a,b,c\},\{a,d\},\{b,c\},\{d\}\}) \ominus b$

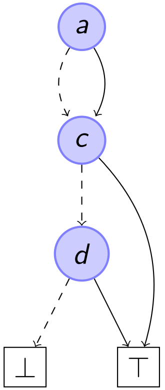


Set Family Decision Diagrams
Removal

Example: $\text{enc}(\{\{a,b,c\},\{a,d\},\{b,c\},\{d\}\}) \ominus b$



Example: enc($\{\{a, b, c\}, \{a, d\}, \{b, c\}, \{d\}\}$) $\ominus$ b



Encodes the sets:

- $\{a, c\}$
- $\{a, d\}$
- $\{c\}$
- $\{d\}$

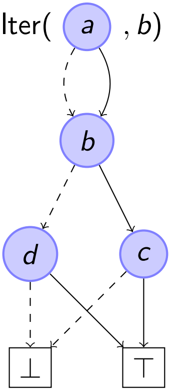
filter : $\mathbb{S}, T \to \mathbb{S}$ filters out the sets that don't contain a term $t \in T$:

$$\text{filter}(\bot, a) = \bot$$
$$\text{filter}(\top, a) = \bot$$
$$\text{filter}(\langle t, \tau, \sigma\rangle, a) = \begin{cases} \langle t, \text{filter}(\tau, a), \text{filter}(\sigma, a)\rangle & \text{if } t < a \\ \langle t, \tau, \bot\rangle & \text{if } t = a \\ \bot & \text{if } t > a \end{cases}$$

NB1: filter is an homomorphism.

Set Family Decision Diagrams
Filtering

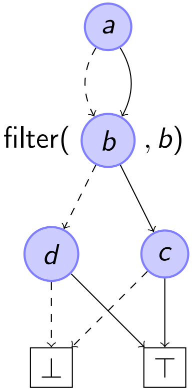
Example: filter(enc($\{\{a, b, c\}, \{a, d\}, \{b, c\}, \{d\}\}$), b)

filter( , b)

Set Family Decision Diagrams

Filtering

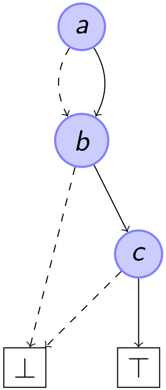
Example: filter(enc($\{\{a, b, c\}, \{a, d\}, \{b, c\}, \{d\}\}$), b)



Set Family Decision Diagrams

Filtering

Example: filter(enc($\{\{a, b, c\}, \{a, d\}, \{b, c\}, \{d\}\}$), b)



Encodes the sets:

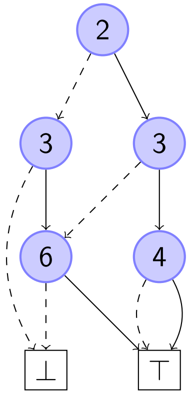
- $\{a, b, c\}$
- $\{b, c\}$

An inductive homomorphism is a tuple $\phi = \langle S, i \rangle$ where:

- $S \in \mathbb{S}$
- $i(A) = \langle \phi_\tau, \phi_\sigma \rangle$ where ϕ_τ, ϕ_σ are homomorphisms and $A \in \mathbb{S} \setminus \{\bot, \top\}$

Let $\phi = \langle S, i \rangle$, its application on $A \in \mathbb{S}$ is given by:

$$\phi(A) = \begin{cases} \bot & \text{if } A = \bot \\ S & \text{if } A = \top \\ \langle t, \phi_\tau(\tau), \phi_\sigma(\sigma) \rangle & \text{if } A = \langle t, \tau, \sigma \rangle, i(A) = \langle \phi_\tau, \phi_\sigma \rangle \end{cases}$$

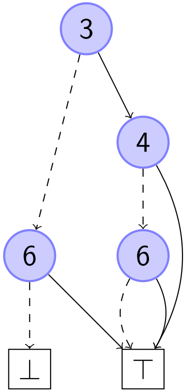


Example: removing values smaller than of 4

$$\phi = \langle \top, i \rangle$$

$$i(\langle t, \tau, \sigma \rangle) = \begin{cases} \langle h[\perp], \phi \circ (h[\tau] + \mathrm{id}) \rangle & \text{if } t < 4 \\ \langle \mathrm{id}, \mathrm{id} \rangle & \text{otherwise} \end{cases}$$

where $\forall S, \mathrm{id}(S) = S$ and $\forall S, h[K](S) = K$.



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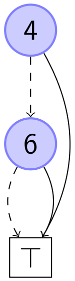
Set Family Decision Diagrams
Inductive Homomorphisms



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where $\forall S, \mathrm{id}(S) = S$ and $\forall S, h[K](S) = K$.

The count of members in a family is the operation size:

$$\mathrm{size}(S) = \begin{cases} 0 & \text{if } S = \bot \\ 1 & \text{if } S = \top \\ \mathrm{size}(\tau) + \mathrm{size}(\sigma) & \text{if } S = \langle t, \tau, \sigma \rangle \end{cases}$$

NB1: size is not an homomorphism.

Set Family Decision Diagrams

Algorithm 1: State space computation on individual states

```
Input: $s_0$ : initial state.
Input: $T$ : set of transition.
Result: set of reachable states
begin
    $s_{rem}, s$ : set of states ;
    $m, mt$ : states ;
    $s_{rem} \leftarrow \{s_0\}$ ; $s \leftarrow \{\}$;
    repeat
        $m \leftarrow choose(s_{rem})$ ;
        $s_{rem} \leftarrow s_{rem}/\{m\}$ ;
        foreach $t \in T$ do
            if $fireable(t,m)$ then
                $mt \leftarrow t(m)$ ;
                if $m \notin s$ then $s \leftarrow s \cup \{mt\}$; $s_{rem} \leftarrow s_{rem} \cup \{mt\}$;
    until $s_{rem} = \emptyset$;
    return $s$;
```

Algorithm 2: Global state space computation

```
Input: $s_0$ : initial state.
Input: $\Phi$ : set of transition homomorphisms.
Result: set of reachable states
begin
    $s, s_{old}, temp$ : set of states ;
    $s \leftarrow \{s_0\}$ ;
    repeat
        $s_{old} \leftarrow s$ ;
        foreach $t \in \Phi$ do
            $temp \leftarrow t(s)$ ;
            $s \leftarrow s \cup temp$ ;
    until $s = s_{old}$;
    return $s$;
```

What about $t(s)$?

$$t(m) = m + post(t) - pre(t)$$

If *pre* and *post* are functions on transition and markings.

$$t(m) = post(t, pre(t, m))$$

Extended to set of states:

$$\begin{aligned} t(s \cup \{m\}) &= t(s) \cup \{post(t, pre(t, m))\} \\ t(\varnothing) &= \varnothing \end{aligned}$$

Petri nets are defined as $\langle P, T, Pre, Post\rangle$ where:

- P and T are finite disjoint sets.
- Pre and $Post$ are functions $P \times T \to \mathbb{N}$

The state of a Petri net is the marking $M : P \to \mathbb{N}$.

A transition $t \in T$ is fireable if and only if

$$\forall p \in P, Pre(p,t) \leq M(p)$$

The firing of a transition modifies the marking (i.e. state):

$$\forall p \in P, M'(p) = M(p) + Post(p,t) - Pre(p,t)$$

Encoding a safe Petri net marking M with S_M can be done with sets using simply P as terms:

$$S_M = \bigcup_{p \in P, M(p)=1} \{p\}$$

which is encoded directly in SFDD as: $enc(\bigcup_{p \in P, M(p)=1} \{p\})$ with the total order $P = \{p_1, p_2, ..., p_k\}$ and $p_1 < p_2 < \cdots < p_k$

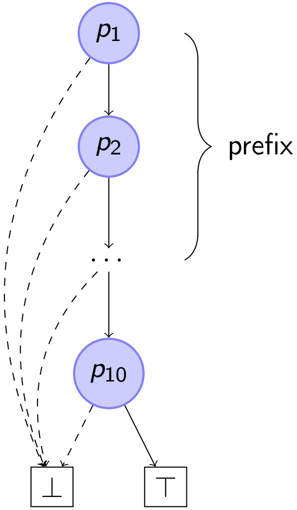
Set Family Decision Diagrams
Encoding safe PN pre and post conditions

$$\begin{aligned}
t &= post(t) \circ pre(t) \\
pre(t) &= pre(t, p_1) \circ pre(t, p_2) \circ \cdots \circ pre(t, p_n) \\
post(t) &= post(t, p_1) \circ post(t, p_2) \circ \cdots \circ post(t, p_1) \\
pre(t, p_i) &= \begin{cases} \ominus(p_i) \circ \text{filter}(p_i) & \text{if } Pre(t, p_i) \neq 0 \\ (id) & \text{otherwise} \end{cases} \\
post(t, p_i) &= \begin{cases} \oplus(p_i) & \text{if } Post(t, p_i) \neq 0 \\ (id) & \text{otherwise} \end{cases}
\end{aligned}$$

Set Family Decision Diagrams
Optimizations

Homomorphisms may involve uncessary operations on large prefixes:

$$\mathrm{filter}(p_{10})(\mathrm{enc}(\bigcup_{1 \leq i \leq 10} p_i))$$



The idea is to dive as deep as possible before applying an homomorphism:

$$\mathrm{dive}(k,\phi)(\bot) = \bot$$
$$\mathrm{dive}(k,\phi)(\top) = \top$$
$$\mathrm{dive}(k,\phi)(\langle t,\tau,\sigma\rangle) = \begin{cases} \langle t, \mathrm{dive}(k,\phi)(\tau), \mathrm{dive}(k,\phi)(\sigma)\rangle & \text{if } t < k \\ \phi(\langle t,\tau,\sigma\rangle) & \text{if } t = k \\ \langle t,\tau,\sigma\rangle & \text{if } t > k \end{cases}$$

Grouping homomorphisms that work on close variables can avoid processing long prefixes multiple times:

$$\mathsf{filter}(p_8) \circ \mathsf{filter}(p_{10}) \equiv \mathrm{dive}(p_8, \mathsf{filter}(p_8) \circ \mathsf{filter}(p_{10}))$$

Some homomorphism may be reordered so they can be grouped:

$$\mathsf{filter}(p_i) \circ \mathsf{filter}(p_j) \equiv \mathsf{filter}(p_j) \circ \mathsf{filter}(p_i)$$

As in BDD we need to proceed as:

- encoding the Kripke structure
- Define homomorphisms Pre and Post on states encoded using $post(t) \circ pre(t)$
- Define homomorphism $PreE(S)$ of predecessors
- Fixpoint computations using CTL model checking algorithms

Kripke Structure
definition

Definition

A Kripke structure of a set of atomic propositions AP is a tuple $K = \langle S, S_0, R, L \rangle$ where:

- **S** is a finite set of states

Kripke Structure
definition

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- $\mathbf{S_0} \subseteq \mathbf{S}$ is a non-empty set of initial states

Kripke Structure
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- $\mathbf{S}$ is a finite set of states
- $\mathbf{S_0} \subseteq \mathbf{S}$ is a non-empty set of initial states
- $\mathbf{R} \subseteq \mathbf{S} \times \mathbf{S}$ is a left-total binary relation on S representing the transitions

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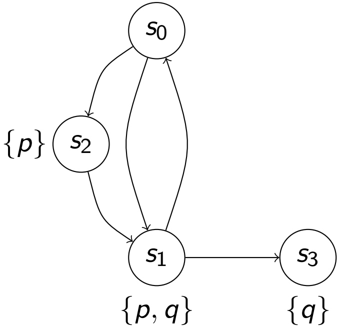
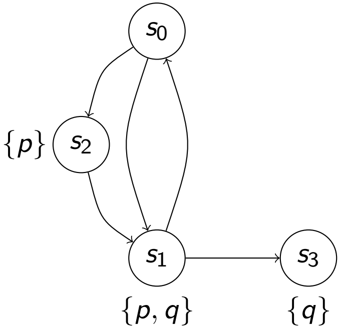
- $\mathbf{S_0} \subseteq \mathbf{S}$ is a non-empty set of initial states
- $\mathbf{R} \subseteq \mathbf{S} \times \mathbf{S}$ is a left-total binary relation on S representing the transitions
- $\mathbf{L} : \mathbf{S} \rightarrow \mathcal{P}(\mathbf{AP})$ labels each state with a set of atomic propositions that hold on that state

Kripke Structure
example



$K = \langle S, S_0, R, L \rangle$ where:

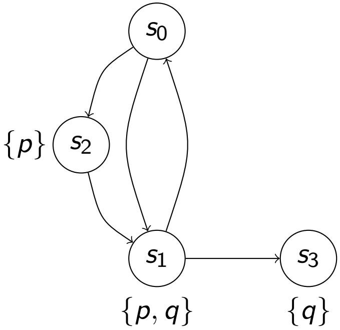
Kripke Structure
example



$K = \langle S, S_0, R, L \rangle$ where:

- $S = \{s_0, s_1, s_2, s_3\}$

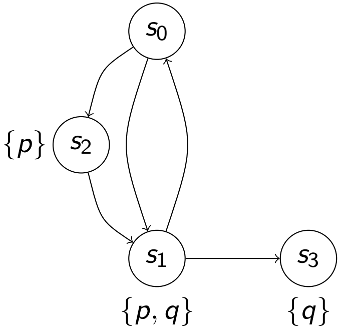
Kripke Structure
example



$K = \langle S, S_0, R, L \rangle$ where:

- $S = \{s_0, s_1, s_2, s_3\}$
- $S_0 = \{s_0\}$

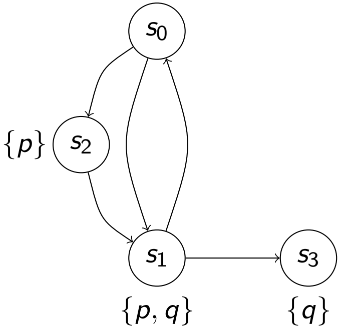
Kripke Structure
example



$K = \langle S, S_0, R, L \rangle$ where:

- $S = \{s_0, s_1, s_2, s_3\}$
- $S_0 = \{s_0\}$
- $R = \{(s_0, s_1), (s_1, s_0), (s_0, s_2), (s_2, s_1), (s_1, s_3)\}$

Kripke Structure
example



$K = \langle S, S_0, R, L\rangle$ where:

- $S = \{s_0, s_1, s_2, s_3\}$
- $S_0 = \{s_0\}$
- $R = \{(s_0, s_1), (s_1, s_0), (s_0, s_2), (s_2, s_1), (s_1, s_3)\}$
- $L(s_0) = \emptyset, L(s_1) = \{p, q\}, L(s_2) = \{p\}, L(s_3) = \{q\}$

From Kripke Structure to SFDD

- Given AP, we create a sibling set AP' different from AP: $AP \cap AP' = \emptyset$ and a bijective function $sib : AP \rightarrow AP'$
- We create also an order on $AP \cup AP'$ $<'$ from the order $<$ such as $\forall s_a$ and $s_b \in AP$:
  - $s_a < s_b \Rightarrow s_a <' sib(s_a) <' s_b$
  - $sib(s_a) <' sib(s_b) \Rightarrow sib(s_a) <' s_b <' sib(s_b)$

We can prove that $\forall s_a$ and $s_b \in AP$:
$s_a < s_b \Leftrightarrow s_a <' s_b \Leftrightarrow sib(s_a) <' sib(s_b)$

Example

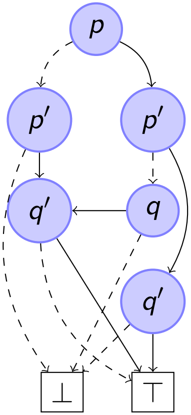
$AP = \{p, q\}$, $AP' = \{p', q'\}$ and $sib(p) = p'$ and $sib(q) = q'$.[a]
We also have the orders:
$p < q$ and $p <' p' <' q <' q'$

[a] sib is naturally extended to $sib : \mathcal{P}(AP) \rightarrow \mathcal{P}(AP')$ and $sib^{-1} : AP' \rightarrow AP$

From Kripke Structure to SFDD(2)

Given $K = \langle S, S_0, R, L\rangle$ the SFDD that we will build is:

$$G_K = \bigcup_{(s_a,s_b)\in R} enc_{AP\cup AP'}(\{L(s_a) \cup sib(L(s_b))\})$$



$S = \{\{p', q'\}, \{p, q\}, \{p'\}, \{p, p', q'\}, \{p, q, q'\}\}$

Shannon decomposition on sets

Let's do Shannon decomposition

$$S = \{\{\overline{p}, \overline{q}, p', q'\}, \{p, q, \overline{p'}, \overline{q'}\}, \{\overline{p}, \overline{q}, p', \overline{q'}\}, \{p, p', \overline{q}, q'\}, \{p, \overline{p'}, q, q'\}\}$$

$$S = \{\{\overline{p}, \overline{q}, p', q'\}, \{\overline{p}, \overline{q}, p', \overline{q'}\}\} \cup \{\{p, q, \overline{p'}, \overline{q'}\}, \{p, p', \overline{q}, q'\}, \{p, \overline{p'}, q, q'\}\}$$

$$S = \{\overline{p}\} \otimes \{\{\overline{q}, p', q'\}, \{\overline{q}, p', \overline{q'}\}\} \cup \{p\} \otimes \{\{q, \overline{p'}, \overline{q'}\}, \{p', \overline{q}, q'\}, \{\overline{p'}, q, q'\}\}$$

$$S = \{\overline{p}\} \otimes \{p'\} \otimes \{\{\overline{q}, q'\}, \{\overline{q}, \overline{q'}\}\} \cup \{p\} \otimes (\{\overline{p'}\} \otimes \{\{q, \overline{q'}\}, \{q, q'\}\} \cup \{\{p', \overline{q}, q'\}\})$$

$$S = \{\overline{p}\} \otimes \{p'\} \otimes \{\overline{q}\} \otimes \{\{q'\}, \{\overline{q'}\}\} \cup \{p\} \otimes (\{\overline{p'}\} \otimes \{q\} \otimes \{\{\overline{q'}\}, \{q'\}\} \cup \{\{p', \overline{q}, q'\}\})$$

CTL

- Only need algorithms for EX, EU, EG since:
  - $AX\phi \iff \neg EX(\neg\phi)$
  - $AF\phi \iff \neg EG(\neg\phi)$
  - $AG\phi \iff \neg EF(\neg\phi)$
  - $EF\phi \iff E[true \cup \phi]$
  - $A[\phi \cup \theta] \iff \neg E[\neg\theta \cup (\neg\phi \wedge \neg\theta)] \wedge \neg EG(\neg\theta)$

$EX(\phi)$

- Let F be the set of states ($\in$ *SFDD*) satisfying ϕ:
    - S := *PreE*(F)
    - Return S

Function PreE

$$reduce_T(H, T') = H \ominus (T - T')$$

$$PreE(F) = reduce_{AP \cup AP'}(G_K \cap (enc(\mathcal{P}(AP)) \otimes sib(F)), AP)$$

$E(\phi Until \theta)$

- Let F(resp. G) be the set of states ($\in SFDD$) satisfying ϕ (resp. θ):

```
S := G
N := enc($\emptyset$)
While( N $\neq$ S)
do
    N := S;
    S := $S \cup (F \cap PreE($S$))$
done
Return S
```

$EG(\phi)$

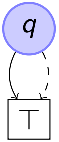
- Let F be the set of states ($\in SFDD$) satisfying ϕ:

```
S := F
N := enc($\emptyset$)
While( N $\neq$ S)
do
    N := S;
    S := S $\cap$ $preE$(S)
done
Return S
```

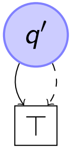
$EX(\neg p)$

- Let's compute the set of states that satisfy $EX(\neg p)$:

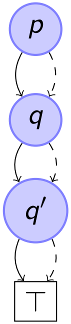
The states satisfying $\neg p$ are: s_0, s_3 which are the states where the $\{\emptyset, \{q\}\}$ atomic propositions are valid



transformed by: sib

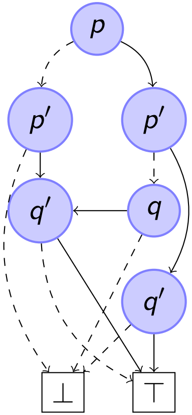


and extend by: $enc\mathcal{P}(AP)$

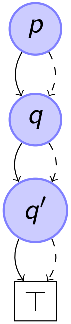


$EX(\neg p)$

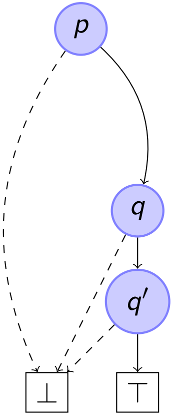
- Let's compute the intersection:



$\cap$

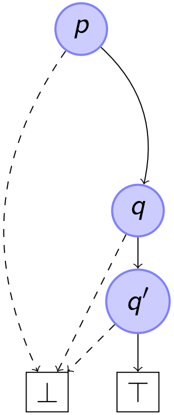


$=$

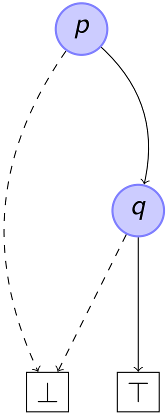


$EX(\neg p)$

- Let's compute the reduction to AP:



reduced into



which means that $\{p, q\}$ is the only state s_1 satisfying $EX(\neg p)$.

- SFDD encoding of sets
- SFDD properties such as canonization
- Homomorphic operations on SFDD
- Inductive homomorphisms as pattern of computation
- Encoding of PN markings and set of markings for safe nets
- Encoding of PN fire functions
- Computation of PN state space
- Computation of CTL Formulae